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EQUILIBRIUM IN A REINSURANCE MARKET

BY KARL BORCH

This paper investigates the possibility of generalizing the classical theory of commodity markets to include uncertainty. It is shown that if uncertainty is considered as a commodity, it is possible to define a meaningful price concept, and to determine a price which makes supply equal to demand. However, if each participant seeks to maximize his utility, taking this price as given, the market will not in general reach a Pareto optimal state. If the market shall reach a Pareto optimal state, there must be negotiations between the participants, and it seems that the problem can best be analysed as an n-person cooperative game.

The paper is written in the terminology of reinsurance markets. The theoretical model studied should be applicable also to stock exchanges and other markets where the participants seek to reach an optimal distribution of risk.

1. INTRODUCTION

1.1. THE WALRAS-CASSEL system of equations which determines a static equilibrium in a competitive economy is certainly one of the most beautiful constructions in mathematical economics. The mathematical rigour which was lacking when the system was first presented has since been provided by Wald [**10**] and Arrow and Debreu [**4**]. For more than a generation one of the favourite occupations of economists has been to generalize the system to dynamic economies. The mere volume of the literature dealing with this subject gives ample evidence of its popularity.

1.2. The present paper investigates the possibilities of generalizing the Walras-Cassel model in another direction. The model as presented by its authors assumes complete certainty, in the sense that all consumers and producers know exactly what will be the outcome of their actions. It will obviously be of interest to extend the model to markets where decisions are made under uncertainty as to what the outcome will be. This problem seems to have been studied systematically only by Allais [**1**] and Arrow [**3**] and to some extent by Debreu [**7**] who includes uncertainty in the last chapter of his recent book. It is surprising that a problem of such obvious and fundamental importance to economic theory has not received more attention. Allais ascribes this neglect of the subject to *son extrême difficulté*.

1.3. The subject does not appear inherently difficult, however, at least not when presented in Allais' elegant manner. What seems to be forbiddingly difficult is to extend his relatively simple model to situations in the real world where uncertainty and attitude toward risk play a decisive part, for instance in the determination of interest rates, share prices, and supply and demand for risk capital. Debreu's abstract treatment also seems very remote from such familiar problems. There are further difficulties of which Allais, particularly, seems acutely aware, such as the psychological problems connected with the elusive concepts of "subjective probabilities" and "rational behaviour." In the present paper we shall put these latter difficulties aside. It then appears fairly simple to construct a model of a competitive market which seems reasonably close to the situations in real life where rational beings exchange risk and cash among themselves. The problem still remains difficult, but it seems that the difficulty is the familiar one of laying down assumptions which lead to a determinate solution of an n-person game.

1.4. The reason why neither Allais nor Arrow has followed up his preliminary study of the problem is probably that their relatively simple models appear too remote from any really interesting practical economic situation. However, the model they consider gives a fairly accurate description of a *reinsurance market*. The participants in this market are insurance companies, and the commodity they trade is risk. The purpose of the deals which the companies make in this market is to redistribute the risks which each company has accepted by its direct underwriting for the public. The companies which gain from this redistribution of risks are ready to pay compensation in cash to the other companies. This is a real life example of just the situation which Allais and Arrow have studied in rather artificial models.

It seems indeed that the reinsurance market offers promising possibilities of studying how attitudes toward risk influence decision making and the interaction between the decisions made by the various participants. This problem has so far been studied mainly in the theory of investment and capital markets where one must expect that a large number of "disturbing factors" are at play. It is really surprising that economists have overlooked the fact that the problem can be studied, almost under laboratory conditions, in the reinsurance market.

2. A MODEL OF THE REINSURANCE MARKET

2.1. Consider n insurance companies, each holding a portfolio of insurance contracts.

The *risk situation* of company i ($i = 1, 2, \ldots, n$) is defined by the following two elements: (i) The *risk distribution*, $F_i(x_i)$, which is the probability that the total amount of claims to be paid under the contracts in the company's portfolio shall not exceed x_1. (ii) The *funds*, S_i, which the company has available to pay claims.

We shall assume that $x_1, \ldots, x_n$ are stochastically independent. To this risk situation the company attaches a utility $U_i(S_i, F_i(x_i))$. From the so-called "Bernoulli hypothesis" it follows that

$$U_i(S_i, F_i(x_i)) = \int_0^\infty u_i(S_i - x_i)\, dF_i(x_i) .$$

Here $u_i(S) = U_i(S, \varepsilon(x))$, where $\varepsilon(x)$ is the degenerate probability distribution defined by

$$\begin{aligned} \varepsilon(x) &= 0 \qquad \text{for } x < 0 , \\ \varepsilon(x) &= 1 \qquad \text{for } 0 \leqslant x . \end{aligned}$$

Hence $u_i(S)$ is the utility attached to a risk situation with funds S and probability of 1 that claims shall be zero. In the following we shall refer to the function $u_i(S)$ as the "utility of money to company i." We shall assume that $u_i(S)$ is continuous and that its first derivative is positive and decreases with increasing S.

2.2. Von Neumann and Morgenstern [**9**] proved the Bernoulli hypothesis as a theorem, derived from a few simple axioms. Since then there has been considerable controversy over the plausibility of the various formulations which can be given to these axioms. There is no need to take up this question here, since it is almost trivial that the Bernoulli hypothesis must hold for a company in the insurance business.

2.3. In the initial situation company i is committed to pay x_i, the total amount of claims which occur in its own portfolio. The commitments of company i do not depend on the claims which occur in the portfolios of the other companies. In the reinsurance market the companies can conclude agreements, usually referred to as *treaties* which redistribute the commitments that the companies had in the initial situation.

