physics lab

EQUATION SHEET & STUDY GUIDES

EQUATION SHEET

Circle Geometry

Diameter

$$d = 2r$$

Circumference

$$C = 2\pi r = \pi d$$

Area

$$A = \pi r^2$$

$$1 \text{ revolution} = 360° = 2\pi \text{ radians}$$

Triangle Geometry

Area

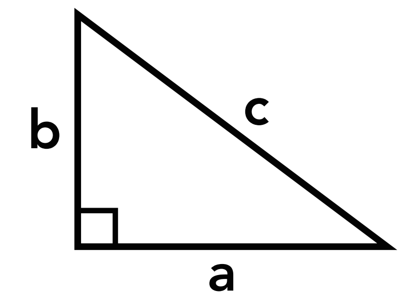
$$A = \frac{1}{2}bh$$

Angles

$$\theta_1 + \theta_2 + \theta_3 = 180°$$

Pythagorean theorem

$$c^2 = a^2 + b^2$$



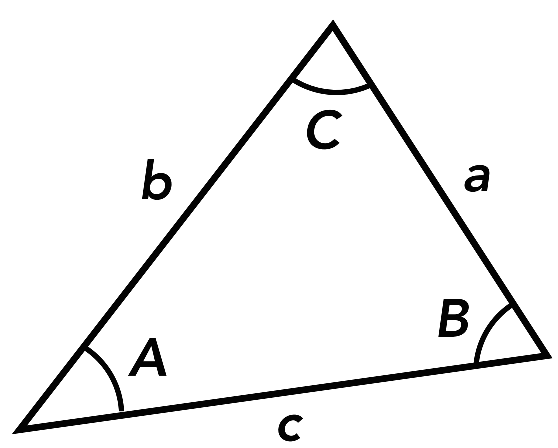
Trig identities:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

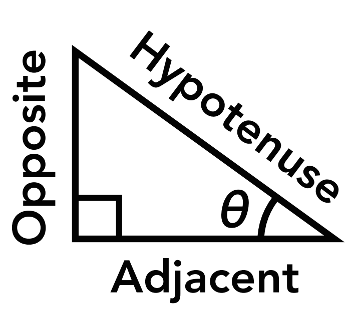


Law of sines

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

Law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$



$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\theta = \sin^{-1}\left(\frac{\text{opposite}}{\text{hypotenuse}}\right) \qquad \theta = \cos^{-1}\left(\frac{\text{adjacent}}{\text{hypotenuse}}\right) \qquad \theta = \tan^{-1}\left(\frac{\text{opposite}}{\text{adjacent}}\right)$$

Quadratic Formula

Quadratic formula for

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1D and 2D Kinematics

Average speed

$$v_{avg} = \frac{\text{total distance}}{\text{total time}}$$

delta

$$\Delta = \text{final} - \text{initial}$$

$$\Delta x = x_f - x_i$$

or

$$\Delta x = x - x_0$$

Variables		SI Unit
t	time	s
x	horizontal position	m
y	vertical position	m
v	velocity	$\frac{m}{s}$
a	acceleration	$\frac{m}{s^2}$

	Horizontal motion:	Vertical motion:
Displacement:	$\Delta x = x_f - x_i$	$\Delta y = y_f - y_i$
Velocity:	$v_x = \frac{\Delta x}{\Delta t}$	$v_y = \frac{\Delta y}{\Delta t}$
Velocity (rearranged):	$x_f = x_i + v_x \Delta t$	$y_f = y_i + v_y \Delta t$
Acceleration:	$a_x = \frac{\Delta v_x}{\Delta t}$	$a_y = \frac{\Delta v_y}{\Delta t}$
Acceleration (rearranged):	$v_{xf} = v_{xi} + a_x \Delta t$	$v_{yf} = v_{yi} + a_y \Delta t$
Kinematic equations for constant acceleration:	$x_f = x_i + v_{xi} t + \frac{1}{2} a_x t^2$	$y_f = y_i + v_{yi} t + \frac{1}{2} a_y t^2$
	$v_{xf}^2 = v_{xi}^2 + 2 a_x (x_f - x_i)$	$v_{yf}^2 = v_{yi}^2 + 2 a_y (y_f - y_i)$

Subscripts		
i	0	initial
f	_	final
x		horizontal
y		vertical

Projectile Motion

Range

$$\Delta x = v_i \cos(\theta) \frac{v_i \sin(\theta) + \sqrt{(v_i \sin(\theta))^2 + 2 g y_i}}{g}$$

Range (if $y_i = y_f$)

$$\Delta x = \frac{v_i^2 \sin(2\theta)}{g}$$

Circular and Rotational Motion

Variables		SI Unit
s	tangential position	m
Δs	tangential displacement	m
v_t	tangential velocity	$\frac{m}{s}$
a_t	tangential acceleration	$\frac{m}{s^2}$

Variables		SI Unit
θ	angular position	rad
$\Delta\theta$	angular displacement	rad
ω	angular velocity	$\frac{rad}{s}$
α	angular acceleration	$\frac{rad}{s^2}$

Conversion
(Angular variable must use radians)

	Circular motion (tangential description)	$\longleftrightarrow$	**Rotational motion (angular description)**
Position:	s m	$s = r\,\theta$	θ rad
Displacement:	$\Delta s = s_f - s_i$ m	$\Delta s = r\,\Delta\theta$	$\Delta\theta = \theta_f - \theta_i$ rad
Velocity:	$v_t = \frac{\Delta s}{\Delta t}$ $\frac{m}{s}$	$v_t = r\,\omega$	$\omega = \frac{\Delta\theta}{\Delta t}$ $\frac{rad}{s}$
Acceleration:	$a_t = \frac{\Delta v_t}{\Delta t}$ $\frac{m}{s^2}$	$a_t = r\,\alpha$	$\alpha = \frac{\Delta\omega}{\Delta t}$ $\frac{rad}{s^2}$
Kinematic equations with acceleration:	$s_f = s_i + v_{ti}t + \frac{1}{2}a_t t^2$		$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$
	$v_{tf}^2 = v_{ti}^2 + 2a_t(s_f - s_i)$		$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$

Newton's 2nd Law of Motion

Newton's 2nd law of motion

$\vec{F}_{net} = m\vec{a}$ or $\sum\vec{F} = m\vec{a}$

$\sum$: the sum of __

Variables		SI Unit
F	force	$N = \frac{kg \cdot m}{s^2}$
m	mass	kg
a	acceleration	$\frac{m}{s^2}$
v	velocity	$\frac{m}{s}$

Gravitational Force & Weight

Constants		Unit	Name
G	6.67×10^{-11}	$\frac{m^3}{kg \cdot s^2}$	gravitational constant

Newton's Law of Universal Gravitation (gravitational force)

$$F_g = \frac{G m_1 m_2}{r^2} = F_{1 \text{ on } 2} = F_{2 \text{ on } 1}$$

Gravitational field strength or acceleration due to gravity

$$g = \frac{GM}{r^2}$$

Gravitational force on mass in gravitational field

$$F_g = mg \quad = \quad F_g = \frac{GMm}{r^2}$$

Variables		SI Unit
F_g	gravitational force	N
w	weight force	N
m	mass	kg
M	mass producing a field	kg
r	distance between centers	m
g	gravitational acceleration	$\frac{m}{s^2}$

Weight force

$$F_g = mg \quad \text{or} \quad w = mg$$

Friction

Maximum static friction force

$$f_{s\,max} = \mu_s F_n$$

μ_s : coefficient of static friction

Kinetic friction force

$$f_k = \mu_k F_n$$

μ_k : coefficient of kinetic friction

Rolling friction force

$$f_r = \mu_r F_n$$

μ_r : coefficient of kinetic friction

Variables		SI Unit
f_s	static friction force	N
f_k	kinetic friction force	N
f_r	rolling friction force	N
μ_s	coefficient of static friction	
μ_k	coefficient of kinetic friction	
μ_r	coefficient of rolling friction	
F_n	normal force	N

Spring Force

Spring force (Hooke's Law)

$$F_{sp} = k\Delta x$$

Equivalent spring constant for springs in series

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots$$

Equivalent spring constant for springs in parallel

$$k_{eq} = k_1 + k_2 + \ldots$$

Variables		SI Unit
F_{sp}	spring force	N
Δx	displacement	m
k	spring constant	$\frac{N}{m}$

Elasticity of Materials

"Spring constant" for a material

$$k = \frac{YA}{L}$$

Elastic Force

$$F = \frac{YA}{L}\Delta L$$

Stress

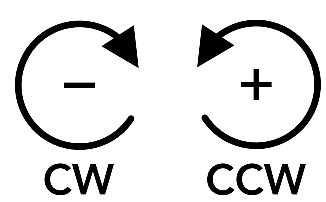
$$\frac{F}{A} = Y\frac{\Delta L}{L}$$

Variables		SI Unit
F	force	N
k	spring constant	$\frac{N}{m}$
Y , E	Young's modulus	$\frac{N}{m^2}$
A	cross-sectional area	m^2
L	length	m

Torque

Torque

$$\tau = rF_{\perp} \quad \text{or} \quad \tau = r_{\perp}F$$



Variables		SI Unit
τ	torque	N·m
F	force	N
r	distance from rotation axis	m

Rotational Dynamics

Newton's 2nd law of motion applied to rotation

$$\tau_{net} = I\alpha \quad \text{or} \quad \sum\tau = I\alpha$$

$\sum$: the sum of __

Rotational inertia for a system of masses

$$I = \sum m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots$$

Variables		SI Unit
τ	torque	N·m
I	rotational inertia	$kg\cdot m^2$
α	angular acceleration	$\frac{rad}{s^2}$
m	mass	kg
r	distance from rotation axis	m

Rotational inertia for common shapes:

Solid sphere (center)	Sphere shell (center)	Solid cylinder (center)	Cylinder shell (center)	Solid rod (center)	Solid rod (end)
$I = \frac{2}{5}mR^2$	$I = \frac{2}{3}mR^2$	$I = \frac{1}{2}mR^2$	$I = mR^2$	$I = \frac{1}{12}mL^2$	$I = \frac{1}{3}mL^2$

Center of Mass

x coordinate of center of mass of a system

$$x_{COM} = \frac{m_1x_1 + m_2x_2 + \dots}{m_1 + m_2 + \dots}$$

y coordinate of center of mass of a system

$$y_{COM} = \frac{m_1y_1 + m_2y_2 + \dots}{m_1 + m_2 + \dots}$$

Variables		SI Unit
x	x position	m
y	y position	m
m	mass	kg

Uniform Circular Motion

Frequency

$$f = \frac{1}{T}$$

Tangential velocity

$$v = \frac{2\pi r}{T}$$

$$v = 2\pi r f$$

Variables		SI Unit
v	velocity	$\frac{m}{s}$
r	radius	m
T	period	s
f	frequency	$Hz = \frac{cycles}{s}$
ω	angular velocity	$\frac{rad}{s}$

Centripetal Acceleration and Force

Centripetal acceleration

$$\vec{a}_c = \frac{v^2}{r} \text{ (towards center of circle)}$$

v : tangential speed (m/s)
r : radius of circular path (m)

Variables		SI Unit
a_c	centripetal acceleration	$\frac{m}{s^2}$
a	acceleration	$\frac{m}{s^2}$
v	velocity	$\frac{m}{s}$
r	radius	m
t	time	s

Centripetal acceleration
(other variables substituted for speed)

$$a_c = \frac{v^2}{r} = \omega^2 r = (2\pi f)^2 r = \left(\frac{2\pi}{T}\right)^2 r$$

ω : angular speed (rad/s)
f : frequency (Hz = rev/s)
T : period (s)

Centripetal force

$$\vec{F}_c = m\frac{v^2}{r} \text{ (towards center of circle)}$$

Orbital Motion

Constants		Unit	Name
G	6.67×10^{-11}	$\frac{m^3}{kg \cdot s^2}$	gravitational constant

Variables		SI Unit
M	planet mass	kg
m	object mass	kg
R	planet radius	m
r	orbital radius	m
v	orbital speed	$\frac{m}{s}$
T	orbital period	s
F_g	gravitational force	N
F_c	centripetal force	N

Orbital velocity

$$v = \sqrt{\frac{GM}{r}}$$

Orbital period

$$T = 2\pi\sqrt{\frac{r^3}{GM}}$$

Orbital period for elliptical orbit

$$T = 2\pi\sqrt{\frac{a^3}{G(M + m)}}$$

Orbital period for elliptical orbit (assuming M is much larger than m)

$$T = 2\pi\sqrt{\frac{a^3}{GM}}$$

Kinetic energy of object in a circular orbit

$$K = \frac{1}{2}mv^2 = \frac{GMm}{2r}$$

Gravitational potential energy of two-mass system

$$U_g = -\frac{GMm}{r}$$

Variables		SI Unit
E	total energy	J
K	kinetic enerrgy	J
U_g	potential energy	J

Total energy of object in a circular orbit

$$E = K + U_g = -\frac{GMm}{2r}$$

Kinetic Energy

Kinetic energy

$$K = \frac{1}{2}mv^2$$

Variables		SI Unit
K	kinetic energy	J
m	mass	kg
v	speed	$\frac{m}{s}$

Rotational kinetic energy

$$K_{rot} = \frac{1}{2}I\omega^2$$

Variables		SI Unit
K_{rot}	rotational kinetic energy	J
I	rotational inertia	$kg \cdot m^2$
ω	angular speed	$\frac{rad}{s}$

Gravitational Potential Energy

Constants		Unit	Name
G	6.67×10^{-11}	$\frac{m^3}{kg \cdot s^2}$	gravitational constant

Gravitational potential energy of a two-mass system

$$U_g = -\frac{GMm}{r}$$

$U_g = 0$ at $r = \infty$

Change in gravitational potential energy of an object-earth system

$$\Delta U_g = mg\Delta y$$

Gravitational potential energy of an object-earth system
*relative to a reference point

$$U_g = mgy \qquad U_g = 0 \text{ at } y = 0$$

Variables		SI Unit
U_g	gravitational potential energy	J
M	planet mass	kg
m	object mass	kg
r	distance between centers	m
y	height	m
g	gravitational acceleration	$\frac{m}{s^2}$

Spring Potential Energy

Spring potential energy

$$U_{sp} = \frac{1}{2}k\Delta x^2$$

Δx or Δy

Variables		SI Unit
U_{sp}	spring potential energy	J
k	spring constant	$\frac{N}{m}$
Δx	displacement	m

Conservation of Energy

Conservation of energy
(universe and isolated systems)

$$\Delta E_{total} = 0 \quad , \quad E_{total\ i} = E_{total\ f}$$

Variables		SI Unit
E	energy	J
K	kinetic energy	J
U_g	gravitational potential energy	J
U_{sp}	spring potential energy	J

Work

Work

$$\Delta E_{system} = W$$

Work

$$W = F_{\parallel} d$$

$F_{\parallel}$: component of force parallel to d
*F is an external force
d : displacement of the system

Variables		SI Unit
W	work	$J = N \cdot m$
E	energy	J
F	force	N
d	displacement	m

Power

Power

$$P = \frac{\Delta E}{\Delta t}$$

Power

$$P = \frac{W}{\Delta t} = F_{\parallel} v$$

$F_{\parallel}$: component of force parallel to v

*F is an external force

v : velocity of the system

Variables		SI Unit
P	power	$W = \frac{J}{s}$
E	energy	J
W	work	J
F	force	N
v	velocity	$\frac{m}{s}$

Momentum

Momentum

$$\vec{p} = m\vec{v}$$

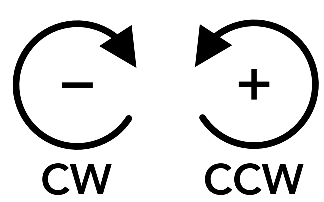
Momentum vector components

$$p_x = mv_x \qquad p_y = mv_y$$

Variables		SI Unit
p	momentum	$\frac{kg \cdot m}{s}$
m	mass	kg
v	velocity	$\frac{m}{s}$

Angular momentum

$$L = I\omega$$



Variables		SI Unit
L	angular momentum	$\frac{kg \cdot m^2}{s}$
I	rotational inertia	$kg \cdot m^2$
ω	angular velocity	$\frac{rad}{s}$

Impulse

Impulse

$$\vec{J} = \Delta\vec{p} = \vec{F}_{avg}\Delta t$$

F_{avg} : average force over time

Variables		SI Unit
J	impulse	$\frac{kg \cdot m}{s} = N \cdot s$
p	momentum	$\frac{kg \cdot m}{s}$
F	force	N
t	time	s

Rotational impulse

$$\Delta L = \tau_{avg} \Delta t$$

τ_{avg} : average torque over time

Variables		SI Unit
τ	torque	$N \cdot m$
L	angular momentum	$\frac{kg \cdot m^2}{s}$
I	rotational inertia	$kg \cdot m^2$
ω	angular velocity	$\frac{rad}{s}$

Conservation of Momentum

Law of conservation of momentum
(universe and isolated systems)

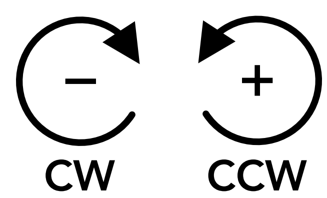
$$\Delta\vec{p}_{\text{total}} = 0 \quad , \quad \vec{p}_{\text{total i}} = \vec{p}_{\text{total f}}$$

$$\Delta p_{\text{x total}} = 0 \quad , \quad p_{\text{xi total}} = p_{\text{xf total}}$$
$$\Delta p_{\text{y total}} = 0 \quad , \quad p_{\text{yi total}} = p_{\text{yf total}}$$

Variables		SI Unit
p	momentum	$\frac{\text{kg}\cdot\text{m}}{\text{s}}$
m	mass	kg
v	velocity	$\frac{\text{m}}{\text{s}}$
J	impulse	$\frac{\text{kg}\cdot\text{m}}{\text{s}}$
F	force	N
t	time	s

Law of conservation of angular momentum
(universe and isolated systems)

$$\Delta\vec{L}_{\text{total}} = 0 \quad , \quad \vec{L}_{\text{total i}} = \vec{L}_{\text{total f}}$$



Variables		SI Unit
L	angular momentum	$\frac{\text{kg}\cdot\text{m}^2}{\text{s}}$
I	rotational inertia	$\text{kg}\cdot\text{m}^2$
ω	angular velocity	$\frac{\text{rad}}{\text{s}}$

Simple Harmonic Motion

Period of a
mass-spring oscillation

$$T_{\text{sp}} = 2\pi\sqrt{\frac{m}{k}}$$

Frequency of a
mass-spring oscillation

$$f_{\text{sp}} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Maximum velocity of a
mass-spring oscillation

$$v_{\text{max}} = A\sqrt{\frac{k}{m}}$$

Variables		SI Unit
T	period	s
f	frequency	$\text{Hz} = \frac{\text{cycles}}{\text{s}}$
A	amplitude	m
m	mass	kg
k	spring constant	$\frac{\text{N}}{\text{s}}$
U_{sp}	spring potential energy	J
K	kinetic energy	J

Period of a
pendulum oscillation

$$T_{\text{p}} = 2\pi\sqrt{\frac{L}{g}}$$

Frequency of a
pendulum oscillation

$$f_{\text{p}} = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

Maximum velocity of a
pendulum oscillation

$$v_{\text{max}} = \theta_{\text{max}}\sqrt{gL}$$

Variables		SI Unit
T	period	s
f	frequency	$\text{Hz} = \frac{\text{cycles}}{\text{s}}$
θ	angle	rad
L	length	m
g	grav. acceleration	$\frac{\text{m}}{\text{s}^2}$
U_{g}	grav. potential energy	J
K	kinetic energy	J

Waves

Wave speed

$$v = \lambda f = \frac{\lambda}{T}$$

Linear density

$$\mu = \frac{m}{L}$$

Speed of a wave in a string

$$v_{\text{string}} = \sqrt{\frac{T_s}{\mu}}$$

Variables		SI Unit
λ	wavelength	m
T	period	s
f	frequency	$\text{Hz} = \frac{\text{cycles}}{\text{s}}$
A	amplitude	m, ...
v	velocity	$\frac{\text{m}}{\text{s}}$

Sound

Constants		Unit	Name
I_0	1×10^{-12}	$\frac{\text{W}}{\text{m}^2}$	threshold of hearing

Constants		Unit	Name
R	8.3145	$\frac{\text{J}}{\text{mol}\cdot\text{K}}$	ideal gas constant

Speed of sound in a gas

$$v_{\text{sound}} = \sqrt{\frac{\gamma R T}{M}}$$

Variables		SI Unit
v	velocity	$\frac{\text{m}}{\text{s}}$
γ	adiabatic index	
T	temperature	K
M	molar mass	$\frac{\text{kg}}{\text{mol}}$

Sound intensity

$$I = \frac{P_{\text{source}}}{4\pi r^2}$$

Sound intensity level

$$\beta = (10\ \text{dB}) \log_{10}\left(\frac{I}{I_0}\right)$$

Variables		SI Unit
I	sound intensity	$\frac{\text{W}}{\text{m}^2}$
P	power	$\frac{\text{J}}{\text{s}}$
r	distance from source	m
β	sound intensity level	dB

Observed frequency, receding sound source

$$f_o = \frac{f_s}{1 + (v_s/v)}$$

Observed frequency, approaching sound source

$$f_o = \frac{f_s}{1 - (v_s/v)}$$

Observed frequency, receding observer

$$f_o = \left(1 - \frac{v_o}{v}\right) f_s$$

Observed frequency, approaching observer

$$f_o = \left(1 + \frac{v_o}{v}\right) f_s$$

Variables		SI Unit
f_s	source frequency	Hz
f_o	observed frequency	Hz
v_s	source speed	$\frac{\text{m}}{\text{s}}$
v_o	observer speed	$\frac{\text{m}}{\text{s}}$
v	speed of sound	$\frac{\text{m}}{\text{s}}$

Wave Interference

Variables		SI Unit
d	in-line path length	m
r	radial path length	m
λ	wavelength	m
m	number of wavelengths	

Beat frequency

$$f_{beat} = |f_1 - f_2|$$

In-line interference:

Constructive interference

$$\Delta d = m\lambda \quad m = 0, 1, 2, \ldots$$

Destructive interference

$$\Delta d = \left(m + \frac{1}{2}\right)\lambda \quad m = 0, 1, 2, \ldots$$

Radial interference:

Constructive interference (point C)

$$\Delta r = m\lambda \quad m = 0, 1, 2, \ldots$$

Destructive interference (point D)

$$\Delta r = \left(m + \frac{1}{2}\right)\lambda \quad m = 0, 1, 2, \ldots$$

Standing Waves

Variables		SI Unit
λ	wavelength	m
f	frequency	Hz
L	length	m
v	velocity	$\frac{m}{s}$
m	mode	

Both ends are either nodes or antinodes:

Wavelengths

$$\lambda_m = \frac{2L}{m} \quad m = 1, 2, 3, \ldots$$

Frequencies

$$f_m = \frac{v}{\lambda_m} = m\left(\frac{v}{2L}\right) \quad m = 1, 2, 3, \ldots$$

One end is a node, one end is an antinode:

Wavelengths

$$\lambda_m = \frac{4L}{m} \quad m = 1, 3, 5, \ldots$$

Frequencies

$$f_m = \frac{v}{\lambda_m} = m\left(\frac{v}{4L}\right) \quad m = 1, 3, 5, \ldots$$

BASICS

SI Units and Prefixes

SI Units		
Length	meter	m
Mass	kilogram	kg
Time	second	s
Temperature	kelvin	K
Amount of substance	mole	mol
Electrical current	amp	A
Light intensity	candela	cd

Prefix	Symbol	Exponent	Decimal	Word
tera-	T	10^{12}	1,000,000,000,000	trillion
giga-	G	10^{9}	1,000,000,000	billion
mega-	M	10^{6}	1,000,000	million
kilo-	k	10^{3}	1,000	thousand
hecto-	h	10^{2}	100	hundred
deka-	da	10^{1}	10	ten
—	—	10^{0}	1	one
deci-	d	10^{-1}	0.1	tenth
centi-	c	10^{-2}	0.01	hundredth
milli-	m	10^{-3}	0.001	thousandth
micro-	μ	10^{-6}	0.000 001	millionth
nano-	n	10^{-9}	0.000 000 001	billionth
pico-	p	10^{-12}	0.000 000 000 001	trillionth

Big

Small

Unit Conversion

Example: Convert 1 day into units of seconds

1. Find the equal amounts (the relationships) for the units that you're working with
2. Write the starting amount
3. Multiply by the equal amounts as fractions
4. Cross out units that are on both the top and bottom of the list of fractions
5. Multiply the numbers to get the final amount, which will have the units that are remaining

$$\underbrace{\frac{1\ \cancel{\text{day}}}{}}_{\text{Starting Amount}} \times \underbrace{\frac{24\ \cancel{\text{hours}}}{1\ \cancel{\text{day}}} \times \frac{60\ \cancel{\text{minutes}}}{1\ \cancel{\text{hour}}} \times \frac{60\ \text{seconds}}{1\ \cancel{\text{minute}}}}_{\text{Equal Amounts}} = \underbrace{\frac{86{,}400\ \text{seconds}}{}}_{\text{Final Amount}}$$

Scientific Notation

How to write a number in Scientific Notation	3800	0.00024
1. Move the decimal until there is only 1 number to the left of it	3.8.0.0. (3 2 1)	0.0.0.0.2.4 (1 2 3 4)
2. Write down the new number and "x 10"	3.8×10	2.4×10
3. Count how many times you moved the decimal	3.8.0.0. (3 2 1) left + ←	0.0.0.0.2.4 (1 2 3 4) → right -
4. Write that number as your exponent - If you moved the decimal left, the exponent is positive - If you moved the decimal right, the exponent is negative	3.8×10^{3}	2.4×10^{-4}

Order of Operations

PEMDAS

Please Excuse My Dear Aunt Sally

1. Parentheses	$(1+2)$
2. Exponents	3^2
3. Multiplication	$(2)(4)$
4. Division	$\frac{4}{2}$
5. Addition	$3+2$
6. Subtraction	$3-2$

Parentheses $$x = \frac{(2-1)(1+3)}{2} + \frac{(5-2)^2}{(8-5)} - 1$$

Exponents $$x = \frac{(1)(4)}{2} + \frac{3^2}{3} - 1$$

Multiplication $$x = \frac{(1)(4)}{2} + \frac{9}{3} - 1$$

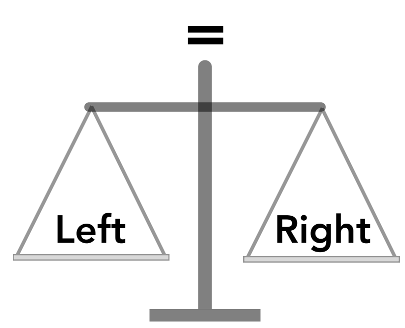
Division $$x = \frac{4}{2} + \frac{9}{3} - 1$$

Addition, Subtraction $$x = 2 + 3 - 1$$

$$x = 4$$

Solving Equations

Do the same thing to both sides of the equation:



$$5 + x = 12$$

Subtract 5 $\quad 5 + x - 5 = 12 - 5 \quad$ Subtract 5

$$x = 7$$

$$2x = 8$$

Divide by 2 $\quad 2x = 8 \quad$ Divide by 2

$$x = 4$$

$$1 = 1$$

Add 5 $\quad 6 = 6 \quad$ Add 5

Multiply by 3 $\quad 18 = 18 \quad$ Multiply by 3

Divide by 2 $\quad 9 = 9 \quad$ Divide by 2

Example: Solve for m
Given: $F = ma$, $F = 10$, $a = 2$

A Rearrange the equation, then plug in numbers

$$F = ma$$

Rearrange equation $\quad \frac{F}{a} = m \quad$ Divide both sides by a

Plug in numbers $\quad \frac{(10)}{(2)} = m$

$$5 = m$$

B Plug in numbers, then rearrange the equation

$$F = ma$$

Plug in numbers $\quad (10) = m(2)$

Rearrange equation $\quad 5 = m \quad$ Divide both sides by 2

The Quadratic Formula

If an equation is in this form: $ax^2 + bx + c = 0$

$\boldsymbol{x}$: any unknown variable
a, b, c : constants

the two solutions (values) of x are:

Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

or

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

- Some physics equations require you to solve for an unknown variable in a quadratic equation. If you need to solve for the variable by hand (without having a calculator solve it for you), use the quadratic formula.
- Rearrange the equation so it matches the form shown above where each term is added together. If there is no constant in the place of **a** or **b** then the constant would be 1. If a term is being subtracted then change the equation so you're adding a negative term.
- There will be two solutions: one solution when using the "+" and one solution when using the "-" of the "±". Both solutions may not be possible values for the physical quantity being represented, so double check the solutions.

Example:

$$8 = 2 + 5t - t^2$$

$$t^2 - 5t + 6 = 0$$

$$(1)t^2 + (-5)t + (6) = 0$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$at^2 + bt + c = 0$$

$$t = \frac{-(-5) + \sqrt{(-5)^2 - 4(1)(6)}}{2(1)}$$

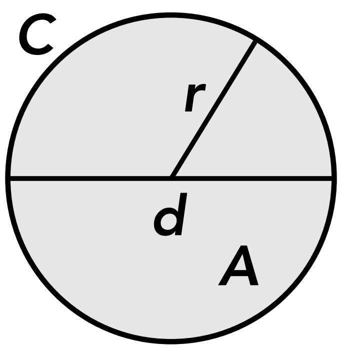
$$t = 3$$

or

$$t = \frac{-(-5) - \sqrt{(-5)^2 - 4(1)(6)}}{2(1)}$$

$$t = 2$$

Circle and Triangle Geometry

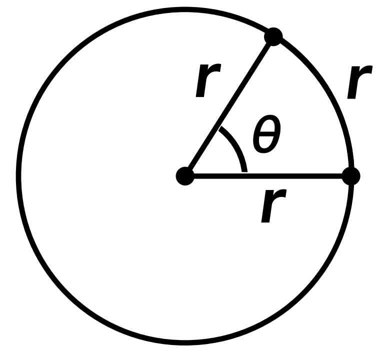


$d = 2r$

$C = 2\pi r = \pi d$

$A = \pi r^2$

r : radius C : circumference
d : diameter A : area

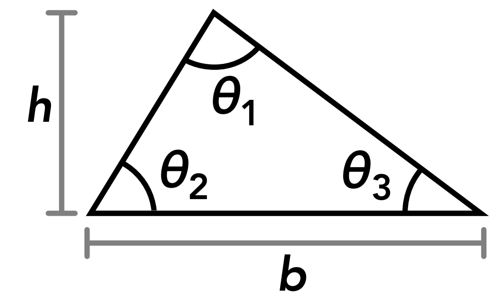


$\theta = 1$ radian

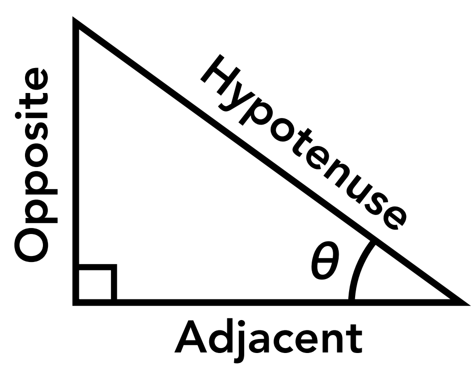
1 revolution = 360° = 2π radians

$\theta_1 + \theta_2 + \theta_3 = 180°$

$\text{Area} = \frac{1}{2}bh$



Right Triangle Trigonometry



SOH - CAH - TOA

SOH	CAH	TOA
Sine	Cosine	Tangent
Opposite	Adjacent	Opposite
Hypotenuse	Hypotenuse	Adjacent

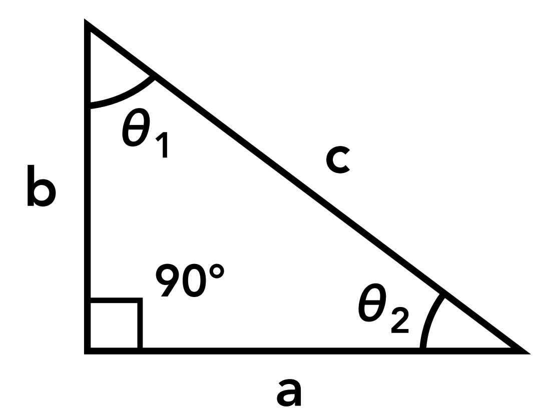
$$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \theta = \sin^{-1}\left(\frac{\text{Opposite}}{\text{Hypotenuse}}\right) = \text{arcsin( )}$$

$$\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \theta = \cos^{-1}\left(\frac{\text{Adjacent}}{\text{Hypotenuse}}\right) = \text{arccos( )}$$

$$\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}} \qquad \theta = \tan^{-1}\left(\frac{\text{Opposite}}{\text{Adjacent}}\right) = \text{arctan( )}$$

Pythagorean Theorem

$c^2 = a^2 + b^2$



$\theta_1 + \theta_2 = 90°$

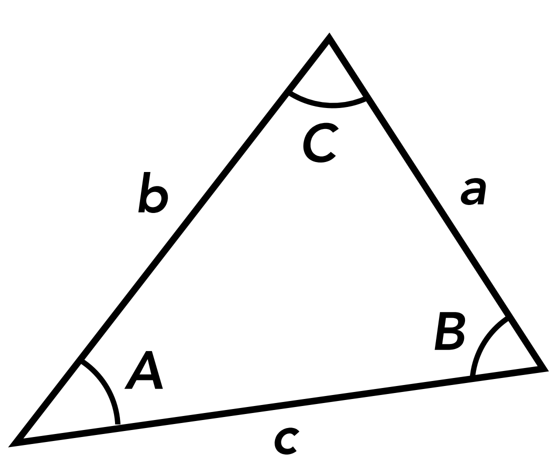
Trigonometric Identities and Laws

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$



Law of sines

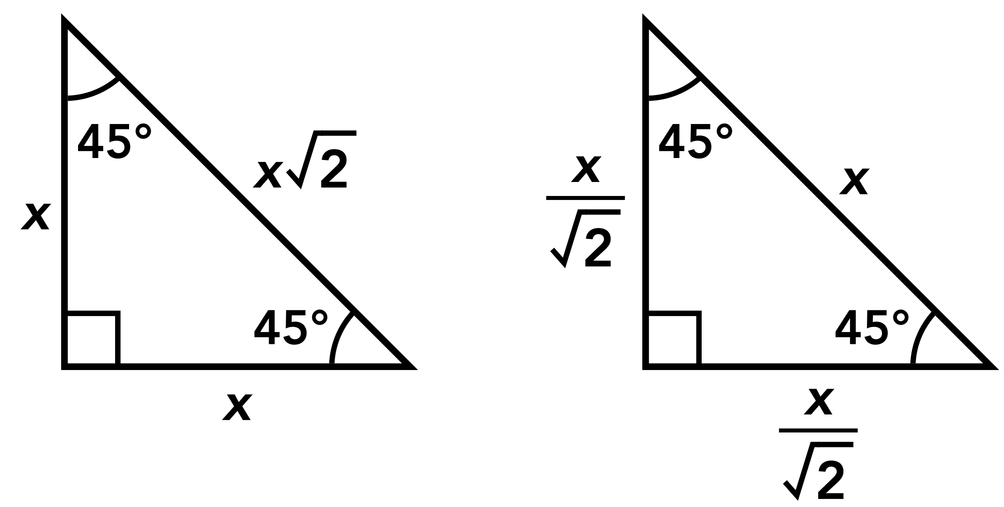
$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

Law of cosines

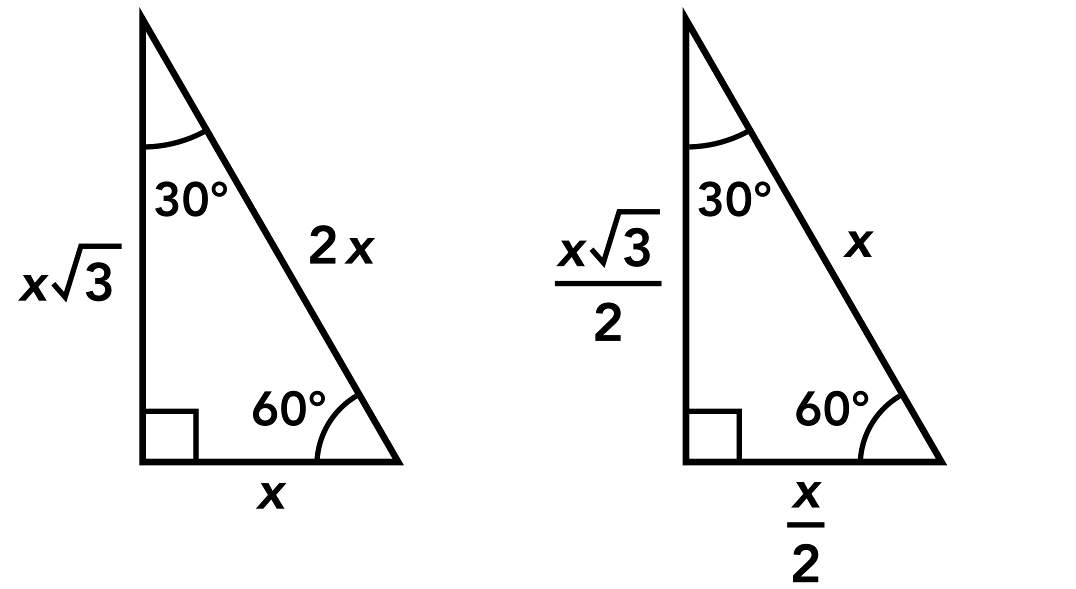
$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

Special Triangles

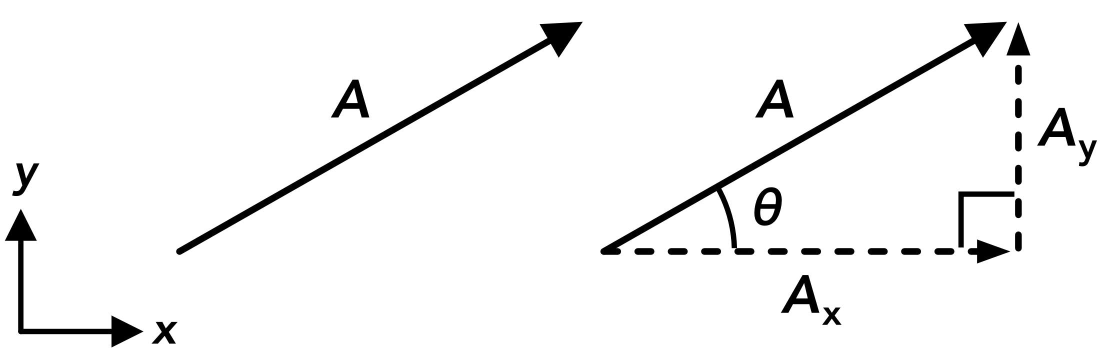
45-45-90 triangle



30-60-90 triangle

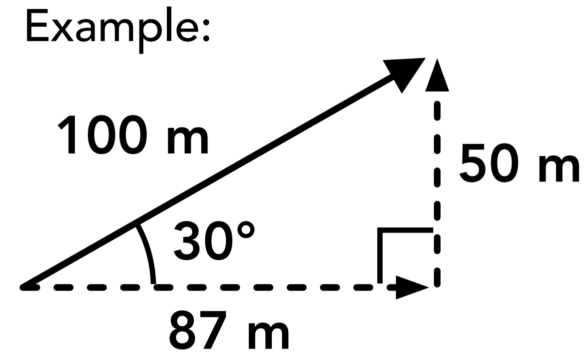


Vectors



A : magnitude
A_x: x component
A_y: y component

Example:



Magnitude and direction:

100 m, 30°

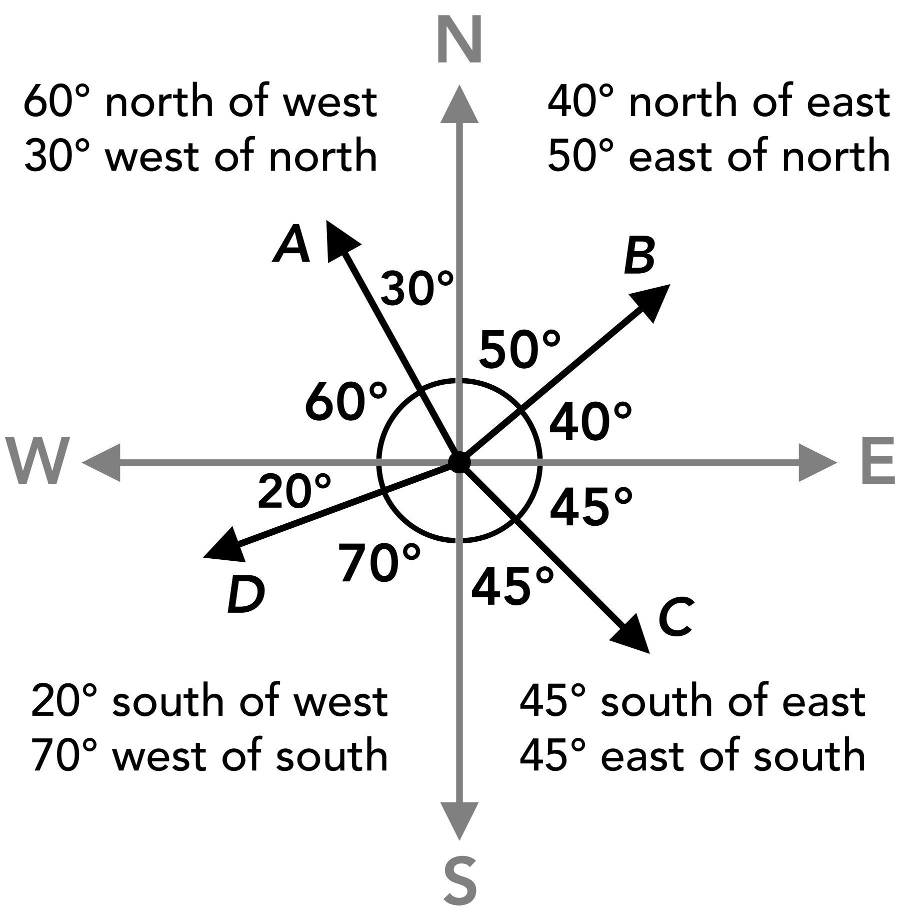
Components:

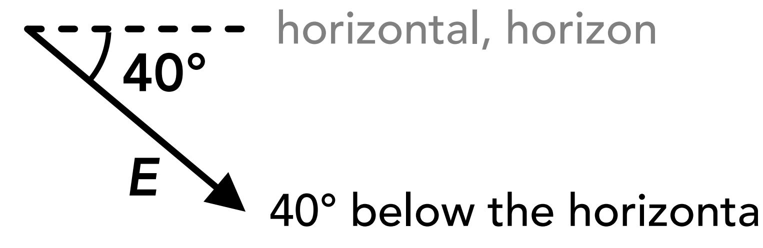
(87 m, 50 m)

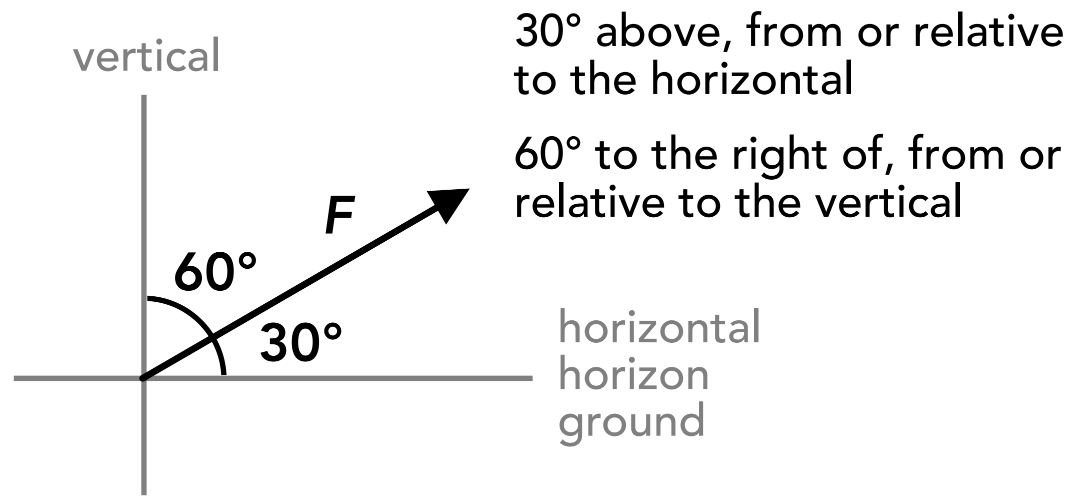
$87\ \text{m}\ \hat{i} + 50\ \text{m}\ \hat{j}$

- A **vector** is a quantity that includes a **magnitude** and a **direction**. Some examples of vector quantities are displacement, velocity, acceleration and force.
- Vectors can be represented graphically as arrows and the ***x*** and ***y* components** represent the amount of the vector that points in the *x* and *y* directions.
- The **magnitude** is the value of the vector (which is always positive) and is represented by the length of the vector arrow.
- The **direction** of a vector is usually described as an angle.
- A vector and its components form a right triangle so we can use right triangle geometry to find the magnitude, angle and components.
- Each vector can be fully described using either the magnitude and direction, or the combination of the *x* and *y* components.

