Trigonometric derivatives

There are six trigonometric functions shown in the first column of the chart below. Each one of them has its own unique derivative, which is shown in the second column.

It's important to remember that we always have to apply chain rule whenever we take the derivative of a trigonometric function. Remember that chain rule tells us to take the derivative of the "outside" function first, leaving the inside function completely untouched, and then multiply our result by the derivative of the inside function.

If we have any of the most basic trigonometric functions (those listed in the first column of the table below), then the inside function is just x, and since the derivative of x is 1, we take the derivative of these functions and we get

$$y = \sin x \quad \to \quad y' = \cos x \cdot (1) \quad \to \quad y' = \cos x$$

$$y = \cos x \quad \to \quad y' = -\sin x \cdot (1) \quad \to \quad y' = -\sin x$$

$$y = \tan x \quad \to \quad y' = \sec^2 x \cdot (1) \quad \to \quad y' = \sec^2 x$$

$$y = \cot x \quad \to \quad y' = -\csc^2 x \cdot (1) \quad \to \quad y' = -\csc^2 x$$

$$y = \sec x \quad \to \quad y' = \sec x \tan x \cdot (1) \quad \to \quad y' = \sec x \tan x$$

$$y = \csc x \quad \to \quad y' = -\csc x \cot x \cdot (1) \quad \to \quad y' = -\csc x \cot x$$

And so we can see that the chain rule becomes somewhat irrelevant with these basic trigonometric functions, because we just end up multiplying by 1, which doesn't change the function. But what happens when we have a value other than x inside the trigonometric function?

If we have a function $g(x)$ inside our trigonometric function that's more than just x, we can think about the trigonometric functions in the third column of the chart below. Then according to chain rule, the derivatives of the third column appear in the fourth column, and we notice that they're exactly the same as the derivatives we found in the chart above, except that we're multiplying by $g'(x)$, which will be some value other than 1 if $g(x)$ is more than just x.

$$y = \sin x \qquad y' = \cos x \qquad y = \sin[g(x)] \qquad y' = \cos[g(x)] \cdot [g'(x)]$$

$$y = \cos x \qquad y' = -\sin x \qquad y = \cos[g(x)] \qquad y' = -\sin[g(x)] \cdot [g'(x)]$$

$$y = \tan x \qquad y' = \sec^2 x \qquad y = \tan[g(x)] \qquad y' = \sec^2[g(x)] \cdot [g'(x)]$$

$$y = \cot x \qquad y' = -\csc^2 x \qquad y = \cot[g(x)] \qquad y' = -\csc^2[g(x)] \cdot [g'(x)]$$

$$y = \sec x \qquad y' = \sec x \tan x \qquad y = \sec[g(x)] \qquad y' = \sec[g(x)]\tan[g(x)] \cdot [g'(x)]$$

$$y = \csc x \qquad y' = -\csc x \cot x \qquad y = \csc[g(x)] \qquad y' = -\csc[g(x)]\cot[g(x)] \cdot [g'(x)]$$

Example

Find the derivative.

$$y = 4\cos 2x$$

Remember, according to chain rule, we take the derivative of the outside function first (cosine, in this case), and then we multiply by the derivative of the inside function ($2x$, in this case).

$$y' = 4(-\sin 2x)(2)$$

$$y' = -8\sin 2x$$

Let's try an example with a trigonometric function occurring as part of a larger function.

Example

Find the derivative.

$$y = 8x^5 - 9\cot 7x^4$$

Dealing with one term at a time, and remembering to use chain rule to handle the derivative of $-9\cot 7x^4$, we get

$$y' = 40x^4 - 9(-\csc^2 7x^4)(28x^3)$$

$$y' = 40x^4 + 252x^3 \csc^2 7x^4$$

$$y' = 4x^3(10x + 63\csc^2 7x^4)$$

Let's try a more complex example.

Example

Find the derivative.

$$y = \sec 7x^3 - 7x^5 \sin x + 3\csc 5x^7$$

Again, we'll take the derivative one term at a time, focusing first on $\sec 7x^3$, then on $-7x^5 \sin x$, and then on $3\csc 5x^7$. We have to remember to use chain rule for each of these and multiply the derivative of the outside function by the derivative of the inside function. Additionally, we'll need product rule to find the derivative of $-7x^5 \sin x$.

The derivative is

$$y' = \left(\sec 7x^3 \tan 7x^3\right)\left(21x^2\right) - \left[\left(35x^4\right)\left(\sin x\right) + \left(7x^5\right)\left(\cos x\right)\right] + 3\left(-\csc 5x^7 \cot 5x^7\right)\left(35x^6\right)$$

$$y' = 21x^2 \sec 7x^3 \tan 7x^3 - \left(35x^4 \sin x + 7x^5 \cos x\right) - 105x^6 \csc 5x^7 \cot 5x^7$$

$$y' = 21x^2 \sec 7x^3 \tan 7x^3 - 35x^4 \sin x - 7x^5 \cos x - 105x^6 \csc 5x^7 \cot 5x^7$$

$$y' = 7x^2\left(3\sec 7x^3 \tan 7x^3 - 5x^2 \sin x - x^3 \cos x - 15x^4 \csc 5x^7 \cot 5x^7\right)$$