In general these treaties can be represented by a set of functions:

$$y_i(x_1, x_2, \ldots, x_n) \qquad (i = 1, 2, \ldots, n)$$

where $y_i(x_1, x_2, \ldots, x_n)$ is the amount company i has to pay if claims in the respective portfolios amount to $x_1, x_2, \ldots, x_n$. Since all claims have to be paid, we must obviously have

$$\sum_{i=1}^{n} y_i(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i .$$

These treaties will change the utility of company i from

$$U_i(x) = \int_0^\infty u_i(S_i - x_i)\, dF_i(x_i)$$

to

$$U_i(y) = \int_R u_i(S_i - y_i(x))\, dF(x) ,$$

where $F(x)$ is the joint probability distribution of $x_1, \ldots, x_n$, and where R stands for the positive orthant in the n-dimensional x-space.

For simplicity we have written x and y respectively for the vectors $\{x_1, \ldots, x_n\}$ and $\{y_1(x), \ldots, y_n(x)\}$

2.4. If the companies act rationally, they will not conclude a set of treaties represented by a vector y if there exists another set of treaties with a corresponding vector $\bar{y}$, such that

$$U_i(y) \leqslant U_i(\bar{y}) \qquad \text{for all } i,$$

with at least one strict inequality. y will in this case clearly be inferior to $\bar{y}$. If there exists no vector $\bar{y}$ satisfying the above condition, the set of treaties represented by y will be referred to as *Pareto optimal.* If the companies act rationally, the treaties they conclude must obviously constitute a Pareto optimal set.

2.5. It has been proved in a previous paper [**6**] that a necessary and sufficient condition that a vector y is Pareto optimal is that its elements, the functions $y_1(x), \ldots, y_n(x)$ satisfy the relations:

$$u_i'(S_i - y_i(x)) = k_i u_1'(S_1 - y_1(x)), \tag{1}$$

$$\sum_{i=1}^{n} y_i(x) = \sum_{i=1}^{n} x_i, \tag{2}$$

where $k_2, k_3, \ldots, k_n$ are positive constants which can be chosen arbitrarily.

The proof is elementary. It will not be repeated here since a rigorous statement is lengthy and rather tedious. Heuristically it is almost self-evident that if the condition is fulfilled, a change in y cannot increase the utility of all the companies, i.e., that the condition is sufficient. The proof that it is necessary is slightly less transparent.

2.6. Differentiation of the equations in the preceding paragraph with respect to x_j gives

$$u_i''(S_i - y_i(x)) \frac{\partial y_i}{\partial x_j} = k_i u_1''(S_1 - y_1(x)) \frac{\partial y_1}{\partial x_j}$$

and

$$\sum_{i=1}^{n} \frac{\partial y_i}{\partial x_j} = 1 .$$

Dividing the first equation by $u_i''(S_i - y_i(x))$ and summing over all i, we obtain

$$u_1''(S_1 - y_1(x)) \frac{\partial y_1}{\partial x_j} \sum_{i=1}^{n} \frac{k_i}{u_i''(S_i - y_i(x))} = 1$$

where $k_1 = 1$.

It then follows that for any i and j we must have

$$\frac{\partial y_1}{\partial x_i} = \frac{\partial y_1}{\partial x_j}.$$

This implies that the vector function $y_1(x)$ is a scalar function of one single variable

$$z = \sum_{i=1}^{n} x_i .$$

It is easy to verify that in general we have

$$\frac{dy_i(z)}{dz} = \frac{\dfrac{k_i}{u_i''(S_i - y_i(z))}}{\sum_{j=1}^{n} \dfrac{k_j}{u_j''(S_j - y_j(z))}} .$$

This means that the amount $y_i(z)$ which company i has to pay will depend only on $z = x_1 + \ldots + x_n$, i.e., on the total amount of claims made against the insurance industry. Hence any Pareto optimal set of treaties is equivalent to a pool arrangement, i.e., all companies hand their portfolios over to a pool, and agree on some rule as to how payment of claims against the pool shall be divided among the companies. In general there will be an infinity of such rules, since the $n - 1$ positive constants $k_2, k_3, \ldots, k_n$ can be chosen arbitrarily. In general the utility of company i will decrease with increasing $k_i (i \neq 1)$. Since the company will not be party to a set of treaties unless $U_i(y) \geqslant U_i(x)$ there must be an upper limit to k_i. We shall return to this question in Section 4.

2.7. The results reached in the preceding paragraphs correspond very well to what one could expect on more intuitive grounds. If all companies are averse to risk, it was to be expected that the best arrangement would be to spread the risks as widely as possible. It was also to be expected that the solution should be indeterminate, since no assumptions were made as to how the companies should divide the gain resulting from the greater spread of risks.

In the Walras-Cassel model there is a determinate equilibrium, i.e., unique Pareto optimal distribution of the goods in the market. The basic assumption required to reach this result is that each participant considers the market price as given, and then buys or sells quantities of the various goods so that his utility is maximized. In the following section we shall investigate the possibility of finding some equally simple assumptions which will bring a reinsurance market into an equilibrium.

3. THE PRICE CONCEPT IN A REINSURANCE MARKET

3.1. In insurance circles it is generally assumed that there exists a well defined market price, at least for some particular forms of reinsurance. It is also generally believed that Lloyd's in London is willing to quote a price for any kind of reinsurance cover.