Vector Angles - Using Compass and Other Directions

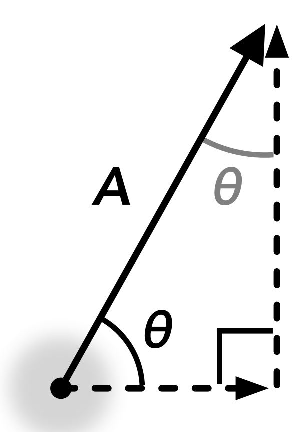




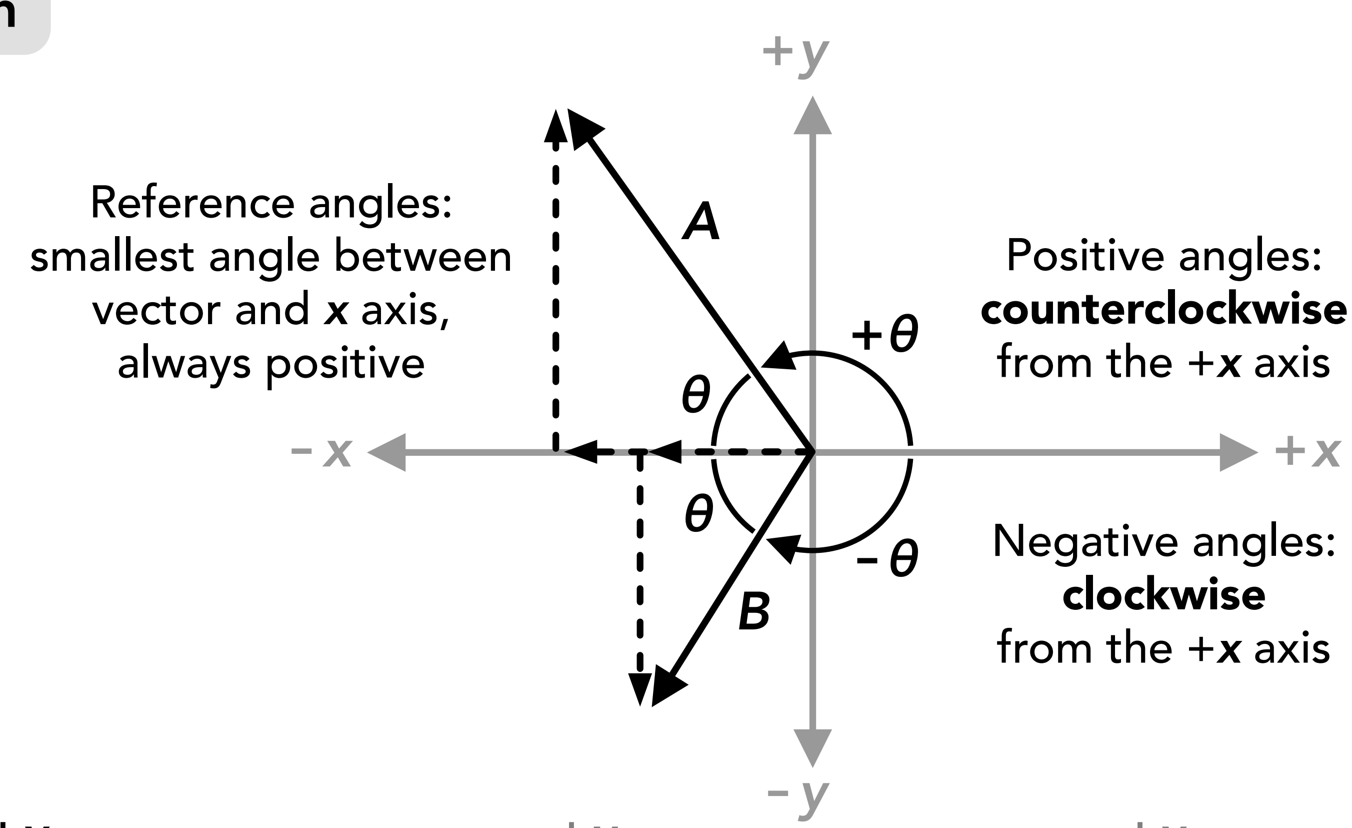


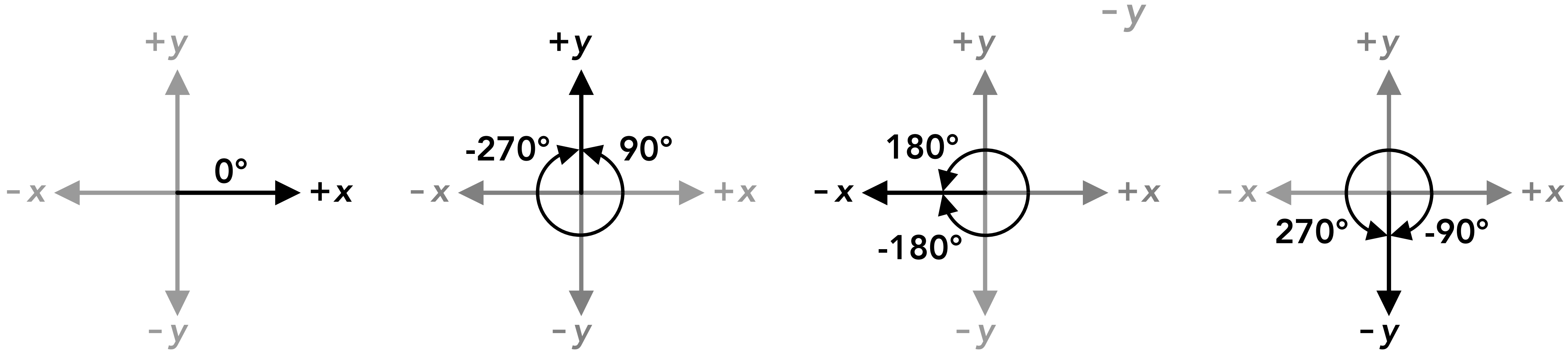
- If an angle is described as "40° north of east" we can imagine a vector that points in the east direction and then rotates 40° so it also points towards the north direction.
- An angle can be described relative to a horizontal line referred to as "the horizontal", "the horizon" or sometimes "the ground" depending on the scenario. An angle can also be described relative to a vertical line or "the vertical".

Vector Angles - Using Convention



A vector's angle is measured at the start of the vector

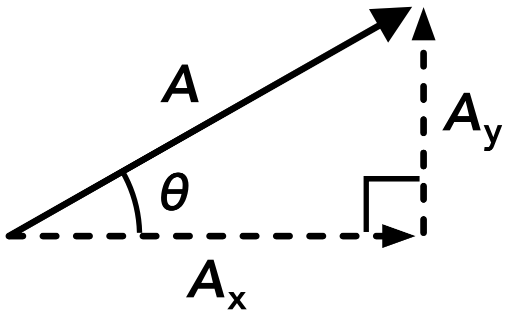




- The conventional way to describe the angle of a vector is counterclockwise from the positive ***x*** axis (0° to 360°).
- If an angle is negative then the angle is clockwise from the positive ***x*** axis (0° to -360°).
- If a vector's angle is described using a single value with no other information (such as "60°") or if the angle is greater than 90°, then the angle is likely the conventional angle.
- This conventional angle (positive or negative, -360° to 360°) can be used with the **sin**( ) and **cos**( ) functions to find the ***x*** and ***y*** components of the vector, and it will result in the correct +/- signs for the component directions.

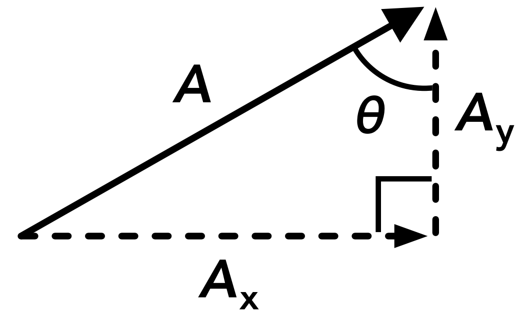
Finding the Components of a Vector

Using a reference angle:



$A_x = A\cos(\theta)$

$A_y = A\sin(\theta)$

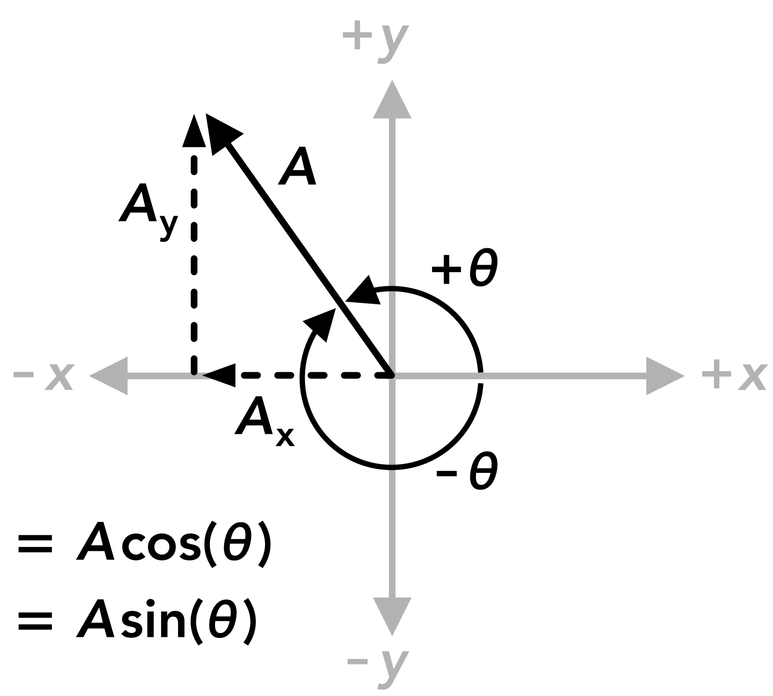


$A_x = A\sin(\theta)$

$A_y = A\cos(\theta)$

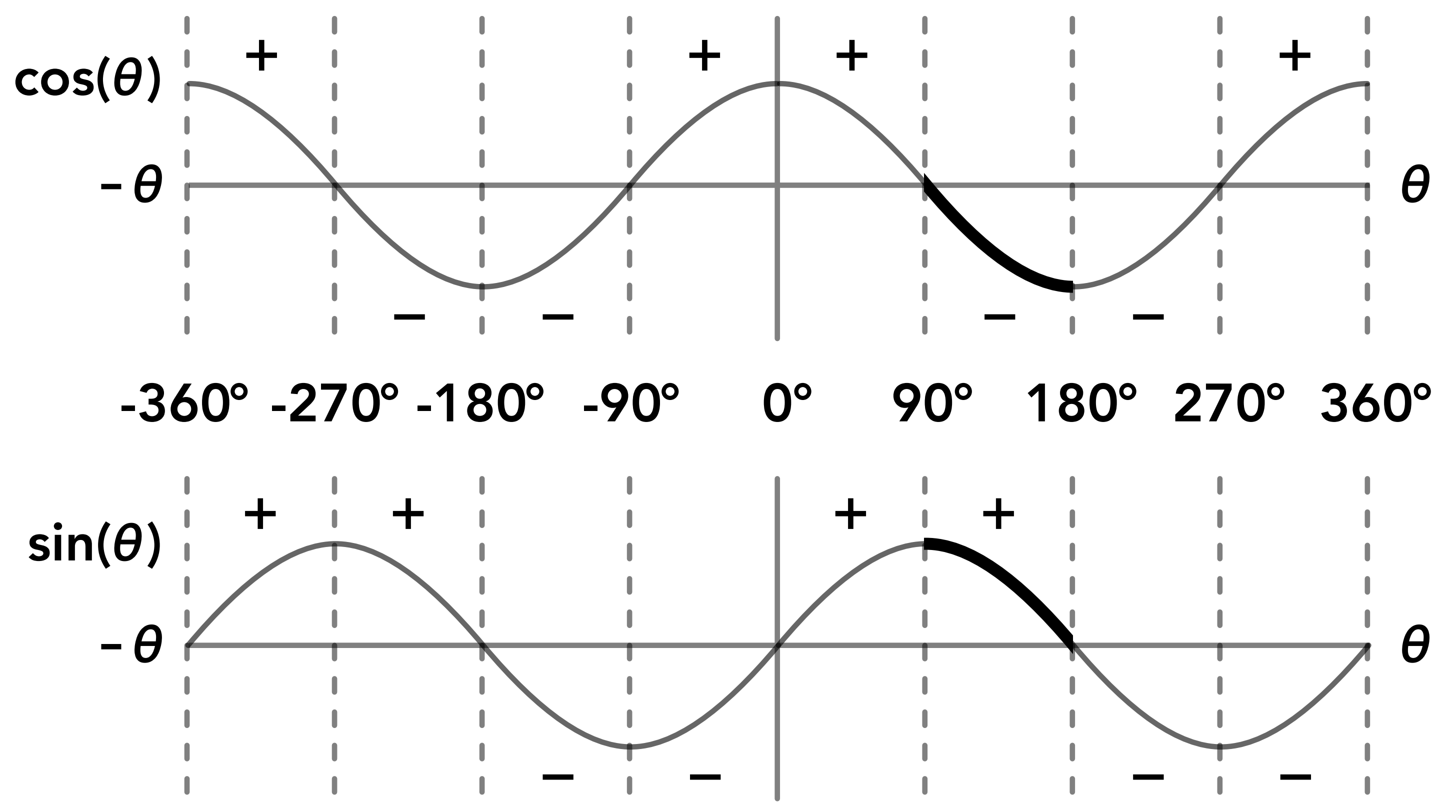
- A vector and its components form a right triangle: the magnitude of a vector is the length of the hypotenuse, and the ***x*** and ***y*** components are the two legs.
- The angle between the vector and the ***x*** component is often used but not always, so don't memorize if the ***x*** or ***y*** components go with **sin**( ) or **cos**( ), just remember how to use the right triangle trig functions.

Using the conventional angle:



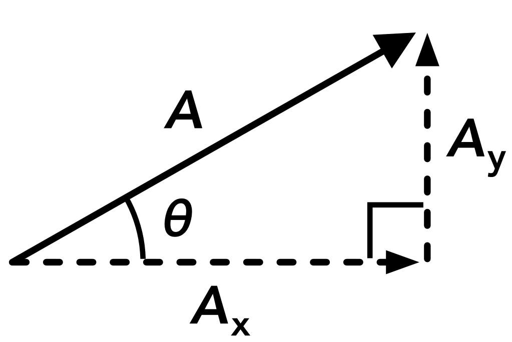
$A_x = A\cos(\theta)$

$A_y = A\sin(\theta)$



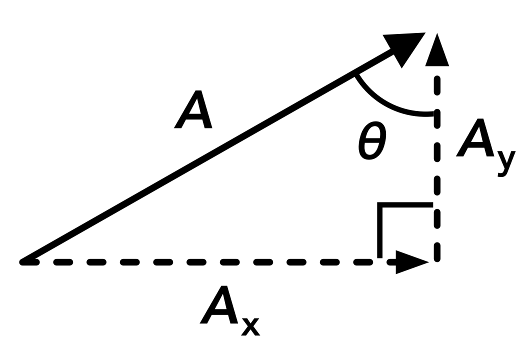
- The conventional angle (-360° to 360°) can be used with the **sin**( ) and **cos**( ) functions to find the components of the vector with the correct +/- signs for the component directions, regardless of the vector's angle.
- The ***x*** component uses **cos**( ) and the ***y*** component uses **sin**( ).

Finding the Magnitude and Angle of a Vector



$$A = \sqrt{A_x^2 + A_y^2}$$

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$



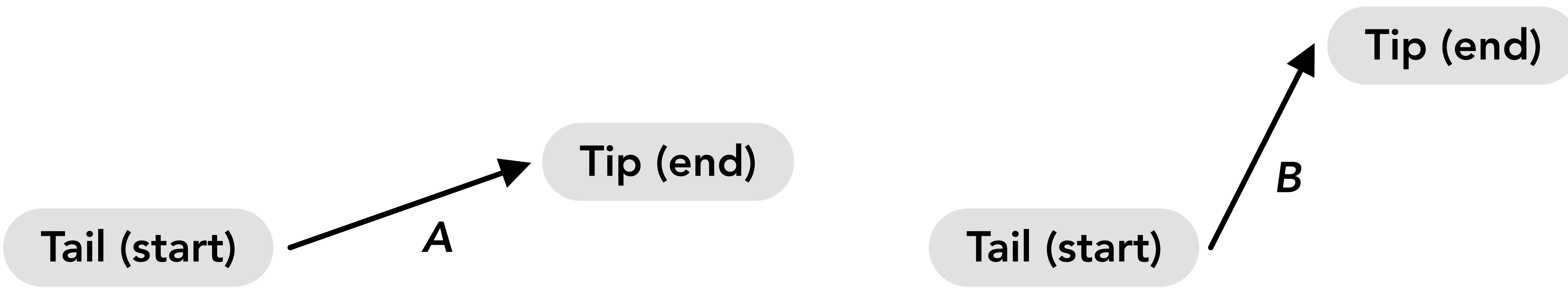
$$A = \sqrt{A_x^2 + A_y^2}$$

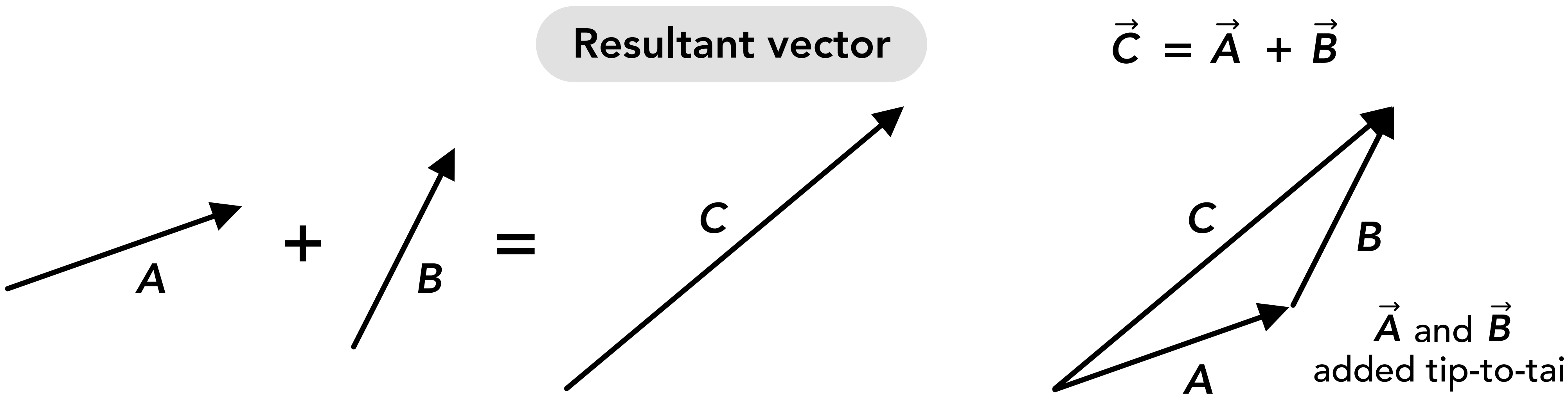
$$\theta = \tan^{-1}\left(\frac{A_x}{A_y}\right)$$

Note: Plug positive values into the **$\tan^{-1}$( )** function and the result will be a positive reference angle

- Use the Pythagorean Theorem to find the magnitude of the vector which is the length of the hypotenuse.
- The inverse **tan**( ) relationship can always be used to find the angle, but once we know the components we can also use one of the other inverse trig relationships.

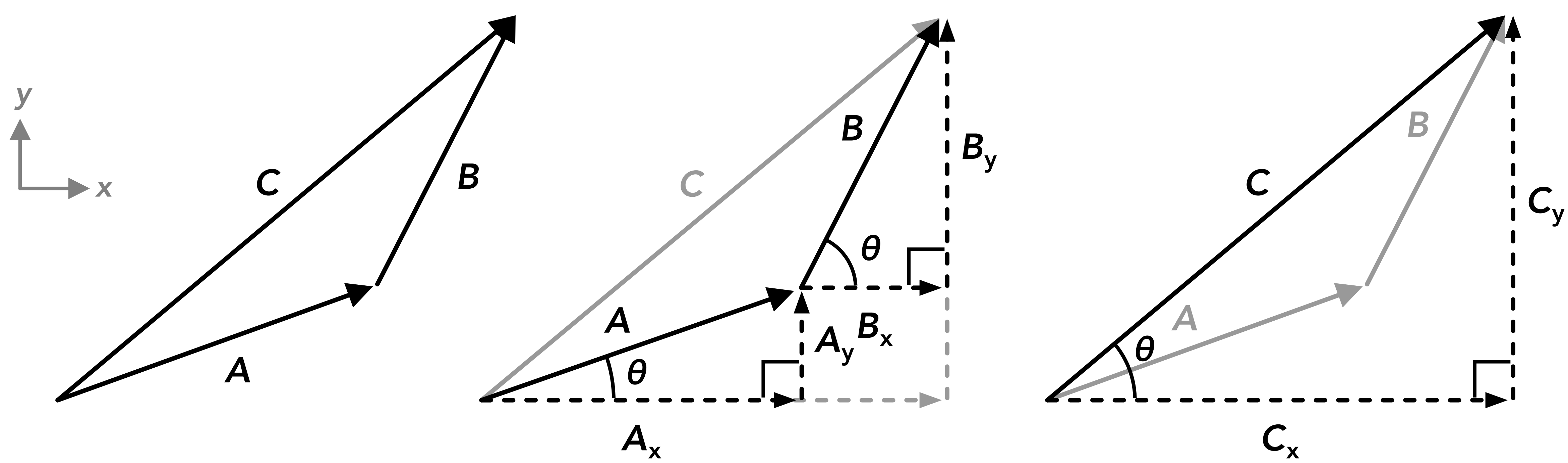
Adding Vectors Graphically Using the Tip-to-Tail Method





- We can add vectors graphically by drawing them out using the **tip-to-tail method**.
- The **tail** is the start of the vector and the **tip** is the end of the vector (the tip of the arrow).
- Each new vector to be added starts at the tip (end) of the previous vector.
- The **resultant** vector is the sum of the other vectors and it points from the tail (start) of the first vector to the tip (end) of the last vector. Any number of vectors can be added together in this way.

Adding Vectors Using Components



- Find the x and y components of each individual vector, add the x components together, then add the y components together.
- Note that a vector is the sum of its 2 component vectors.

$$\vec{C} = \vec{C}_x + \vec{C}_y$$

$$\vec{C} = \vec{A} + \vec{B}$$

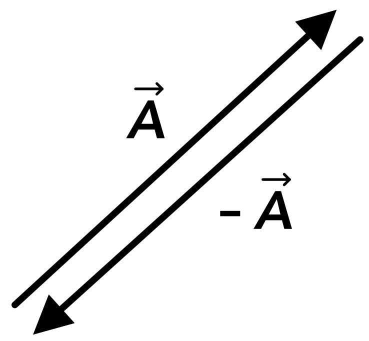
$$\vec{C}_x = \vec{A}_x + \vec{B}_x$$

$$\vec{C}_y = \vec{A}_y + \vec{B}_y$$

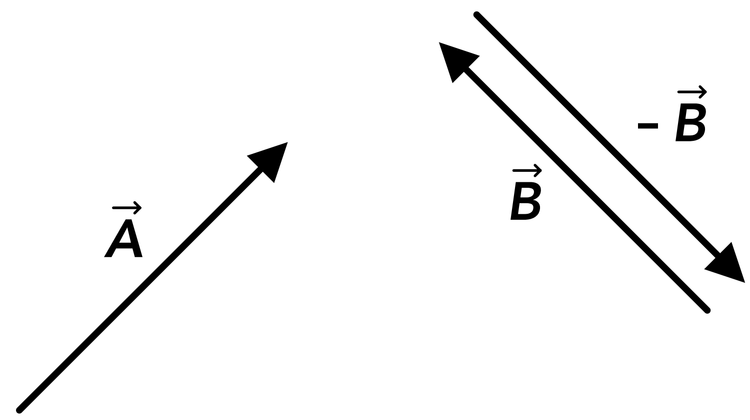
$$C = \sqrt{C_x^2 + C_y^2}$$

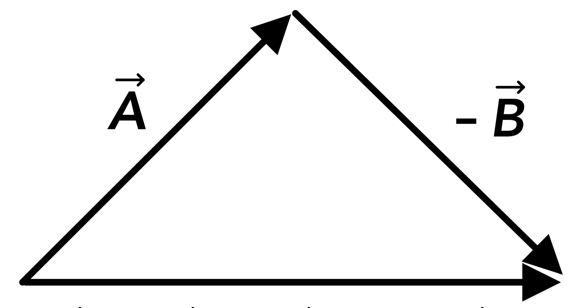
$$\theta = \tan^{-1}\left(\frac{C_y}{C_x}\right)$$

Negative Vectors and Subtracting Vectors



$$\vec{A} + (-\vec{A}) = \vec{0}$$

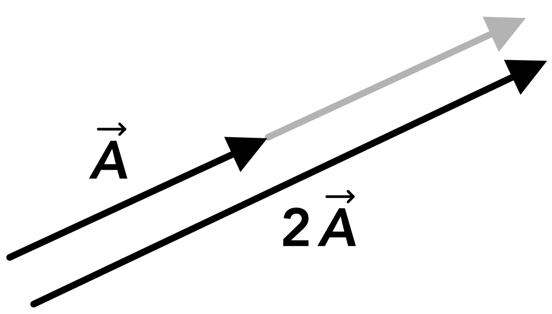


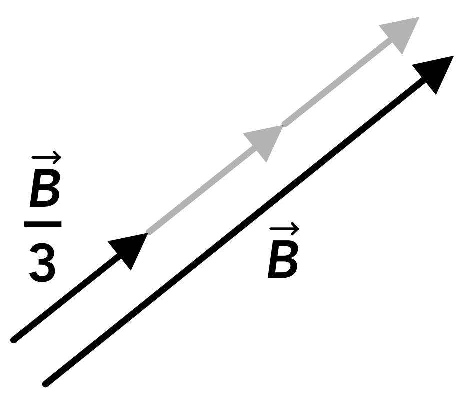


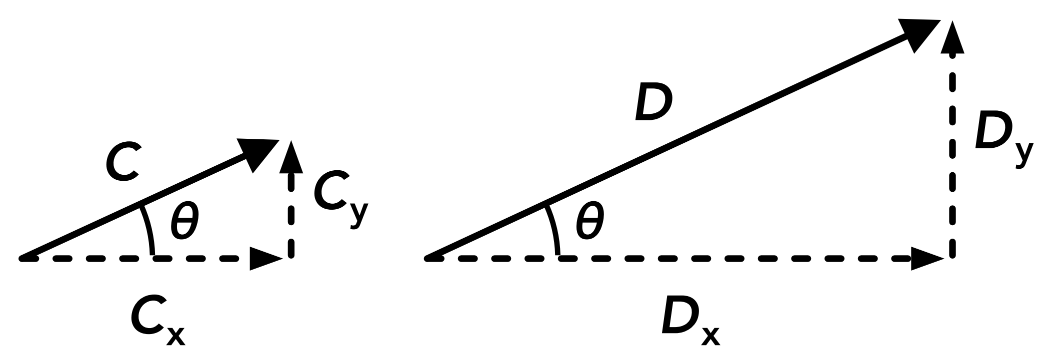
$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

- The negative of a vector has the same magnitude but the opposite direction as the original vector.
- Adding a vector with its negative results in the **0 vector** so they "cancel" each other.
- Subtracting a vector is the same as adding its negative vector.

Multiplying and Dividing Vectors by a Scalar Value







$$\vec{D} = 2\vec{C} \quad \vec{D}_x = 2\vec{C}_x \quad \vec{D}_y = 2\vec{C}_y$$

- Multiplying or dividing a vector by a scalar (a number) scales the magnitude (length) of the vector but doesn't change its direction (angle).
- The components are each scaled (multiplied or divided) by the same value as the vector.

1D MOTION (LINEAR MOTION)

Variables and Kinematic Equations

Variables		SI Unit
t	time	s
x	horizontal position	m
y	vertical position	m
v	velocity	$\frac{m}{s}$
a	acceleration	$\frac{m}{s^2}$

Variable

$$\mathbf{v_x}$$

Subscript

"horizontal velocity"

Subscripts		
i	0	initial
f	_	final
	x	horizontal
	y	vertical

delta

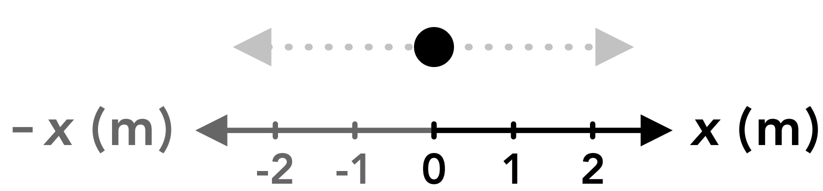
Δ = final − initial

$$\Delta x = x_f - x_i$$

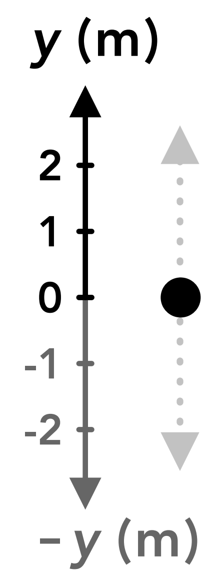
or

$$\Delta x = x - x_0$$

Horizontal motion



Vertical motion

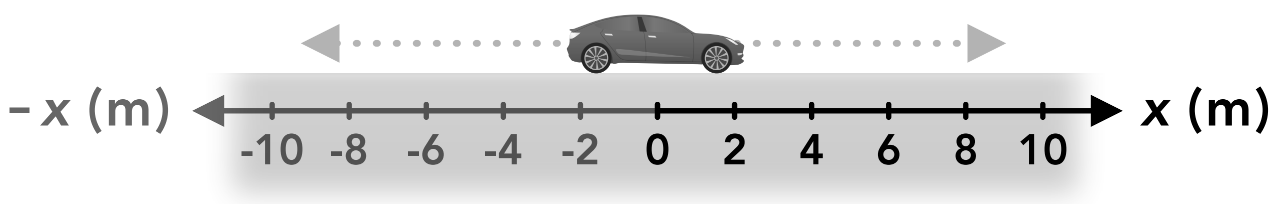


	Horizontal motion	Vertical motion
Displacement:	$\Delta x = x_f - x_i$	$\Delta y = y_f - y_i$
Velocity:	$v_x = \frac{\Delta x}{\Delta t}$	$v_y = \frac{\Delta y}{\Delta t}$
Velocity (rearranged):	$x_f = x_i + v_x \Delta t$	$y_f = y_i + v_y \Delta t$
Acceleration:	$a_x = \frac{\Delta v_x}{\Delta t}$	$a_y = \frac{\Delta v_y}{\Delta t}$
Acceleration (rearranged):	$v_{xf} = v_{xi} + a_x \Delta t$	$v_{yf} = v_{yi} + a_y \Delta t$
Kinematic equations for constant acceleration:	$x_f = x_i + v_{xi} t + \frac{1}{2} a_x t^2$	$y_f = y_i + v_{yi} t + \frac{1}{2} a_y t^2$
	$v_{xf}^2 = v_{xi}^2 + 2 a_x (x_f - x_i)$	$v_{yf}^2 = v_{yi}^2 + 2 a_y (y_f - y_i)$

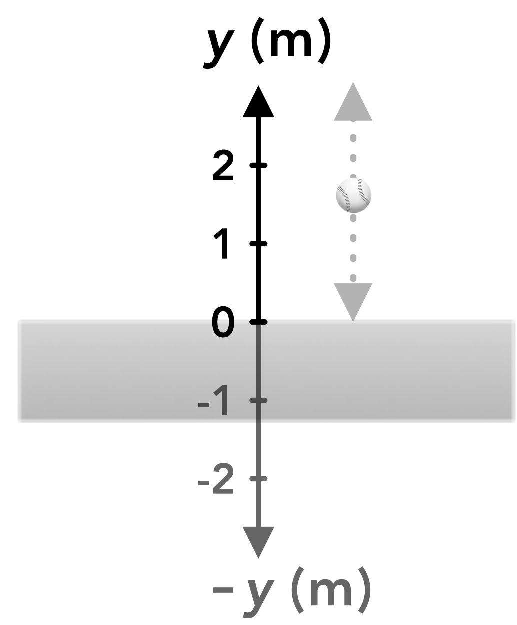
- The kinematic equations for horizontal motion and vertical motion are the same, but we use different variables and subscripts to represent the different directions. Any equation can be rearranged algebraically.
- "Δ_" and "$\_{f} - \_{i}$" are interchangeable. "t" and "Δt" are often used interchangeably.

1D Motion (Linear Motion)

Horizontal motion



Vertical motion

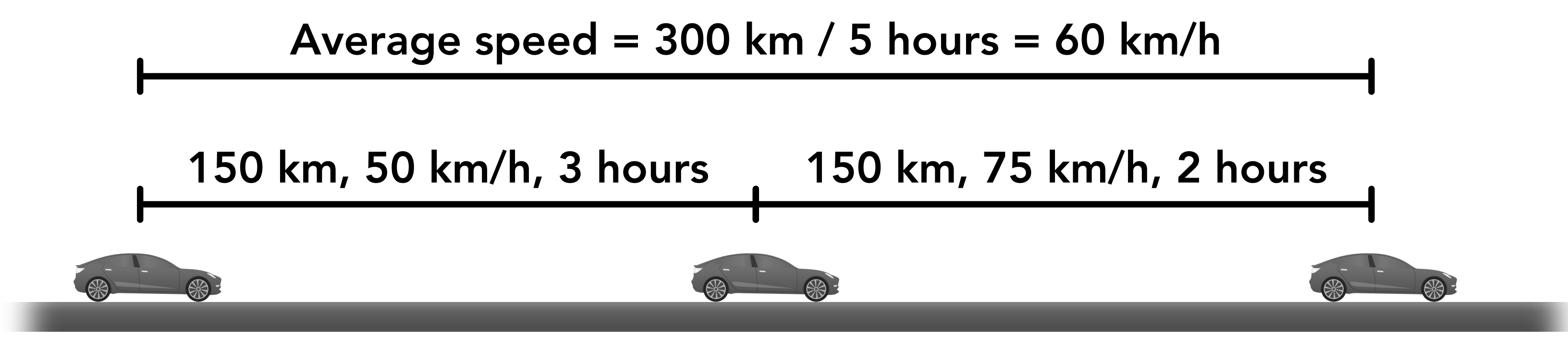


- **1-dimensional motion** (**1D motion** or **linear motion**) is a category of motion where an object only moves along a straight line, in either direction.
- The ***x*** axis is typically used to describe horizontal motion and the ***y*** axis is typically used to describe vertical motion.

Average Speed

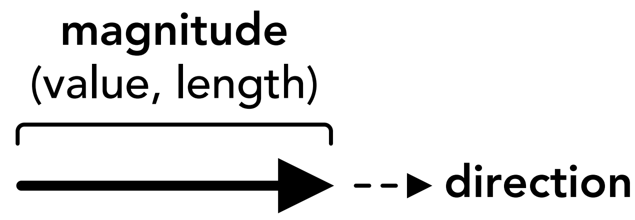
Average speed

$$v_{avg} = \frac{\text{total distance}}{\text{total time}}$$

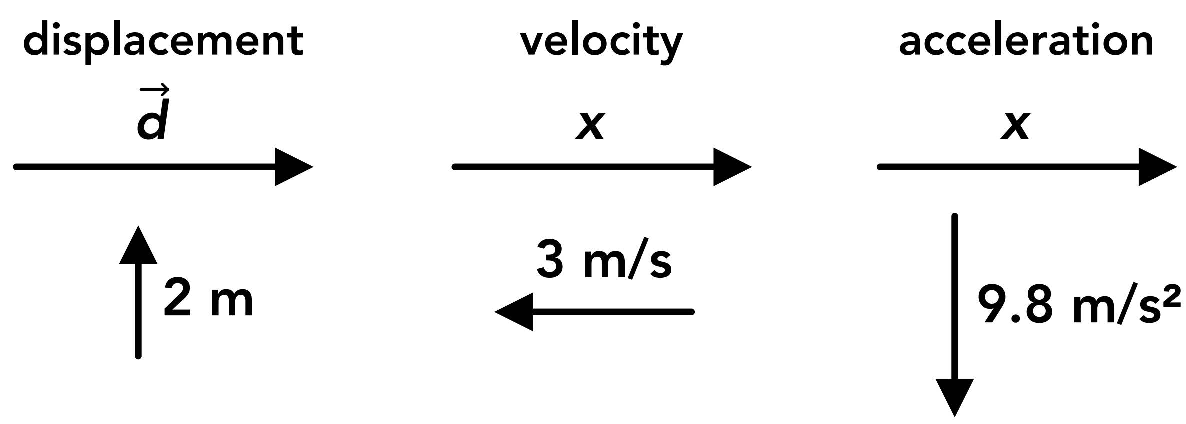


- The **average speed** of an object over some period of time is the total distance traveled divided by the total amount of time. The average speed is not the average or mean of the different speeds during that period.
- Average speed is the "time-weighted-average speed" where each of the different speeds is weighted based on the amount of time spent traveling at that speed (instead of the amount of distance traveled at that speed).

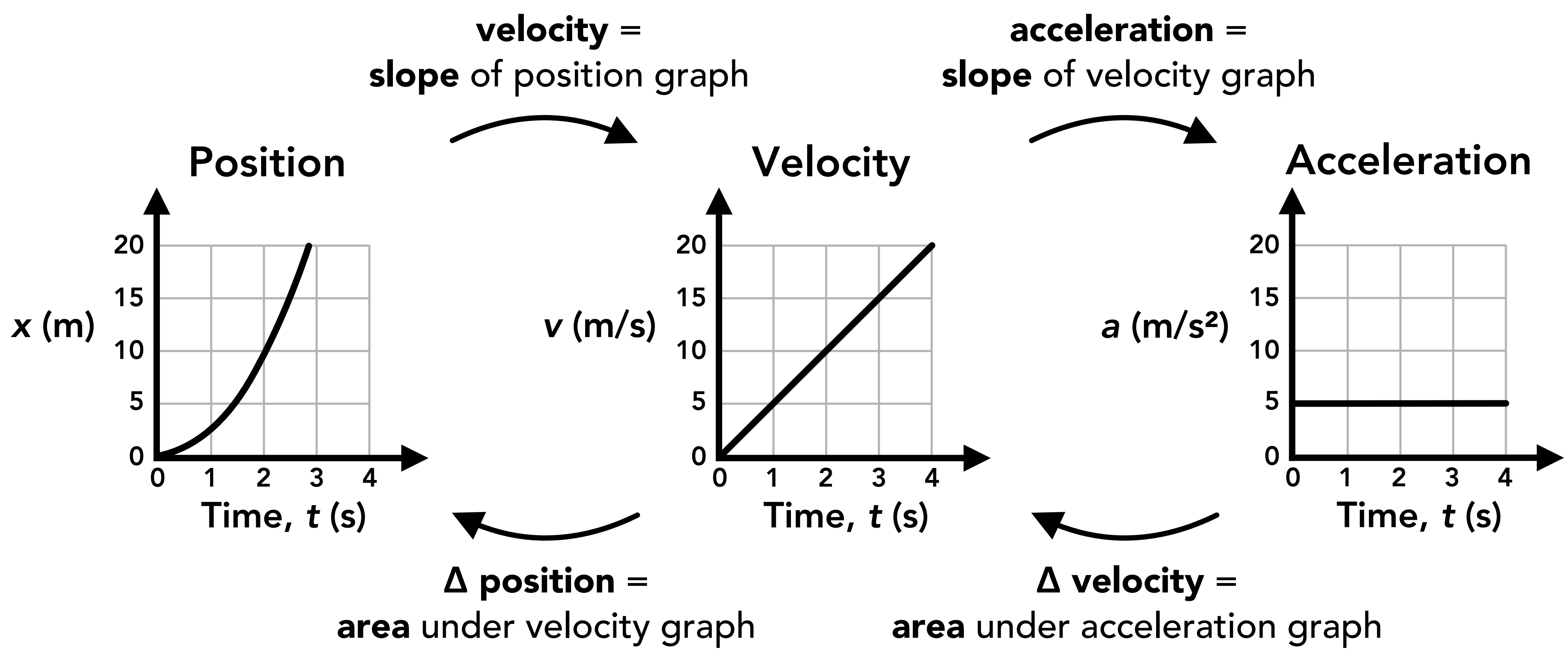
Scalar and Vector Quantities



Scalars	Vectors
distance	**displacement**
5 m	5 m in the north direction
2 km	2 km in the +***x*** direction
speed	**velocity**
8 m/s	8 m/s to the left
60 km/h	60 km/h in the east direction

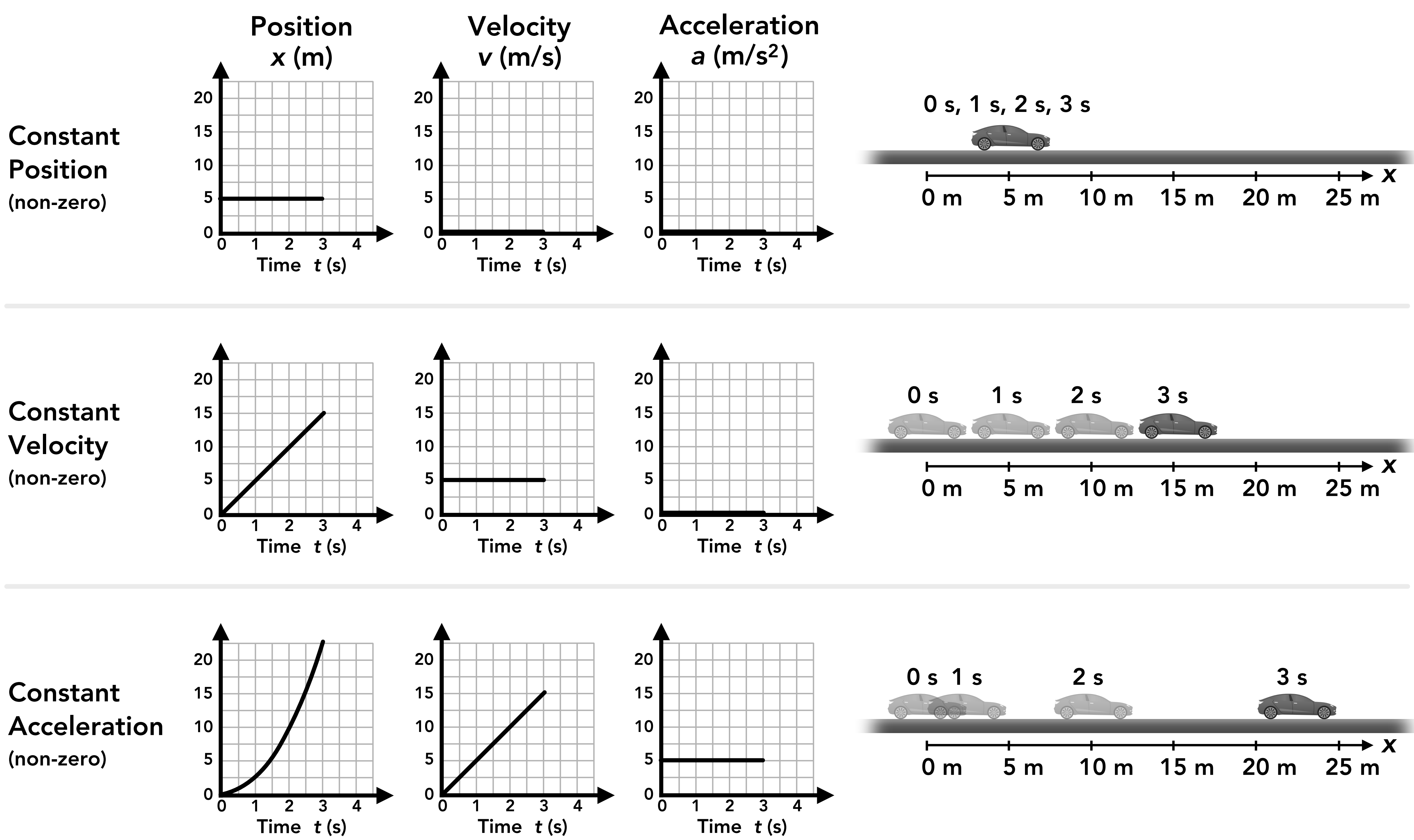


- A **scalar** quantity includes only a **magnitude** (a value) and no direction. Distance and speed are scalars.
- A **vector** quantity includes both a **magnitude** and a **direction**. Displacement and velocity are vectors.
- When a vector quantity is represented using an arrow, the length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector.

