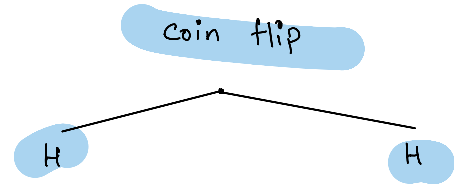
Entropy

$P(H) = 1$



Flip!

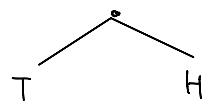
How much information did you get out of it? 0 "bits"

We will define a bit of information as:

"How many Yes/ No questions you need to ask to determine the outcome?"

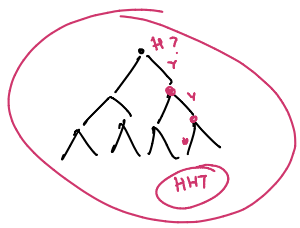
$P(H) = 0.5$

"1 bit"



Now, let's do 3 coin flips

3 - bits



Pick a random letter from A – Z

What question should you ask?

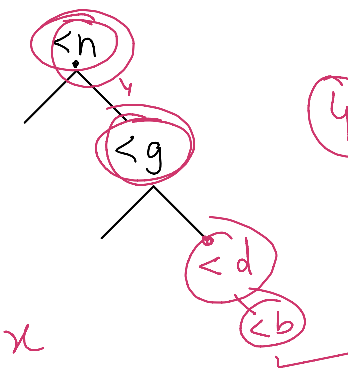
A? B? C? → ~~26~~ questions 25

Y N → Z

a b c d e f g h i j k l m n o p q r s t u v w x y z

no. of questions = $\boxed{\log_2(26)}$

= 4.7 bits



4

a

4.7

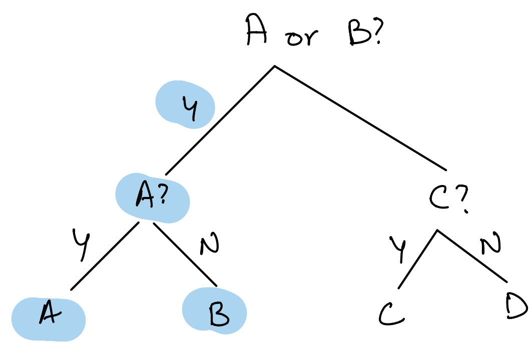
$\log_2 x$

$2^5 = 32$ $2^4 = 16$ $2^{?} = x$

Let's consider just ABCD.

We have a sequence AABBCCDD

What's $P(A) = 2/8 = 1/4$



# of questions to find out it's an A = 2

Same for B, C and D

Total "Expected value" for number of questions

$$= \sum_{\ell} P(L=\ell) \cdot Q$$

"Expected values"

# of questions for this letter

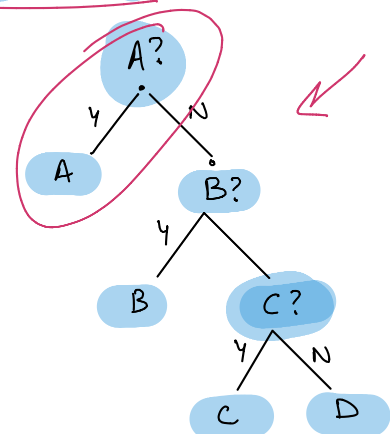
prob. of this letter

Expected number of questions $= \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 2$

$= 2$ questions (on average)

notice this is $\log_2(4)$

Let's change the sequence to AAAABBCD



What's $P(A) = 4/8 = 1/2$

# of questions to find out it's an A = 1

Expected number of questions:

$$\underbrace{\frac{1}{2} \cdot 1}_{A} + \underbrace{\frac{1}{4} \cdot 2}_{B} + \underbrace{\frac{1}{8} \cdot 3}_{C} + \underbrace{\frac{1}{8} \cdot 3}_{D}$$

$= 1.75$ Average number of questions is lower!