If a market price exists, it must mean that it is possible to associate a number $P(F)$ to any probability distribution $F(x)$, so that an insurance company can receive the amount $P(F)$ from the market by undertaking to pay the claims which occur in a portfolio with risk distribution $F(x)$. It must also be possible for the company to be relieved of the responsibility for paying such claims by paying the amount $P(F)$ to the market.

3.2. Assume now that a company accepts responsibility for two portfolios with risk distributions $F_1(x_1)$ and $F_2(x_2)$. Assume further that x_1 and x_2 are stochastically independent and that $x = x_1 + x_2$ has the probability distribution $F(x)$. It is natural to require that the company shall receive the same amount whether it accepts the two portfolios separately or in one single transaction. This means that we must have

$$P(F) = P(F_1) + P(F_2) .$$

This additivity condition is clearly a parallel to the assumption in the classical model that the price per unit is independent of the number of units included in a transaction.

3.3. The additivity condition is obviously satisfied by a number of functionals. It is for instance satisfied by the cumulant generating function

$$\psi(t) = \log \varphi(t)$$

where $\varphi(t)$ is the characteristic function

$$\varphi(t) = \int_0^\infty e^{itx} dF(x) .$$

As it is inconvenient to work with a complex valued function, we shall in the following use the corresponding real functions

$$\varphi(t) = \int_0^\infty e^{-tx} dF(x)$$

and

$$\psi(t) = \log \varphi(t)$$

which exist for any nonnegative value of t. The cumulants are then given by the expansion

$$\psi(t) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\kappa_n}{n!} t^n .$$

3.4. It follows that for any nonnegative value of t, $\psi(t)$ can be interpreted as a price which satisfies the additivity condition. The same will hold for any linear combination of the form:

$$c_1 \psi(t_1) + c_2 \psi(t_2) + \ldots$$

where $c_1, c_2, \ldots$ are constants. Similar expressions containing derivatives of $\psi(t)$ of any order will also satisfy the condition.

It is obvious that any expression of this kind can be written as a sum of cumulants. Hence we can write

$$P(F) = \sum_{n=1}^{\infty} p_n \kappa_n$$

where $p_1, \ldots, p_n$ are constants.

It follows from a theorem by Lukacs [8] that this is the most general expression which satisfies the additivity condition.

3.5. Let now $\varepsilon(x)$ be the degenerate probability distribution defined in paragraph 2.1.

$\varepsilon(x - m)$ can then be interpreted as a risk distribution according to which the amount m will be claimed with probability 1. The price associated with this distribution will be

$$P(\varepsilon(x - m)) = p_1 m$$

since $\kappa_n = 0$ for $1 < n$. We shall therefore require as a continuity condition that $p_1 = 1$.

3.6. We now assume that a market price of this form is given, and we consider a company in the risk situation $(S, F(x))$. The utility of the company in this situation is

$$U(S, F(x)) = \int_0^{\infty} u(S - x)\, dF(x) .$$

If the company undertakes to pay a claim y with probability distribution $G(y)$, it will receive an amount $P(G)$. If x and y are stochastically independent this transaction will change the company's utility to

$$\begin{aligned} U(S + P(G), H(x)) &= \int_0^{\infty} u(S + P(G) - x)\, d\left\{\int_0^{\infty} F(x - y)\, dG(y)\right\} \\ &= \int_0^{\infty} u(S + P(G) - x)\, dH(x) \end{aligned}$$

where $H(x)$ is the convolution of $F(x)$ and $G(y)$.

If the company acts rationally, it will select among the portfolios available in the market one with a risk distribution $G_0(y)$ which maximizes $U(S + P(G), H(x))$. This function $G_0(y)$ can be considered as the amount of reinsurance cover which the company will *supply* at the given price.

3.7. The nature of the maximization problem appears more clearly if we introduce the cumulants explicitly in the formula of the preceding paragraph.

Let $f(t)$ and $g(t)$ be the characteristic functions of $F(x)$ and $G(y)$ respectively. The characteristic function of $H(x)$ is then $f(t)\, g(t)$, and if $H(x)$ has a derivative, we have

$$\frac{dH(x)}{dx} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx} f(t) g(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx} e^{\log f(t) + \log g(t)} dt$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left\{-itx + \sum_{n=1}^{\infty} \frac{(it)^n}{n!} (k_n + \kappa_n)\right\} dt$$

where k_n and κ_n are the nth cumulants of $F(x)$ and $G(y)$, respectively. Hence the problem becomes that of determining the values of $\kappa_1, \kappa_2, \ldots, \kappa_n$ which maximize the expression

$$\int_0^{\infty} u\left(S - x + \sum_{n=1}^{\infty} p_n \kappa_n\right) \int_{-\infty}^{+\infty} \exp\left\{-itx + \sum_{n=1}^{\infty} \frac{(it)^n}{n!} (k_n + \kappa_n)\right\} dt dx .$$

It is interesting to note that the cumulants of different order appear as different commodities, each with its particular price. The "quantities" $\kappa_1, \ldots, \kappa_n$, however, must satisfy certain restraints in order to be the cumulants of a probability distribution. These restraints will be of a complicated nature. A sufficient set of restraints can be derived from the Liapounoff inequalities

$$\frac{1}{n} \log m_n \leqslant \frac{1}{n+1} \log m_{n+1}$$

where m_n is the nth absolute moment about an arbitrary point. Since $G(y) = 0$ for $y < 0$, the inequalities must hold for the moments about zero of $G(y)$. It is easy to see that the sign of equality will hold only in the degenerate case when $G(y) = \varepsilon(y - m)$.