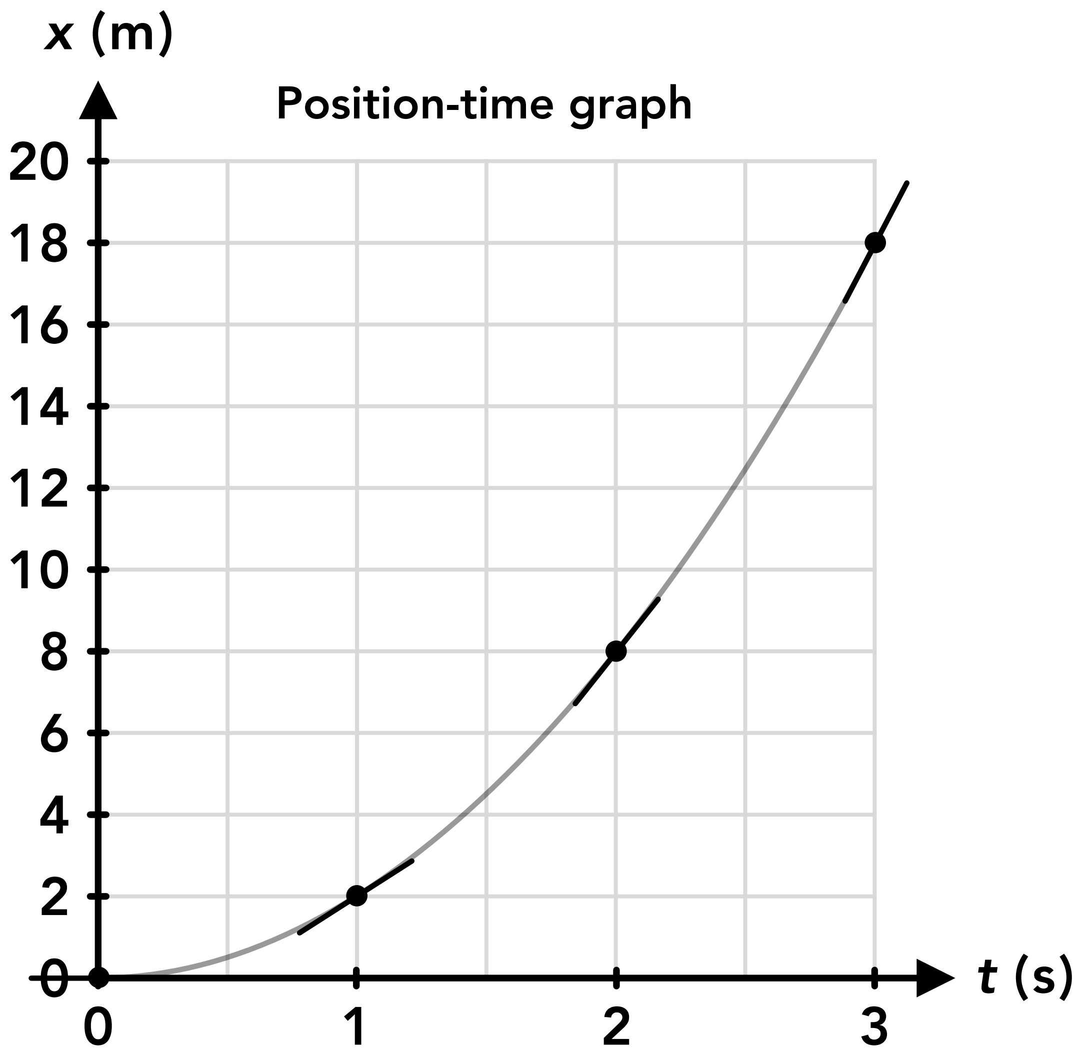


- A **motion graph** shows an object's position, velocity or acceleration over time.
- The instantaneous slope of the position graph (the slope at a single point or instant in time) is the instantaneous velocity of the object at that time. Using calculus, velocity is the derivative of position with respect to time.
- The area under the curve of the velocity graph for a period of time (the area between the graph line and the horizontal axis between two time points) is the displacement or the change in position of the object during that time (areas above the horizontal axis are positive and areas below the horizontal axis are negative). Using calculus, the change in position is the integral of velocity with respect to time.
- The instantaneous slope of the velocity graph is the instantaneous acceleration of the object.
- The area under the curve of the acceleration graph for a period of time is the change in velocity during that time.

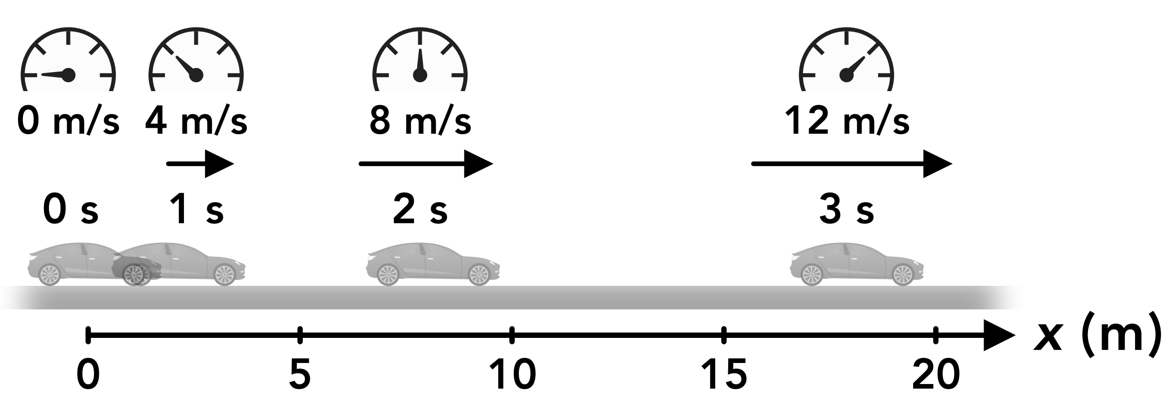
Examples of motion graphs:



Instantaneous velocity

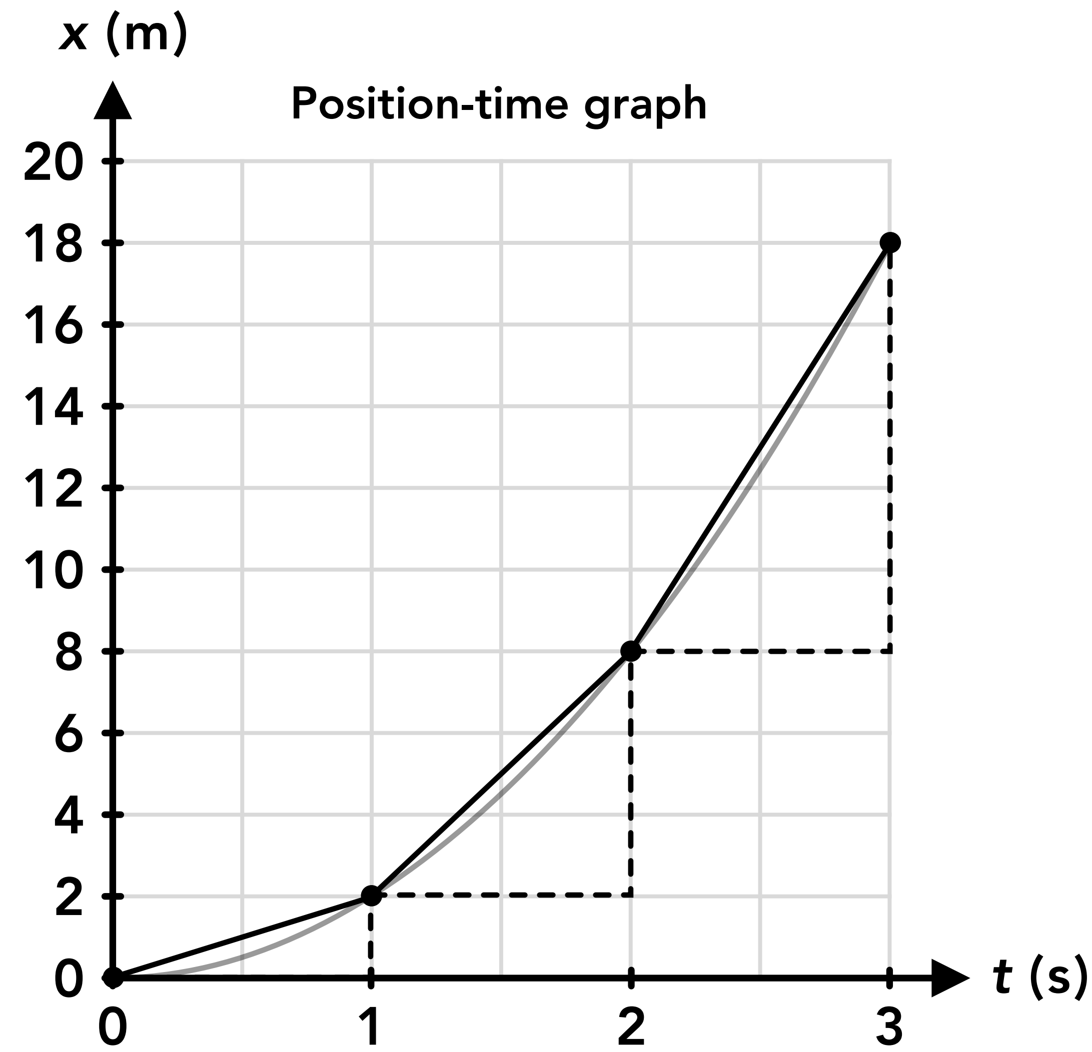


t (s)	x (m)	instantaneous v (m/s)
0	0	**0**
1	2	**4**
2	8	**8**
3	18	**12**

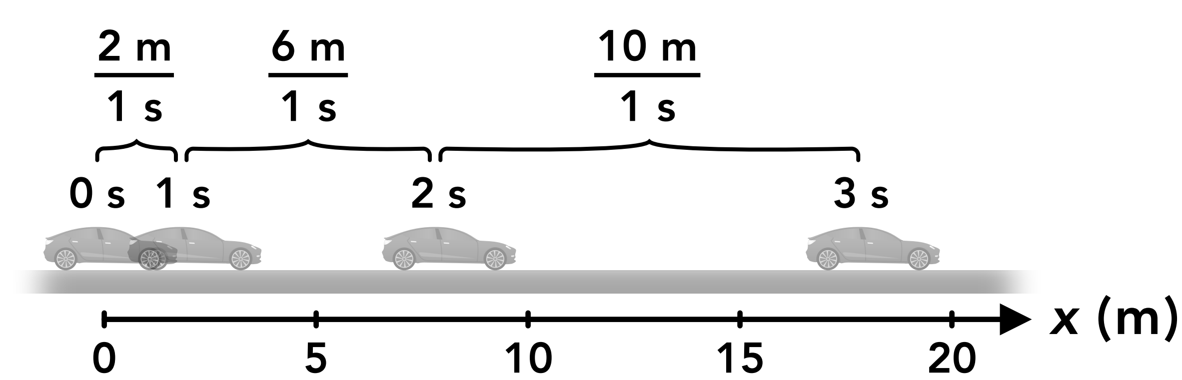


- **Instantaneous velocity** (or instantaneous speed) is the velocity of an object at a single instant in time.
- It is represented as the instantaneous slope of the position-time graph at a single point.

Average velocity



t (s)	x (m)	average v (m/s)
0	0	
1	2	2
2	8	6
3	18	10



- The **average velocity** (or average speed) of an object is the displacement divided by a period of time.
- It is represented as the average slope of the position-time graph for a period of time (between two points).

2D MOTION

Variables and Kinematic Equations

Variables		SI Unit
t	time	s
x	horizontal position	m
y	vertical position	m
v	velocity	$\frac{m}{s}$
a	acceleration	$\frac{m}{s^2}$

Variable

$$v_x$$

Subscript

"horizontal velocity"

Subscripts		
i	0	initial
f	_	final
x		horizontal
y		vertical

delta

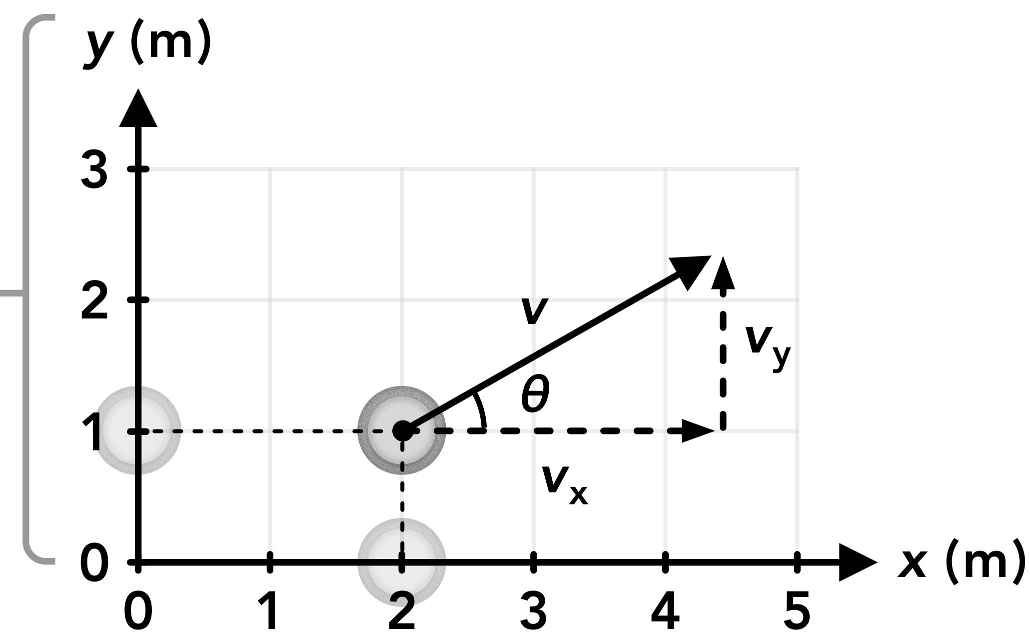
Δ = final − initial

$$\Delta x = x_f - x_i$$

or

$$\Delta x = x - x_0$$

An object's x motion and y motion are <u>independent</u> of each other



	Vertical motion	Horizontal motion
Displacement:	$\Delta y = y_f - y_i$	$\Delta x = x_f - x_i$
Velocity:	$v_y = \frac{\Delta y}{\Delta t}$	$v_x = \frac{\Delta x}{\Delta t}$
Velocity (rearranged):	$y_f = y_i + v_y \Delta t$	$x_f = x_i + v_x \Delta t$
Acceleration:	$a_y = \frac{\Delta v_y}{\Delta t}$	$a_x = \frac{\Delta v_x}{\Delta t}$
Acceleration (rearranged):	$v_{yf} = v_{yi} + a_y \Delta t$	$v_{xf} = v_{xi} + a_x \Delta t$
Kinematic equations for constant acceleration:	$y_f = y_i + v_{yi} t + \frac{1}{2} a_y t^2$	$x_f = x_i + v_{xi} t + \frac{1}{2} a_x t^2$
	$v_{yf}^2 = v_{yi}^2 + 2 a_y (y_f - y_i)$	$v_{xf}^2 = v_{xi}^2 + 2 a_x (x_f - x_i)$

- An object's x motion and y motion are completely independent of each other, so the x and y motions can be described separately. The kinematic equations for an object in 2D motion are just a combination of the 1D kinematic equations for horizontal motion and vertical motion.
- When starting with a 2D displacement, velocity or acceleration vector, the vector can be broken down into its x and y components and the kinematic equations apply to the x and y motions separately.
- When starting with separate x and y motions, a 2D displacement, velocity or acceleration vector can be found by combining the x and y motion components.

2D Position and Coordinates

Coordinates

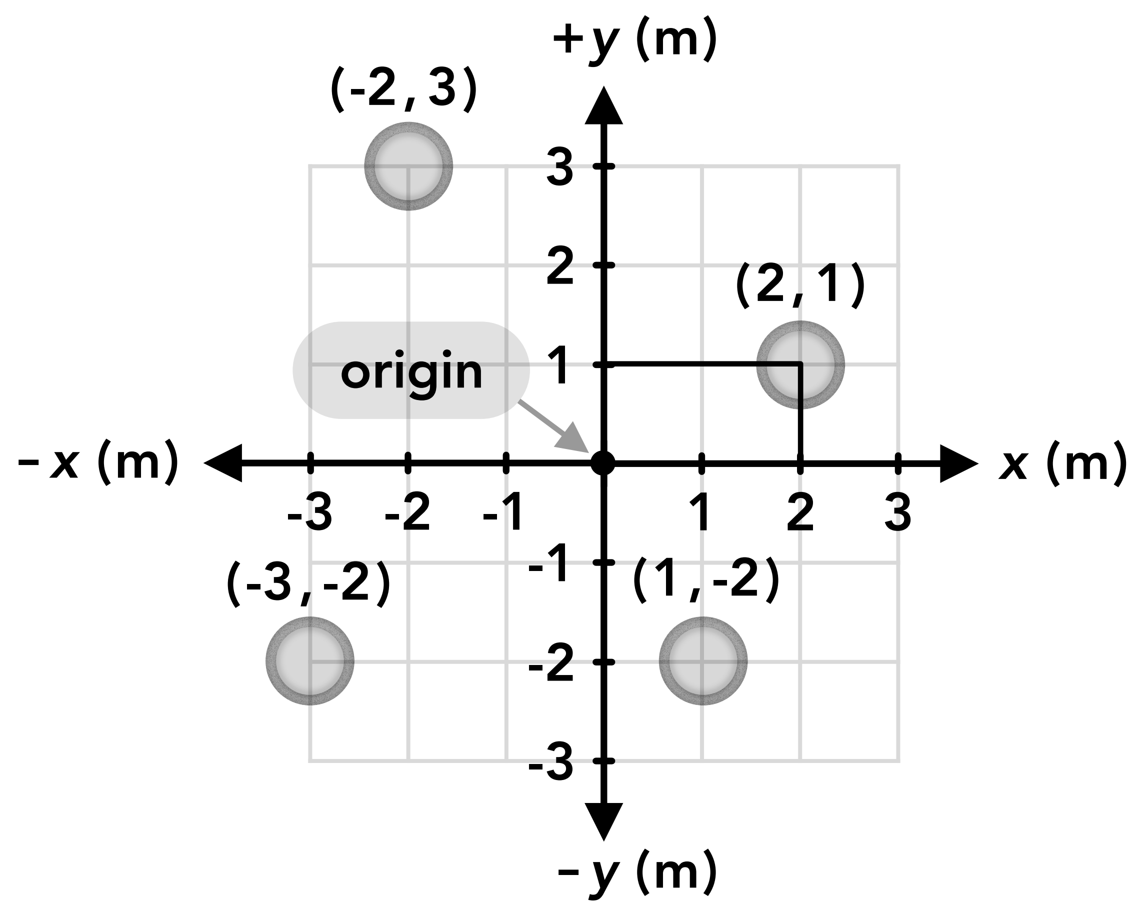
(x, y)

(x position, y position)

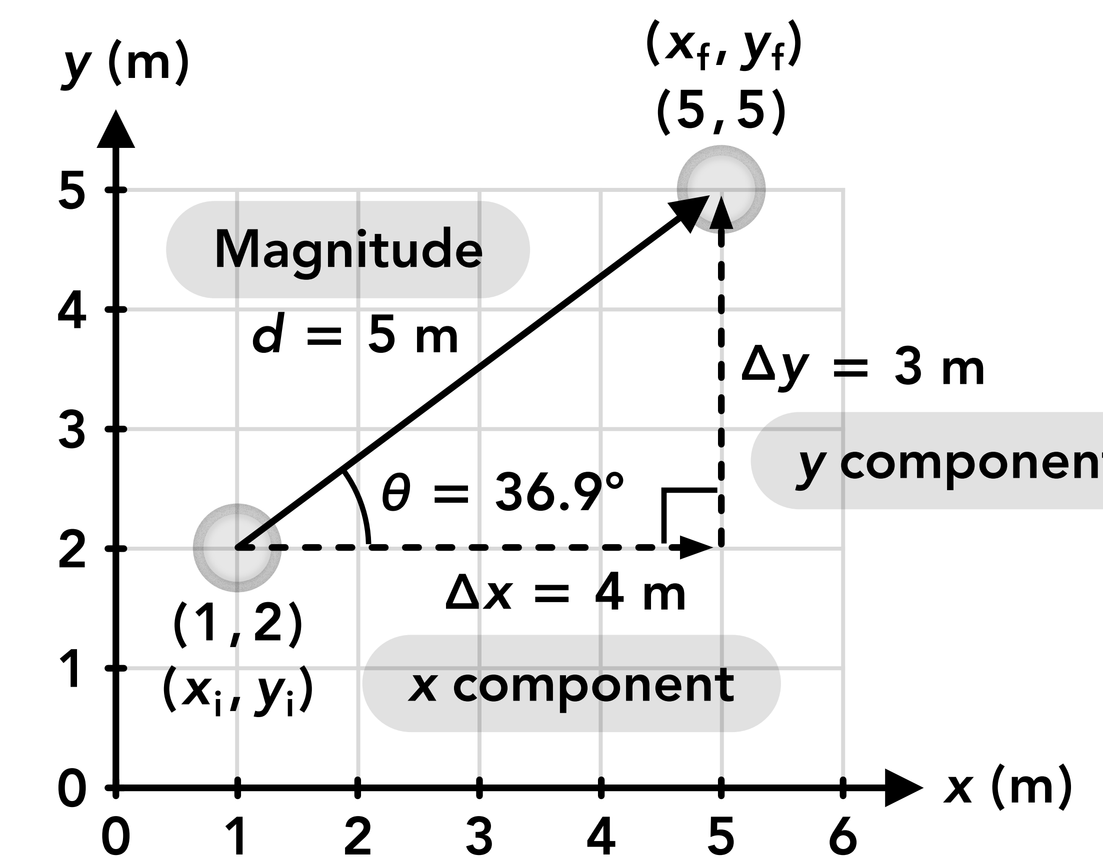
- If an object is in two-dimensional (2D) motion it has an x position and a y position at every moment in time.
- The position of an object or a point in 2D space is described using coordinates on a 2D plane (known as a Cartesian coordinate system).
- **Coordinates** are a pair of values: the first value represents the position of the object along the x axis, the second value represents the position along the y axis.
- The axes of the 2D coordinate system are just like the x and y axes from linear (1D) motion.
- The **origin** of the coordinate system has coordinates (0, 0).

2D coordinate system

(Cartesian coordinate system)



2D Displacement Vectors



Displacement vector

Magnitude and angle:

$d = \sqrt{\Delta x^2 + \Delta y^2}$

$\theta = \tan^{-1}\left(\dfrac{\Delta y}{\Delta x}\right)$

5 m, 36.9°

Components:

$\Delta x = d\cos(\theta)$

$\Delta y = d\sin(\theta)$

$\Delta x = 4\text{ m},\ \Delta y = 3\text{ m}$

x component

x displacement

$\Delta x = x_f - x_i$

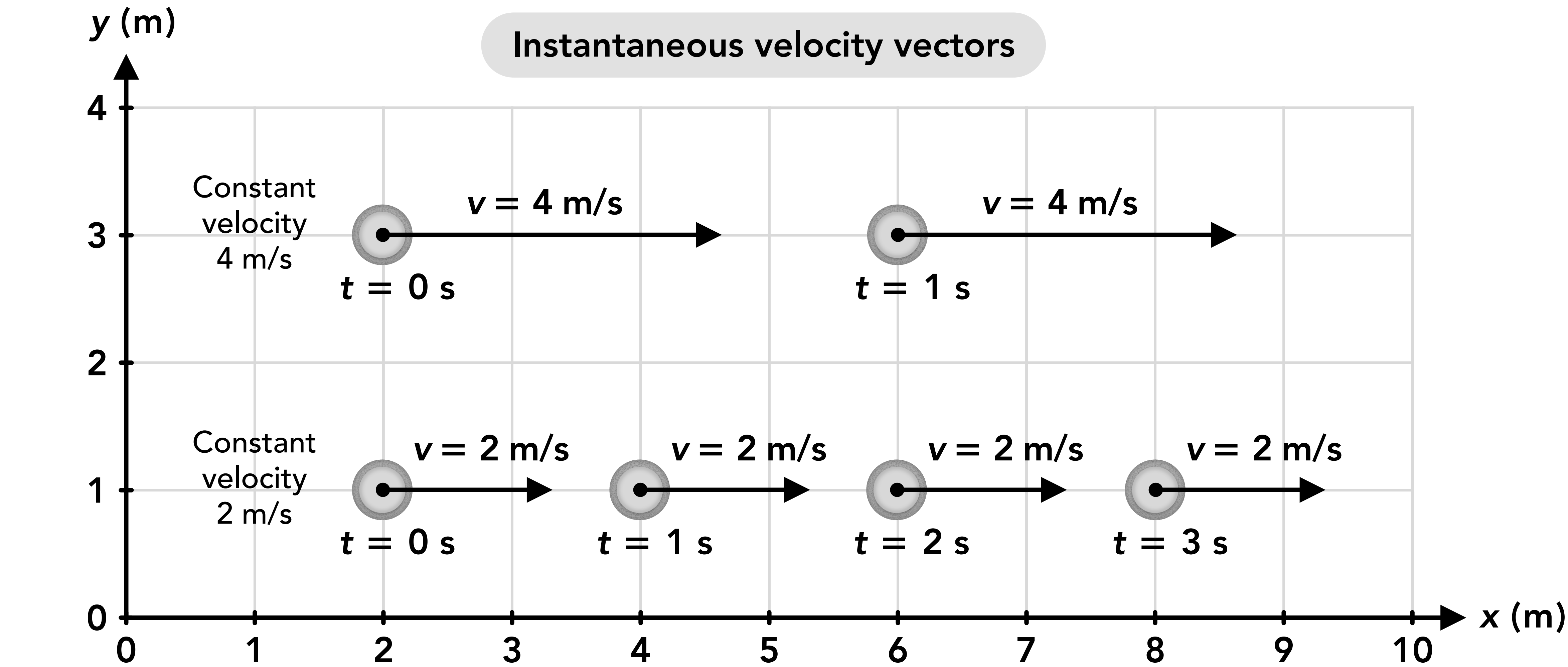
y component

y displacement

$\Delta y = y_f - y_i$

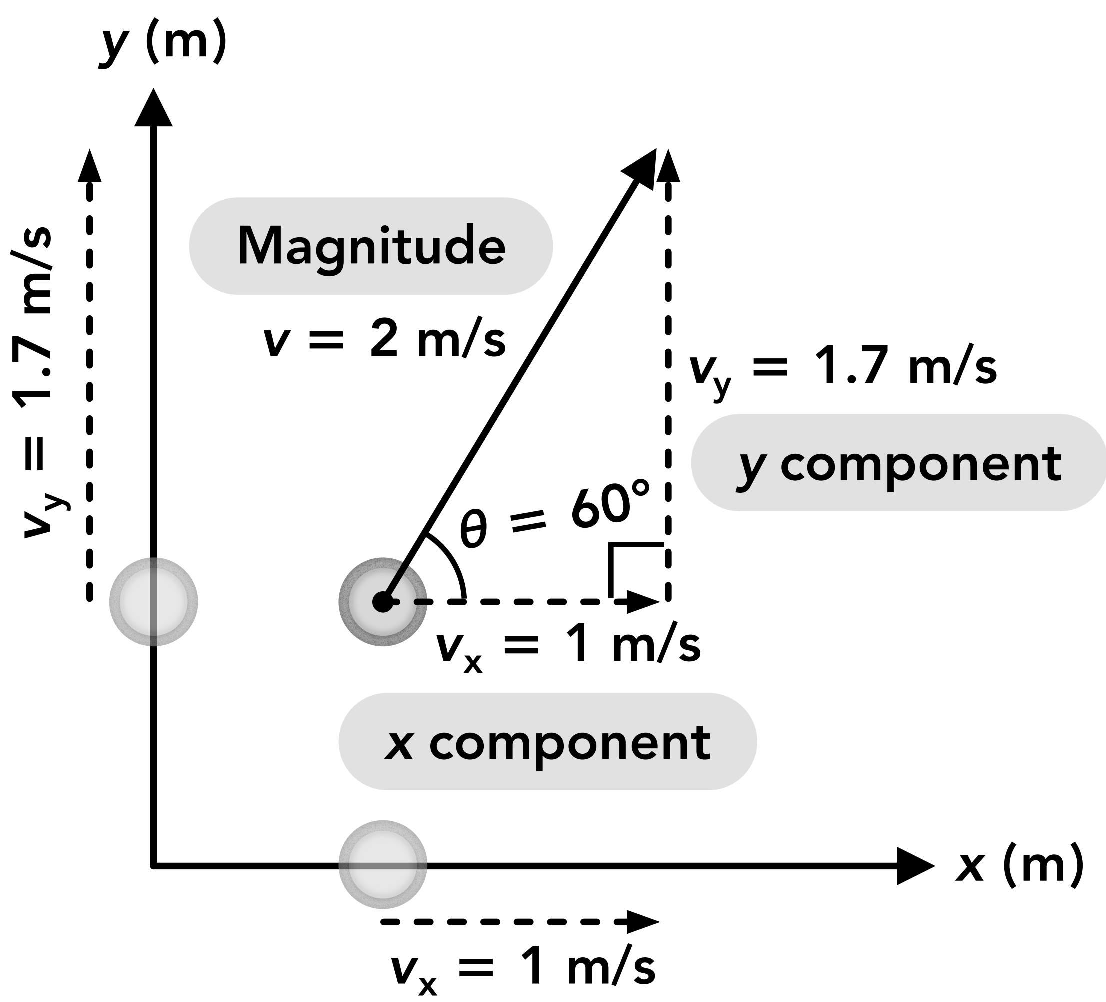
- When an object moves in 2D its x position and y position both change, so it has an x displacement and a y displacement at the same time. The 2D displacement is represented with a vector connecting the initial and final positions, and the x and y displacements are the components of the displacement vector.
- The components of the displacement vector (the x and y displacements) can be calculated using the initial and final coordinates, or using the magnitude and angle of the vector. The magnitude and angle of the vector can be calculated using the x and y components.

Velocity Vectors



- In 1D and 2D motion, the velocity of an object can be represented using a velocity vector.
- This is usually representing the instantaneous velocity of the object: the magnitude (speed) and direction of the velocity at an instant in time, as opposed to an average velocity.
- When comparing the lengths of several vectors, the length of the vector represents the magnitude of the velocity (the speed). Otherwise, the length of the velocity vector is arbitrary.

2D Velocity Vectors



Velocity vector

Magnitude and angle:

$v = \sqrt{v_x^2 + v_y^2}$

$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$

2 m/s, 60°

Components:

$v_x = v\cos(\theta)$

$v_y = v\sin(\theta)$

$v_x = 1$ m/s, $v_y = 1.7$ m/s

x component

x velocity

$$v_x = \frac{\Delta x}{\Delta t}$$

y component

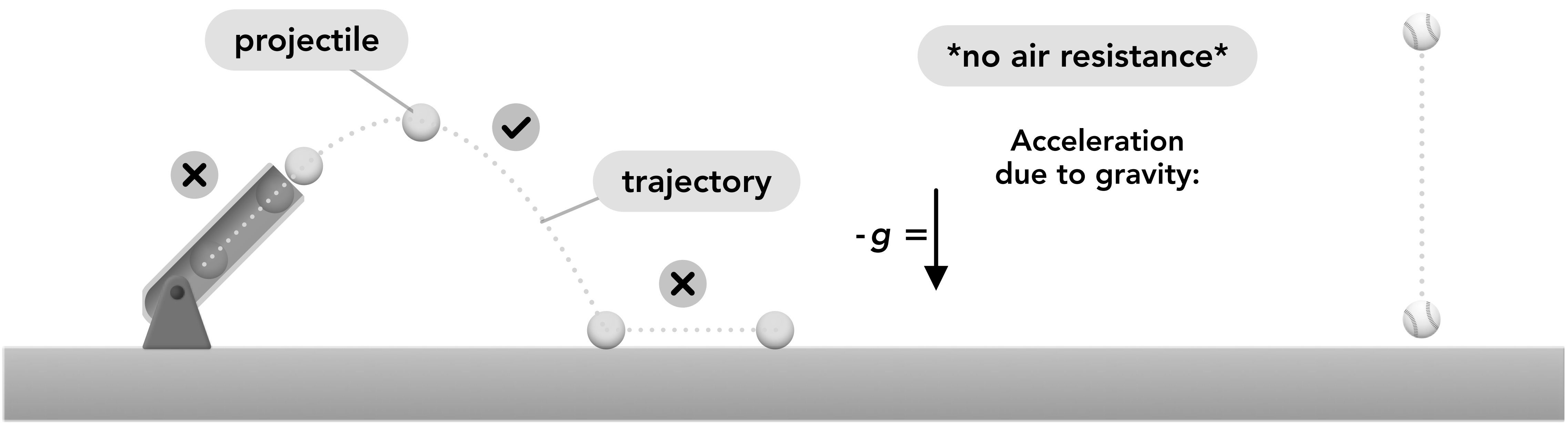
y velocity

$$v_y = \frac{\Delta y}{\Delta t}$$

- When an object moves in 2D its ***x*** and ***y*** positions are changing at the same time, so it has an ***x*** velocity and a ***y*** velocity at every moment. The velocity is represented as a vector, and the ***x*** and ***y*** velocities are the components.
- The velocity components can be thought of as the velocities of the object's shadows along the ***x*** and ***y*** axes.
- The components of the velocity vector (the ***x*** and ***y*** velocities) can be calculated using the magnitude and angle of the vector. The magnitude and angle of the vector can be calculated using the ***x*** and ***y*** components.

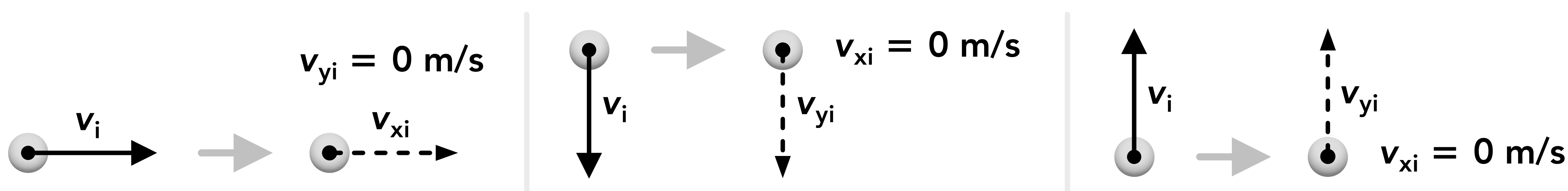
PROJECTILE MOTION

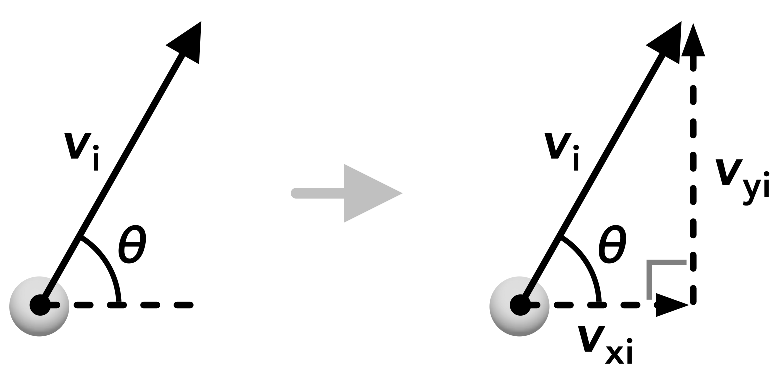
Projectile Motion



- **Projectile motion** is the motion of an object while it's only being affected by gravity and no other forces. An object is only in projectile motion while in the air, not when it's touching the ground or other objects.
- Also referred to as "free fall" (although "free fall" may only refer to 1D projectile motion).
- The vertical acceleration is the acceleration due to gravity, g = 9.8 m/s², which always acts downwards.
- There is no horizontal acceleration so the horizontal velocity is constant.
- The object in projectile motion is called a **projectile** and the path is called the **trajectory**.

Initial Velocity Vector



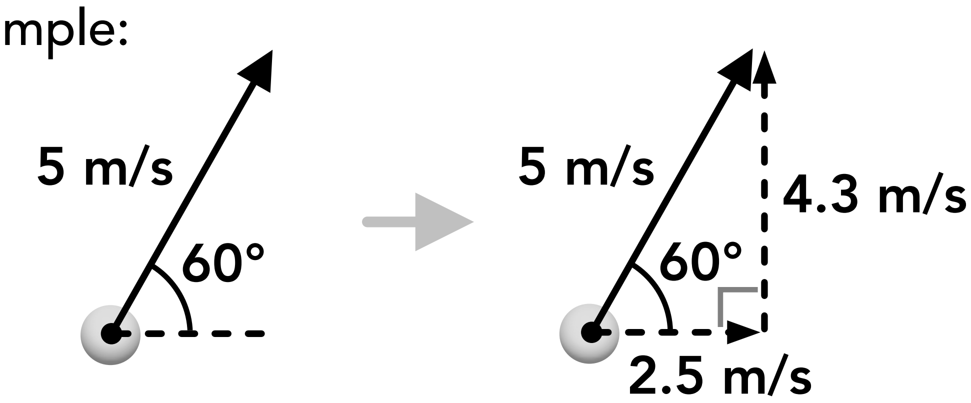


$$\cos(\theta) = \frac{v_{xi}}{v_i} \qquad \sin(\theta) = \frac{v_y}{v_i}$$

$$v_{xi} = v_i \cos(\theta) \qquad v_{yi} = v_i \sin(\theta)$$

*cos and sin are reversed if the other angle is used

Example:



$$v_{xi} = v_i \cos(\theta) \qquad v_{yi} = v_i \sin(\theta)$$

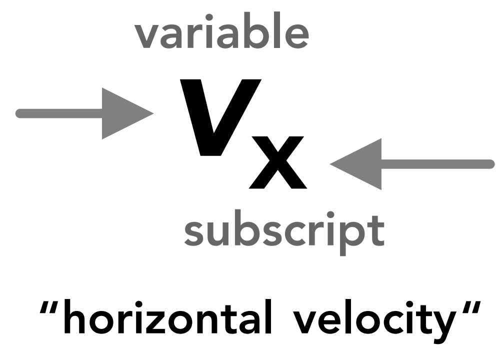
$$v_{xi} = (5\text{ m/s})\cos(60°) \qquad v_{yi} = (5\text{ m/s})\sin(60°)$$

$$v_{xi} = 2.5\text{ m/s} \qquad v_{yi} = 4.3\text{ m/s}$$

- The components of the initial velocity vector are the initial horizontal velocity and the initial vertical velocity.
- The angle is usually between the vector and the horizontal component but double check which angle is given before using the trig functions (cosine for the adjacent component and sine for the opposite component).
- If the vector is vertical or horizontal then the parallel component is equal to the vector (and the other is zero).
- The initial horizontal velocity will be the horizontal velocity for the entire motion.
- The initial vertical velocity can be used with the vertical motion kinematic equations.

Kinematic Equations and Variables

Variables		SI Unit
t	time	s
x	horizontal position	m
y	vertical position	m
v	velocity	$\frac{m}{s}$
a	acceleration	$\frac{m}{s^2}$



Subscripts		
i	0	initial
f	_	final
x		horizontal
y		vertical

delta

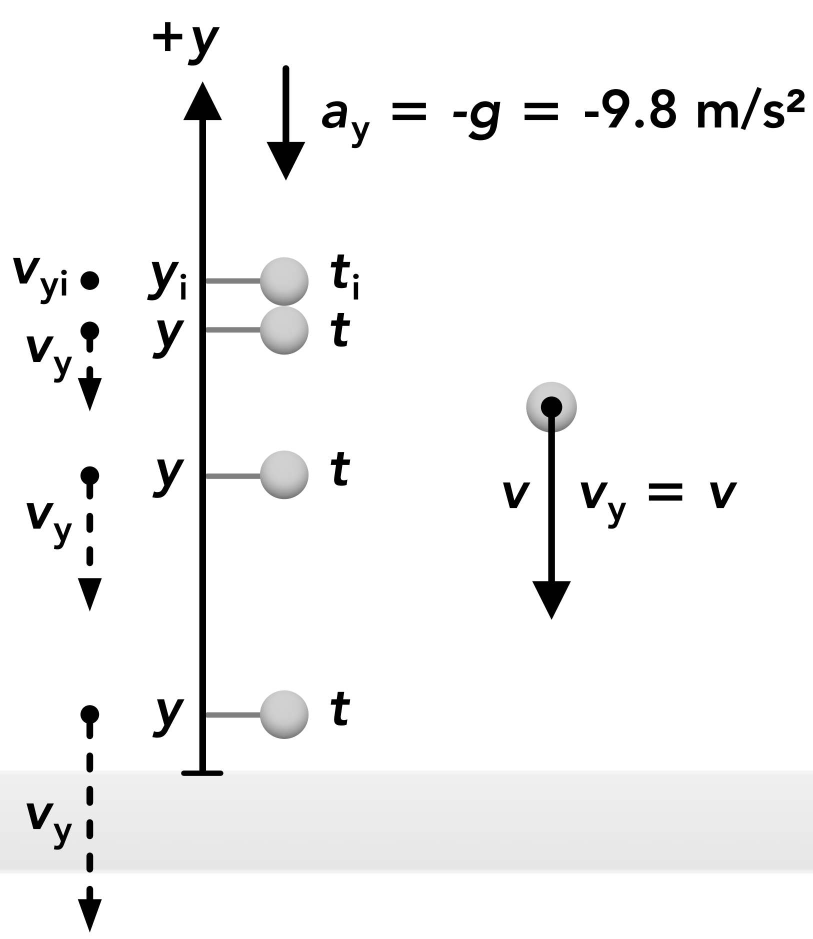
Δ = final − initial

$\Delta x = x_f - x_i$

or

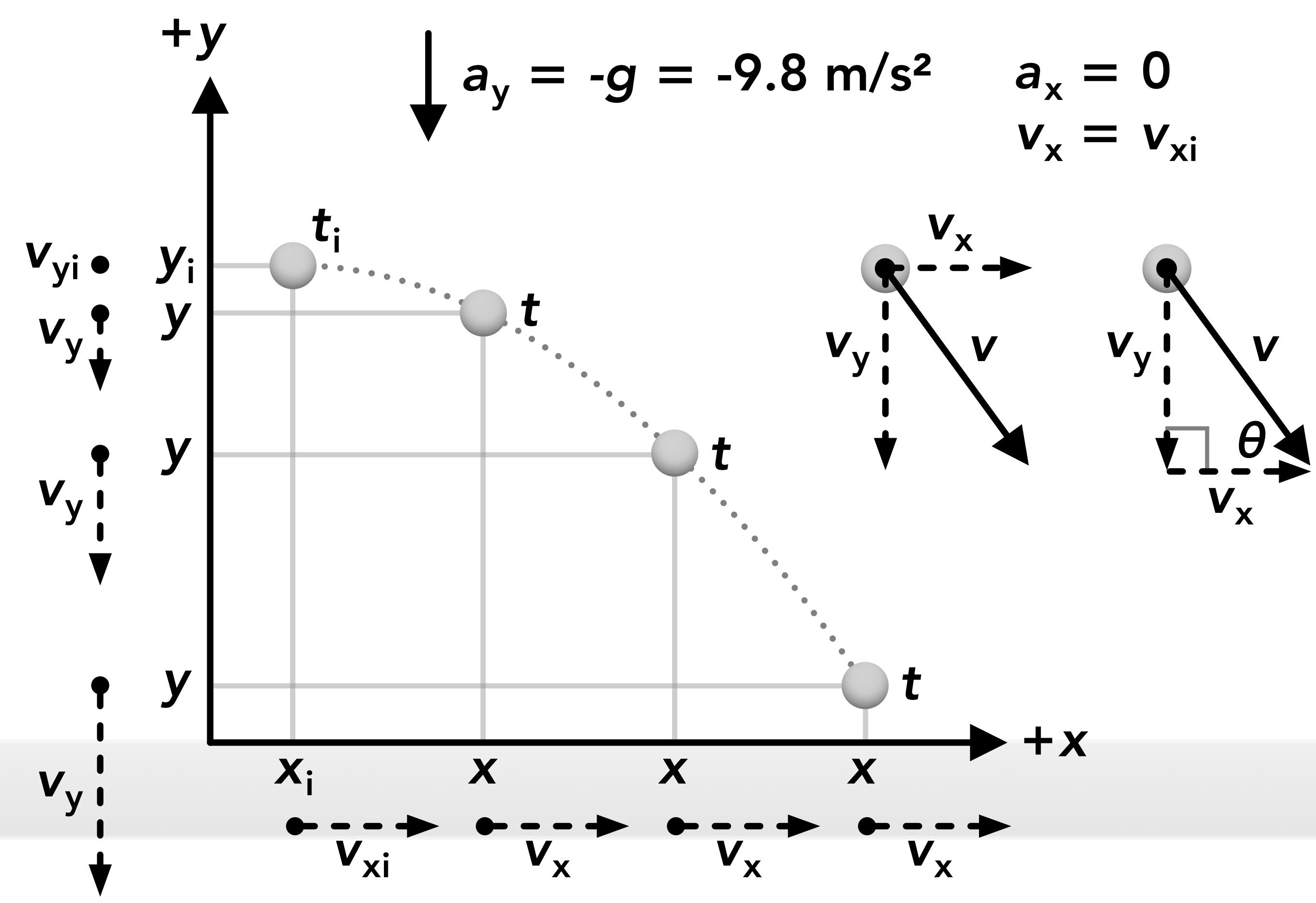
$\Delta x = x - x_0$

1D projectile motion



2D projectile motion

*The vertical and horizontal motion are **independent**



Vertical motion:

Displacement: $\Delta y = y_f - y_i$

Acceleration: $a_y = \dfrac{\Delta v_y}{\Delta t}$

Acceleration (rearranged): $v_{yf} = v_{yi} + a_y \Delta t$

Kinematic equations for constant acceleration:

$$y_f = y_i + v_{yi} t + \frac{1}{2} a_y t^2$$

$$v_{yf}^2 = v_{yi}^2 + 2 a_y (y_f - y_i)$$

Horizontal motion:

Displacement: $\Delta x = x_f - x_i$

Velocity: $v_x = \dfrac{\Delta x}{\Delta t}$

Velocity (rearranged): $x_f = x_i + v_x \Delta t$

- 1D projectile motion only includes motion in the vertical (***y***) direction and 2D projectile motion includes motion in the vertical (***y***) direction and the horizontal (***x***) direction.
- Like with any 2D motion, the horizontal and vertical motions (***x*** and ***y*** motions) are independent from each other and we use separate variables and equations for each direction.