AA BB CC DD

2

Higher entropy

unordered

AAAA BB CD

1.75

lower entropy

ordered

AAAAAAA

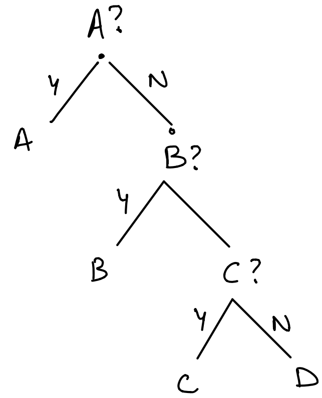
0

Points to note:

- Have to ask more questions to "gain knowledge" if entropy is higher

- If a machine is trying to learn something, we want to minimize the entropy so that it "learns" with fewer questions

- When transmitting information, we want to minimize entropy (# of questions asked)
  That is why Huffman coding works



Total "Expected value" for number of questions

$$= \sum_{\ell} P(L=\ell) \cdot Q$$

expected # of questions

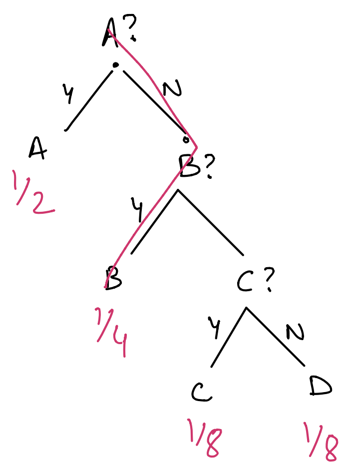
# of questions for this letter (information)

prob. of this letter

Q for A : 1 $= -\log_2(0.5)$ information $-\lg(p(x))$

Q for B : 2 $= -\log_2(0.25)$

Q for C/D : 3 $= -\log_2(0.125)$

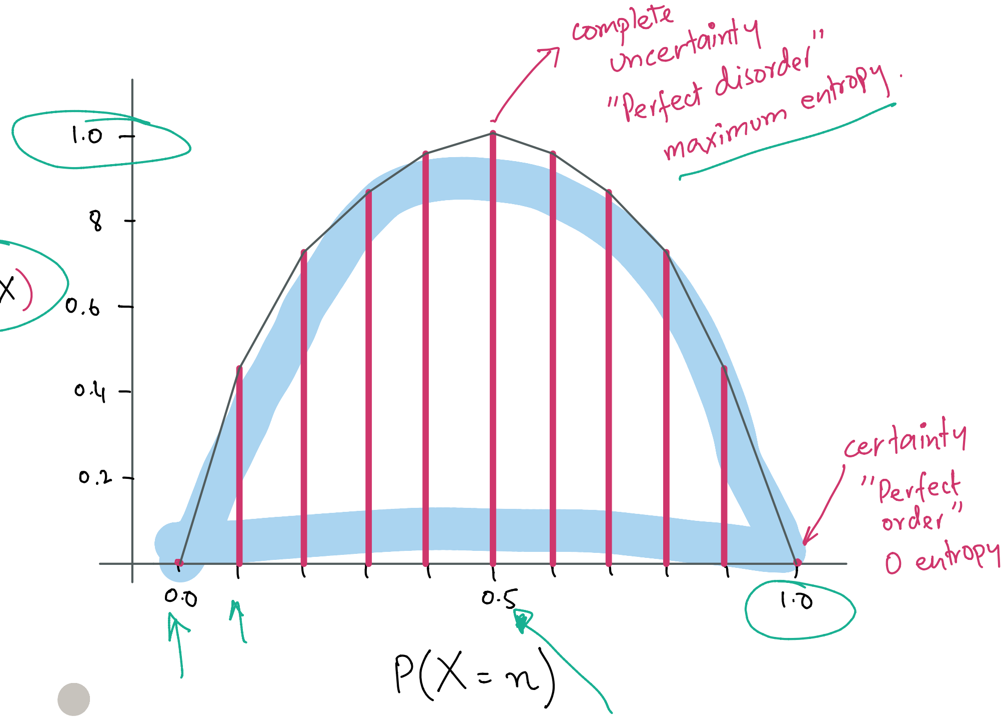


$$\text{Entropy} = H(X) = -\sum_i p_i \cdot \log_2(p_i)$$

So, entropy is "expected information"

Probability affecting entropy of a coin flip

H(X)



$$H(x) = -\left[0.1 \log_2(0.1) + 0.9 \log_2(0.9)\right]$$

H

T

$$\boxed{= 0.46}$$

$P(X = H) = 0.1$ $\qquad$ $H(x) = 0.46$