The problem on the *supply* side of a reinsurance market thus appears to be similar to the problems of maximization under restraints which occur in some production models. It is clear that the problem will have a solution, at least under certain conditions.

3.8. The problems on the *demand* side are more complicated. Assume that with a given price a company demands reinsurance cover corresponding to a probability distribution $G(y)$. This means that in order to be relieved of an obligation to pay a claim with a probability distribution $G(y)$, the company is willing to pay an amount

$$P(G(y)) = \sum_{n=1}^{\infty} p_n \kappa_n$$

where $\kappa_1, \ldots, \kappa_n, \ldots$ are the cumulants of $G(y)$.

Assume now that the company can buy its reinsurance cover in two transactions, for instance by placing two portfolios with risk distribution $G(\frac{1}{2}y)$ with two different reinsurers. If the market price is applied to both transactions, the company will have to pay

$$2P(G(\tfrac{1}{2}y)) = \sum_{n=1}^{\infty} \frac{p_n}{2^{n-1}} \kappa_n$$

for the reinsurance cover. $2P(G(\frac{1}{2}y))$ will generally be different from $P(G(y))$. Hence the reinsurance arrangement which maximizes the company's utility will depend not only on the given price, but also on the number of reinsurers who are willing to deal at this price. This makes it doubtful if any meaning can be given to the term "market price" in a reinsurance market. We shall not at present discuss this problem in further detail. We shall, however, consider it again for a special case in Section 4.

4. EXISTENCE OF AN EQUILIBRIUM PRICE

4.1. In the preceding section we studied separately the demand and supply of reinsurance cover. It is fairly obvious, however, that if the companies shall reach the Pareto optimum which we found in Section 2.5, each company must act *both* as seller and buyer of reinsurance cover. In a previous paper [**5**] it was proved by a more direct approach that it will in general be to the advantage of a company to act in both capacities at the same time.

In this section we shall study whether a price mechanism can bring supply and demand into an equilibrium which also represents a Pareto optimal distribution of the risks.

4.2. Since the problem is rather complex, we shall analyse only a special case. We assume that the utility of money to all companies can be represented by a function of the form:

$$u_i(x) = -a_i x^2 + x\,, \qquad \text{for } i = 1, 2, \ldots, n.$$

We assume that a_i is positive and so small that $u_i(x)$ is an increasing function over the whole range which enters into consideration.

a_i can evidently be interpreted as a measure of the company's "risk aversion." If $a_i = 0$, the company will be indifferent to risk. Its sole objective will then be to maximize expected profits, ignoring all risk of deviations from the expected value. The greater a_i is, the more concerned will the company be about the possibility of suffering great losses.

From the formulae in section 2.6 we find

$$\frac{dy_i(z)}{dz} = \frac{k_i/a_i}{\Sigma\, k_j/a_j} = q_i\,.$$

Hence the optimum arrangement is that company i shall pay a fixed quota q_i of the amount of claims z made against the pool. It is easily verified that

$$y_i(z) = q_i z + q_i \sum_{j=1}^{n} \left(\frac{1}{2a_j} - S_j\right) - \left(\frac{1}{2a_i} - S_i\right) = q_i z + q_i \sum_{j=1}^{n} A_j - A_i\,.$$

For $z = 0$ we find

$$y_i(0) = q_i \sum_{j=1}^{n} A_j - A_i\,.$$

$y_i(0)$ is the amount (positive or negative) that company i has to pay if there are no claims. Hence $y_i(0)$ must be the difference between the amount the company pays for the reinsurance cover it buys and the amount the company receives for the reinsurance cover it sells.

4.3. If $u(x) = -ax^2 + x$, the utility of the company in the initial situation is

$$\begin{aligned} U(0) &= \int_0^\infty u(S-x)\,dF(x) = \int_0^\infty \{-a(S-x)^2 + (S-x)\}\,dF(x) \\ &= -a(S-\kappa_1)^2 + (S-\kappa_1) - a\kappa_2 \end{aligned}$$

where κ_1 and κ_2 are the two first cumulants, i.e., the mean and the variance of $F(x)$. We see that in this case the utility which the company attaches to a risk situation will depend only on the two first cumulants of the risk distribution. If the utility function $u(x)$ is of the form $-ax^2 + x$ for all companies, the cumulants of higher order can have no effect of the optimal arrangement. They will appear as "free goods" in the market, i.e., with price zero. Hence, in the expression for price we must have $p_n = 0$ for all $n > 2$. The amount paid for reinsurance cover of a risk distribution $F(x)$ will then be

$$P(F) = \kappa_1 + p_2\kappa_2 = m + pV$$

if we drop the index of p_2, and write m and V for the mean and variance of $F(x)$, respectively.

4.4. We now consider two companies, i and j, with risk distributions $F_i(x_i)$ and $F_j(x_j)$ where x_i and x_j are stochastically independent. In a Pareto optimal set of reinsurance treaties the two companies will have to pay fixed quotas, q_i and q_j, of the claims made against the pool $z = \sum_{j=1}^{n} x_j$.

It is evident that a Pareto optimal arrangement will result if every pair of companies concludes a reciprocal treaty, according to which company i undertakes to pay $q_i x_j$ if claims against company j amount to x_j, and company j in return pays $q_j x_i$ if claims x_i are made against company i (i.e., q_i is the same for every j).