Range

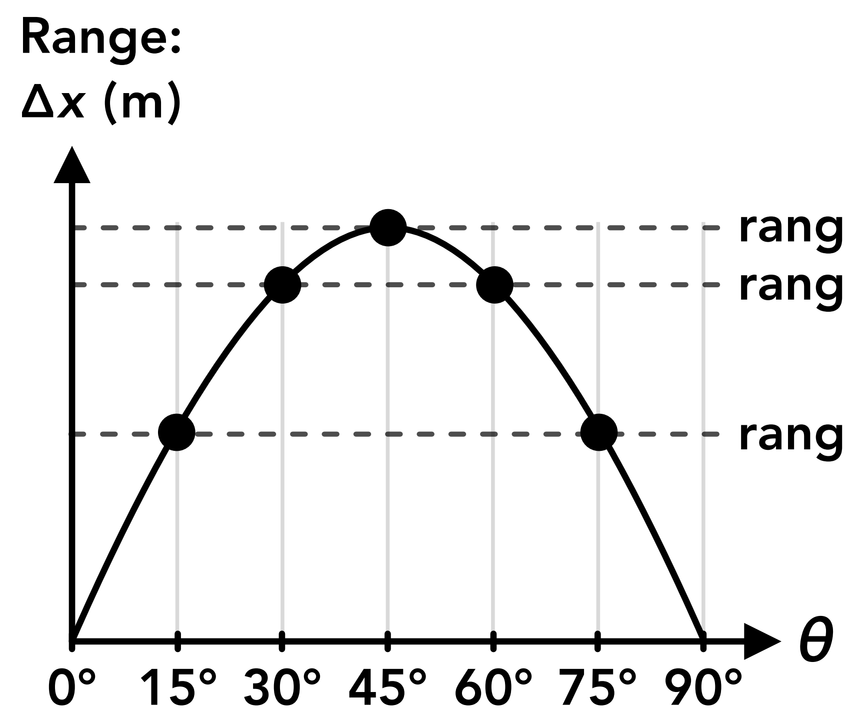
Steps for finding the range:

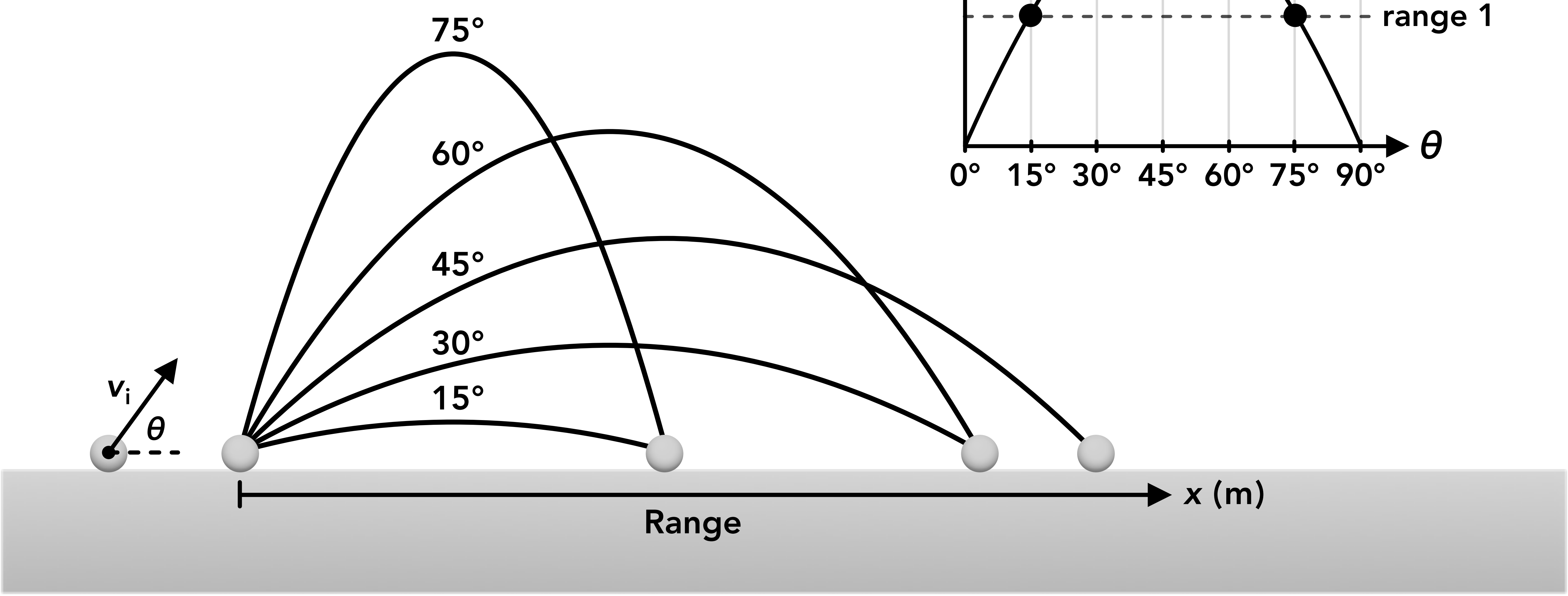
1. Find the initial horizontal and vertical velocity components, $\boldsymbol{v}_{xi}$ and $\boldsymbol{v}_{yi}$
2. Find the time in the air from the y motion using this equation: $y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2$
2. Use that time to find the range (horizontal displacement) using this equation: $\Delta x = v_x \Delta t$

Range:
(if $y_i = y_f$)

$$\Delta x = \frac{v_i^2 \sin(2\theta)}{g}$$

$\theta_{\text{max range}} = 45°$
(if $y_i = y_f$)





- The **range** of a projectile motion is the horizontal distance (Δx) traveled by the projectile.
- The range depends on the initial speed, initial angle, initial height and final height.
- If the initial and final heights are the same (like if a projectile starts and ends on the ground) then an initial launch angle of 45° will result in the maximum range (for any given initial speed). The range decreases as the angle moves farther from 45°. Two angles that are the same amount greater than and less than 45° (such as 30° and 60°) will result in the same range as each other.
- If the initial height is greater than the final height, the angle corresponding to the maximum range is less than 45°.

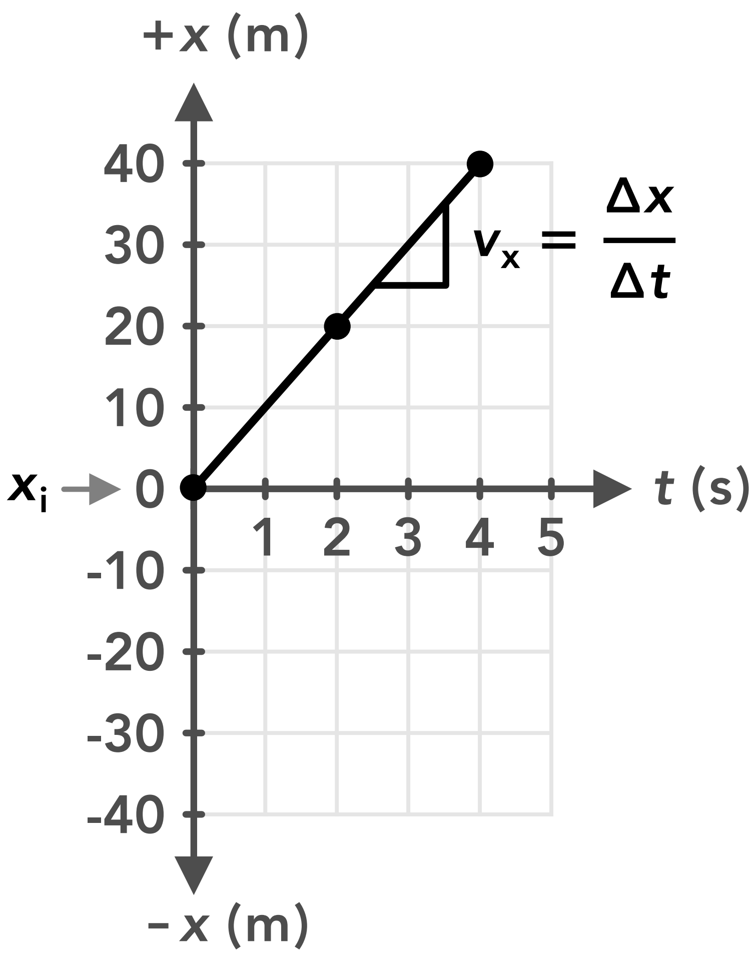
Here are some example graphs for the projectile motion shown below. The graphs for each projectile motion are different but there are some common things for every motion:

- There is no horizontal acceleration so a_x is always 0 m/s².
- The slope of the velocity graph is the acceleration so v_x is a flat line and is always the same as v_{xi} .
- The slope of the position graph is the velocity so x is a straight line with a constant slope.

x

$$\boxed{x_f = x_i + v_x \Delta t}$$

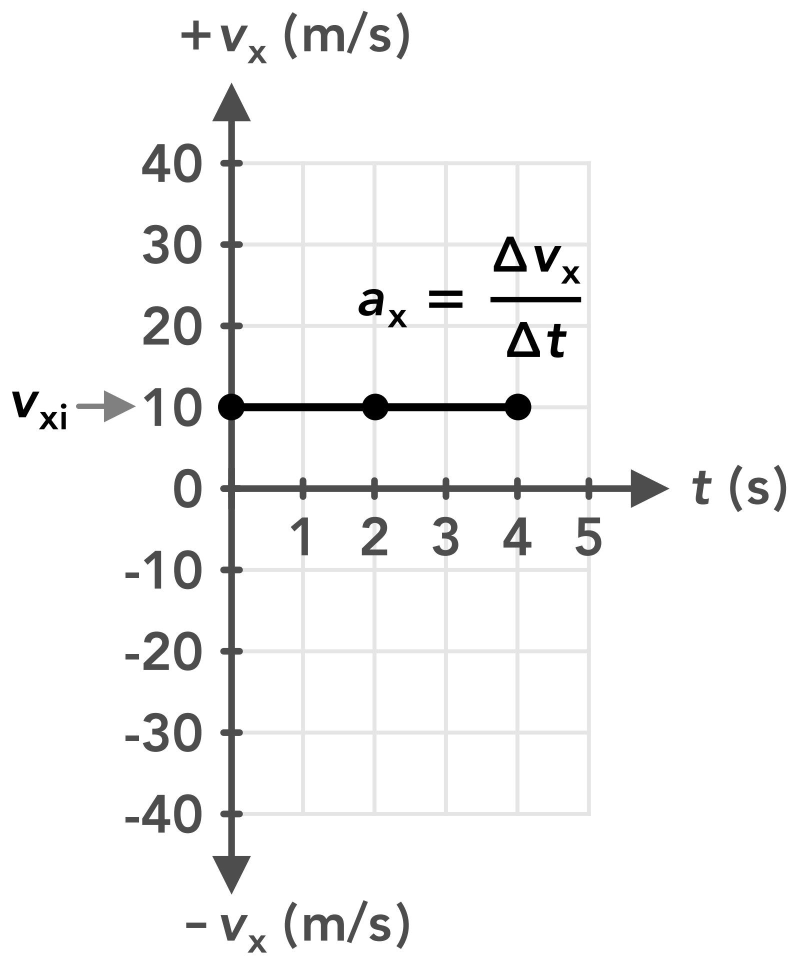
$$x(t) = 10t$$



v_x

$$v_x = v_{xi}$$

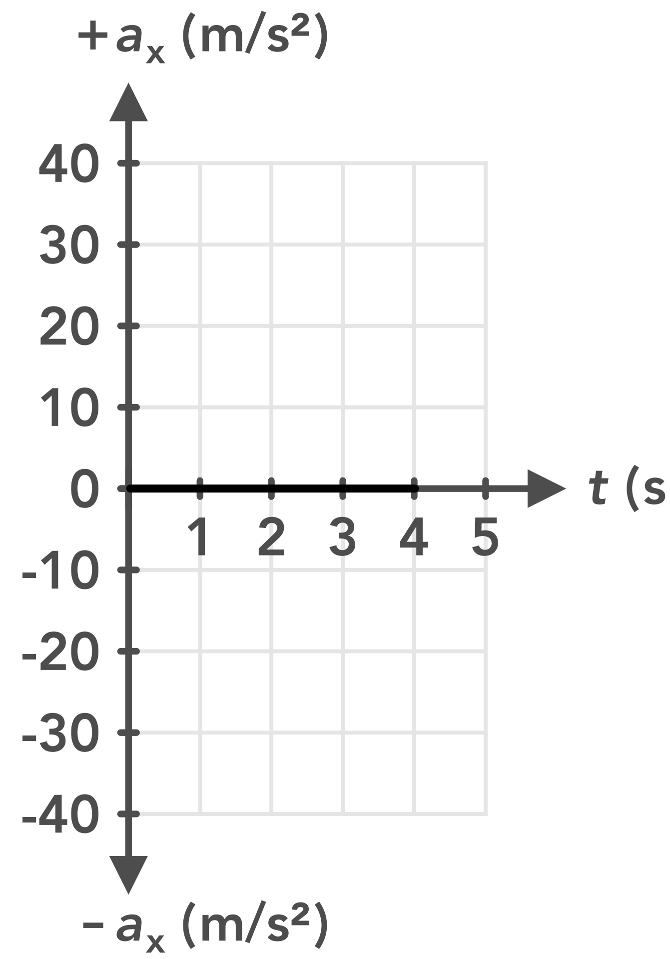
$$v_x(t) = 10$$

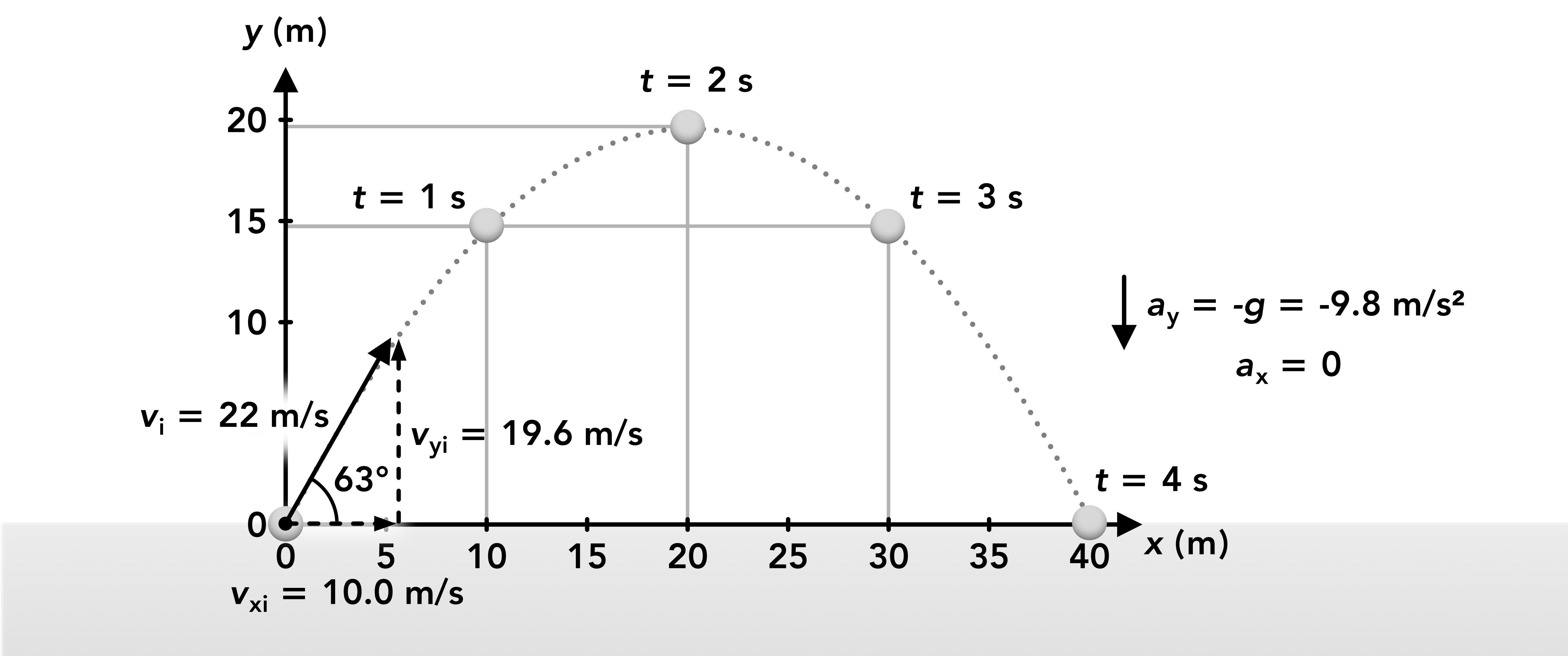


a_x

$$a_x = 0$$

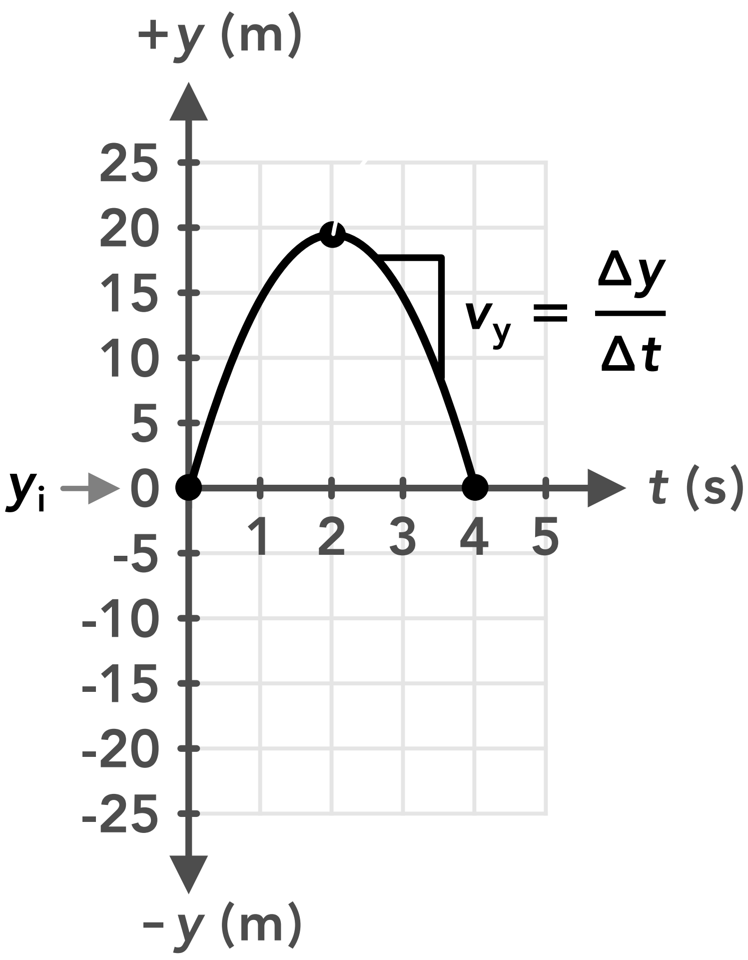
$$a_x(t) = 0$$

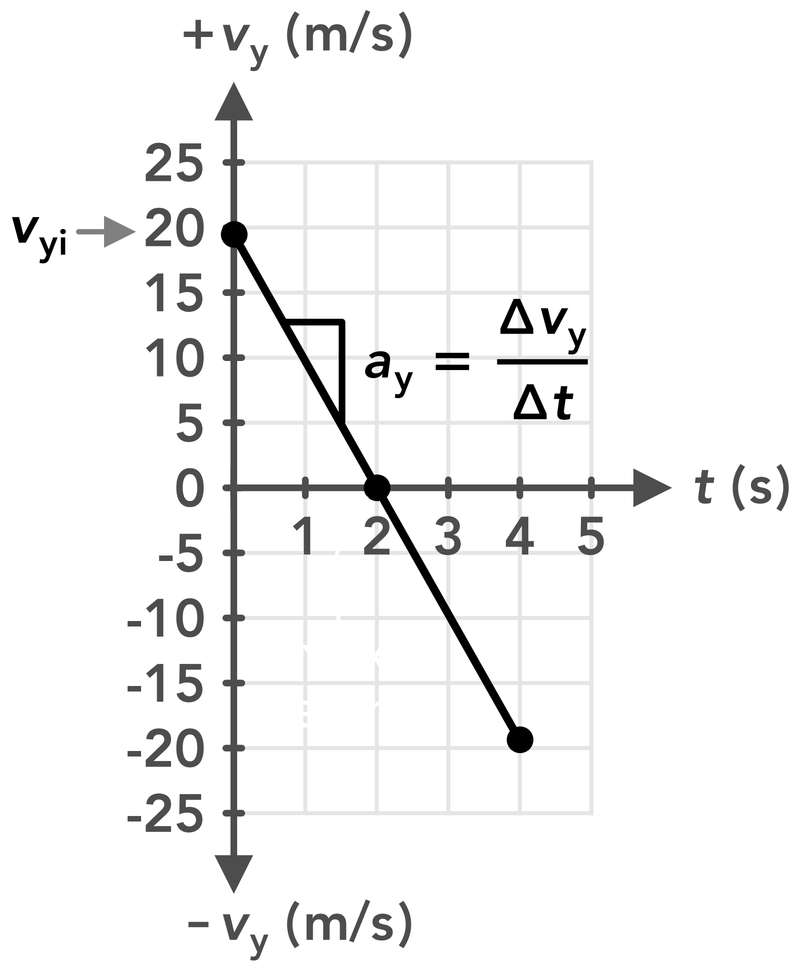


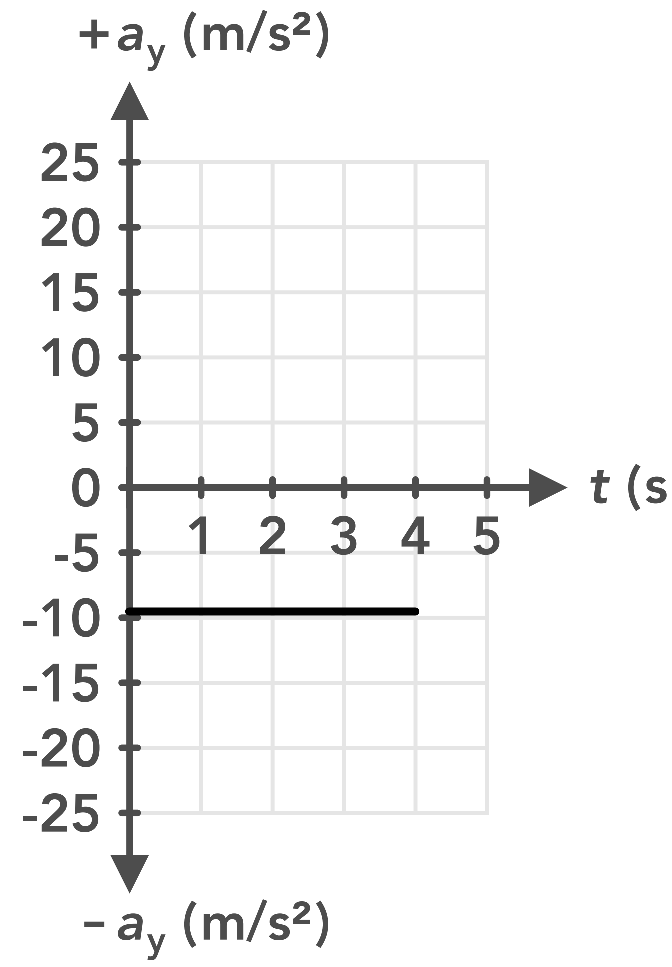


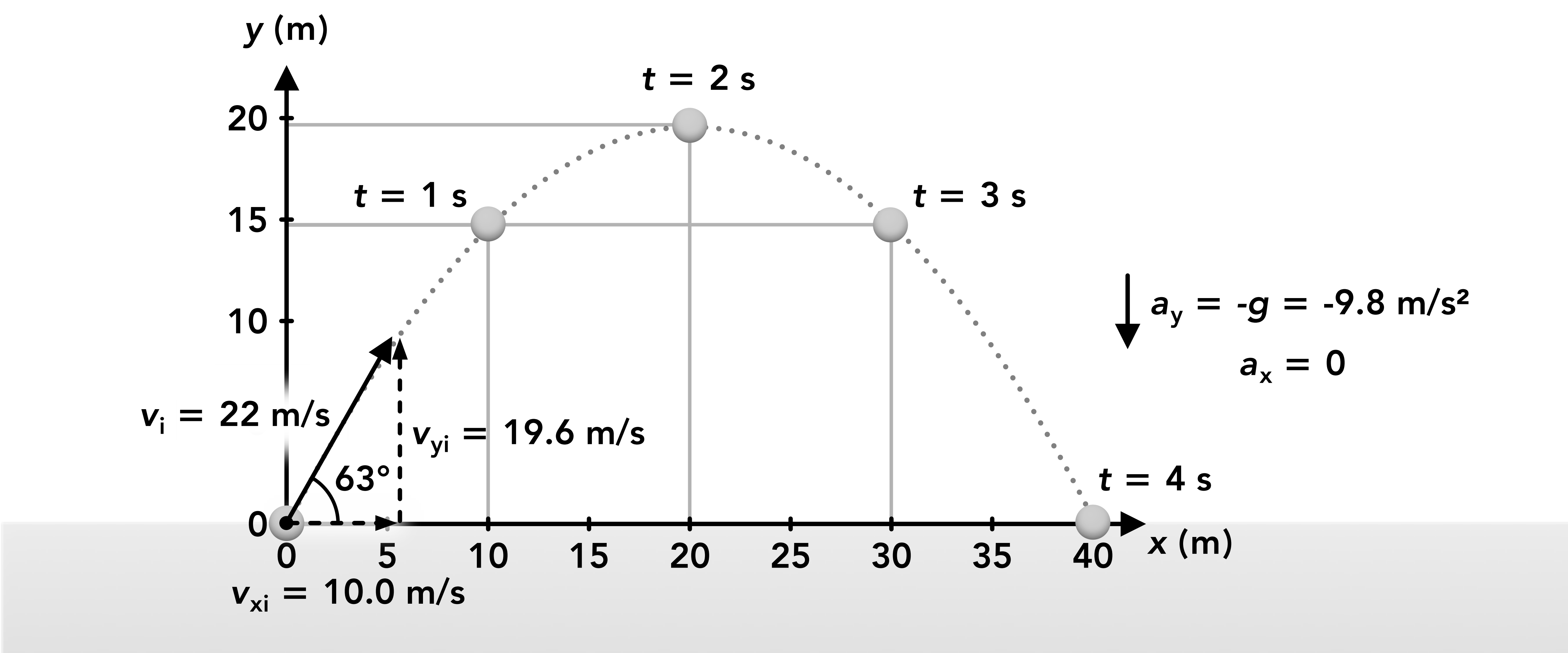
- The vertical acceleration is always 9.8 m/s² downwards so a_y is -9.8 m/s² (if up is the positive direction).
- The slope of the velocity graph is the acceleration so v_y is a straight line with a constant slope. A projectile is at the maximum height in the trajectory when v_y is 0 m/s.
- The slope of the position graph is the velocity so the ***y*** graph is a curved line (parabola) because the velocity is changing (there is acceleration). ***y*** is at its maximum when the slope (velocity) is zero.

y	v_y	a_y
$\boxed{y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2}$	$\boxed{v_{yf} = v_{yi} + a_y\Delta t}$	$a_y = -g = -9.8 \text{ m/s}^2$
$y(t) = 19.6t + \frac{1}{2}9.8t^2$	$v_y(t) = 19.6 - 9.8t$	$a_y(t) = -9.8$





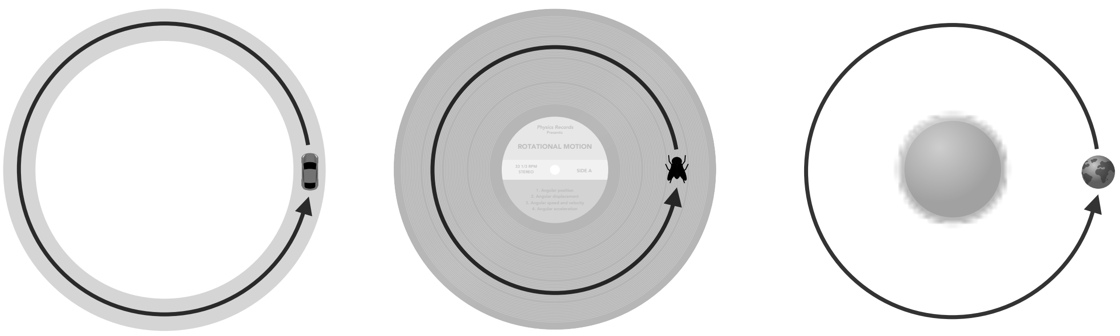




CIRCULAR & ROTATIONAL MOTION

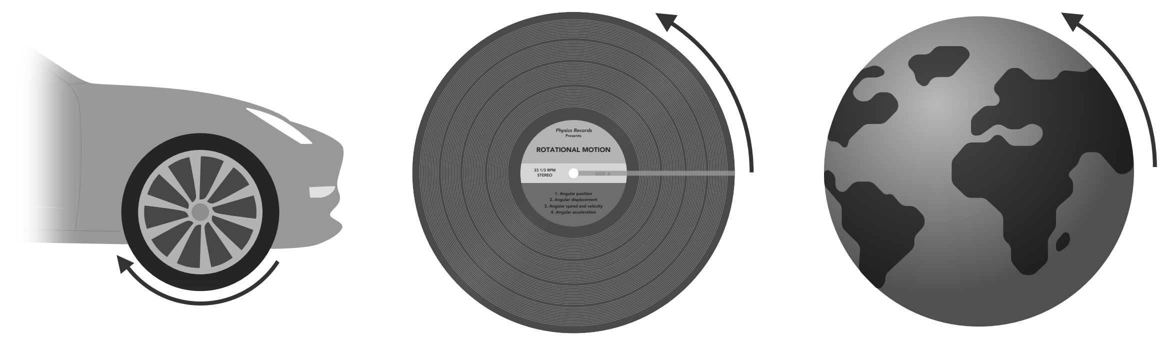
Circular vs Rotational Motion

Circular Motion



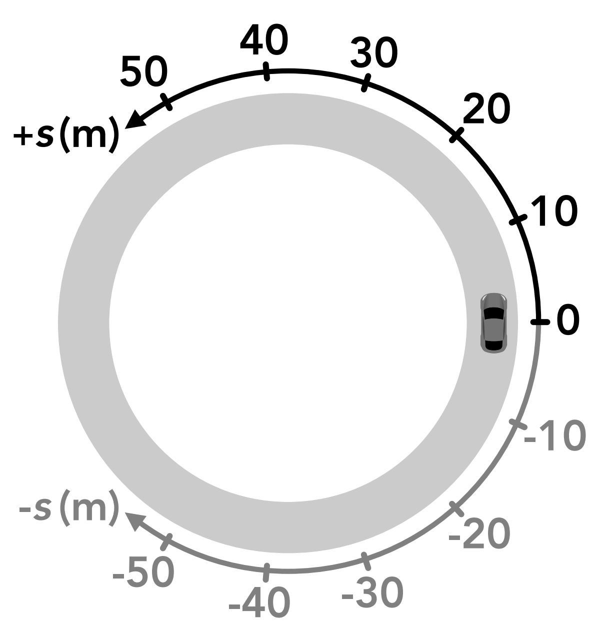
- Object travels along a circular path (circumference of a circle whose center lies outside of the object).
- A point on a rotating object is in circular motion.
- Typically uses the **tangential description of motion**.

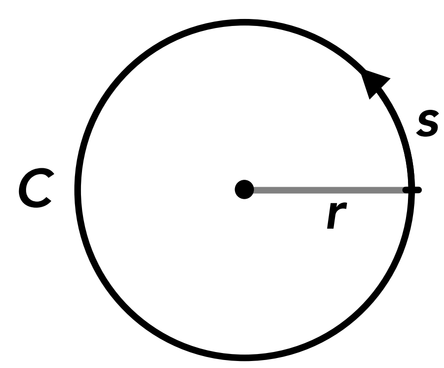
Rotational Motion



- Object rotates about its own center (a point or axis that passes through the object).
- Typically uses the **angular description of motion**.
- All points on the object have the same angular motion.

Circular Motion (Tangential Description)





$$C = 2\pi r$$

Variables		SI Unit
s	tangential position	m
Δs	tangential displacement	m
v_t	tangential velocity	$\frac{m}{s}$
a_t	tangential acceleration	$\frac{m}{s^2}$

$$\Delta s = s_f - s_i$$

Tangential displacement

$$v_t = \frac{\Delta s}{\Delta t}$$

Tangential velocity

$$a_t = \frac{\Delta v_t}{\Delta t}$$

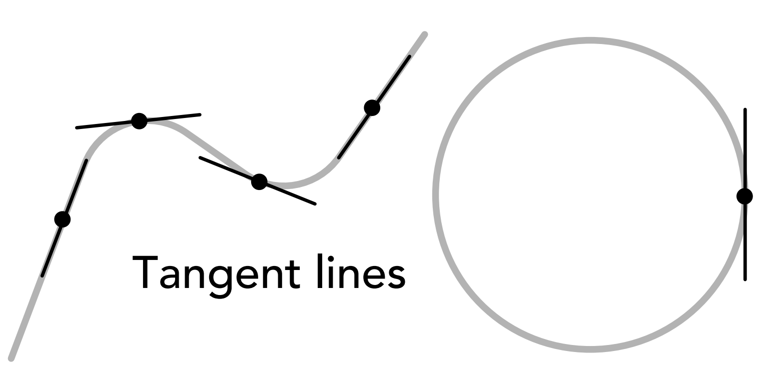
Tangential acceleration

$$s_f = s_i + v_{ti} t + \frac{1}{2} a_t t^2$$

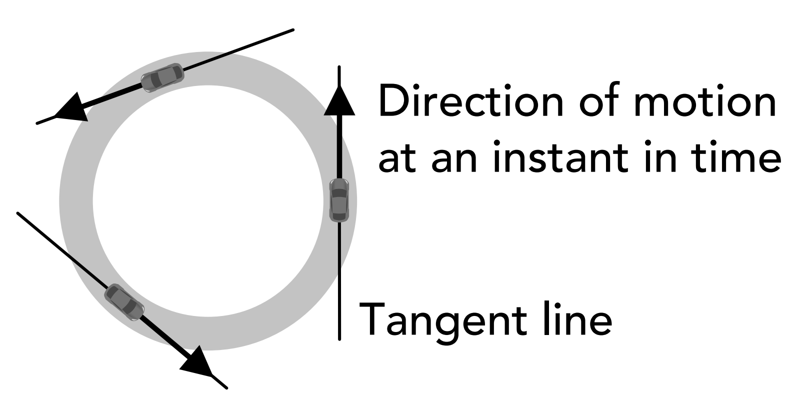
$$v_{tf}^2 = v_{ti}^2 + 2a_t(s_f - s_i)$$

Kinematic equations with constant acceleration

- **Circular motion** typically uses the **tangential** desciption of motion.
- The value of the position will continue to increase past 1 revolution (or decrease in the negative direction).
- Tangential motion is sometimes referred to as the "linear" motion of an object in circular motion because the displacement, velocity and acceleration are directed along a **tangent line**.

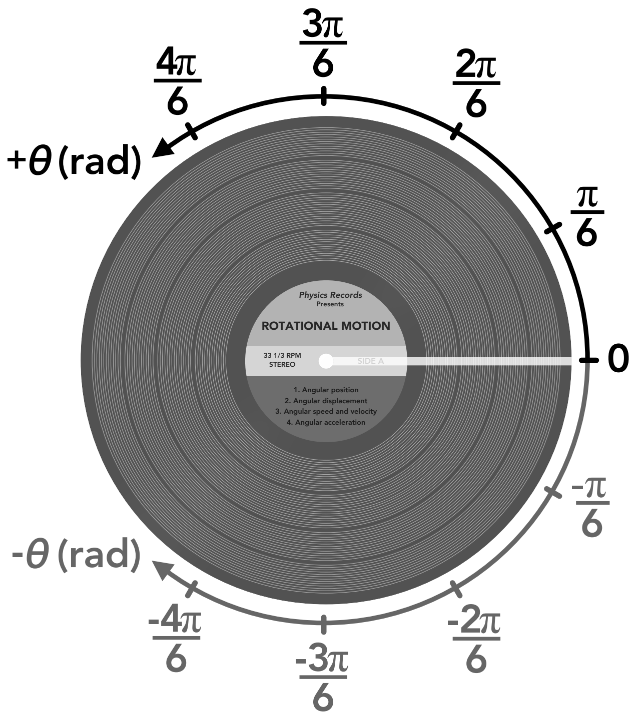


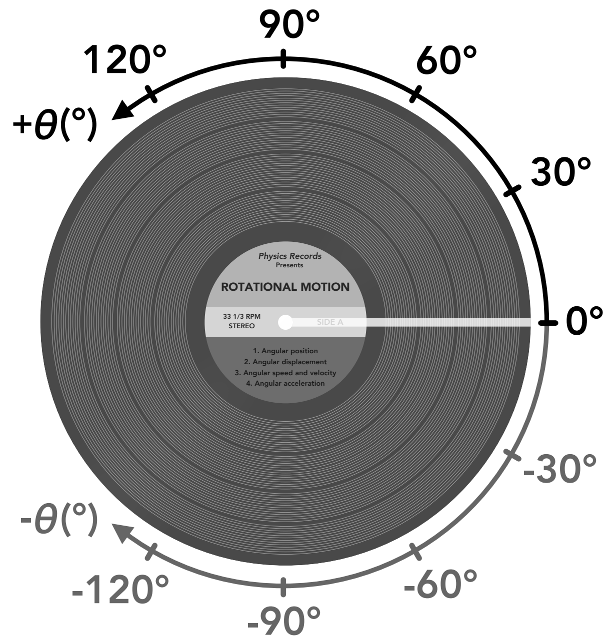
- At a point on a curve, the **tangent line** passing through it matches the curvature or "slope" of the curve.
- For a circle, a tangent line only touches one point.



- For an object in circular motion, the instantaneous direction of the motion is always tangent to the circle.

Rotational Motion (Angular Description)





Variables		SI Unit
θ	angular position	rad
$\Delta\theta$	angular displacement	rad
ω	angular velocity	$\frac{\text{rad}}{\text{s}}$
α	angular acceleration	$\frac{\text{rad}}{\text{s}^2}$

θ "theta" ω "omega" α "alpha"

RPM: $\frac{\text{revolutions}}{\text{minute}}$

$$\Delta\theta = \theta_f - \theta_i$$

Angular displacement

$$\omega = \frac{\Delta\theta}{\Delta t}$$

Angular velocity

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

Angular acceleration

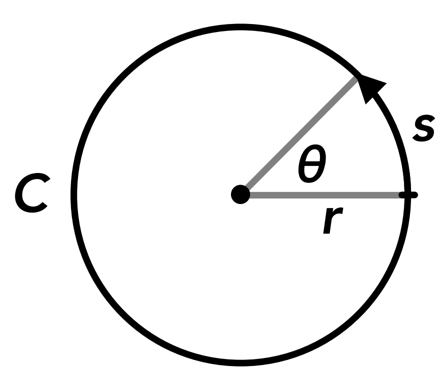
$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$$

$$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$$

Kinematic equations with constant acceleration

- **Rotational motion** typically uses the **angular** description of motion.
- Can also be used to describe the angle that is "swept out" by an object in circular motion.
- All points on a rotating object have the same angular motion because they rotate together (but they may have different tangential motions depending on their distance from the center).
- The value of the position will continue to increase past 1 revolution (or decrease in the negative direction).

Converting Between Tangential & Angular Descriptions



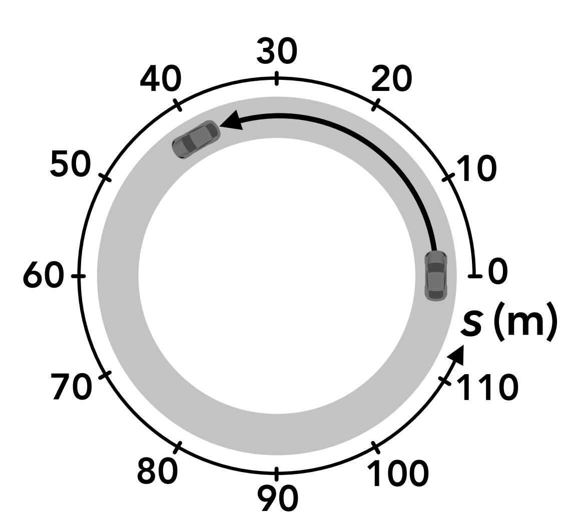
Circumference: $C = 2\pi r$

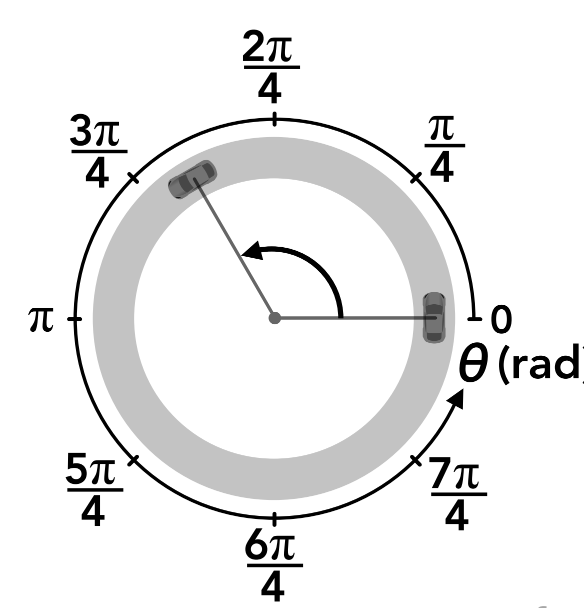
1 circumference ⟷ 2π radians
1 circumference ⟷ 360°
1 circumference ⟷ 1 revolution
1 circumference ⟷ 1 cycle

Conversion (Angular variable must use *radians)*

	Tangential description		⟷	Angular description	
Position:	s	m	$s = r\,\theta$	θ	rad
Displacement:	$\Delta s = s_f - s_i$	m	$\Delta s = r\,\Delta\theta$	$\Delta\theta = \theta_f - \theta_i$	rad
Velocity:	$v_t = \frac{\Delta s}{\Delta t}$	$\frac{\text{m}}{\text{s}}$	$v_t = r\,\omega$	$\omega = \frac{\Delta\theta}{\Delta t}$	$\frac{\text{rad}}{\text{s}}$
Acceleration:	$a_t = \frac{\Delta v_t}{\Delta t}$	$\frac{\text{m}}{\text{s}^2}$	$a_t = r\,\alpha$	$\alpha = \frac{\Delta\omega}{\Delta t}$	$\frac{\text{rad}}{\text{s}^2}$

- In some cases, we need to convert from one description to another.
- This conversion is based on the definition of a radian, or the relationship between the circumference and the number of radians in a circle.





