Complex Numbers – three different forms

Notes: In IB mathematics a complex number z can be written in three different forms.

$z = a + \mathrm{i}b$ **Cartesian** form

$z = r(\cos\theta + \mathrm{i}\sin\theta) = r\,\mathrm{cis}\,\theta$ **modulus-argument** form* (also known as polar form, or trigonometric form)

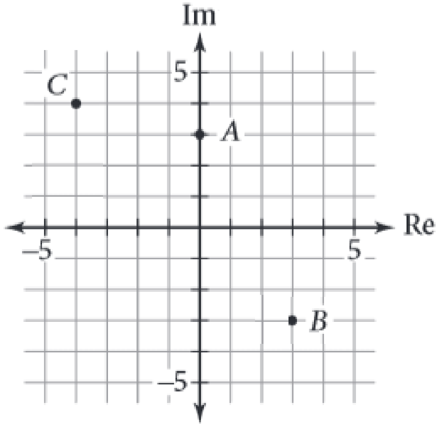
$z = re^{\mathrm{i}\theta}$ **Euler's** form* (also known as exponential form)

* the argument θ can have multiple values, but best to give the principle value such that $-\pi < \theta \le \pi$

♦ do **not** use a calculator ♦

Exercises

1. For each complex number represented by the letters A, B & C plotted in the complex plane, write it in each of the three forms: Cartesian form, modulus-argument form and Euler's form.



2. Write each of the following complex numbers in modulus-argument form.

(a) $w = -\sqrt{3} + \mathrm{i}$ (b) $w = 2 + 2\mathrm{i}\sqrt{3}$ (c) $w = -\dfrac{1}{2} - \dfrac{\mathrm{i}}{2}$

3. Write each of the following complex numbers in Cartesian form.

(a) $z = 5e^{\frac{\pi}{2}\mathrm{i}}$ (b) $z = 8e^{-\frac{5\pi}{6}\mathrm{i}}$ (c) $z = 2e^{\frac{2\pi}{3}\mathrm{i}}$

4. For each of the two expressions below, first write it as a complex number in the form $a + \mathrm{i}b$ and then write it in the form $r\,\mathrm{cis}\,\theta$.

(a) $\dfrac{4}{1+\mathrm{i}}$ (b) $\dfrac{2\mathrm{i}}{1-\mathrm{i}}$

5. Consider the complex numbers $z_1 = 1 + \mathrm{i}$ and $z_2 = \dfrac{\sqrt{6}}{2} - \mathrm{i}\dfrac{\sqrt{2}}{2}$.

(a) Show that, in Cartesian form, $z_2 = \sqrt{2}\,\mathrm{cis}\left(-\dfrac{\pi}{6}\right)$.

(b) For two complex numbers in modulus-argument form, $w_1 = r_1\,\mathrm{cis}\,\theta_1$ and $w_2 = r_2\,\mathrm{cis}\,\theta_2$, it can be shown that $\dfrac{w_1}{w_2} = \dfrac{r_1}{r_2}\mathrm{cis}(\theta_1 - \theta_2)$. Use this to express $\dfrac{z_1}{z_2}$ in modulus-argument form.

(c) Express $\dfrac{z_1}{z_2}$ in Cartesian form.

(d) Use the results from (b) and (c) to write down the exact values of $\sin\dfrac{5\pi}{12}$ and $\cos\dfrac{5\pi}{12}$.

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Answers

1. **A**: 3i ; $3\text{cis}\frac{\pi}{2}$; $3e^{\frac{\pi}{2}\text{i}}$ **B**: $3-3\text{i}$; $3\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$; $3e^{-\frac{\pi}{4}\text{i}}$ **C**: $-4+4\text{i}$; $4\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$; $4e^{\frac{3\pi}{4}\text{i}}$

2. (a) $w=2\,\text{cis}\frac{5\pi}{6}$ (b) $w=2\sqrt{2}\,\text{cis}\frac{\pi}{3}$ (c) $w=\frac{3\sqrt{2}}{2}\text{cis}\left(-\frac{\pi}{4}\right)$

3. (a) $z=5\text{i}$ (b) $z=-4\sqrt{3}-4\text{i}$ (c) $z=-1+\text{i}\sqrt{3}$

4. (a) $2-2\text{i}$; $2\text{cis}\left(-\frac{\pi}{4}\right)$ (b) $-1+\text{i}$; $\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$

5. (b) $\text{cis}\frac{5\pi}{12}$ (c) $\frac{z_1}{z_2}=\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}+\text{i}\left(\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}\right)$ (d) $\sin\frac{5\pi}{12}=\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}$, $\cos\frac{5\pi}{12}=\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}$