If m_i and V_i are the mean and variance of $F_i(x_i)$, company i will receive an amount $q_i m_j + p q_i^2 V_j$ for the reinsurance cover it gives company j. Similarly company i will have to pay out $q_j m_i + p q_j^2 V_i$ for the cover it receives from company j.

Hence the net payment from company i to company j will be

$$q_j m_i + p q_j^2 V_i - q_i m_j - p q_i^2 V_j .$$

Summing this for all $j \neq i$, we obtain

$$m_i \sum q_j - q_i \sum m_j + p\{V_i \sum q_j^2 - q_i^2 \sum V_j\}$$

which is equal to

$$m_i - q_i \sum_{j=1}^{n} m_j + p\left\{V_i \sum_{j=1}^{n} q_j^2 - q_i^2 \sum_{j=1}^{n} V_j\right\} .$$

This expression, however, must be equal to the total net payment of company i, which according to Section 4.2 is

$$y_i(0) = q_i \sum_{j=1}^{n} A_j - A_i .$$

Hence we must have

$$p\left\{V_i \sum_{j=1}^{n} q_j^2 - q_i^2 \sum_{j=1}^{n} V_j\right\} - q_i \sum_{j=1}^{n} (A_j + m_j) + (A_i + m_i) = 0 .$$

This expression for $i = 1, 2, \ldots, n$, together with $\sum_{j=1}^{n} q_j = 1$ gives a system of $n + 1$ equations for the determination of the $n + 1$ unknowns $q_1, \ldots, q_n$ and p.

These equations are not independent, however, since the last one can be obtained by adding together the first n. Hence the system will give $q_1, \ldots, q_n$ as functions of p.

For $p = 0$ we find

$$q_i(0) = \frac{A_j + m_j}{\sum_{j=1}^{n}(A_j + m_j)} .$$

Differentiating the equations with respect to p, we find

$$\left[\frac{dq_i(p)}{dp}\right]_{p=0} \sum_{j=1}^{n} (A_j + m_j) = V_i \sum_{j=1}^{n} q_j^2 - q_i^2 \sum_{j=1}^{n} V_j .$$

Hence it follows from considerations of continuity that $q_i(p)$ will be real and positive when p lies in some interval containing zero.

4.5. We shall now assume that a price p is given, and study how company 1 can increase its utility by dealing in the market at this price.

(i) The company can *sell* reinsurance cover, i.e., it can accept responsibility for paying a claim with mean m_0 and variance W_1. For giving this cover the company will receive the amount $m_0 + pW_1$.

According to the formulae in Section 4.3, this transaction will change the utility of the company from

$$-a_1(S_1 - m_1)^2 + (S_1 - m_1) - a_1 V_1 = U_1(S_1 - m_1, V_1) = U_1(R_1, V_1)$$

to

$$\begin{aligned} -a_1(S_1 - m_1 + pW_1)^2 + (S_1 - m_1 + pW_1) - a_1(V_1 + W_1) \\ = U_1(R_1 + pW_1, V_1 + W_1) \,. \end{aligned}$$

Here $R_1 = S_1 - m_1$, which in insurance terminology is called the "free reserves" of the company, i.e., funds in excess of expected amount of claims. We see that the utility does not depend on m_0, but only on free reserves and variance.

(ii) The company can *buy* reinsurance cover from the $n-1$ other companies, i.e. by paying the amounts $pv_2, \ldots, pv_n$ of its free reserves to the other companies, it can "get rid of" variances $v_2, \ldots, v_n$.

These transactions will leave the company with a variance

$$v_1 = V_1 - \sum_{i=2}^{n} v_i - 2 \sum_{i \neq j} C_{ij}$$

where C_{ij} is the covariance between claims in the portfolios taken by companies i and j.

Since the utility of company 1 will increase with decreasing v_1, the company will seek to arrange its purchases so that C_{ij} is as great as possible, i.e., so that

$$C_{ij} = (v_i v_j)^{\frac{1}{2}} \,.$$

This clearly means that there must be perfect positive correlation between claims in the part of the original portfolio which the company retains and the parts which are reinsured. Hence we must have $v_i = q_i^2 V_1$ and $\sum_{j=1}^{n} q_i = 1$. This is the same as the result which we in Section 4.2 derived from the general condition for Pareto optimality of Section 2.5.

4.6. If the company buys and sells reinsurance cover in this way, its utility will become

$$U_1\left(\left\{R_1 + p\,(W_1 - \sum_{i=2}^{n} v_i)\right\} \,,\, \left\{W_1 + (V_1^{\frac{1}{2}} - \sum_{i=2}^{n} v_i^{\frac{1}{2}})^2 - V_1\right\}\right).$$

The company will then seek to determine W_1, and $v_2, \ldots, v_n$ so that this expression is maximized.

The first order conditions for a maximum are

$$\frac{\partial U_1}{\partial W_1} = -p\left\{2a_1\left(R_1 + p(W_1 - \sum_{j=2}^{n} v_j)\right) - 1\right\} - a_1 = 0\,,$$

$$\frac{\partial U_1}{\partial v_i} = p\left\{2a_1\left(R_1 + p(W_1 - \sum_{j=2}^{n} v_j)\right) - 1\right\} + a_1 \frac{V_1^{\frac{1}{2}} - \sum_{j=2}^{n} v_j^{\frac{1}{2}}}{v_i^{\frac{1}{2}}} = 0 \; (i = 2, 3 \ldots n)\,.$$

Adding the first of these equations to the one obtained by differentiating with respect to v_i, we obtain

$$V_1^{\frac{1}{2}} - \sum_{j=2}^{n} v_j^{\frac{1}{2}} = v_i^{\frac{1}{2}}\,.$$

Since this must hold for all i, we must have

$$v_i = \frac{1}{n^2} V_1 \qquad \text{for all } n.$$

This means that regardless of what the price is, the company will seek to divide its portfolio into n identical parts, and reinsure $n - 1$ of these with the other companies.