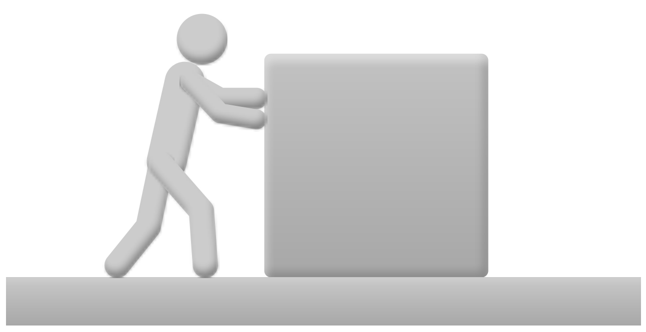
NEWTON'S 1ST & 2ND LAWS

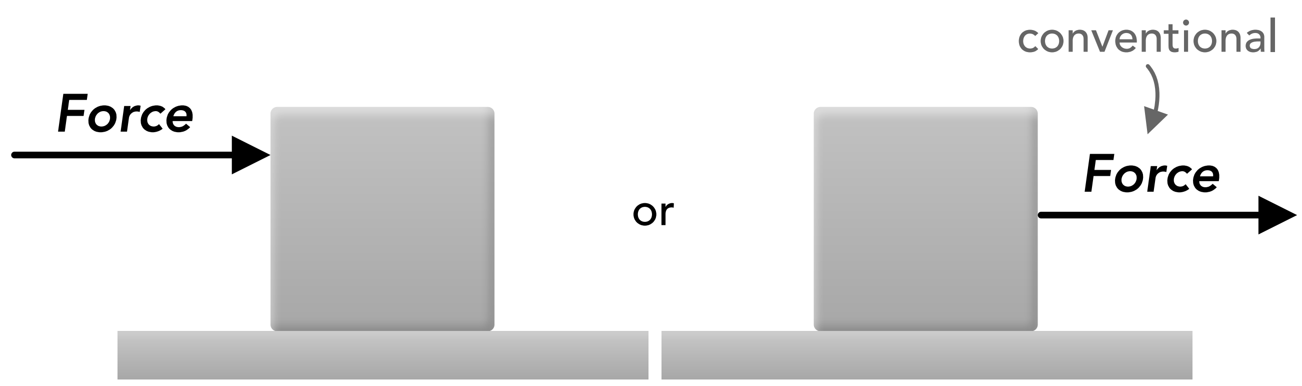
Forces

- A **force** is a push or a pull that acts on an object and is caused by something else. For example, if you push on a box then you are applying a force on the box. The force is acting on the box and the force is caused by you.
- Multiple forces can be acting on an object at the same time.
- Forces are **vector** quantities which means they have a magnitude (a strength or value) and a direction. Forces are often described using vector arrows which have a length (representing the magnitude) and a direction. The force vector arrow typically starts on the object that the force is acting on and points in the direction of the push or pull.
- Forces are not visible, but you can often see the effect of a force, like the motion (or lack of motion) of an object.
- The forces acting on an object are related to the object's motion (or lack of motion) as described by Newton's laws of motion. However, a force exists on its own regardless of how an object is moving.
- The SI unit of force is a **Newton (N)** which is derived from other SI units: N = kg·m/s². 1 Newton (1 N) is equal to approximately 0.22 pounds of force (lbf).

A person pushes a box to the right

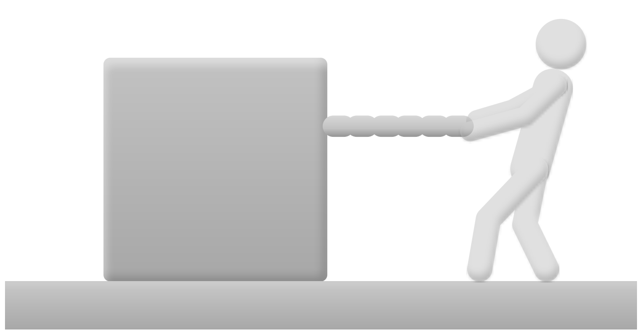
- A force is acting **on** the box
- The force is caused **by** the person
- The force vector can be drawn in different places

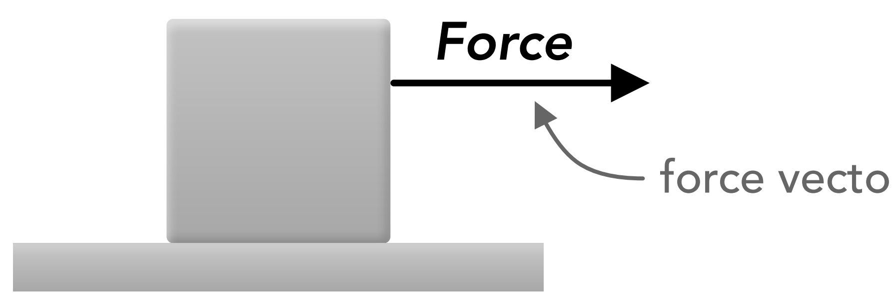




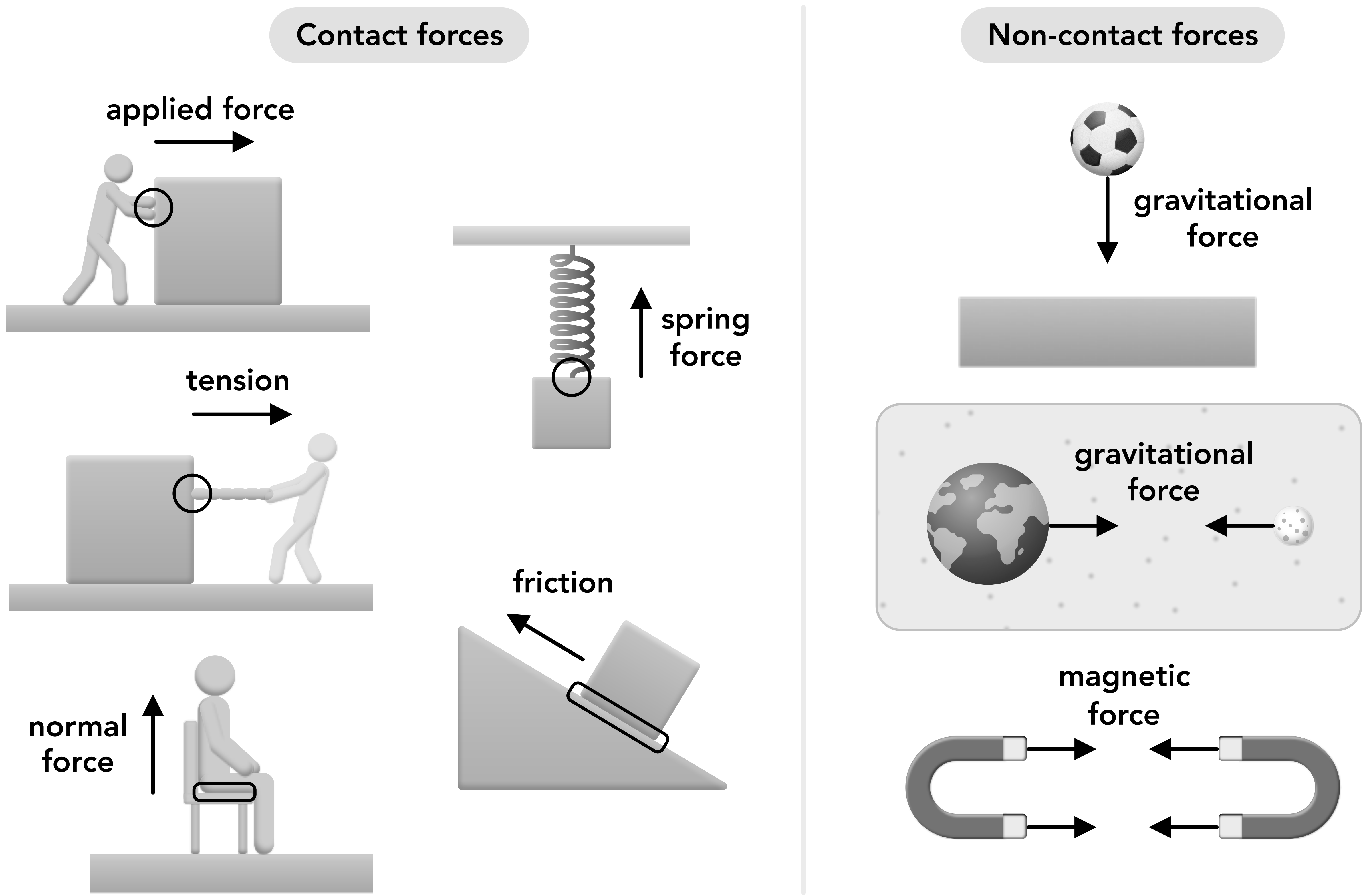
A person pulls a rope attached to a box to the right

- A force is acting **on** the box
- The force is caused **by** the rope



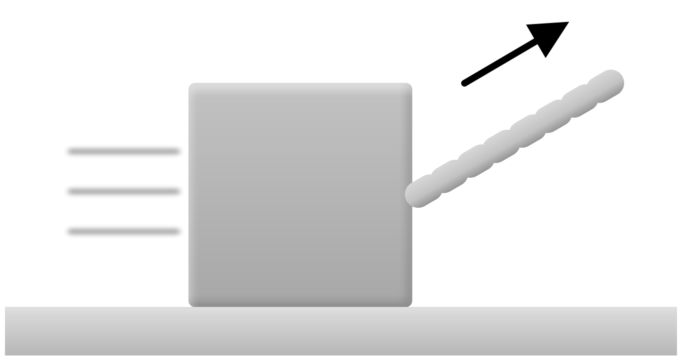


- There are different types of forces which can be grouped into 2 categories: contact forces and non-contact forces.
- **Contact forces** are when the object and the thing causing the force are in contact with each other. These forces are usually easier to "see" and they include any push or pull using contact, a friction force, a tension force, a spring force and a normal force (or reaction force).
- **Non-contact forces** are when the object and the thing causing the force are not in contact with each other. These forces are harder to "see" because the force is acting from a distance. These include the gravitational force (or weight force), magnetic force and electrostatic force.

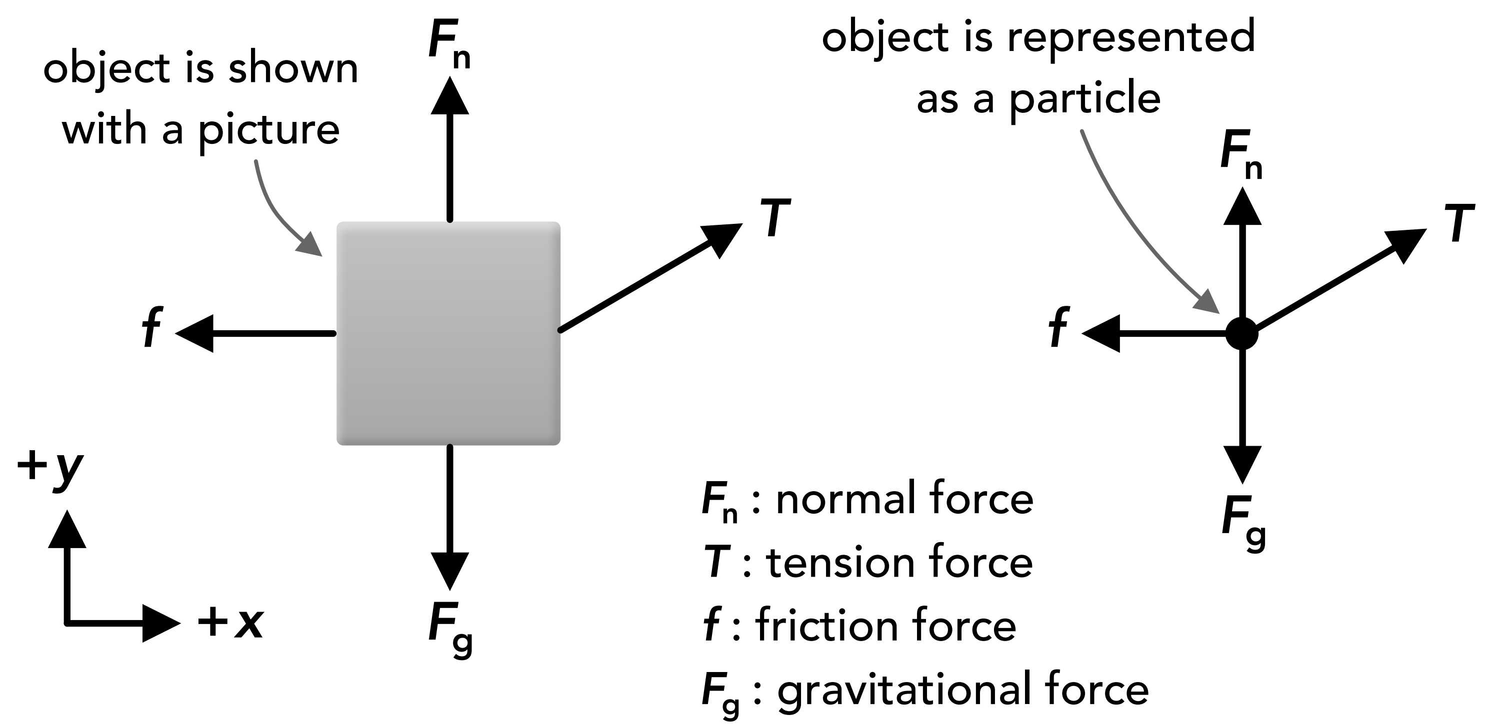


- A **free body diagram (FBD)** or a **force diagram** is a picture that shows a single object or system (a body) and all of the forces acting on that object. We can draw the object itself or represent the object as a **particle** using a dot.
- We need to include a coordinate system which establishes the positive ***x*** and ***y*** directions.
- We do not include the things that are causing the forces, only the forces themselves.
- We do not include any forces that are caused by this object on other things, only the forces acting on this object.
- A free body diagram is used with Newton's 2nd law of motion to analyze the net force and motion of the object.

Picture of a box on the ground being pulled by a rope



Free body diagram of the box and the forces acting on the box



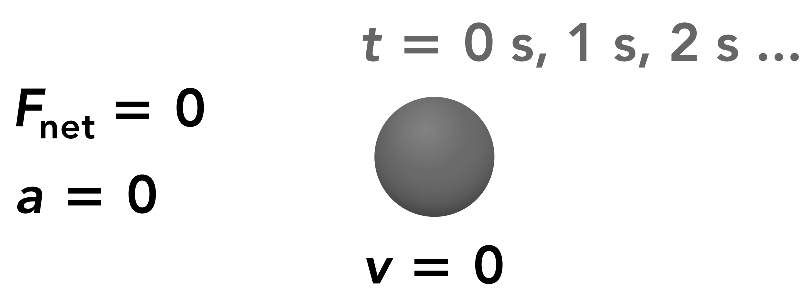
Newton's 1st Law of Motion

Variables		SI Unit
F	force	$N = \frac{kg \cdot m}{s^2}$
m	mass	kg
a	acceleration	$\frac{m}{s^2}$
v	velocity	$\frac{m}{s}$

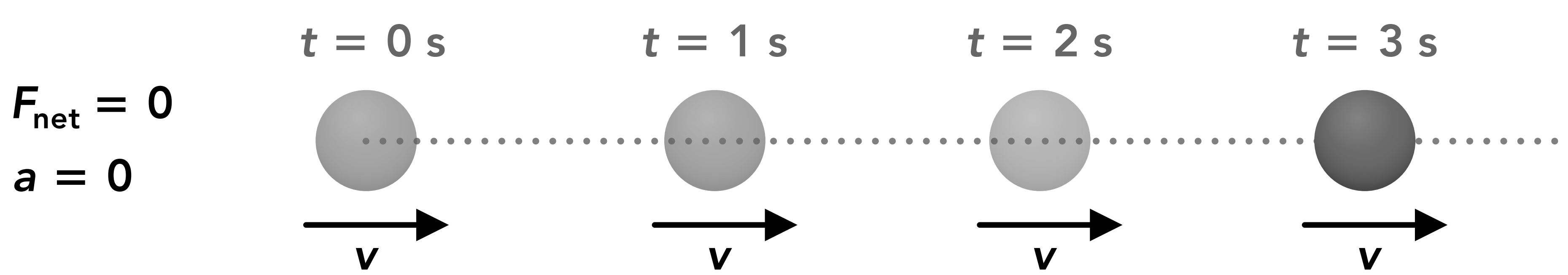
- Isaac Newton's three laws of motion describe the relationship between an object's motion and the forces that are acting on that object. These laws are the foundation for what's known as Newtonian mechanics and they describe why and how an object moves (or doesn't move), which may not be intuitive at first. They're often written in different ways, but the fundamental principle behind each law is simple and very specific as it applies to physics.

- **Newton's 1st law of motion:** An object at rest (with zero velocity) will remain at rest and an object in motion will maintain its velocity (continue moving in a straight line at a constant speed) unless there is a net force acting on the object (the vector sum of all of the forces acting on the object is not zero).
- A simpler but less descriptive version: An object will maintain its state of motion unless acted on by a net force.
- This law also provides the definition of **inertia**. Inertia is the tendency of an object to remain at rest (if at rest) or to remain in motion (if in motion), or the tendency of an object to resist a change to its current state (at rest or in motion). All objects have inertia, which is proportional to their **mass**.

An object at rest (with zero velocity) will remain at rest
if there is no net force acting on it



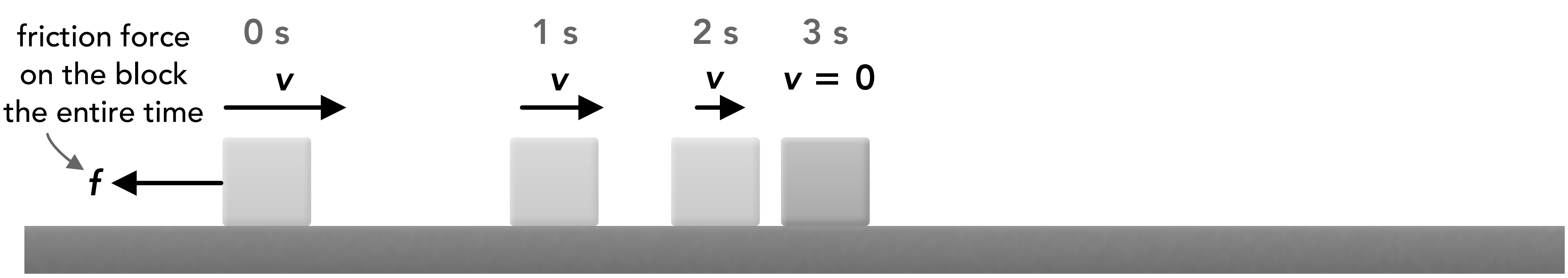
An object in motion will maintain its velocity (move in a straight line at a constant speed)
if there is no net force acting on it



- This law means that **an object which is already in motion does not require any force to continue moving**. The forces acting on an object do affect its motion (see Newton's 2nd law of motion), but nothing causes an object to **continue** moving at a constant velocity, it will do that on its own.
- This may not be intuitive because we often see moving objects appear to slow down and stop on their own, and it seems they would require a force to keep moving. In reality, most objects are experiencing a friction force from any surface they're touching and from the air. The friction force is causing the object to slow down, and if the friction was removed the object would continue moving forever with a constant velocity.
- When thinking about an object's motion, it might help to imagine the object is sliding on ice (with zero friction) or the object is floating in outer space (with zero friction or air resistance, and assuming no force of gravity). In the absence of any forces, the object will remain at rest or will continue moving with a constant velocity forever. Then we can add back the forces acting on the object to analyze its motion.

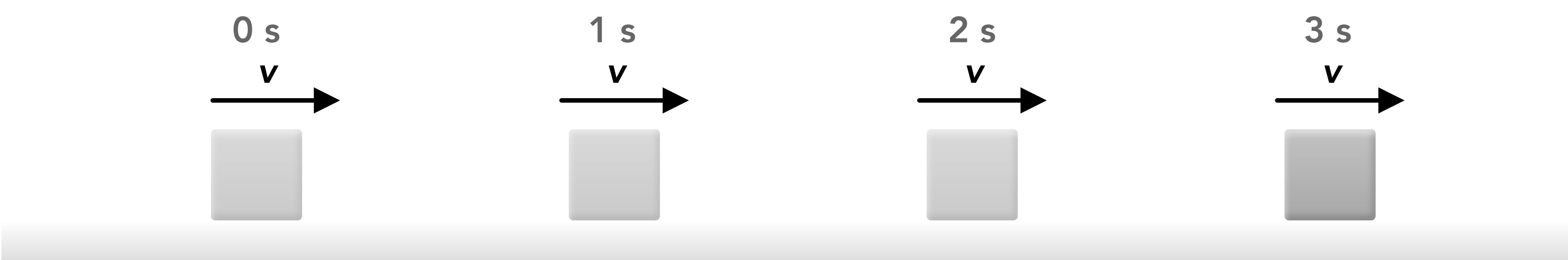
If a block is sliding on a surface with friction, the friction force causes the block to slow down and stop

$F_{net} \neq 0 \quad a \neq 0$



If a block is sliding on ice with no friction force, the block would keep moving forever

$F_{net} = 0 \quad a = 0$ (assuming no friction or air resistance)



If a block is moving freely in space with no forces acting on it, the block will keep moving forever

$F_{net} = 0 \quad a = 0$ (assuming no air resistance or other forces)

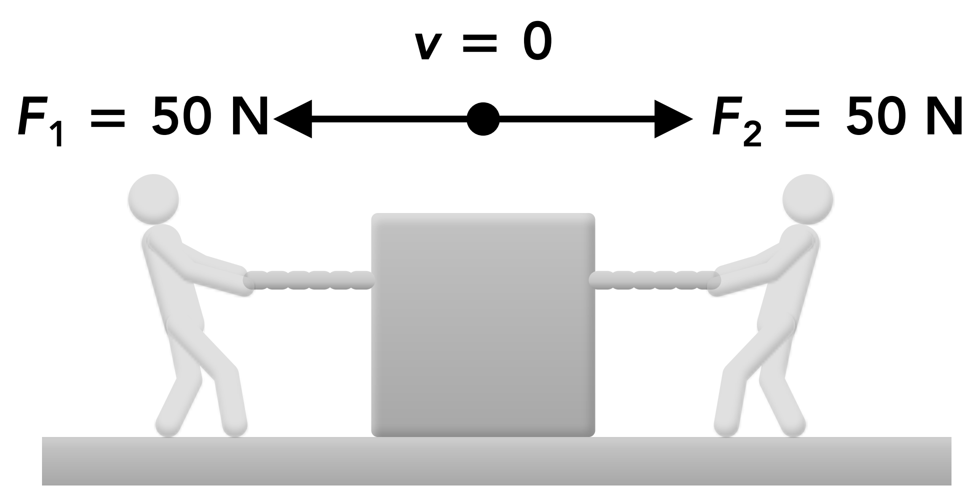
- This law also means that **an object at rest may have forces acting on it**, and **an object that is moving at a constant velocity may have forces acting on it**.
- Newton's 1st law of motion says that an object will remain at rest or maintain a constant velocity if there is no **net force** or no **unbalanced force** acting on it. If there are multiple forces acting on the object but they "cancel out" (the vector sum of all of the forces is zero) then there is no net force.
- For example, imagine an object is being pulled to the left and to the right, like in a game of tug-of-war. If the two forces are equal in magnitude (or strength) then they "cancel" each other out because they act in opposite directions. The net force on the object would be zero and the object would remain at rest (if it was already at rest), or the object would continue moving at a constant velocity (if it was already moving).

Two people pull on a box with equal force in opposite directions
- The net force acting on the box is zero (there is no unbalanced force)
- If the box is at rest, it will remain at rest and won't move

$F_1 = F_2$

$F_{net} = 0 = F_2 - F_1$

$a = 0$

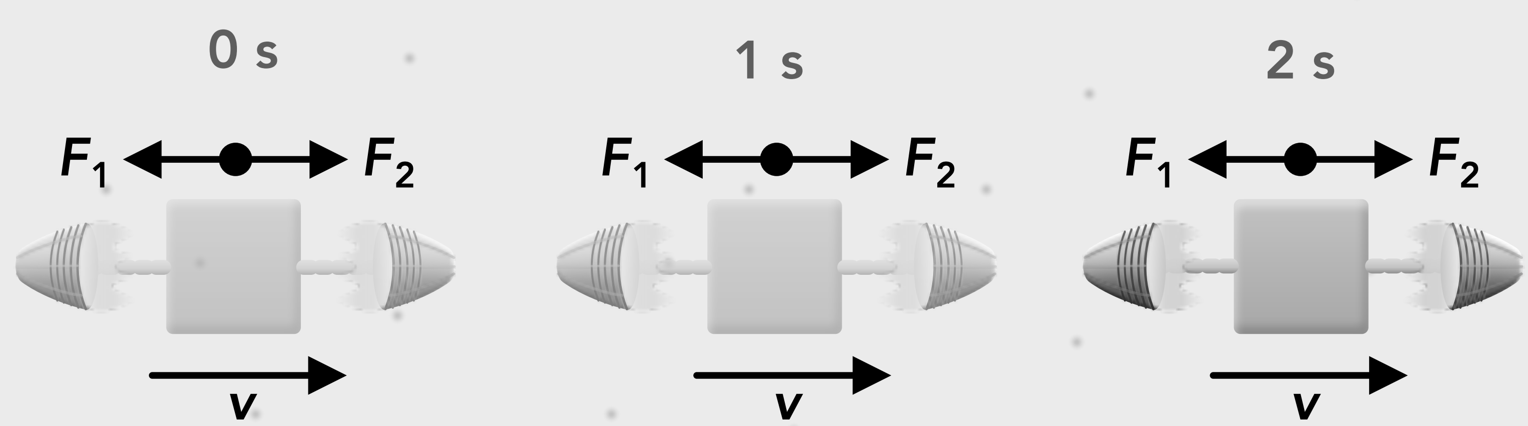


Two rocket thrusters pull on a box with equal force in opposite directions
- The net force acting on the box is zero (there is no unbalanced force)
- If the box was moving it will continue moving at a constant velocity

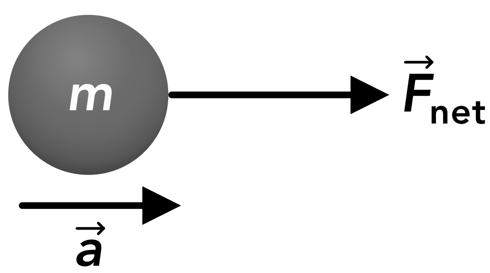
$F_1 = F_2$

$F_{net} = 0 = F_2 - F_1$

$a = 0$



- **Newton's 2nd law of motion:** A net force $\boldsymbol{F}_{\text{net}}$ (the vector sum of all forces) acting on an object of mass $\boldsymbol{m}$ will cause it to accelerate at a rate of $\boldsymbol{a}$ in the same direction as the net force, and the net force is equal to the mass multiplied by the acceleration: $\boldsymbol{F}_{\text{net}} = \boldsymbol{ma}$
- It's important to remember that the forces acting on an object are related to its **acceleration**, not its velocity.
- In a way, Newton's 2nd law also covers Newton's 1st law. Acceleration is the change in an object's velocity. If the net force acting on an object is zero, the acceleration is zero and the velocity will remain the same. If the object is at rest (has zero velocity) it will remain at rest. If the object is moving (has a velocity) it will maintain that velocity.
- The relationship $\boldsymbol{F} = \boldsymbol{ma}$ is the source of the base SI units used in the unit of force, the Newton (N).



Newton's 2nd law of motion

$$\boxed{\vec{F}_{\text{net}} = m\vec{a}} \quad \text{or} \quad \boxed{\sum \vec{F} = m\vec{a}}$$

$\sum$: the sum of __

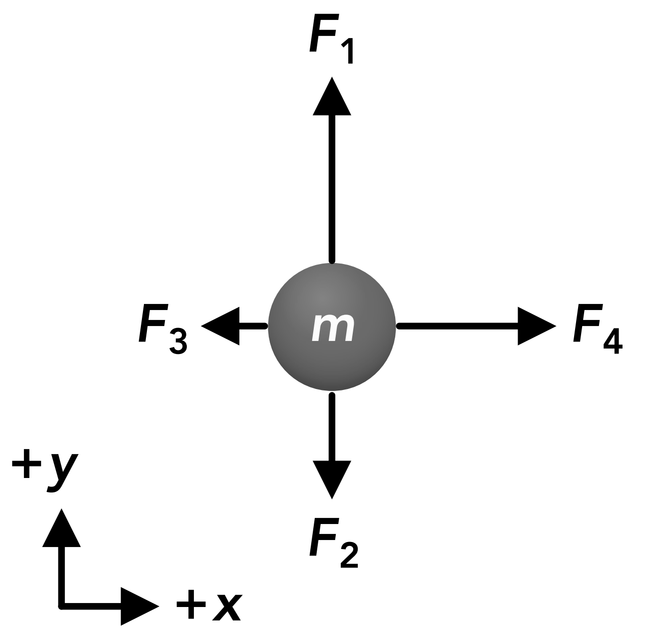
Units:

$$\text{N} = \text{kg}\,\frac{\text{m}}{\text{s}^2}$$

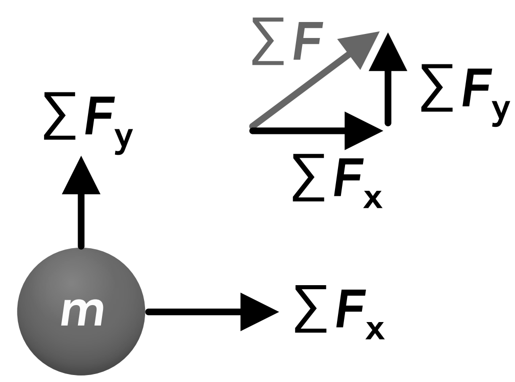
- This may be the first time we're discussing ***mass***. All objects have mass, and there are several ways to define or think about mass:
- As a simple starting point, you can think of an object's mass as the "weight" of the object. Heavier objects have more mass and lighter objects have less mass. It's important to know that mass and weight are two separate things, which will be covered later, but in the presence of gravity an object's weight is proportional to its mass.
- Mass is the amount of matter contained in an object.
- Mass is related to the inertia of an object (for now it's fair to say they are the same thing). An object with more mass has more inertia and will resist a change to its current state (at rest or in motion) more than an object with less mass. This is essentially Newton's 2nd law: the amount an object changes its motion (the acceleration) due to a net force is proportional to its mass ($F_{\text{net}} = ma$, $m = F_{\text{net}}/a$, $a = F_{\text{net}}/m$).
- The SI unit of mass is the **kilogram (kg)**.

- An important part of Newton's 1st and 2nd laws is that only a **net force** causes an acceleration, and the forces and the acceleration are **vectors**.
- Free body diagrams are used in combination with Newton's 2nd law to determine the net force vector acting on an object and the acceleration vector.
- In almost every scenario we're going to **analyze the components of the net force and the acceleration in the *x* and *y* directions separately**, just like in kinematics, because the *x* and *y* directions are independent.
- If a force is not parallel to the *x* or *y* axis then we need to find the *x* and *y* components of that force.
- We will end up with two equations (one for each direction) that describe the relationship between the forces and the acceleration. Then we can plug in all of the known values to solve for an unknown value.

Free body diagram of the forces acting on an object

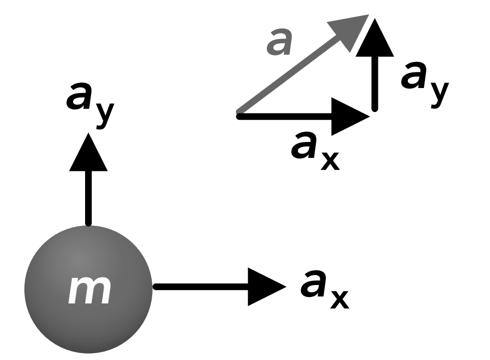


The net force acting on an object is the vector sum of all the forces

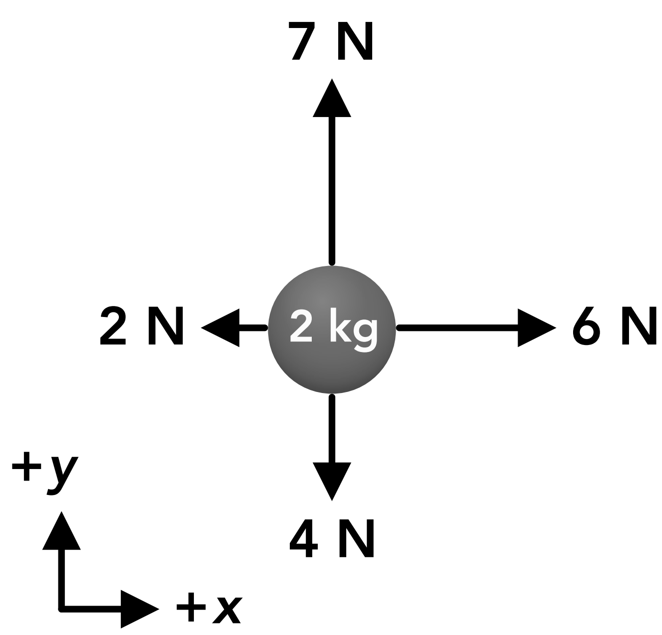


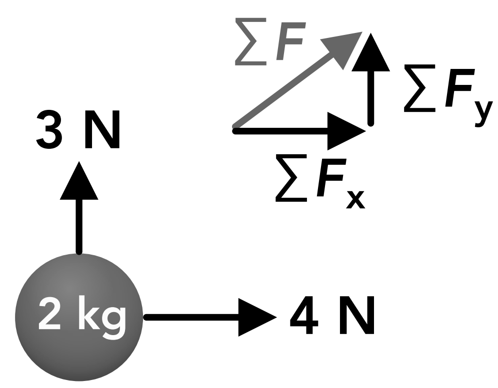
$$\sum F_x = (F_4) + (-F_3)$$
$$\sum F_y = (F_1) + (-F_2)$$

Newton's 2nd law applies to each direction

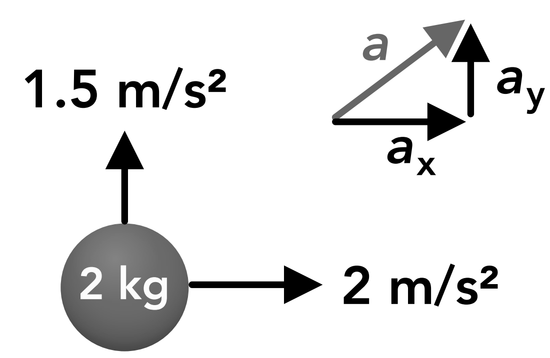


$$\sum F_x = m a_x$$
$$\sum F_y = m a_y$$





$$\sum F_x = (6\text{ N}) + (-2\text{ N}) = 4\text{ N}$$
$$\sum F_y = (7\text{ N}) + (-4\text{ N}) = 3\text{ N}$$

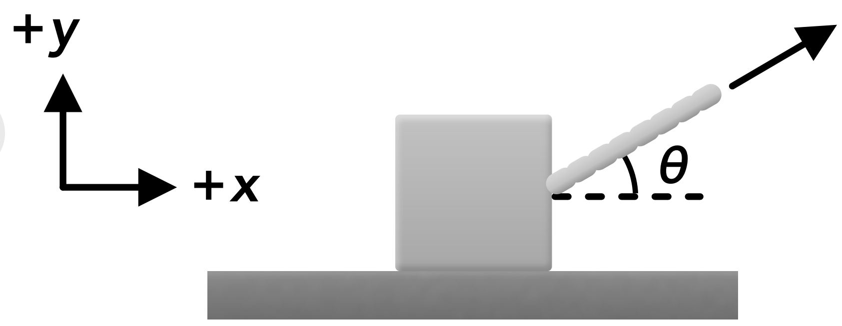


$$4\text{ N} = (2\text{ kg})a_x \quad a_x = 2\text{ m/s}^2$$
$$3\text{ N} = (2\text{ kg})a_y \quad a_y = 1.5\text{ m/s}^2$$

Steps for drawing a free body diagram and applying Newton's 2nd law:

1. Establish the origin and the positive directions of the ***x*** and ***y*** axes. This will determine whether each force is positive or negative when added together. It's useful to set up one of the axes parallel to the direction of the object's motion, or to have the axes parallel to most of the force vectors.
2. Draw a free body diagram of the object and all of the forces acting on the object. If a force is not parallel to one of the axes, find the ***x*** and ***y*** components of the force.
3. Add the forces in the ***x*** direction and apply Newton's 2nd law in the ***x*** direction: $\sum F_x = ma_x$
4. Add the forces in the ***y*** direction and apply Newton's 2nd law in the ***y*** direction: $\sum F_y = ma_y$
5. Use those equations to solve for an unknown variable or answer a question.

1



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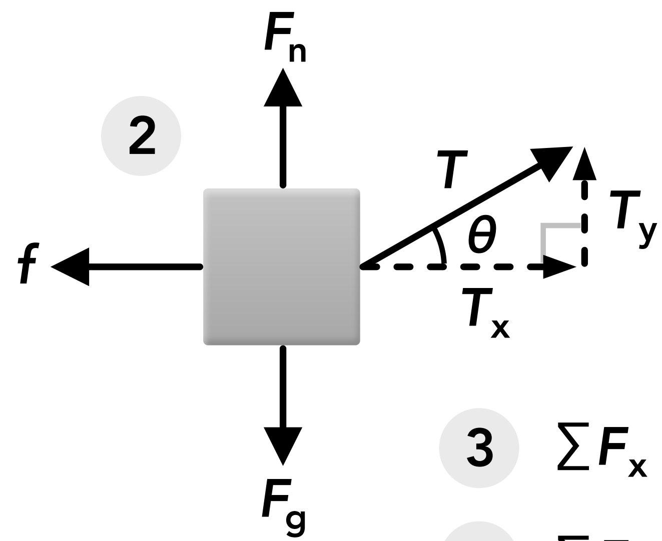


$T_y = T\sin(\theta)$

$T_x = T\cos(\theta)$

3 $\sum F_x = T_x - f = ma_x$

4 $\sum F_y = F_n + T_y - F_g = ma_y$

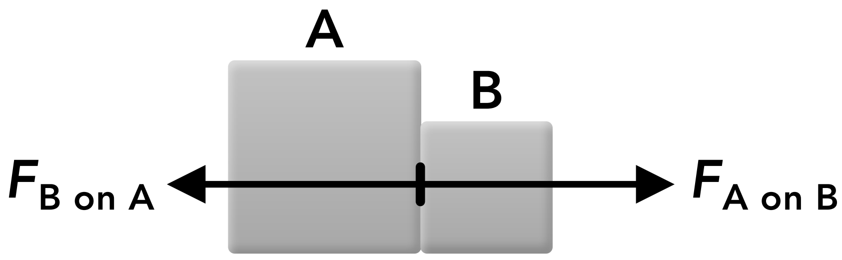
NEWTON'S 3RD LAW & NORMAL FORCE

STUDY GUIDE

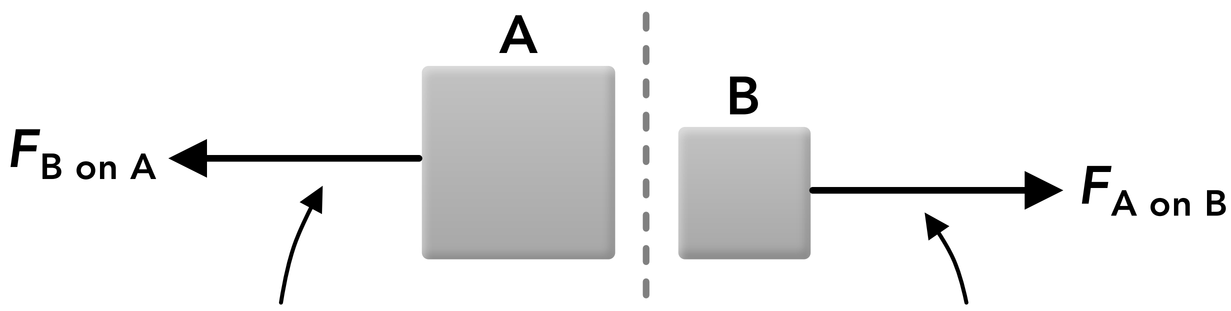
Newton's 3rd Law of Motion

Variables		SI Unit
F	force	N
$F_{\text{A on B}}$	force of A acting on B	N
F_n	normal force	N
F_g	gravitional force	N

$$F_{\text{B on A}} = F_{\text{A on B}}$$



Free body diagram of object A | Free body diagram of object B



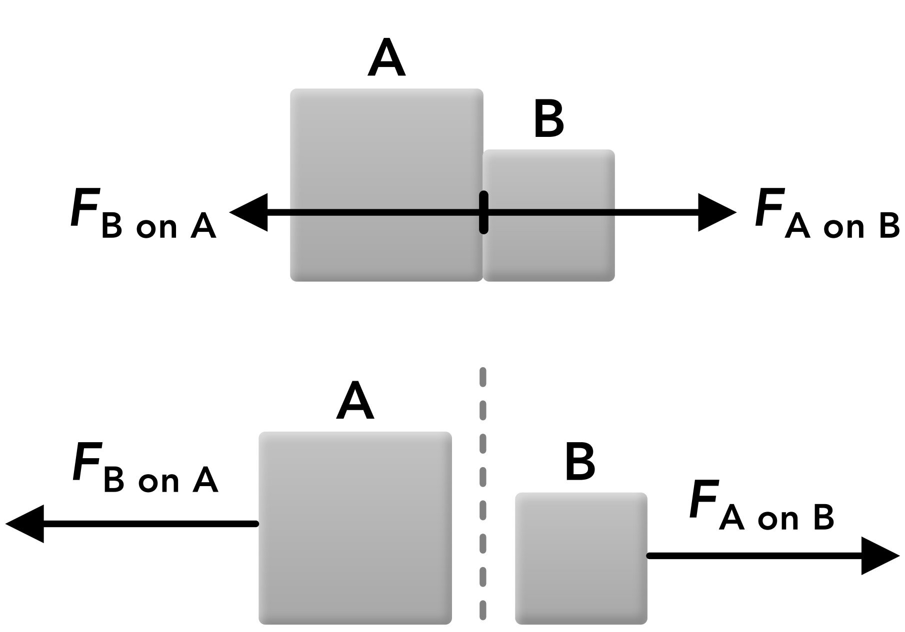
the force acting on object A caused by object B

the force acting on object B caused by object A

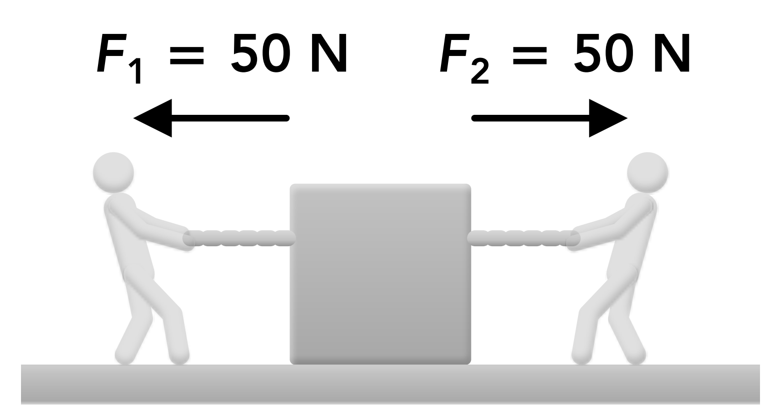
There is a pair of forces that exist when two objects interact with each other. Only one of these forces is included in the free body diagram for each object because the two forces are exerted on two different objects. We do not include both forces in the free body diagram of one object.