Inserting the values of v_i in the first equation, we find

$$W_1 = \frac{n-1}{n^2} V_1 + \frac{2p\left(\frac{1}{2a_1} - R_1\right) - 1}{2p^2}\,.$$

4.7. In general we find that for a given price p, company i is willing to supply reinsurance cover for a variance

$$W_i = \frac{n-1}{n^2} V_i + \frac{2p\left(\frac{1}{2a_i} - R_i\right) - 1}{2p^2}\,.$$

The company will demand cover for a variance

$$W_i' = \frac{n-1}{n^2} V_i$$

regardless of what the price is, provided that this variance can be divided equally between the $n - 1$ other companies.

It is obvious that in this case we cannot determine p by simply requiring that total supply shall be equal to total demand, i.e., from the "market equation"

$$\sum_{i=1}^{n} W_i = \sum_{i=1}^{n} W_i'\,.$$

Instead we have the conditions that supply from company i must equal the sum of $1/(n-1)$ of the demand from the other $n-1$ companies, i.e.,

$$W_i = \frac{1}{n-1} \sum_{j \neq i} W'_i = \frac{1}{n^2} \sum_{j \neq i} V_j \, .$$

Hence p must satisfy the n equations

$$\frac{1}{n^2} \sum_{j=1}^{n} V_j - \frac{1}{n} V_i = \frac{2p\left(\dfrac{1}{2a_i} - R_i\right) - 1}{2p^2} \qquad (i = 1, 2, \ldots, n).$$

This is clearly impossible, except for special values of a_i, R_i and V_i.

4.8. It is obvious from the preceding paragraph that unrestricted utility maximization with a given price has little meaning in our model. The procedure may, however, have some meaning if we introduce restrictions so that it necessarily leads to a Pareto optimal arrangement.

These restrictions can be formulated as follows. For all i and j, $j \neq i$, company i can satisfy its demand for reinsurance cover only by placing a part $q_j^2 V_i$ of its variance with company j.

Company i will then be willing to supply reinsurance cover for a variance

$$W_i = V_i \sum_{j \neq i}^{n} q_j^2 + \frac{2p\left(\dfrac{1}{2a_i} - R_i\right) - 1}{2p^2} \, .$$

The n market equations from Section 4.7 will then take the form

$$q_i^2 \sum_{j=1}^{n} V_j - V_i \sum_{j=1}^{n} q_j^2 = \frac{2p\left(\dfrac{1}{2a_i} - R_i\right) - 1}{2p^2} \qquad (i = 1, 2, \ldots, n).$$

It is easy to see that these n equations, which are linear in q_i^2, have a determinant of rank $n-1$. Hence the equations have a solution only if the sum of the right hand sides is zero. This condition is satisfied: (i) if the right hand sides all vanish, i.e., if p tends to infinity. The corresponding values of q_i are then

$$q_i = \frac{V_i^{\frac{1}{2}}}{\sum_{j=1}^{n} V_j^{\frac{1}{2}}} \, ;$$

(ii) if

$$p = \frac{n}{2 \sum_{j=1}^{n} \left(\dfrac{1}{2a_j} - R_j\right)} \, .$$

This appears to be all that we can get, even from a diluted principle of utility maximization.

The result is not very satisfactory. The general assumptions which lead to these "equilibrium prices" are rather artificial, and it is easy to construct numerical examples where the result becomes meaningless.

From the formulae in Section 4.2 we see that the utility of company i will decrease with increasing q_i. The price we have found may lead to values of q_i which will give some companies a lower utility than they have in the initial situation. These companies will obviously refuse to trade at such a price.

The conditions which q_i must satisfy in order to give a meaningful solution are discussed in the paper [**6**] already referred to, and we shall not pursue the point further in the present paper.

5. THE MODELS OF ALLAIS AND ARROW

5.1. Both Allais [**1**] and Arrow [**3**] have proved that in models very similar to ours, there exists a price such that utility maximization, when this price is considered as given, will lead to a Pareto optimal situation. To explain the apparent contradiction with our result, we shall examine their models in some detail.

5.2. Allais [**1**] studied a model which essentially is a market for lottery tickets. The prize of the tickets is a normally distributed random variable with mean equal to one unit of money, and a given standard deviation. Allais proves that in this model there exists a market price for lottery tickets which will lead to a uniquely determined, optimal distribution of the risks.

The crucial assumption which Allais makes in order to reach this result is that lottery tickets can be bought and sold only in integral numbers, i.e., one can buy one ticket, but not a 50 per cent interest in two tickets. It is obvious that when this assumption is given up, the Pareto optimum is no longer unique. The situation will be similar to the one we found in Section 2.7, which is an example of the familiar problem that an n-person game has an indeterminate solution. To make it determined, one will have to make some assumptions about how the participants form coalitions to buy packages of lottery tickets.

5.3. In the model of Allais there is only one kind of lottery ticket. If tickets are indivisible as Allais assumes, it is almost trivial that there must exist a price which leads to a Pareto optimal situation. The problem will change completely, however, if the model is generalized by the introduction of several kinds of tickets, i.e., tickets where the prize is drawn from different probability distributions. The problem can be handled as we did in the preceding sections if one accepts the Bernoulli hypothesis. Allais [**2**] has emphatically rejected this hypothesis, however, and thus barred the most obvious, and probably the only way to generalize his model.

5.4. The model studied by Arrow [**3**] is far more general. He considers n different commodities, and he assumes that each participant in the market may have his own subjective probabilities. In this paper we shall disregard both these refinements. The generalization to n commodities appears inessential when our main objective is to study the interplay of different attitudes toward risk and uncertainty. The subjective probabilities play a key part in Arrow's model, but it seems unnecessary to introduce them in a study of a reinsurance market. When a reinsurance treaty is concluded, both parties will survey all information relevant to the risks concerned. To hide information from the other party is plain fraud. Whether two rational persons on the basis of the same information can arrive at different evaluations of the probability of a specific event, is a question of semantics. That they may act differently on the same information is well known, but this can usually be explained assuming that the two persons attach different utilities to the event. In some situations, for instance in stock markets, it may be useful to resort *both* to subjective probabilities and different utility functions to explain observed behaviour. This seems, however, to be an unnecessary complication in a first study of reinsurance markets.

5.5. When simplified as indicated in the preceding paragraph, Arrow's model can be described as follows: (i) Company i has a utility of money $u_i(x)$, $i = 1, 2, \ldots, I$.

(ii) As a result of its direct underwriting the company is committed to pay an amount x_{is} if "state of the world" s occurs, $s = 1, 2, \ldots, S$.

(iii) The company has funds amounting to S_i available for meeting the commitments.

(iv) The probability that state of the world s shall occur is p_s ($\sum_{s=1}^{S} p_s = 1$).

The utility of company i in the initial situation is then

$$U_i(0) = \sum_{s=1}^{S} p_s u_i(S_i - x_{is})$$

where x_{is} may be zero for some s.

5.6. It is then assumed that there exists a price vector $g_1, \ldots, g_s, \ldots, g_S$, so that the company can pay an amount $g_s y_{is}$, and then be assured of receiving the amount y_{is} if state of the world s occurs. This means that should this state occur, the company will have to make a net payment of $x_{is} - y_{is}$. If the company makes a series of such contracts, its utility will change to

$$U_1(y) = \sum_{s=1}^{S} p_s u_i\left(\left\{S_i - \sum_{s=1}^{S} g_s y_{is}\right\} - (x_{is} - y_{is})\right)$$

where y_{is} may be positive or negative.

Differentiating with respect to y_{it} we find:

$$\frac{\partial U_i(y)}{\partial y_{it}} = -g_t \sum_{s=1}^{S} p_s u_i'\left(\left\{S_i - \sum_{s=1}^{S} g_s y_{is}\right\} - x_{is} + y_{is}\right) + p_t u_i'\left(\left\{S_i - \sum_{s=1}^{S} g_s y_{is}\right\} - x_{it} + y_{it}\right) .$$

Since we have placed no restrictions on y_{it}, the first order conditions for a maximum will be

$$g_t \sum_{s=1}^{S} p_s u_i'\left(S_i - \sum_{s=1}^{S} g_s y_{is} - x_{is} + y_{is}\right) = p_t u_i'\left(S_i - \sum_{s=1}^{S} g_s y_{is} - x_{it} + y_{it}\right) \qquad (t = 1,2,\ldots,S) .$$

5.7. We now assume that the utility function is of the same simple form as in Section 4, i.e., that

$$u_i(x) = -a_i x^2 + x \qquad (i = 1,2,\ldots,I) .$$

The first order conditions for a maximum will then become

$$2a_i g_t \sum_{s=1}^{S} p_s\left(S_i - \sum_{s=1}^{S} g_s y_{is} - x_{is} + y_{is}\right) - g_t = 2a_i p_t\left(S_i - \sum_{s=1}^{S} g_s y_{is} - x_{it} + y_{it}\right) - p_t .$$

By some rearrangement this system of equations can be written

$$(g_t - p_t)\left(\frac{1}{2a_i} - S_i\right) + g_t \sum_{s=1}^{S} p_s x_{is} - p_t x_{it} = g_t \sum_{s=1}^{S} p_s y_{is} - p_t y_{it} - (g_t - p_t) \sum_{s=1}^{S} g_s y_{is}$$

$$(t = 1,2,\ldots, S \text{ and } i = 1,2,\ldots, I) .$$

y_{is} is the amount (positive or negative) which company i will receive if state of the world s occurs. Since this amount necessarily must be paid out by the other companies, we must have

$$\sum_{i=1}^{I} y_{is} = 0 \qquad \text{for all } s .$$

Hence if we sum the equations over all i, the right hand side will disappear, so that we get the system

$$(g_t - p_t) \sum_{i=1}^{I} \left(\frac{1}{2a_i} - S_i\right) + g_t \sum_{s=1}^{S} p_s \sum_{i=1}^{I} x_{is} - p_t \sum_{i=1}^{I} x_{it} = 0 \qquad (t = 1,2,\dots,S) .$$

From this we obtain

$$g_t = p_t \frac{X_t + A}{X + A}$$

where

$$A = \sum_{i=1}^{I} A_i = \sum_{i=1}^{I} \left(\frac{1}{2a_i} - S_i\right) ,$$

$$X_s = \sum_{i=1}^{I} x_{is} \qquad \text{and} \qquad X = \sum_{s=1}^{S} p_s X_s .$$

The complete solution of the system is given by

$$x_{it} - y_{it} = q_i X_t$$

and

$$q_i = \frac{A_i + \sum_{s=1}^{S} g_s x_{is}}{A + \sum_{s=1}^{S} g_s X_s}$$

where

$$\sum_{i=1}^{I} q_i = 1 .$$

5.8. This solution implies that company i ($i = 1, 2, \dots, I$) shall pay a fixed quota q_i of the total claim payment, regardless of which state of the world may occur. Hence the solution belongs to the set of Pareto optimal arrangements that we found in Section 4.2.