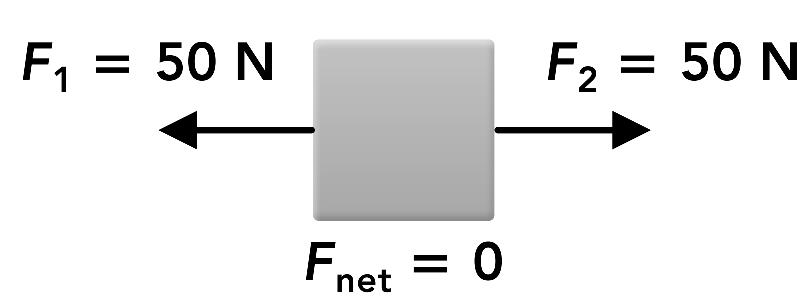
- **Newton’s 3rd law of motion:** If object A exerts a force on object B, then object B exerts an equal and opposite force on object A (the force is equal in magnitude and opposite in direction).
- This law is sometimes stated as “every action has an equal and opposite reaction”. This can be confusing because the words “action” and “reaction” may be misinterpreted as motion or something more complex, but they really just refer to a pair of forces that exist simultaneously (the “action” doesn’t happen before the “reaction”).
- Another common point of confusion is the phrase “equal and opposite forces”. When used to describe the pair of forces from Newton’s 3rd law of motion, it’s accurate to say that two objects exert “equal and opposite forces” on each other when they interact (those two forces are inherently equal in magnitude and opposite in direction). The phrase is also sometimes used to describe two forces that are acting on an object that happen to have the same magnitude and opposite directions, resulting in zero net force and zero acceleration in that direction (due to Newton’s 1st and 2nd laws). In that case, the two forces are entirely separate and have separate causes, and it’s a coincidence that they are equal in magnitude and opposite in direction.
- It may help to remember that the pair of forces described in Newton’s 3rd law must be the same type of force: two normal forces, two gravitational forces, two friction forces, two tension forces, etc.

These two forces are a pair of “equal and opposite” forces according to Newton’s 3rd law, but the forces act on two separate objects



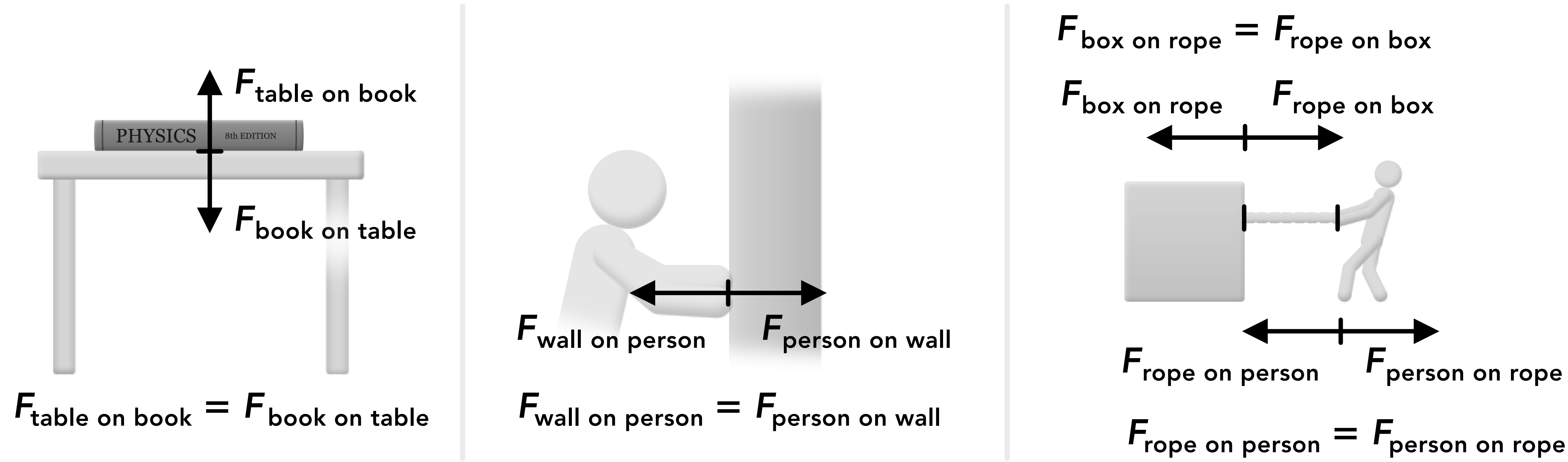
These two forces happen to be “equal and opposite” but they are separate forces that act on the same object (unrelated to Newton’s 3rd law)





- A pair of forces can result from the physical contact between two objects (contact force pairs) or between two objects that are at a distance (non-contact force pairs).
- If two objects are in contact they each apply a force on the other object. This can be a pushing force (such as a normal force) or a pulling force (such as a tension force). This is sometimes referred to as a “reaction force” or simply a “contact force”.

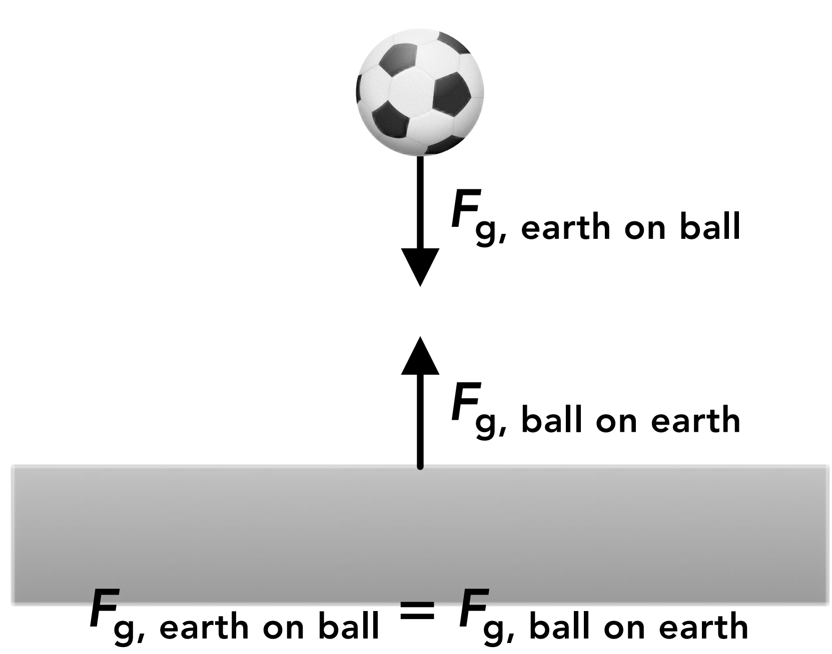
Contact force pairs

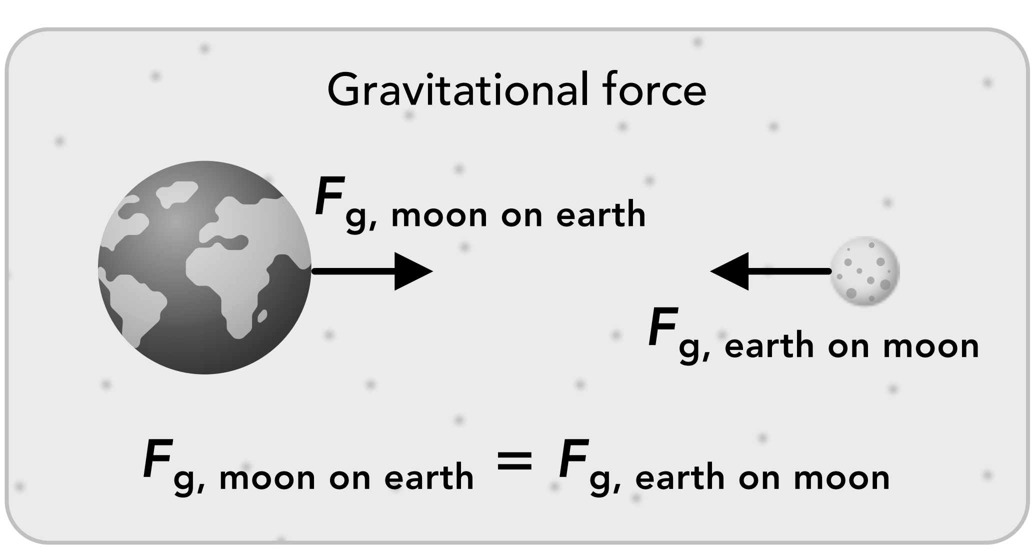


- There is an attractive gravitational force that acts between any two objects due to their mass.
- Although not covered in this course, there is an attractive magnetic force between opposite poles of a magnet and a repulsive force between similar poles. There is also an attractive electric force between two oppositely charged particles, and a repulsive force between two particles with the same charge.

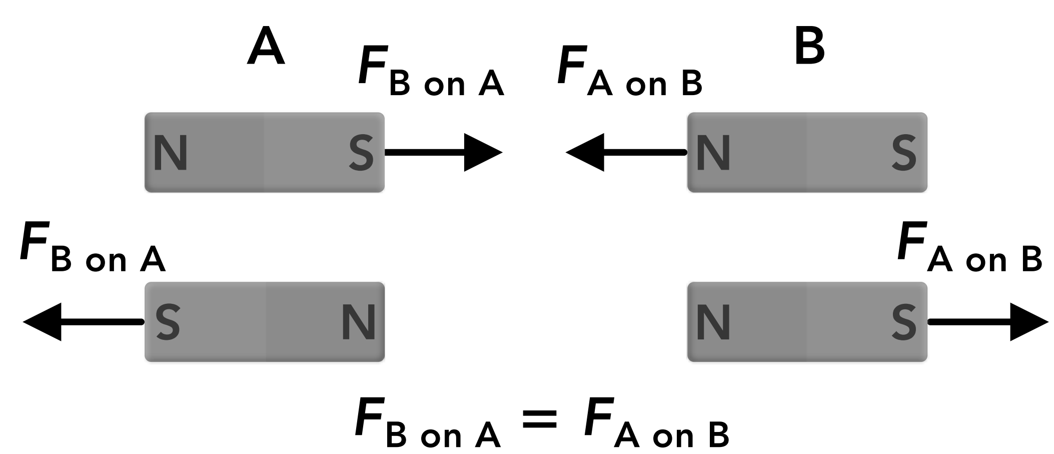
Non-contact force pairs

Gravitational force

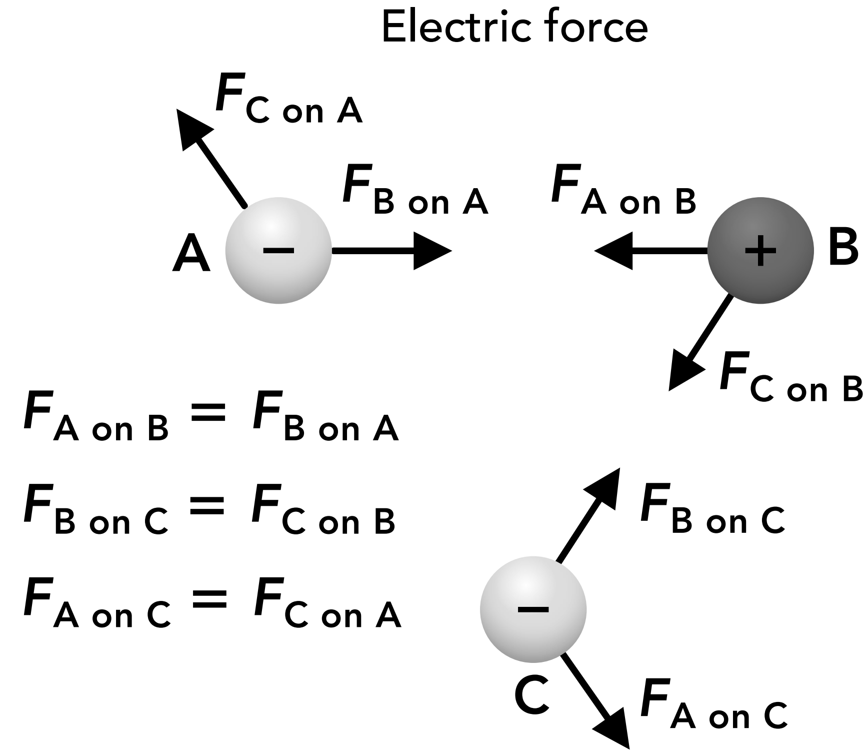




Magnetic force

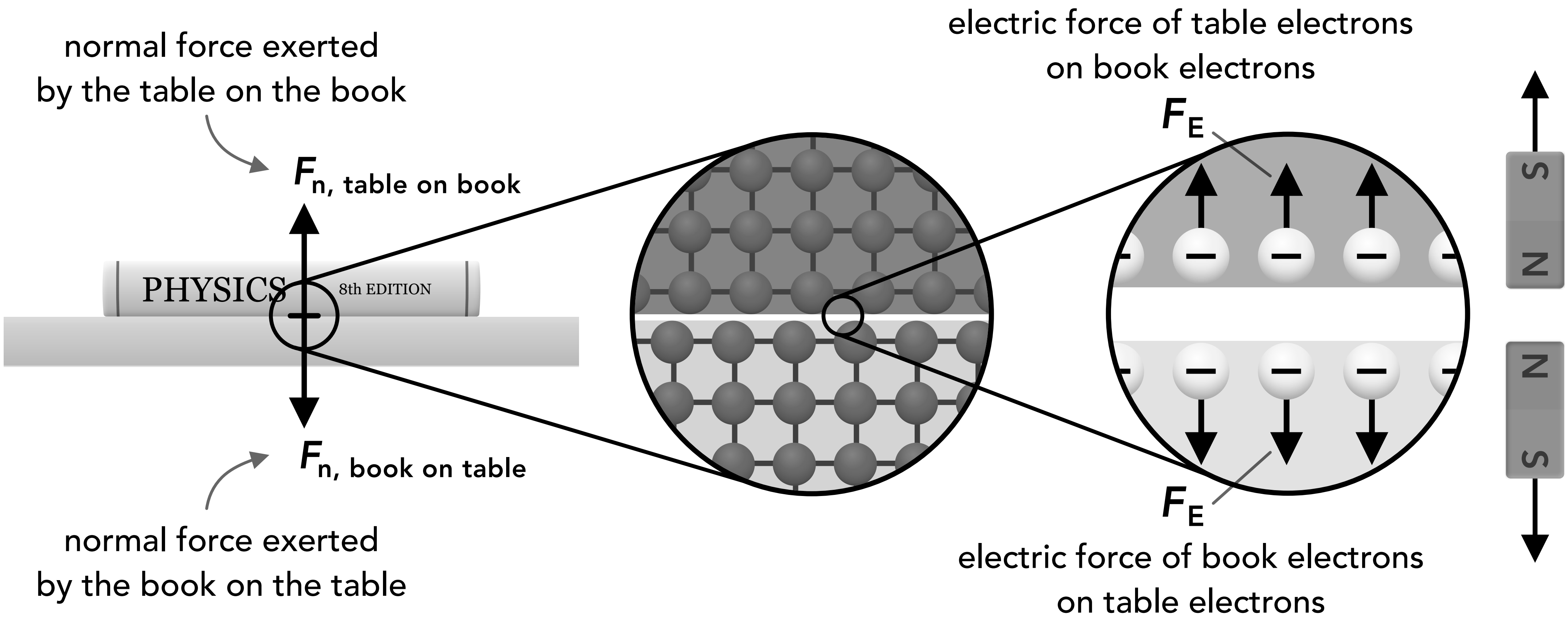


Electric force

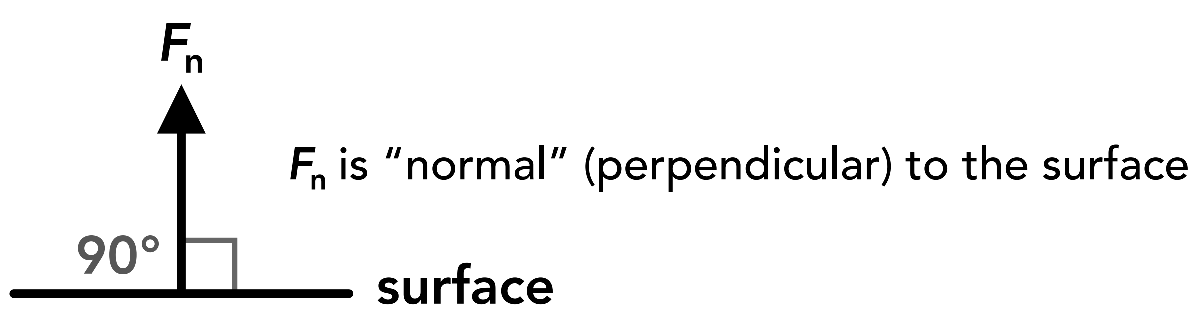


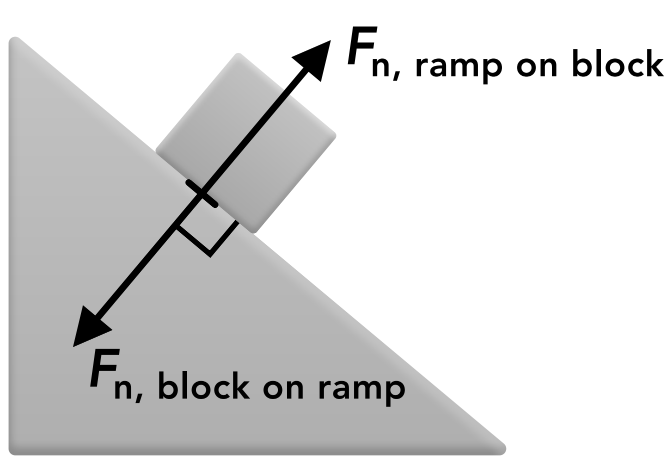
Normal Force

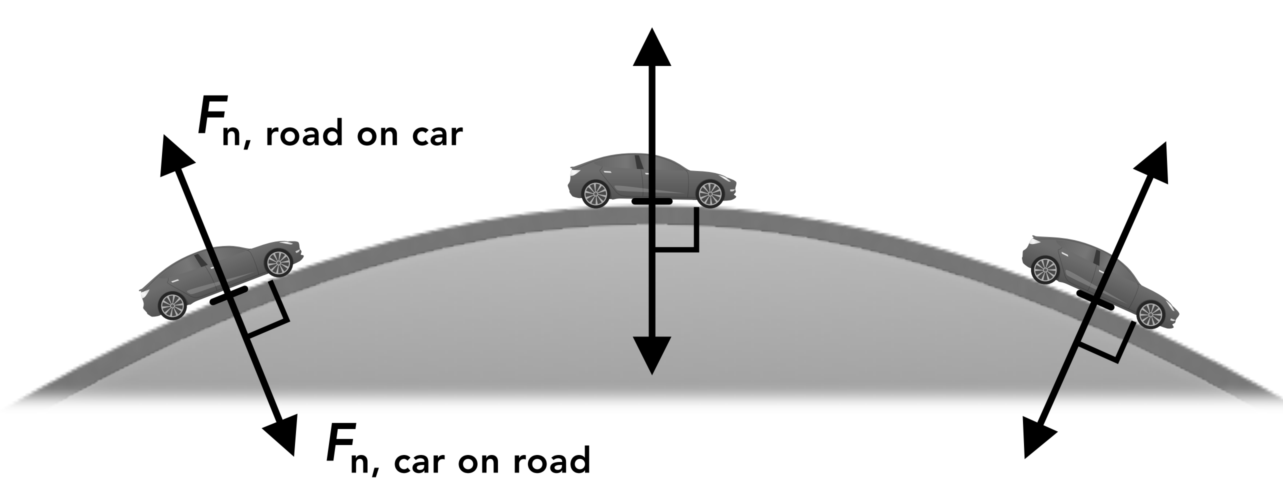
- A **normal force** is just a term for the pushing force that arises when two objects are in contact.
- Normal force is represented as "$\boldsymbol{F}_n$" or "$\boldsymbol{N}$" which should not be confused with a Newton (N), the unit of force.
- At a macroscopic level, a normal force is a contact force that prevents two solid objects from passing through each other. The surfaces of each object may appear to be touching, but at the atomic level the electrons in one object are repelling the electrons in the other object with an electric force (to put it simply). In a way, pushing two objects together is similar to pushing two extremely strong repelling magnets together.



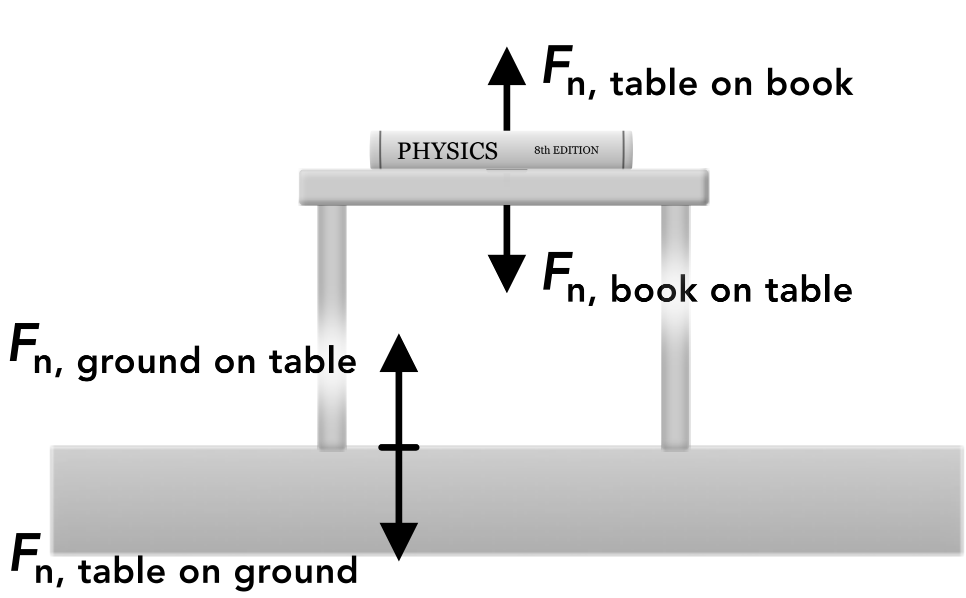
- This is called a normal force because "normal" means **perpendicular** in geometry, and a normal force always acts perpendicular to the surface that the object is contacting.

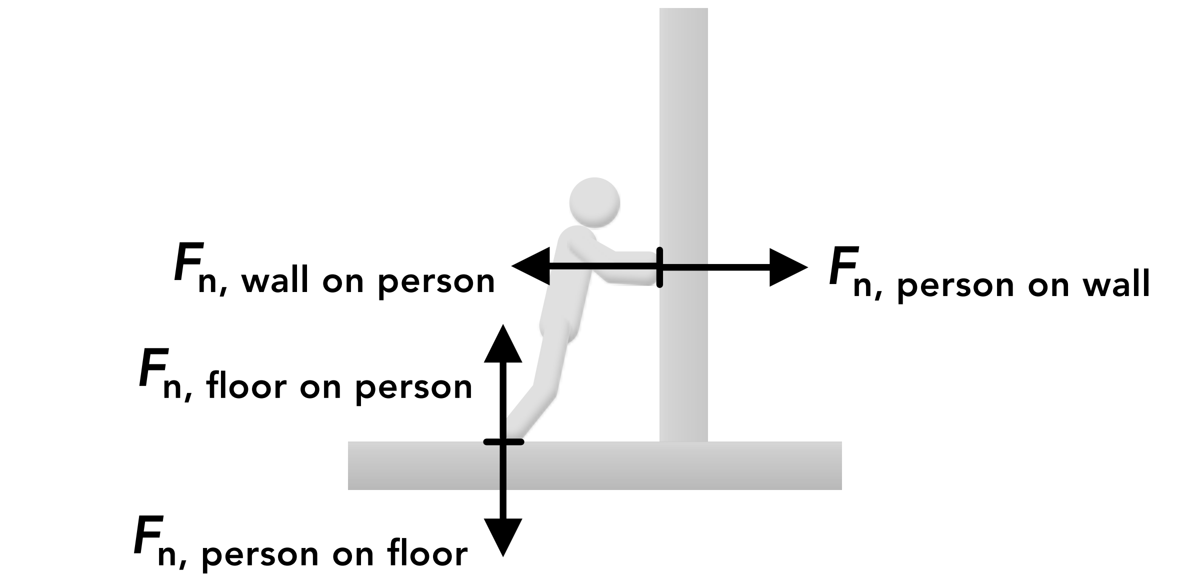


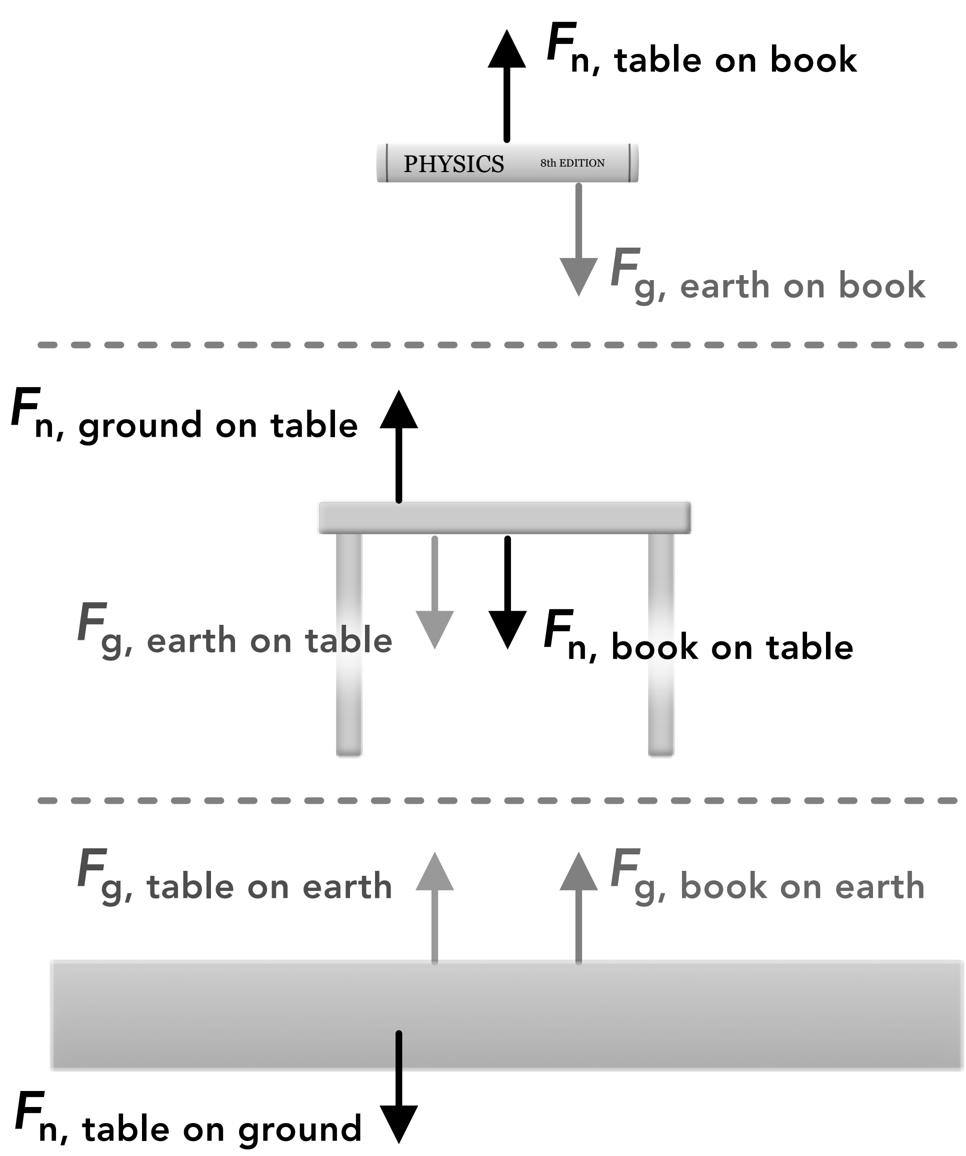


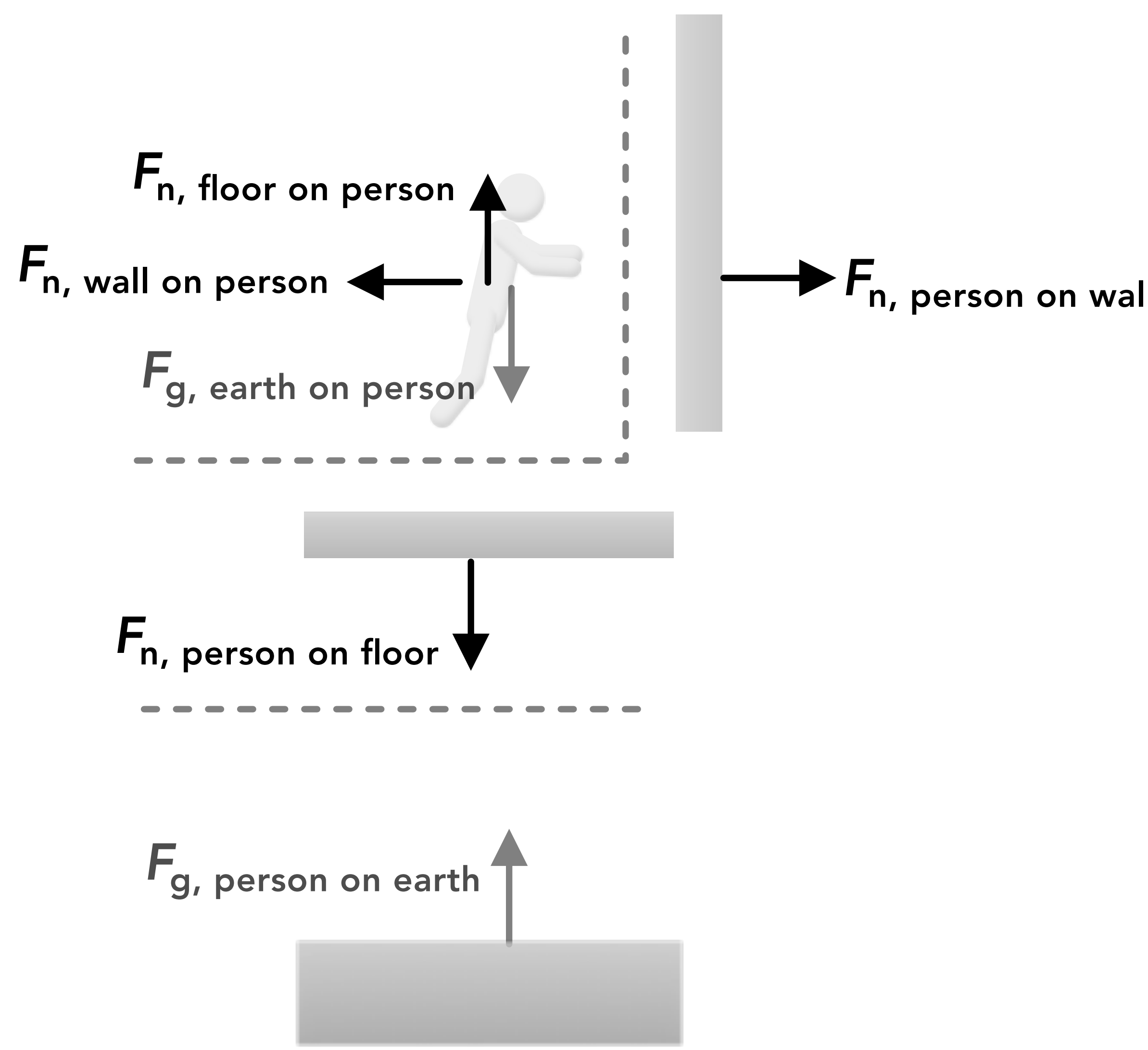


- If a book is resting on a table, the book exerts a downwards normal force on the table and the table exerts an upwards normal force on the book with the same magnitude. Separately, the table exerts a downwards normal force on the ground and the ground exerts an upwards normal force on the table. Note that there are also gravitational force pairs between the earth and the book, and between the earth and the table. Technically there is a gravitational force acting between the book and the table, but it's so weak that it's usually ignored.









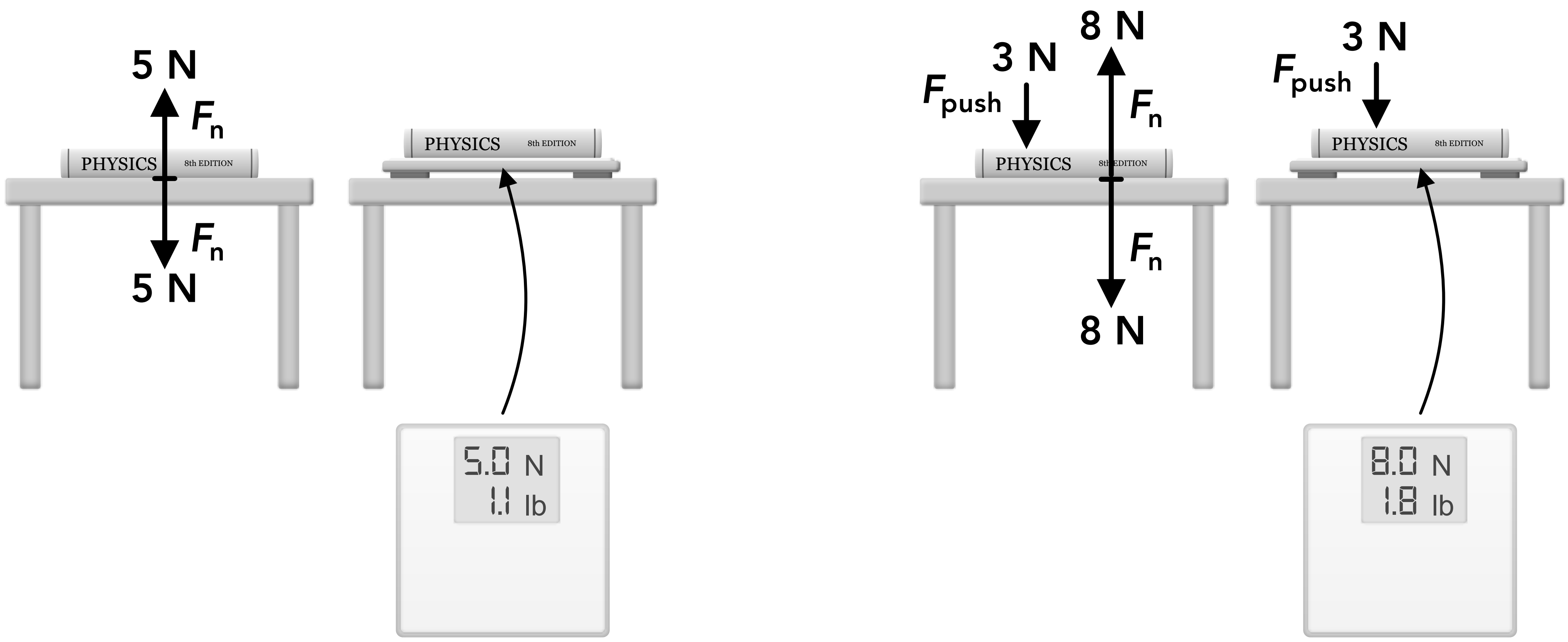
*gravitational force pairs between the floor, the wall and the earth not shown

$$F_{n,\ \text{book on table}} = F_{n,\ \text{table on book}}$$
$$F_{n,\ \text{table on ground}} = F_{n,\ \text{ground on table}}$$

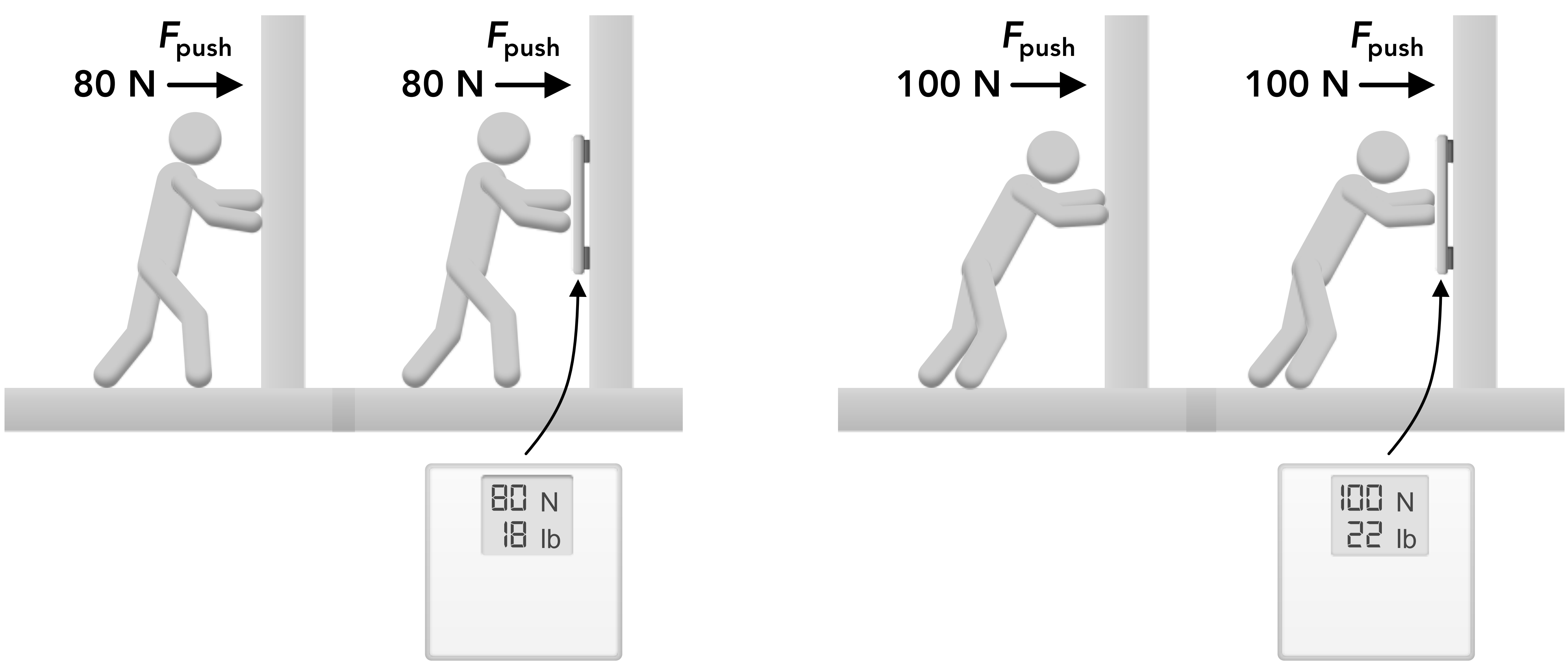
$$F_{n,\ \text{person on wall}} = F_{n,\ \text{wall on person}}$$
$$F_{n,\ \text{person on floor}} = F_{n,\ \text{floor on person}}$$

- The normal force may be less intuitive at first because its magnitude can change based on the other forces being exerted on the object. We usually can't visualize a change in the normal force because the object and the surface don't appear to get closer or farther from each other.
- When thinking about the normal force, we can imagine placing a flat scale or a spring between the object and the surface to visualize the normal force. A scale measures the force acting on both sides (which are equal in magnitude if the scale is not accelerating). A spring will change length when a force is applied to both ends (which are also equal in magnitude if the spring is not accelerating).

A scale would measure the normal force between the book and the table

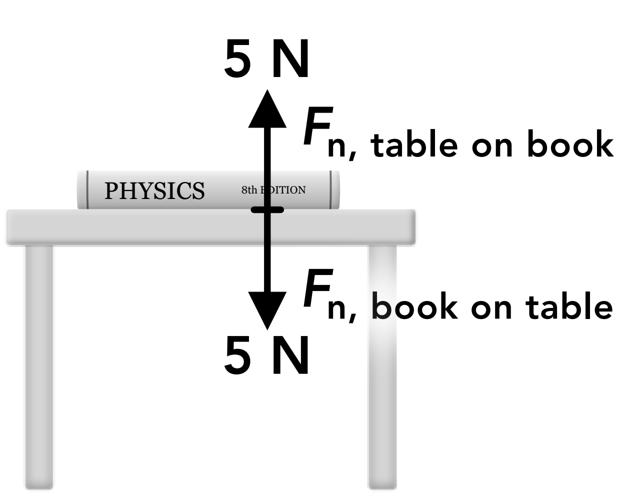


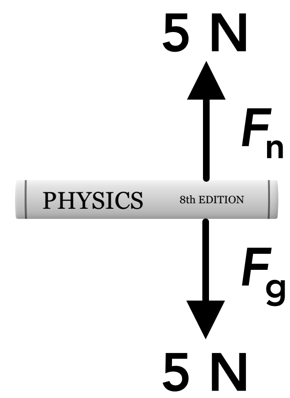
A scale would measure the normal force between the person and the wall



- It's worth noting that the normal force acting upwards on an object is sometimes equal in magnitude to the gravitational force (weight) acting downwards on the object, but not always. This is only true if those are the only two forces acting on the object in those directions and the object is not accelerating, but the normal force depends on other forces being exerted on the object along the same axis.

A book is sitting at rest on a table. The only forces acting on the book are the gravitational force (weight) and the normal force from the table. The net force on the book is zero (it is not accelerating) so the normal force is equal in magnitude to the gravitational force.

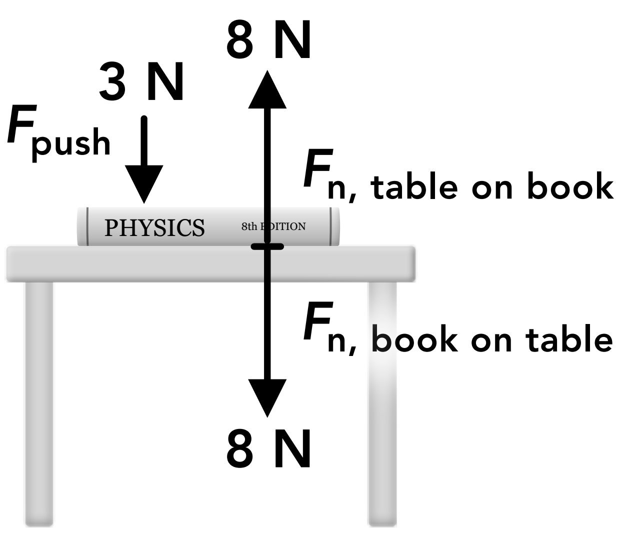


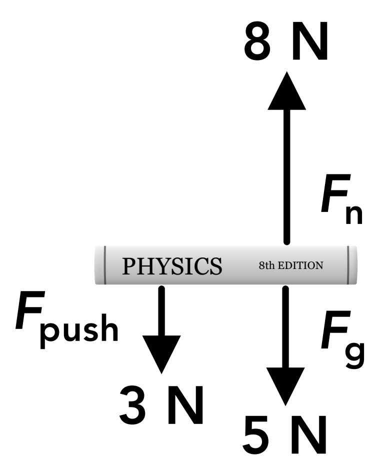


$a_y = 0$

$$\sum F_y = ma_y$$
$$\sum F_y = F_n - F_g = m(0)$$
$$F_n = F_g$$
$$F_n = 5\text{ N}$$

A book is sitting at rest on a table and someone pushes down on the book with a force of 3 N. The net force on the book is still zero (it is not accelerating) so the normal force increases and is equal in magnitude to the gravitational force plus the push force.