Since

$$\sum_{s=1}^{S} g_s x_{is} = \sum_{s=1}^{S} p_s \frac{X_s + A}{X + A} x_{is} = \frac{1}{X + A} \sum_{s=1}^{S} p_s (A x_{is} + X_s x_{is})$$

$$= \frac{1}{X + A} \left\{A m_i + X m_i + \sum_{s=1}^{S} p_s (X_s - X)(x_{is} - m_i)\right\}$$

$$= m_i + \frac{\sum_{s=1}^{S} p_s (X_s - X)(x_{is} - m_i)}{X + A} = m_i + \frac{C_i}{X + A}$$

where $m_i = \sum_{s=1}^{S} p_s x_{is}$, and since $X = \sum_{i=1}^{I} m_i$, we have

$$q_i = \frac{A_i + m_i + \dfrac{C_i}{X + A}}{\sum_{j=1}^{t} (A_j + m_j) + \dfrac{\operatorname{var} X_s}{X + A}} .$$

It is interesting to compare this with the expression which we found in Section 4.4 for the case $p = 0$.

5.9. The difference between Arrow's model and ours obviously lies in the price concept. In Arrow's model there is a price associated with every state of the world. The price will be the same for all states which lead to the same amount of total claim payment.

Our model is essentially a drastic simplification. Instead of the really infinite number of prices considered by Arrow, we have introduced one single price, a specific *price of risk*. We found that this price would have to be a vector with an infinite number of elements. If the utility function has the simple form studied in Section 4, the number of elements is reduced to two. However, in this case a competitive equilibrium cannot in general be a Pareto optimal distribution of risks.

5.10. The price in Arrow's model increases with the probability of a particular state of the world, and with the total amount to be paid if this state occurs. In insurance this means that a reinsurer who is asked to cover a modest amount if a certain person dies, will quote a price increasing with the total amount which is payable on the death of this person.

Such considerations are not unknown in insurance practice. It is well known that it can be difficult, i.e., expensive to arrange satisfactory reinsurance of particularly large risks. Practice seems here to be ahead of insurance theory, however, which still is firmly based on the *principle of equivalence,* i.e., that "net premiums" should be equal to the expected value of claim payments.

To apply Arrow's theory to stock market speculation we just have to reverse the signs of the formulae in this section.

We then find that the price of a certain share will depend not only on its "intrinsic value," but also on the number of such shares in the market. This may seem reasonable, although it implies that one pays more for a chance of getting rich alone, than for an identical chance of getting equally rich together with a lot of other speculators. The implication is that even in a model using essentially classical assumptions, there is a positive price attached to "getting ahead of the Joneses," and this may be a little unexpected.

6. THE PROBLEM SEEN AS AN n-PERSON GAME

6.1. We noted in Section 2.6 that a Pareto optimal set of reinsurance treaties was equivalent to a pool arrangement. Once the pool was established, the companies had to agree on some rule as to how each company should contribute to the payment of claims against the pool. In the special case which we considered in Section 4, this rule was that each company should pay a fixed proportion of these claims, regardless of its size. The quotas which each company should pay remained to be fixed, however.

When the problem is presented in this way, it seems natural to consider it as a problem of bargaining and negotiation which logically should be analysed in the terms of the theory of games. A priori it appears unlikely that there should exist some price mechanism which automatically will lead the companies to such a rather special arrangement as a Pareto optimal set of treaties.

6.2. In general an n-person game has an indeterminate solution. To get a determinate solution we must make *additional assumptions* about how the companies negotiate their way to an agreement.

The point is brought out clearly by the special case studied in Section 4. The usual assumptions of game theory leave the quotas $q_1, \ldots, q_n$ undetermined, except for the restriction $\sum q_i = 1$. The solution has, so to speak, $n-1$ "degrees of freedom." During the negotiations each company will try to get the smallest possible quota for itself.

We then lay down the additional rule that the same price must be applied to all the reciprocal treaties which constitute a Pareto optimal set. This amounts really to a ban on "price discrimination" or a partial ban on coalitions. The rule leaves only the price p to be determined by negotiation, so that the number of degrees of freedom is reduced to one.

6.3. From the expressions in Section 4.4 we can conjecture that a given price p will divide the companies in two groups or coalitions. One group will benefit from a higher price, the other from a lower one. The higher the price, the more companies will be in the latter group. The "equilibrium price" must then be determined so that it divides the companies in two groups, which in some unspecified manner are equal in strength. There are obviously a number of possible ways in which the concept "strength" can be defined, and hence a number of possible determinate solutions. We shall, however, not explore these possibilities in the present paper.

6.4. In real life reinsurance treaties are concluded after lengthy negotiations, often with brokers acting as intermediaries. The concept of prevailing market prices plays a part in the background of these negotiations, but the whole situation is more similar to an n-person game than to a classical market with utility maximization when the price is considered as given.

Little is known about the laws and customs ruling such negotiations in the reinsurance market. It seems, however, that further studies of this subject should be a promising, if not the most promising, way of gaining deeper knowledge of attitudes toward risk and the decisions which rational people make under uncertainty.

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