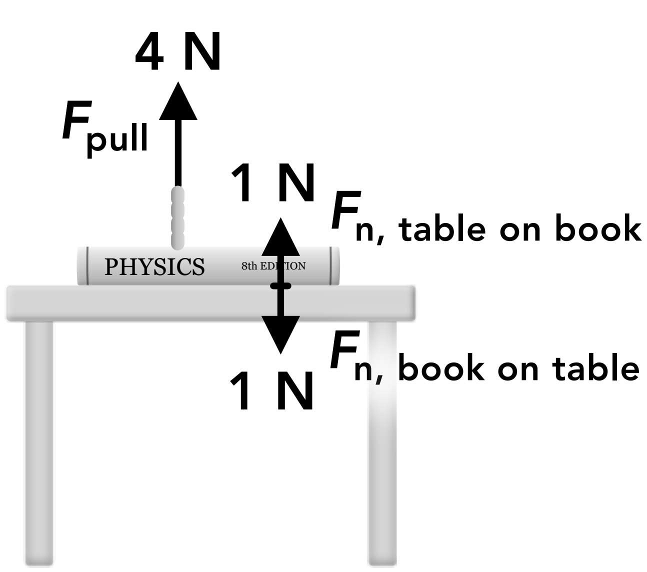


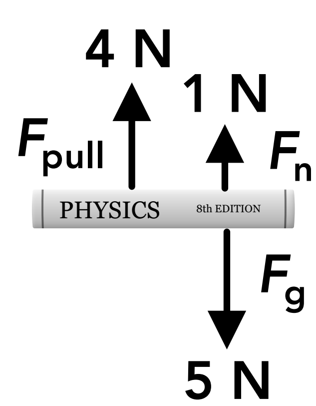


$a_y = 0$

$$\sum F_y = ma_y$$
$$\sum F_y = F_n - F_g - F_{push} = m(0)$$
$$F_n = F_g + F_{push}$$
$$F_n = 5\text{ N} + 3\text{ N}$$
$$F_n = 8\text{ N}$$

A book is sitting at rest on a table and someone pulls up on the book with a force of 4 N. The net force on the book is still zero (it is not accelerating) so the normal force decreases and is equal in magnitude to the gravitational force minus the pull force.





$a_y = 0$

$$\sum F_y = ma_y$$
$$\sum F_y = F_n + F_{pull} - F_g = m(0)$$
$$F_n = F_g - F_{pull}$$
$$F_n = 5\text{ N} - 4\text{ N}$$
$$F_n = 1\text{ N}$$

- An important thing to remember is that a normal force can't cause an object to accelerate if that acceleration means the objects are no longer in contact with each other, because then the normal force would be zero.

STUDY GUIDE

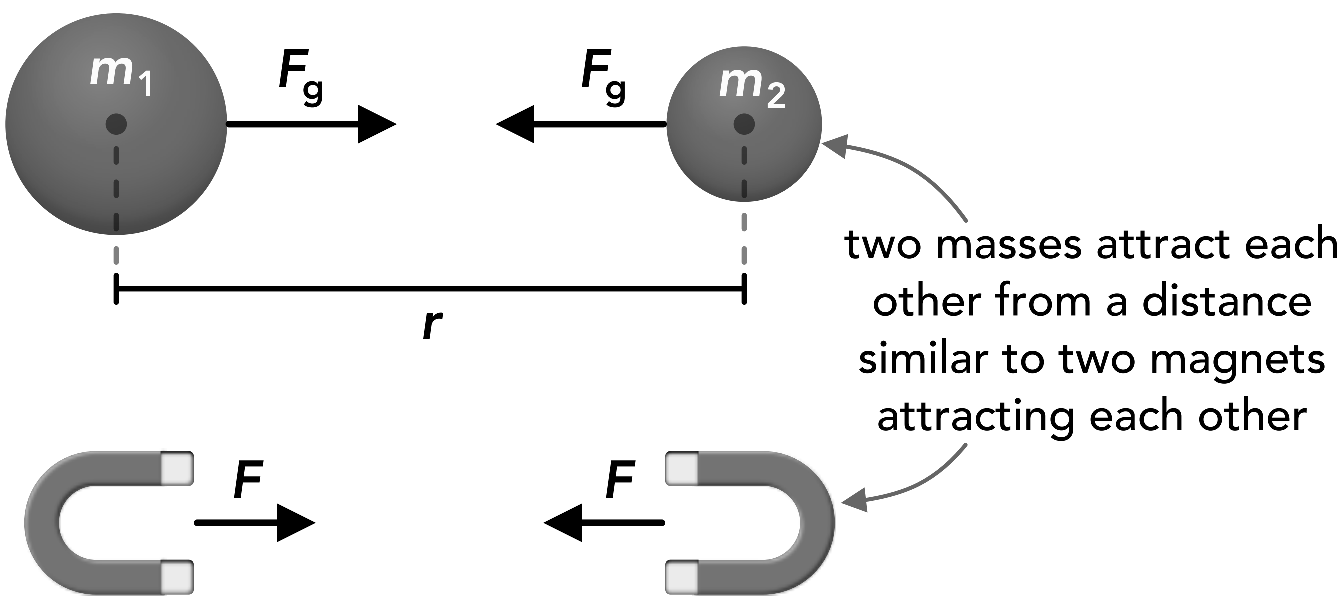
GRAVITY & WEIGHT

Newton's Law of Universal Gravitation

Constants		Unit	Name
G	6.67×10^{-11}	$\frac{m^3}{kg \cdot s^2}$	gravitational constant

Newton's Law of Universal Gravitation (gravitational force)

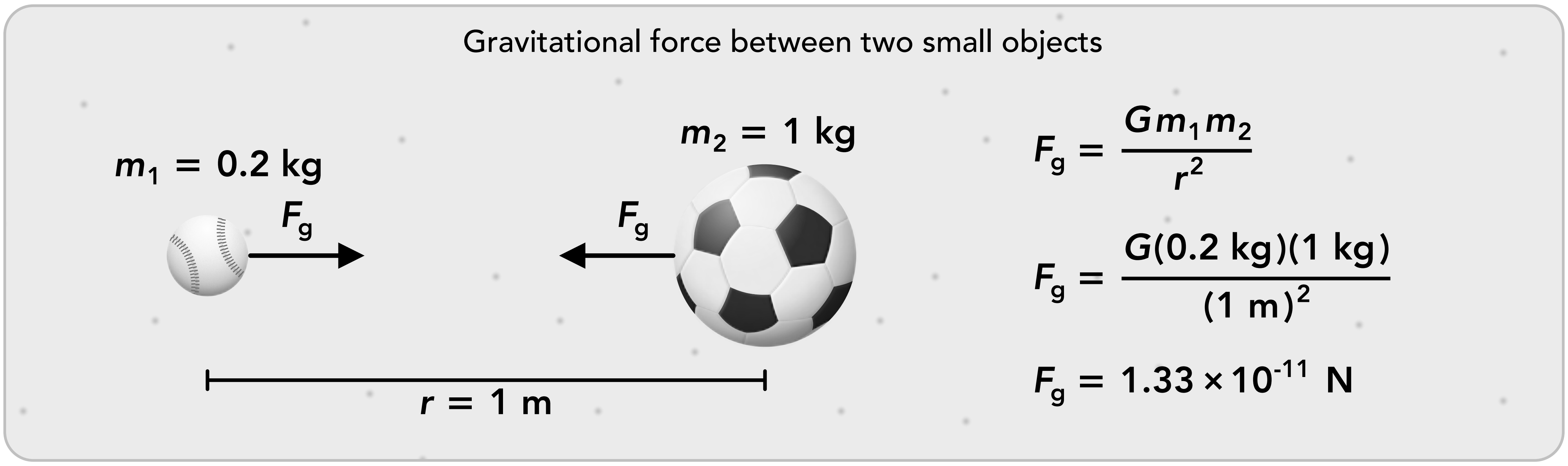
$$F_g = \frac{G m_1 m_2}{r^2} = F_{1\text{ on }2} = F_{2\text{ on }1}$$

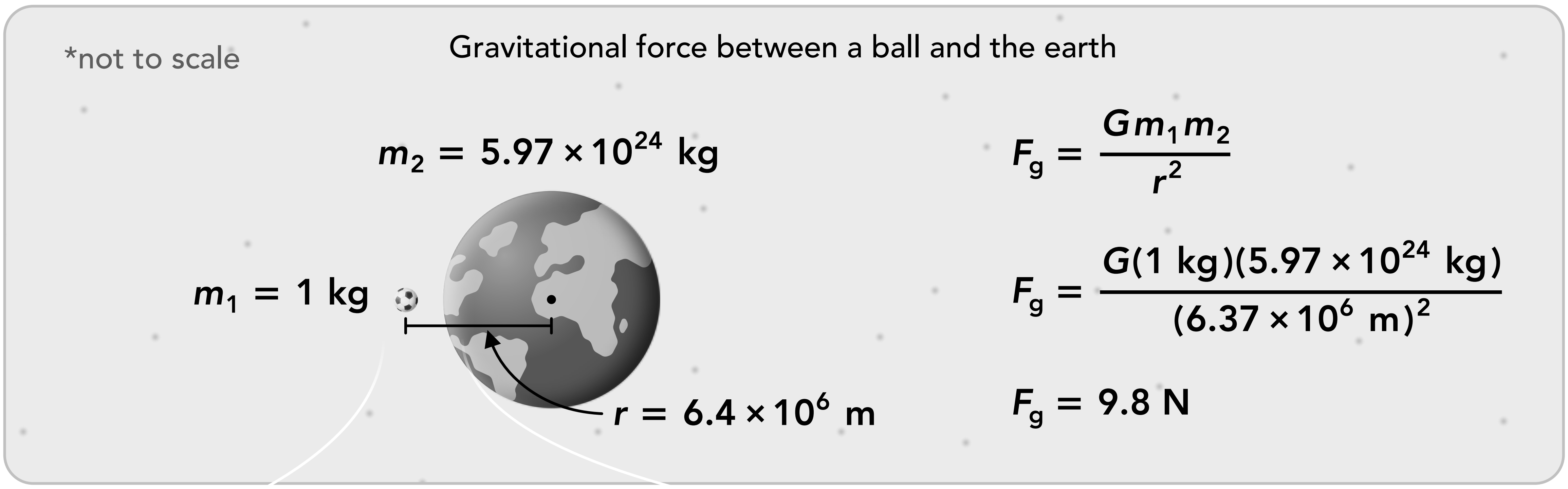


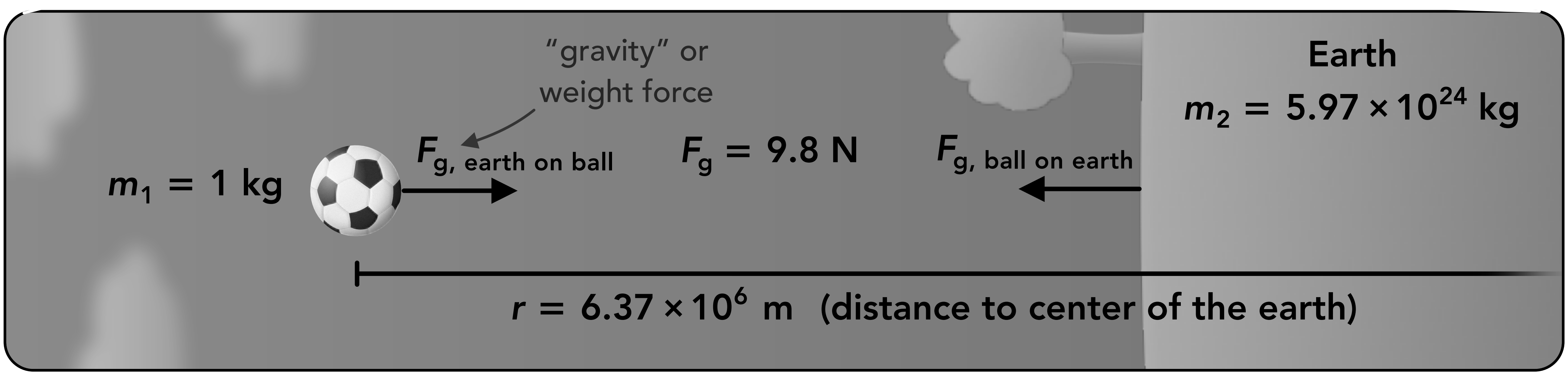
Variables		SI Unit
F_g	gravitational force	N
w	weight force	N
m	mass	kg
M	mass producing a field	kg
r	distance between centers	m
g	gravitational acceleration	$\frac{m}{s^2}$

- **Newton's law of universal gravitation:** Every object in the universe attracts every other object in the universe with a gravitational force that depends on their masses and the distance between their centers.
- This law treats objects as **point masses** which means the gravitational force behaves as if each object's mass is concentrated at a single point (its center of mass, which depends on the object's shape).
- Remember that r is the distance between the centers of the two objects, not between their surfaces.
- A gravitational force is always an **attractive** force that acts towards the center of the other object.
- Based on this equation, the greater the mass of either object the greater the gravitational force. The farther apart the two objects are the smaller the gravitational force.
- The constant G in the equation is the universal gravitational constant whose value is given above.
- It doesn't matter which mass is m_1 and m_2 , and one mass does not need to be larger than the other. It's not the case that only large masses pull on small masses, any two masses pull on each other with the equal force.
- Gravitational forces come in pairs as described in Newton's 3rd law of motion. The gravitational force exerted on mass 1 by mass 2 is equal in magnitude and opposite in direction to the gravitational force exerted on mass 2 by mass 1. Each mass pulls on the other with the same amount of force.

- This gravitational force is what we experience as gravity on earth. However, notice that Newton's law of universal gravitation does not describe gravity using the words "earth", "falling", "down", etc. A gravitational force acts between every two objects in the universe: the earth and the moon attract each other, the earth and a book attract each other, and a book and a cup attract each other because they all have mass.







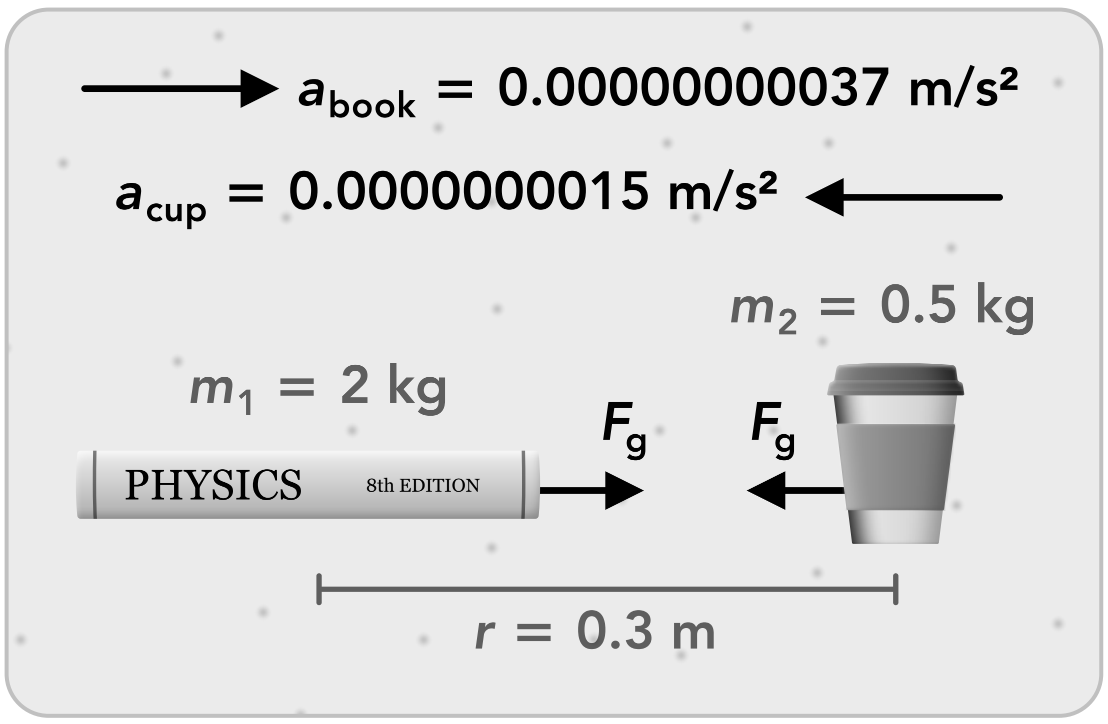
- We call the gravitational force between the earth and an object near the surface of the earth the **weight force** acting on the object. Although we're used to saying objects are pulled "down" by gravity, it's more accurate to say that objects are pulled towards the center of the earth.

- If all objects are attracted to each other by a gravitational force, why don't we experience this? For example, if a book and a cup are sitting next to each other on a table, why don't they move towards each other? There must be one or more forces acting in the opposite direction as the gravitational force so that the net force on each object is zero. In most cases that force is **friction**, but electrostatic forces or other forces may also be involved.

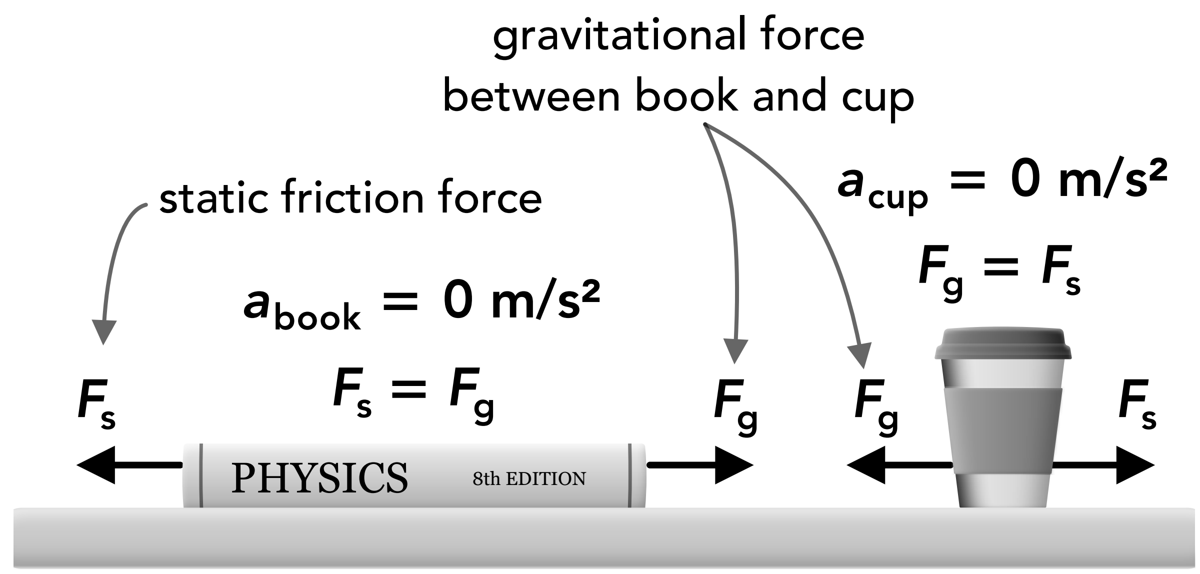
Gravitational force between a book and a cup

$$F_g = \frac{G(2\text{ kg})(0.5\text{ kg})}{(0.3\text{ m})^2} = 7.4\times 10^{-10}\text{ N} = 0.00000000074\text{ N}$$

If there's no friction, the book and the cup will slowly accelerate towards each other



Friction forces prevent the book and the cup from accelerating towards each other



- If the book and the cup were floating in space with no other forces acting on them besides the gravitational force between them, they would slowly accelerate towards each other. In the example above, it would take a few hours for the book and the cup to hit each other (starting from rest).
- In most scenarios there is a static friction force acting between the objects and a surface that opposes the gravitational force and prevents the objects from accelerating towards each other. For a 2 kg book the maximum static friction force could be around 4 N, which means you could push against a resting book with up to 4 N of force before you overcome friction and it begins to slide. That's much more than the gravitational force.
- The gravitational force is very weak compared to the other fundamental forces and the forces we normally experience. Gravity is often associated with the earth and other planets because planets have such a large mass that the gravitational force is significant compared to other forces.

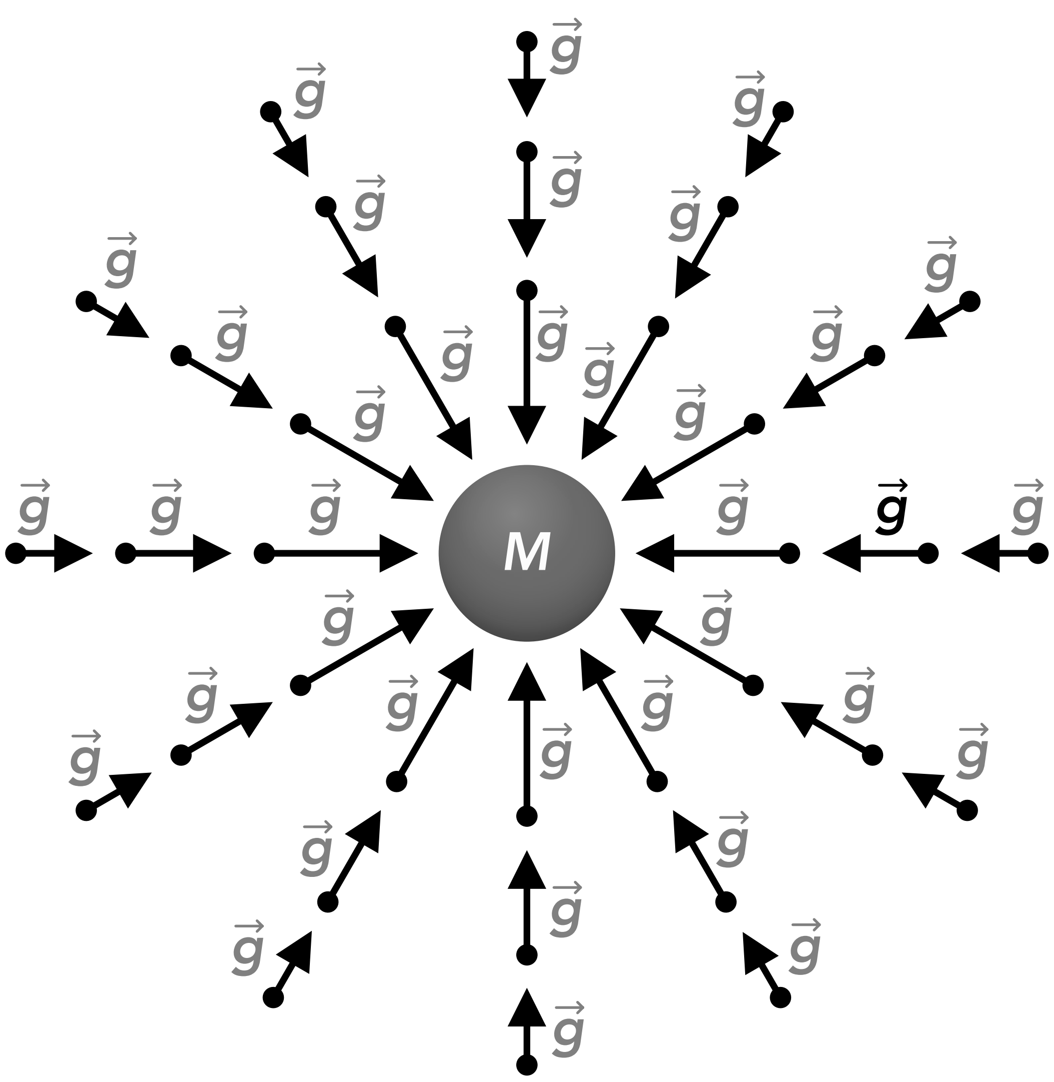
Gravitational Field and Weight

- While Newton's law of universal gravitation treats gravity as a force that exists between two point masses, there is another way to think about gravity: the interaction between a mass and a gravitational field.
- A **gravitational field** exists around every object due to its mass. The field is not visible on its own and is more like a mathematical representation of how a second mass would interact with the mass creating the field at any position in space. A gravitational field is a vector field and is sometimes referred to as a "gravitational acceleration field" because it consists of a vector at every point in space which shows the direction and magnitude of the gravitational acceleration vector at that point.
- The direction of the gravitational field is always **towards the center of the mass producing the field**.
- Note that **a gravitational field is produced by a single mass**. If a second mass is placed in that gravitational field it will experience a gravitational force towards the first mass due to the field. (The second mass also produces its own field which causes the first mass to also experience a gravitational force towards the second mass).
- A mass does not experience a force from its own gravitational field, only the field from another mass.
- In the equations below the variable M represents the mass producing the field and the variable m represents a a second mass in that field, experiencing a force. These two masses can be any size, M does not have to be larger than m, but this is often applied to a planet and a small mass where the planet mass is M.

A gravitational field exists around every mass

g is the **gravitational field strength** and the **acceleration due to gravity** at every point in space

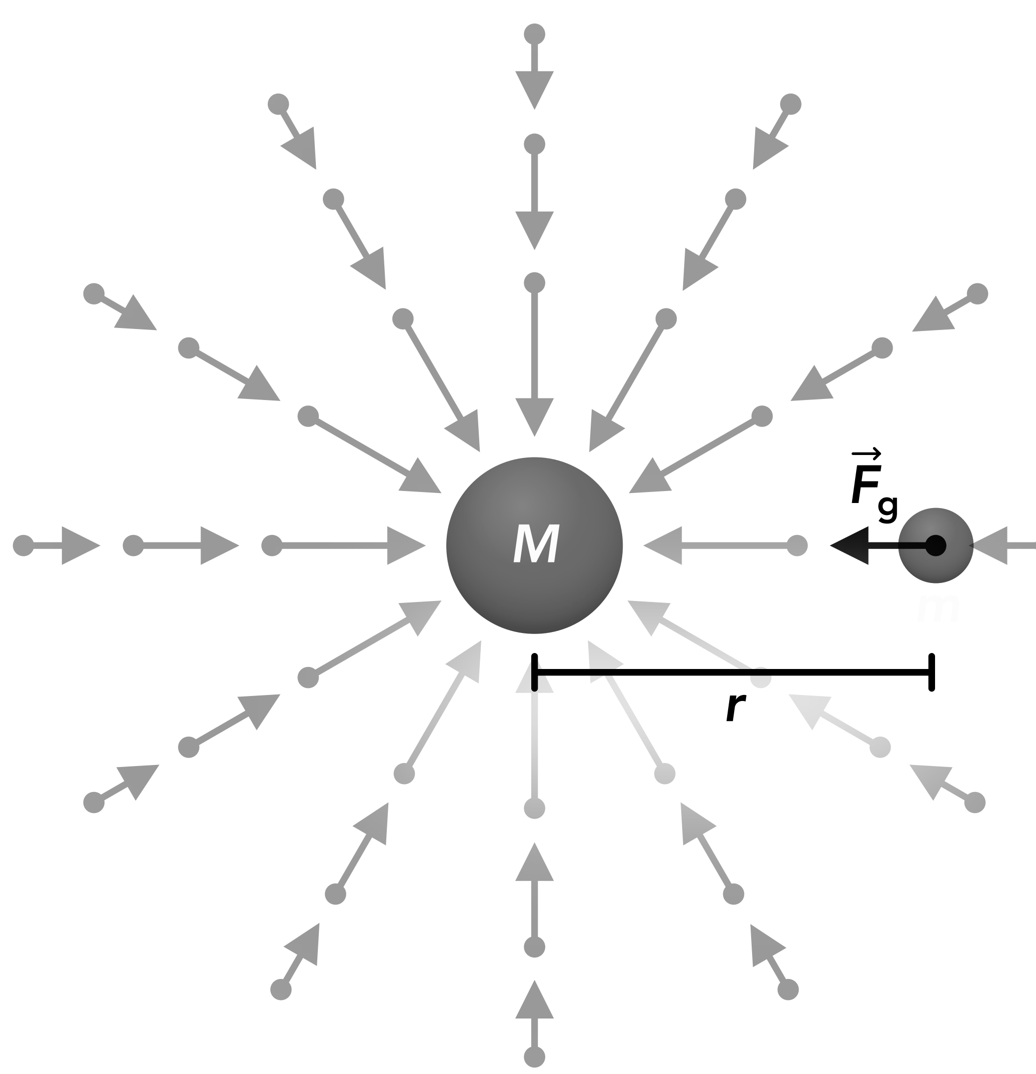
$$\boxed{g = \frac{GM}{r^2}} \qquad \text{Units: } \frac{\text{N}}{\text{kg}} = \frac{\text{m}}{\text{s}^2}$$



A second mass placed in that gravitational field will experience a gravitational force towards the mass that is creating the field

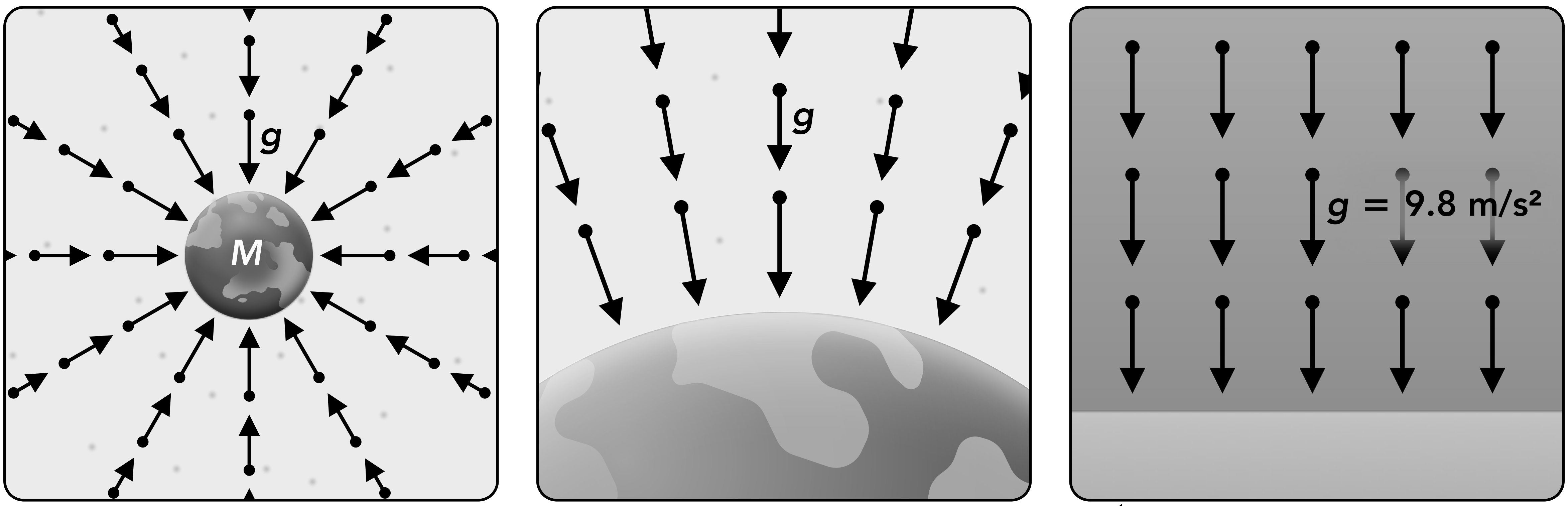
Gravitational force on mass in gravitational field

$$\boxed{F_g = mg} = \boxed{F_g = \frac{GMm}{r^2}}$$



- If the mass producing the gravitational field is the earth, we can look at the strength of the field at different distances from the center of the earth. As we zoom in, the gravitational field lines appear to be more parallel (the earth doesn't appear as curved) and the field strength doesn't vary as much within the smaller window height.
- Near the surface of the earth (where the value of r is equal to the radius of the earth) the value of g is 9.8 N/kg or 9.8 m/s². That's the strength of the gravitational field and the acceleration due to gravity for any object.

The value of g depends on the distance r from the center of the earth



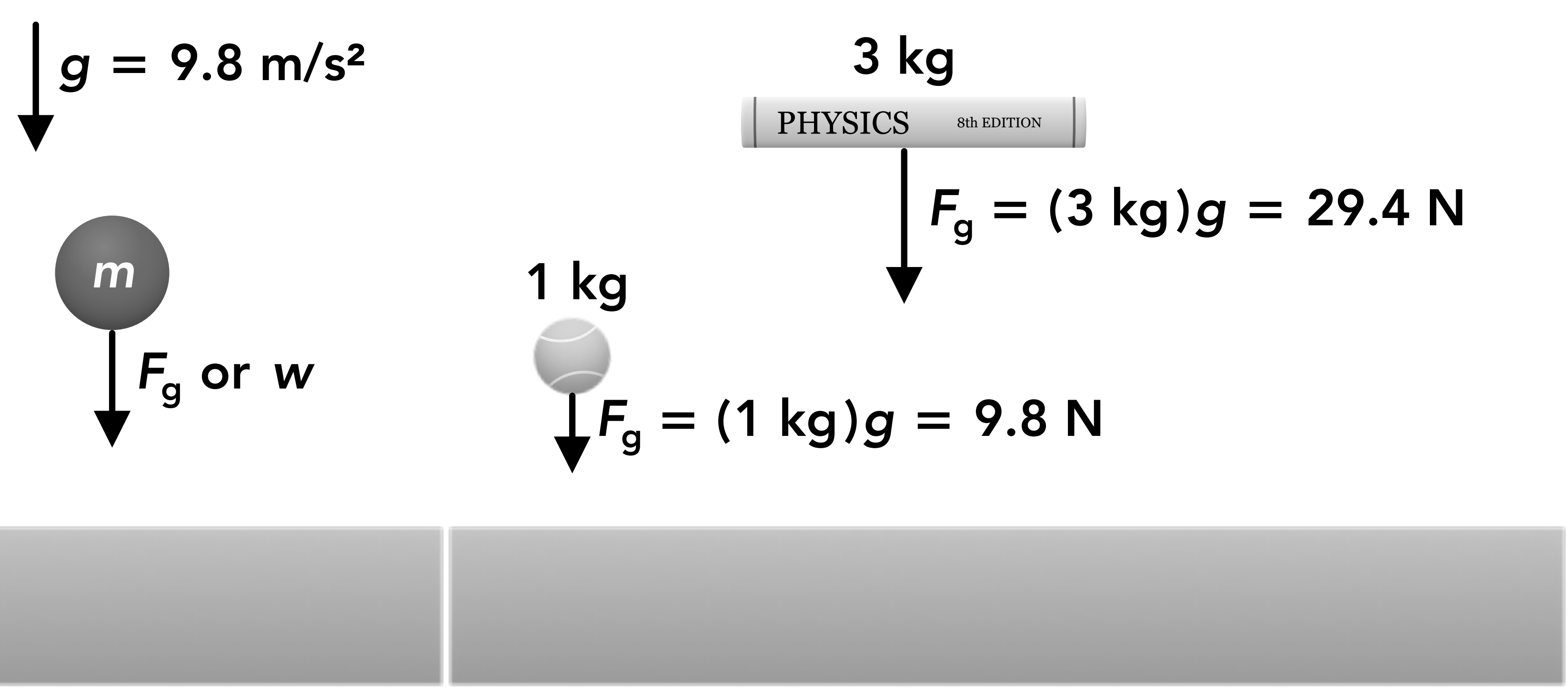
$$g = \frac{G(5.97 \times 10^{24}\ \text{kg})}{(6.37 \times 10^{6}\ \text{m})^2} = 9.8\ \text{m/s}^2$$

Near the surface of the earth the value of g is about 9.8 m/s²

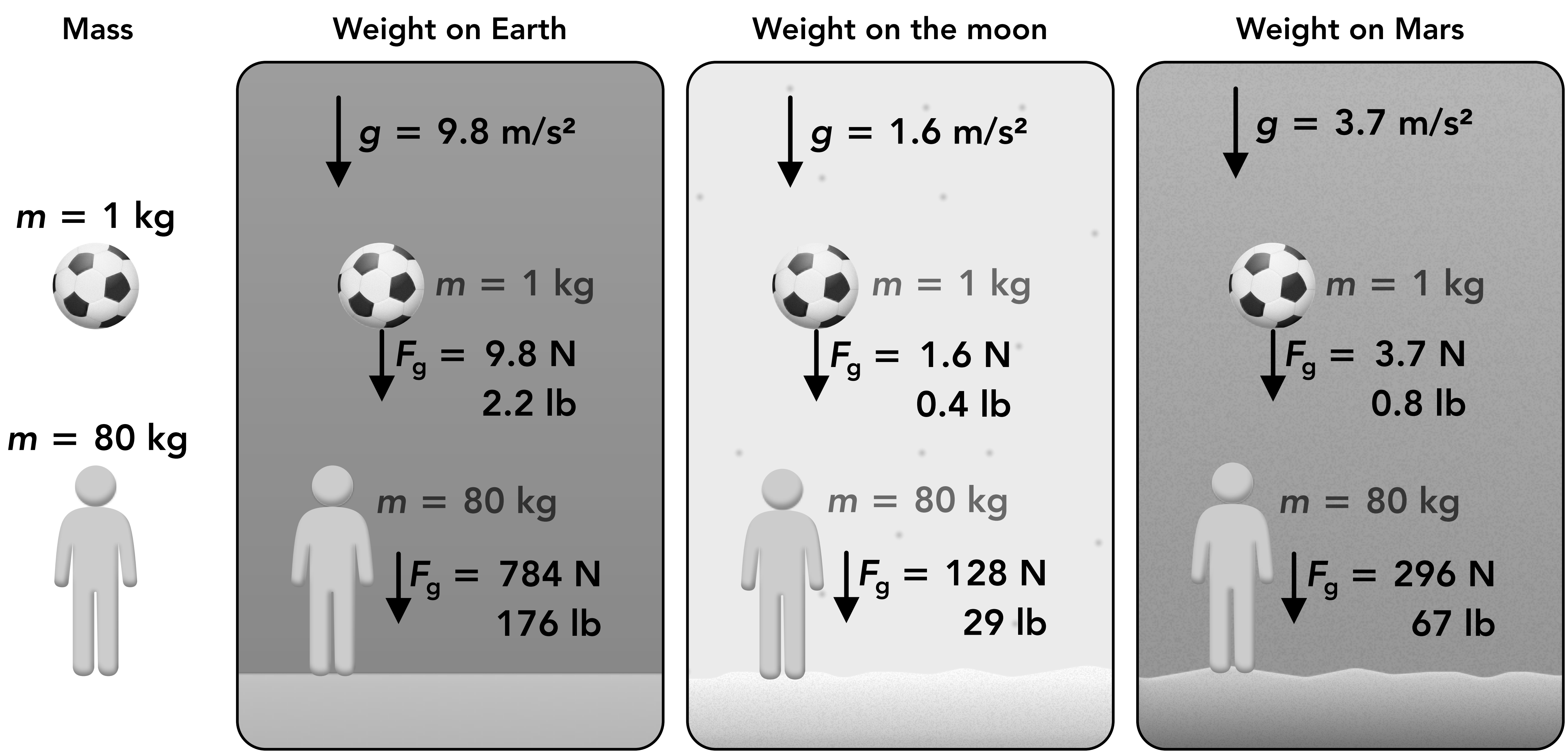
- The **weight force** acting on an object (sometimes referred to as "the weight of an object") is just the gravitational force acting on that object when it's near the earth (or any large body like the moon or another planet).
- Unless a different value is given, **assume the value of g is 9.8 m/s²** when finding the weight of an object on earth.

Weight force

$F_g = mg$ or $w = mg$



- A common confusion is the difference between mass (m) and weight (F_g) because they are sometimes used interchangeably outside of physics.
- An object's mass is its **inertia**, which is how much it resists a change to its state of motion as described in Newton's 1st law of motion. An object's mass doesn't change no matter where it is in the universe.
- An object's weight is the **gravitational force** acting on the object when it's in the gravitational field of another mass (usually the earth, the moon or another planet).
- An object's weight is proportional to its mass so we often measure an object's weight instead of its mass because the value of g is relatively constant near the surface of earth.
- The strength of gravity is different near the surface of the moon and other planets because of the difference in the the planet's mass and radius. Even though the mass of an object is the same everywhere, its weight will change.



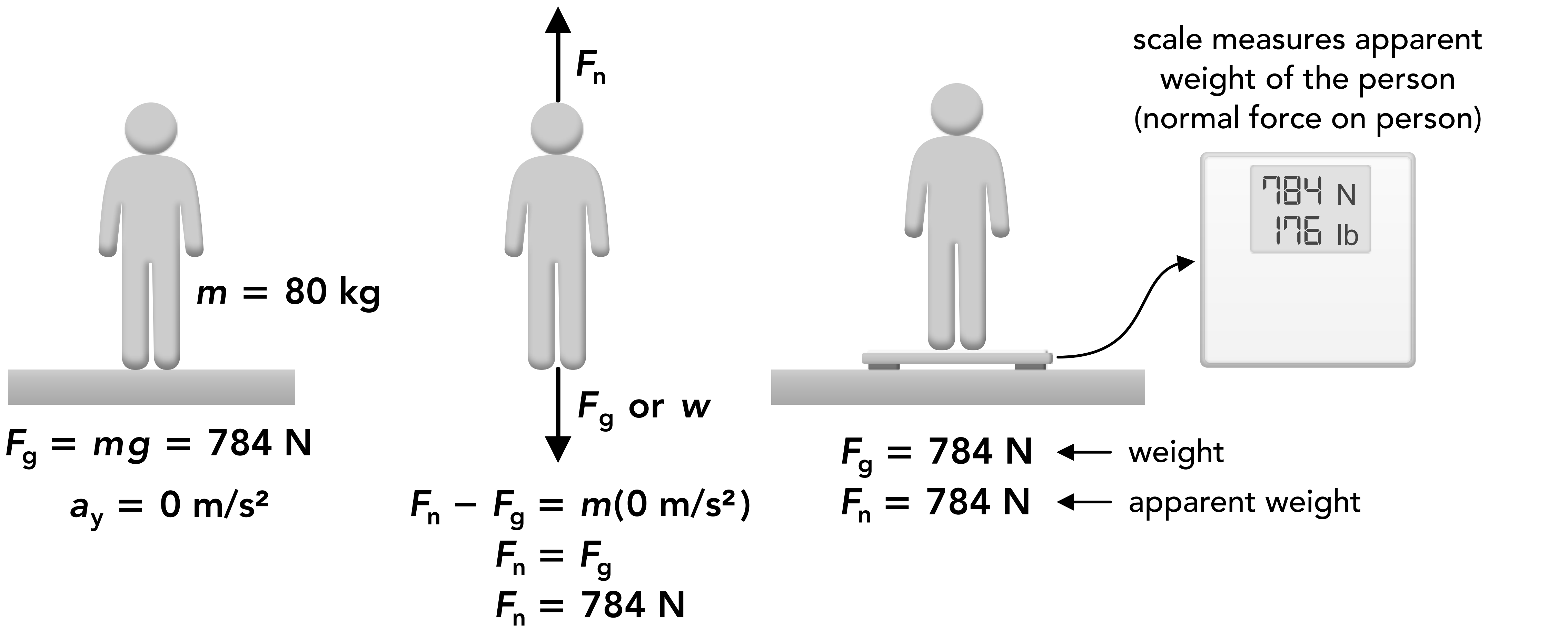
Apparent Weight

- The weight of an object is the gravitational force pulling the object down (towards the center of the planet). The weight does not change due to the motion of the object or other forces acting on it.
- The **apparent weight** of an object is the **normal force** between the object and the surface below it, or the **tension force** in the rope that the object is hanging from.
- Think about how and why you feel your own weight when standing or sitting in a chair. It may seem like you're feeling the force of gravity, but you're actually feeling the **contact forces** that are supporting you from below (the normal force acting upwards on your body from the floor or the chair).
- Remember that a scale placed between two objects (or an object and a surface) measures the normal force between the two objects, so **a scale measures your apparent weight, not your actual weight**.

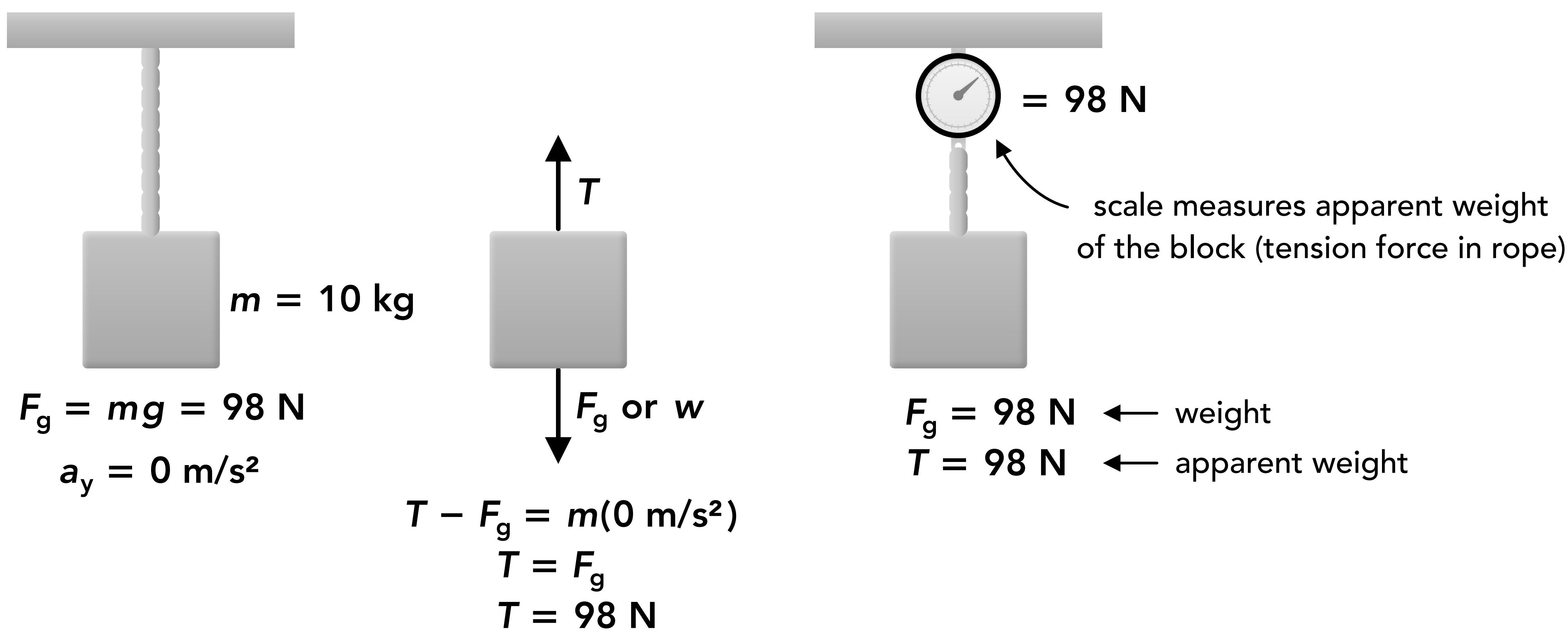
- **Weightlessness** is a term used to describe when an object has **zero apparent weight**. This does not mean the object has zero weight. If there is a gravitational force acting on the object then it still has weight.
- In the fifth elevator example, the elevator is accelerating downwards at 9.8 m/s² and the person's apparent weight is zero. There is still a weight force pulling them down, causing them to accelerate downwards at g, 9.8 m/s². In this example the elevator, the person and the scale are all in **free fall**. The person still has weight but they are experiencing weightlessness, just like if they were falling through the air without an elevator.

- **If you're not accelerating up or down then your weight and apparent weight are equal**. The net force in the vertical direction is zero, and your weight force is equal in magnitude to the normal force acting upwards. This is the case in many "normal" scenarios (like standing on the ground, sitting in a chair) because we're usually not accelerating up or down.

The apparent weight (normal force) is equal to the actual weight if the acceleration is zero



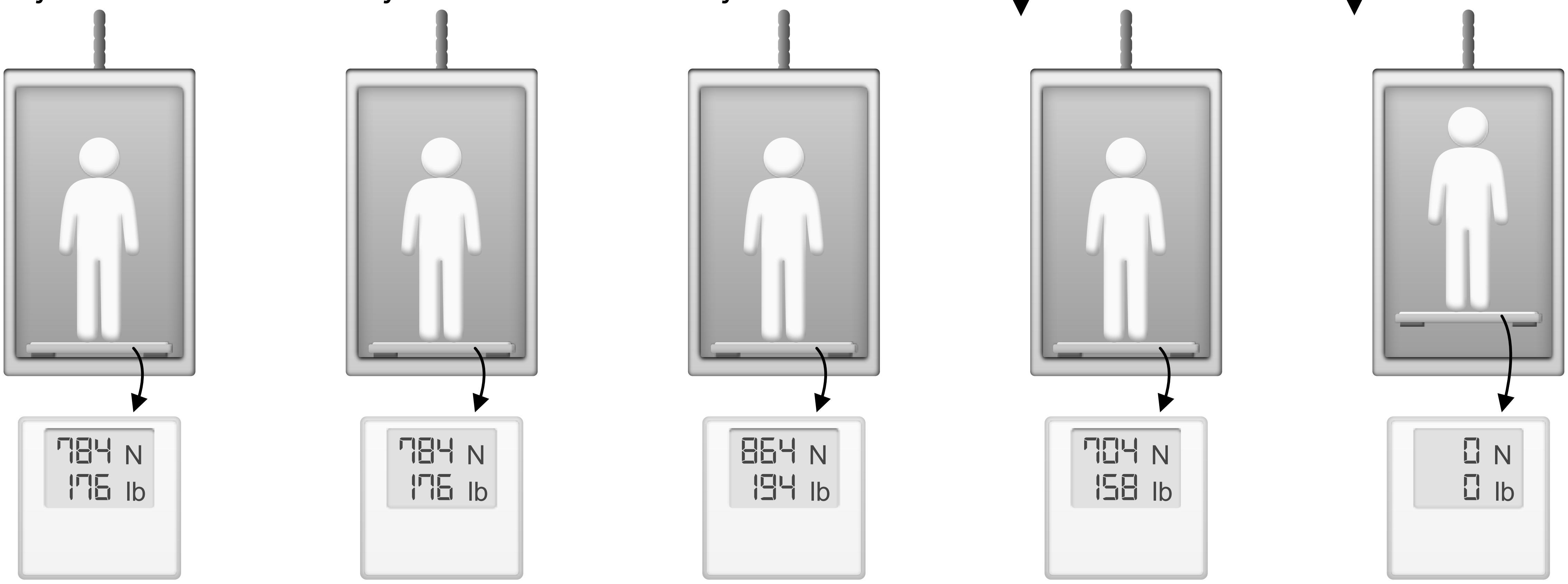
The apparent weight (tension force) is equal to the actual weight if the acceleration is zero



- **If an object is accelerating up or down then the apparent weight is not equal to the weight**. The net force in the vertical direction is not zero so the normal force (or the tension force) is likely not equal to the weight force.
- Imagine you're standing on a scale in an elevator. If the elevator is not moving (the acceleration is zero) your apparent weight is equal to your weight. If the elevator is moving at a constant velocity (the acceleration is still zero) your apparent weight is still equal to your weight. But if you and the elevator are accelerating up or down then your apparent weight is not equal to your actual weight - you'll feel lighter or heavier than your true weight.
- For each of the following examples the elevator has a different motion. Look through the free body diagrams and Newton's 2nd law equations to see how the acceleration affects the apparent weight of the person.

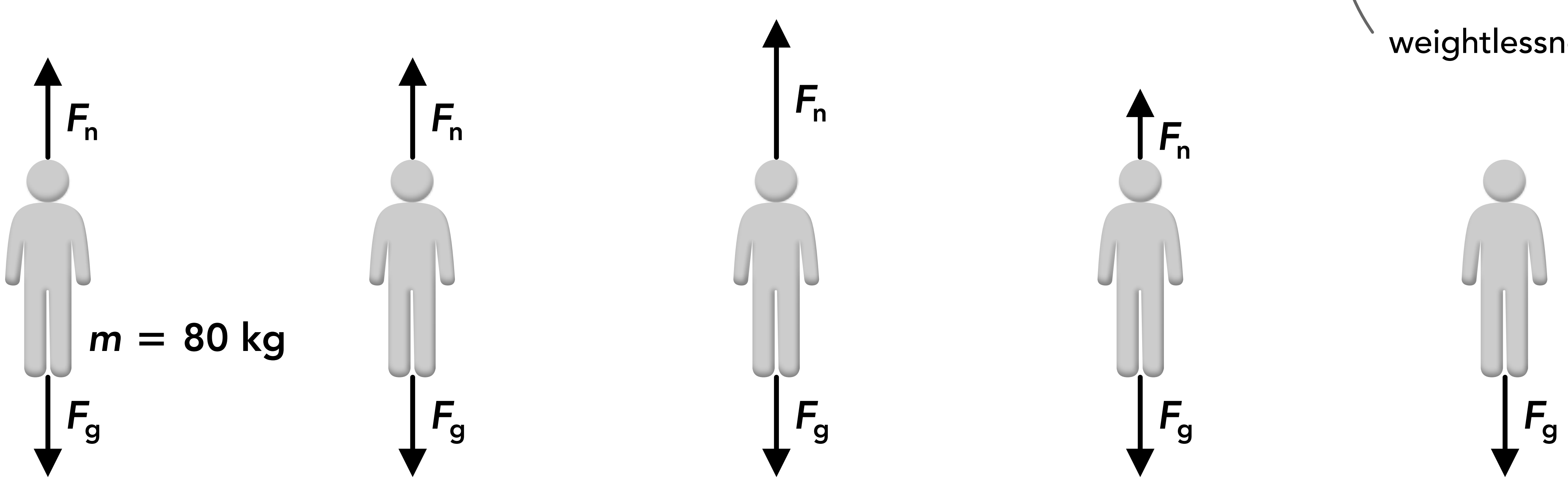
The apparent weight of the person changes if the elevator is accelerating.
The actual weight of person is always: $F_g = mg = 784 \text{ N}$

not moving	moving up at a constant velocity	accelerating up	accelerating down	accelerating down
$v_y = 0 \text{ m/s}$ $a_y = 0 \text{ m/s}^2$	$\uparrow v_y = 2 \text{ m/s}$ $a_y = 0 \text{ m/s}^2$	$\uparrow a_y = 1 \text{ m/s}^2$	$\downarrow a_y = -1 \text{ m/s}^2$	$\downarrow a_y = -9.8 \text{ m/s}^2$



apparent weight: $F_n = 784 \text{ N}$	apparent weight: $F_n = 784 \text{ N}$	apparent weight: $F_n = 864 \text{ N}$	apparent weight: $F_n = 704 \text{ N}$	apparent weight: $F_n = 0 \text{ N}$
weight: $F_g = 784 \text{ N}$	weight: $F_g = 784 \text{ N}$	weight: $F_g = 784 \text{ N}$	weight: $F_g = 784 \text{ N}$	weight: $F_g = 784 \text{ N}$

weightlessness



$$\sum F_y = ma_y$$
$$F_n - F_g = (80 \text{ kg})(0 \text{ m/s}^2)$$
$$F_n = F_g$$
$$F_n = 784 \text{ N}$$

$$\sum F_y = ma_y$$
$$F_n - F_g = (80 \text{ kg})(0 \text{ m/s}^2)$$
$$F_n = F_g$$
$$F_n = 784 \text{ N}$$

$$\sum F_y = ma_y$$
$$F_n - F_g = (80 \text{ kg})(1 \text{ m/s}^2)$$
$$F_n = F_g + 80 \text{ N}$$
$$F_n = 864 \text{ N}$$

$$\sum F_y = ma_y$$
$$F_n - F_g = (80 \text{ kg})(-1 \text{ m/s}^2)$$
$$F_n = F_g - 80 \text{ N}$$
$$F_n = 704 \text{ N}$$

$$\sum F_y = ma_y$$
$$F_n - F_g = (80 \text{ kg})(-9.8 \text{ m/s}^2)$$
$$F_n = F_g - 784 \text{ N}$$
$$F_n = 0 \text{ N}$$

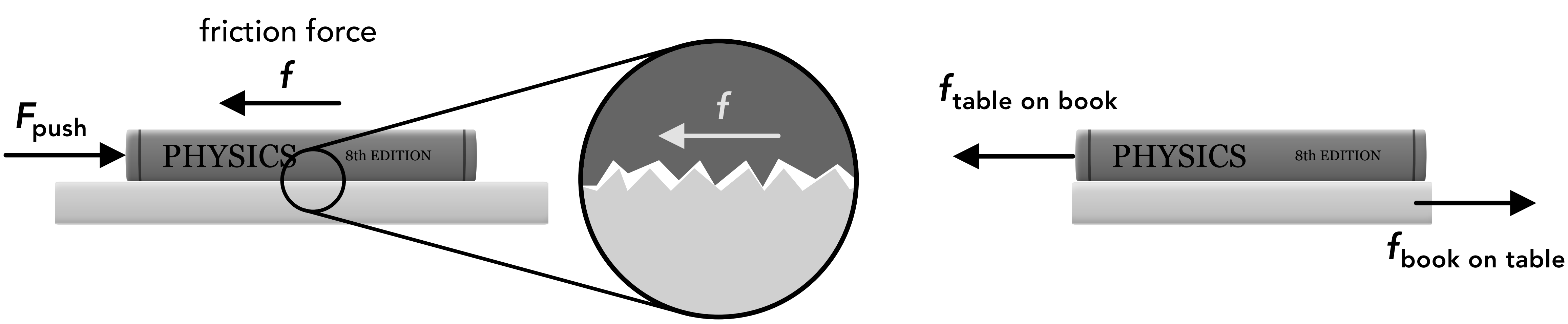
FRICTION

Friction

Variables		SI Unit
f_s	static friction force	N
f_k	kinetic friction force	N
f_r	rolling friction force	N
μ_s	coefficient of static friction	
μ_k	coefficient of kinetic friction	
μ_r	coefficient of rolling friction	
F_n	normal force	N

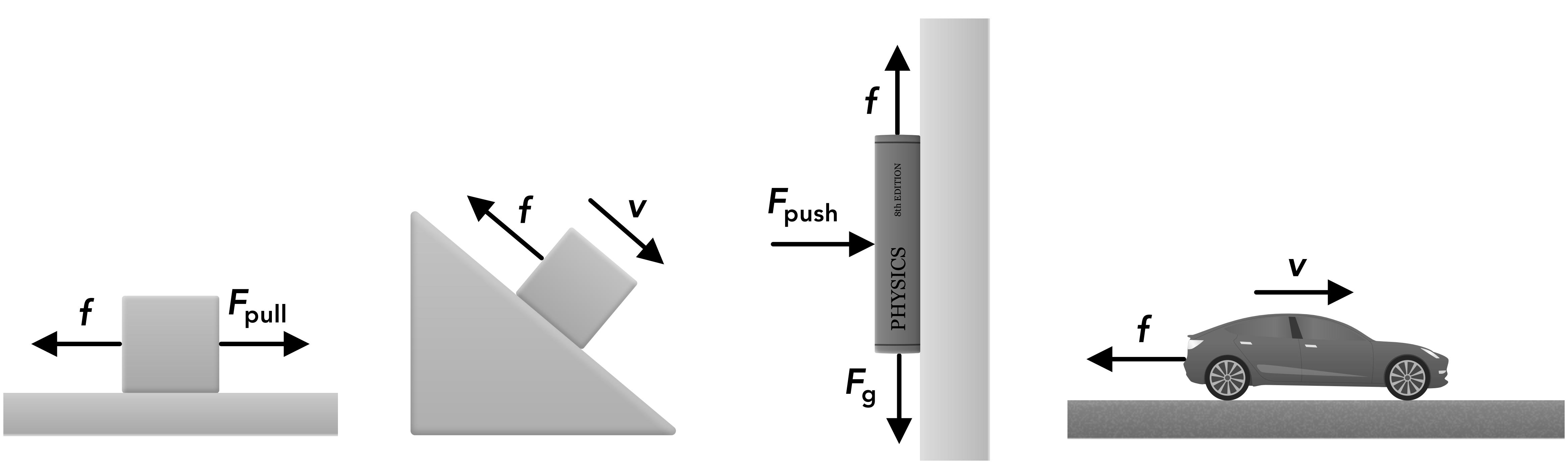
- **Friction** is a force that acts between two objects that are in contact with each other (or between an object and a surface) which acts to oppose or prevent motion.
- The friction force is the reason why moving objects appear to slow down and stop on their own, and why most objects remain at rest. Without the friction force between objects and the surface they're on, things would be sliding everywhere. You also wouldn't be able to walk forward, ride a bike or drive a car.
- The friction force arises between the surfaces of two objects which is rough at the microscopic level. In simple terms, the "mountains" from one surface get stuck in the "valleys" in the other surface, which results in sideways forces that prevent or oppose motion (the force acts parallel to the surface).

A pair of friction forces is caused by surface roughness at a microscopic level



- Friction forces come in pairs as described in Newton's 3rd law of motion. In the example above, a book is being pushed across a table. The table applies a friction force on the book and the book applies a friction force on the table with an equal magnitude in the opposite direction.
- There are several types of friction: static friction, kinetic friction, rolling friction and others. The drag force (air resistance) is also a type of friction between an object and the air around it.
- The friction force on an object always acts **parallel to the surface** and in the direction that **opposes its motion** or **prevents its motion** (if it's not moving).

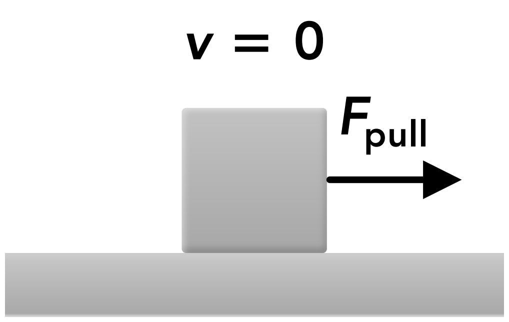
The friction force always acts parallel to the surface

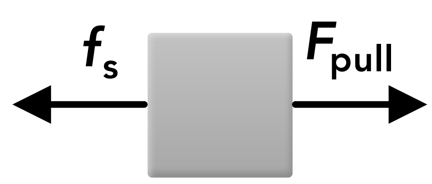


Static Friction

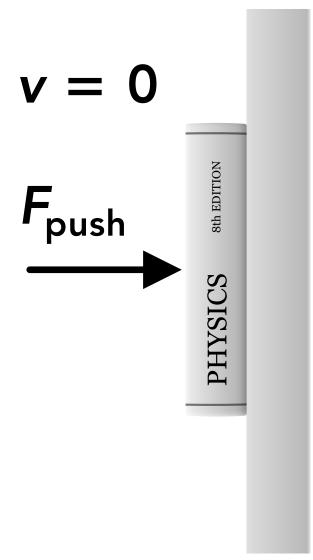
- **Static friction** is a type of friction force that acts on a static (not moving) object to prevent it from moving.
- If you push or pull an object and it doesn't move, there is a static friction force acting in the opposite direction as the force you're applying.
- Notice that the equation below can be used to find the **maximum** static friction force acting on an object. Unlike kinetic friction, the magnitude of a static friction force depends on other forces acting on the object along the same axis, like a normal force does. To find the static friction force on an object the other forces must be known.

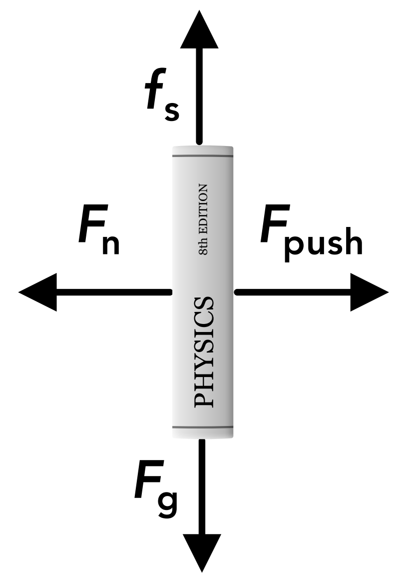
The static friction force depends on the other forces acting on the object along the same axis





$$\sum F_x = ma_x$$
$$F_{pull} - f_s = m(0\ \text{m/s}^2)$$
$$f_s = F_{pull}$$





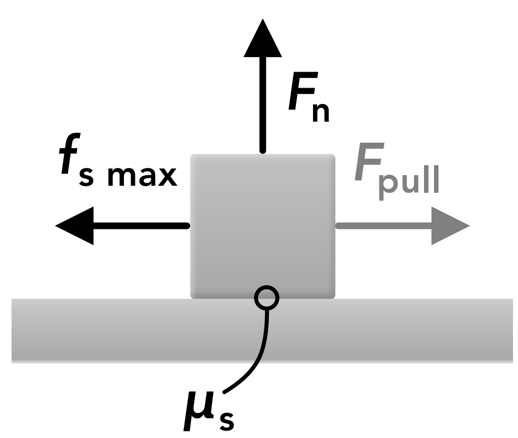
$$\sum F_y = ma_y$$
$$f_s - F_g = m(0\ \text{m/s}^2)$$
$$f_s = F_g$$

- The maximum static friction force that is possible between an object and a surface depends on the coefficient of static friction between the two surfaces and the normal force between the two surfaces.
- The **coefficient of friction** μ (the Greek letter "mu") is a value that depnds on the materials and the conditions of the two surfaces. The value is usually between 0 and 1 and it does not have a unit.

Maximum static friction force

$$\boxed{f_{s\ max} = \mu_s F_n}$$

μ_s : coefficient of static friction



coefficient of static friction between box and ground

Some static friction coefficients:

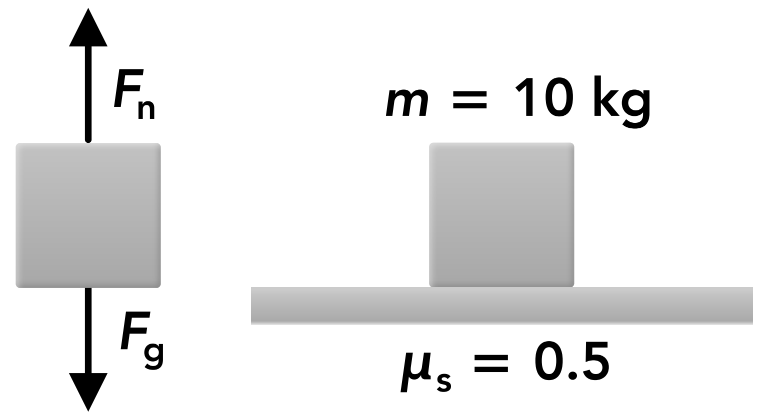
Materials	μ_s
Tire / road (dry)	1.0
Tire / road (wet)	0.2
Skin / metal	0.9
Steel / aluminum	0.6
Steel / ice	0.03

- Again, the equation above can be used to calculate the maximum possible static friction force.
- **The actual static friction force is some value between zero and that maximum value**, and depends on the other forces being applied along the same axis.

Example: A 10 kg box is sitting on the ground at rest. The normal force between the box and the ground is equal to the weight force, 98 N. The coefficient of static friction between the box and the ground is 0.5. The maximum static friction force between the box and the ground is 49 N. A rightwards pulling force is then applied to the box.

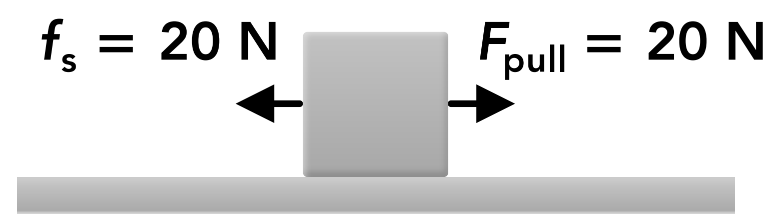
$$\sum F_y = ma_y$$
$$F_n - F_g = m(0\ \text{m/s}^2)$$
$$F_n = F_g = mg = (10\ \text{kg})(9.8\ \text{m/s}^2) = 98\ \text{N}$$

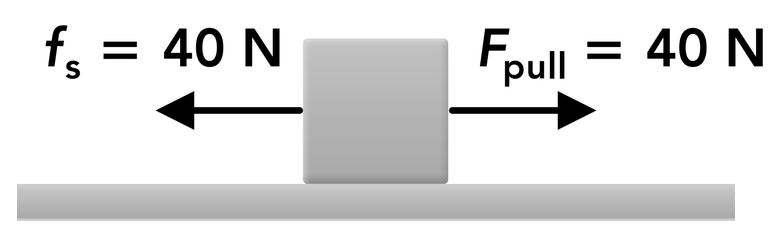
$$f_{s\ max} = \mu_s F_n = (0.5)(98\ \text{N}) = 49\ \text{N}$$



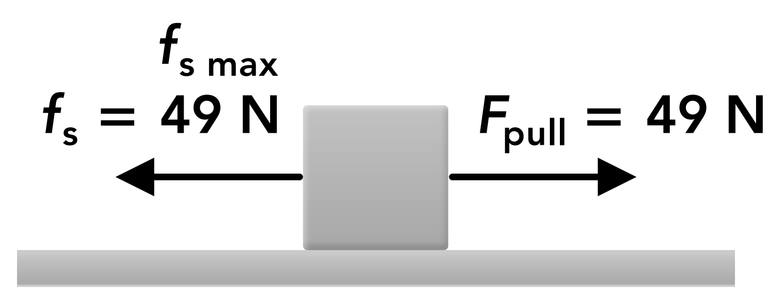
If the applied force is zero (there's no other forces acting parallel to the surface) there is no static friction force because there are no forces for friction to "react to" or any motion to prevent.

As the applied force increases, the static friction "reacts" and also increases so it has the same magnitude in the opposite direction.

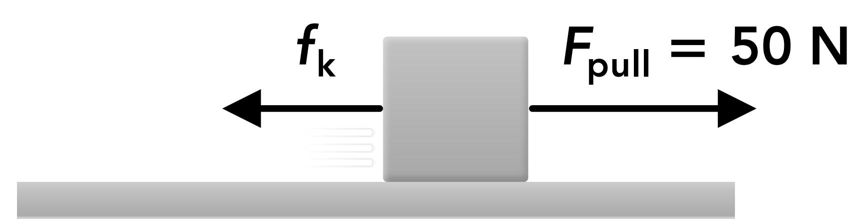




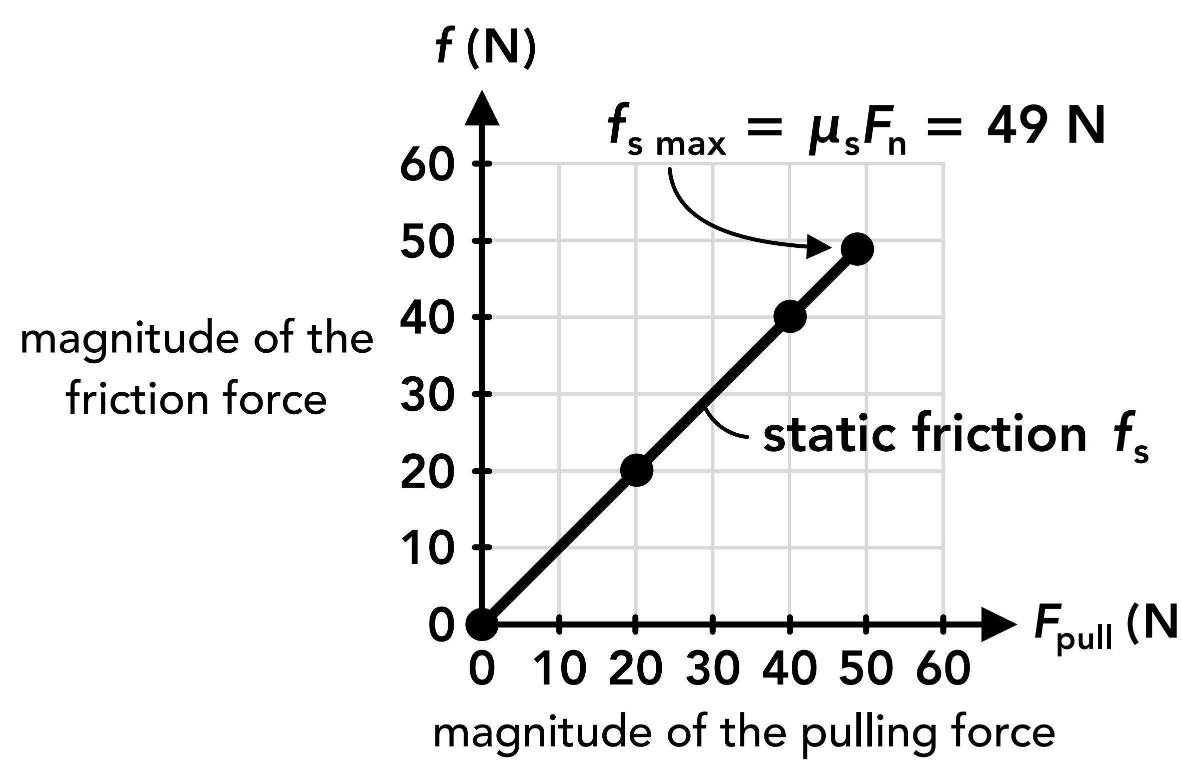
Eventually the applied force equals the maximum static friction force that's possible between the box and the ground.



If the applied force is greater than the maximum static friction force, the box will begin to slide and the friction transitions from static friction to kinetic friction.

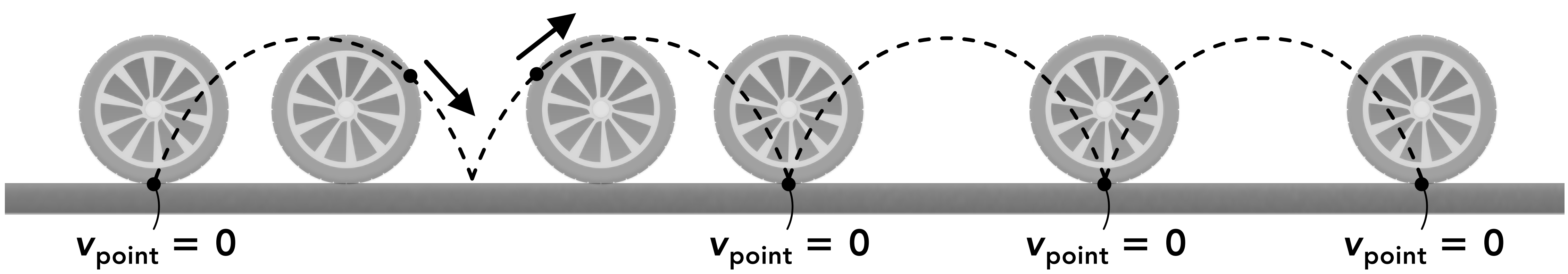


A graph of the static friction force vs the applied force is a straight line with a slope of 1. The static friction force is equal to the applied force until it reaches the maximum static friction force (which is found using the equation above). The points represent the above diagrams.

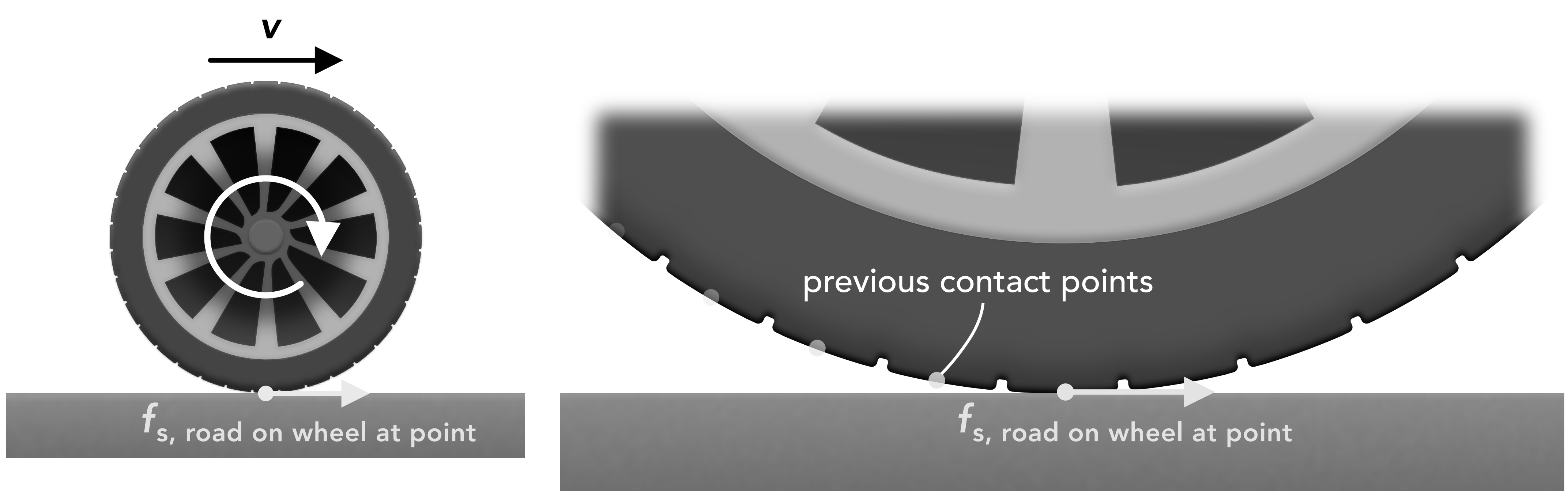


- When an object like a wheel is rolling, there is only a single point of contact between the object and the surface (this is a simplification, there is a small surface area instead of a point if the object deforms).
- As the object rotates, the point on the edge of the object which is in contact with the surface changes.
- At any one moment, the contact point on the rolling object is **not moving relative to the surface**.
- This means that there is a momentary **static friction force** between that point on the edge of the object and the surface that it's rolling on. If this static friction didn't exist for a car tire, the tire would slip and the car would not be able to drive forwards.
- Torque and rotational dynamics are responsible for a car driving forwards as its wheels rotate without slipping, but it's worth noting here that static friction is responsible for the concept of "rolling without slipping".

A point on the edge of a rolling object has no velocity during the moment it's in contact with the ground



There is a momentary static friction force acting on the contact point, causing the object to move forward
***This is not rolling friction, this is just how rolling without slipping occurs**



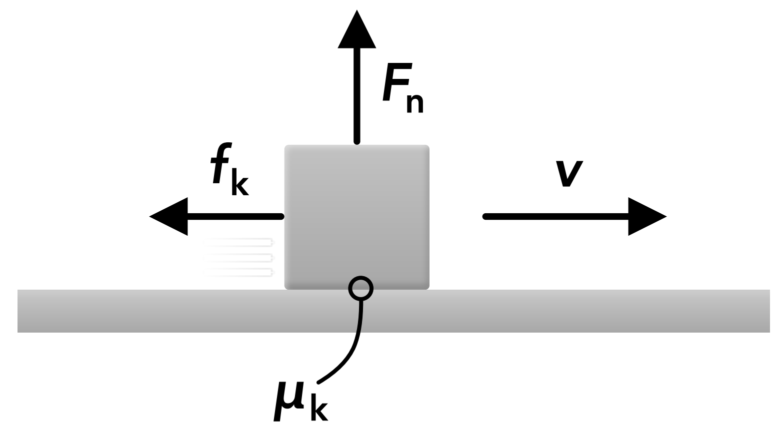
Kinetic Friction

- **Kinetic friction**, also referred to as **sliding friction** is a type of friction force that acts on a moving object in the opposite direction as its motion.
- More specifically, kinetic friction occurs between two surfaces that moving **relative to each other**. For example, the contact point on a tire does not move relative to the road as seen in the static friction section. Even though the car is “moving” that doesn’t mean every friction force involved with the motion is kinetic friction.
- Unlike static friction, **kinetic friction does not depend on other forces along the same axis or the velocity**. It only depends on the normal force acting perpendicular to the surface and the coefficient of kinetic friction.
- For the same object, surface and normal force, **the kinetic friction will be less than the maximum static friction**.

Kinetic friction force

$$\boxed{f_k = \mu_k F_n}$$

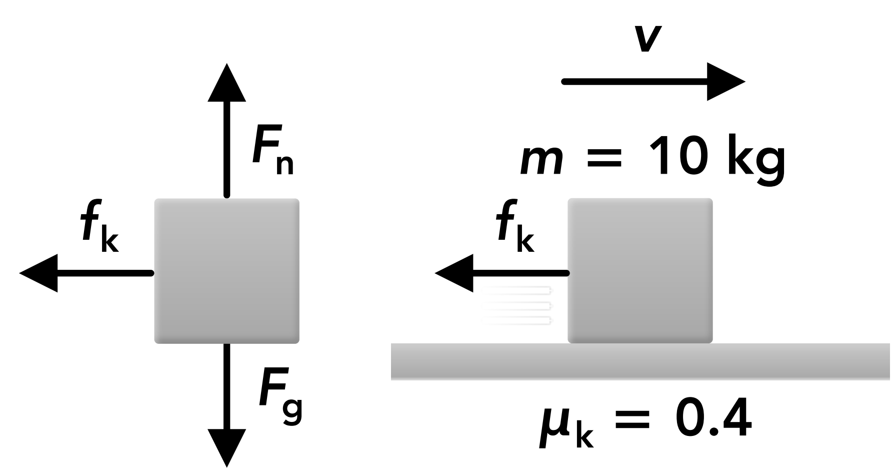
μ_k : coefficient of kinetic friction



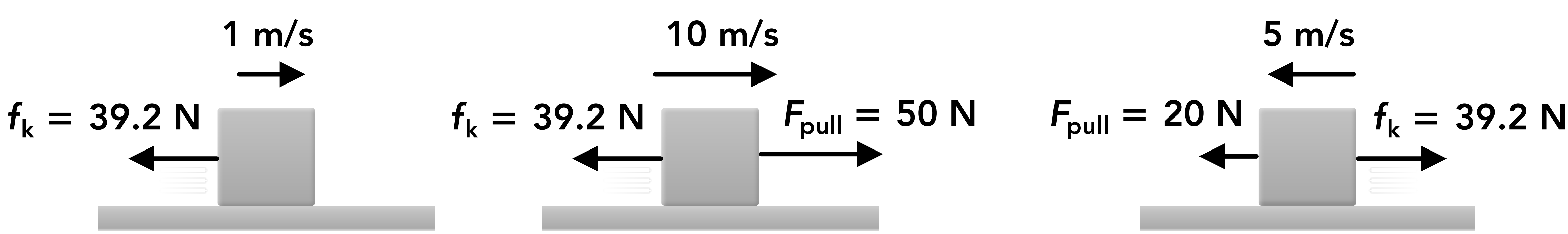
Example: A 10 kg box is sliding along the ground. The normal force between the box and the ground is equal to the weight force, 98 N. The coefficient of kinetic friction between the box and the ground is 0.4. The kinetic friction force is 39.2 N regardless of the velocity and any other horizontal forces.

$$\sum F_y = ma_y$$
$$F_n - F_g = m(0\ \text{m/s}^2)$$
$$F_n = F_g = mg = (10\ \text{kg})(9.8\ \text{m/s}^2) = 98\ \text{N}$$

$$f_k = \mu_k F_n = (0.4)(98\ \text{N}) = 39.2\ \text{N}$$



The kinetic friction force always acts in the opposite direction as the velocity and is always equal to $\mu_k F_n$



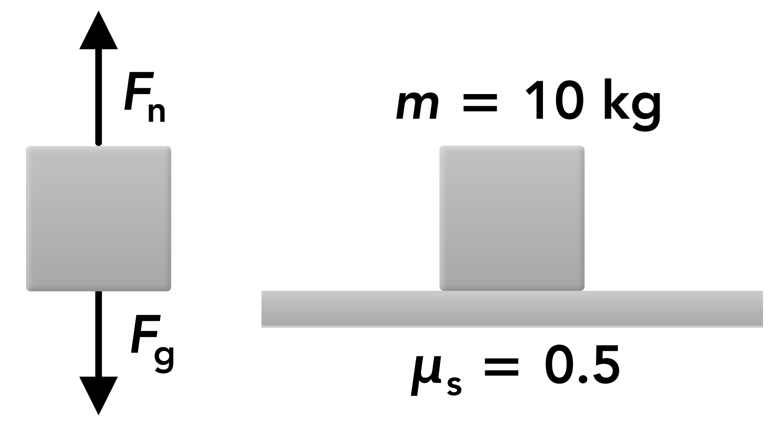
Transition From Static Friction to Kinetic Friction

- If an object begins at rest and a force is applied parallel to the surface, a static friction force acts on the object to prevent it from moving. If the applied force exceeds the maximum possible static friction force, the object begins to slide and the friction force transitions from a static friction force to a kinetic friction force.
- The magnitude of the static friction force changes based on the applied force, but the kinetic friction force is constant while the object is moving.
- The kinetic friction force is always **less than** the maximum static friction force.

Example: A 10 kg box is sitting on the ground at rest. The normal force between the box and the ground is equal to the weight force, 98 N. The coefficient of static friction between the box and the ground is 0.5. The maximum static friction force between the box and the ground is 49 N. A rightwards pulling force is then applied to the box.

$$\sum F_y = ma_y$$
$$F_n - F_g = m(0\ \text{m/s}^2)$$
$$F_n = F_g = mg = (10\ \text{kg})(9.8\ \text{m/s}^2) = 98\ \text{N}$$

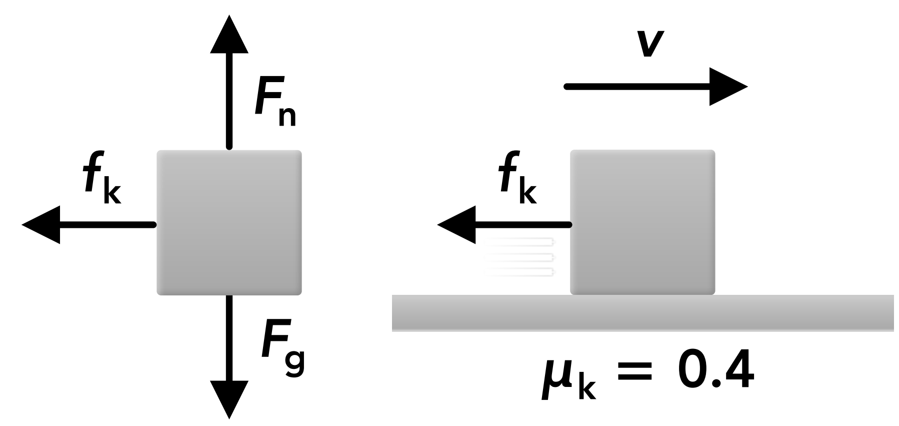
$$f_{s\ max} = \mu_s F_n = (0.5)(98\ \text{N}) = 49\ \text{N}$$



When the box is moving a kinetic friction force replaces the static friction force. The coefficient of kinetic friction between the box and the ground is 0.4. The kinetic friction force is always 39.2 N when the box is moving.

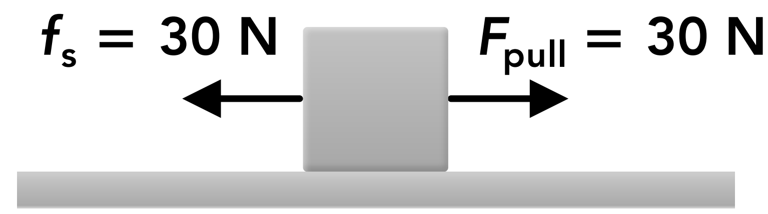
$$\sum F_y = ma_y$$
$$F_n - F_g = m(0\ \text{m/s}^2)$$
$$F_n = F_g = mg = (10\ \text{kg})(9.8\ \text{m/s}^2) = 98\ \text{N}$$

$$f_k = \mu_k F_n = (0.4)(98\ \text{N}) = 39.2\ \text{N}$$

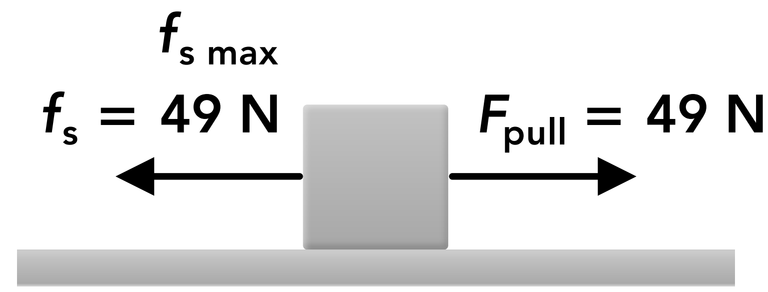


If the applied force is zero (there's no other forces acting parallel to the surface) there is no static friction force because there are no forces for friction to "react to" or any motion to prevent.

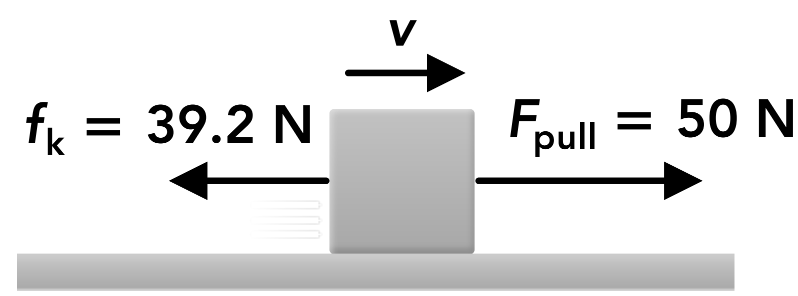
As the applied force increases, the static friction "reacts" and also increases so it has the same magnitude in the opposite direction.



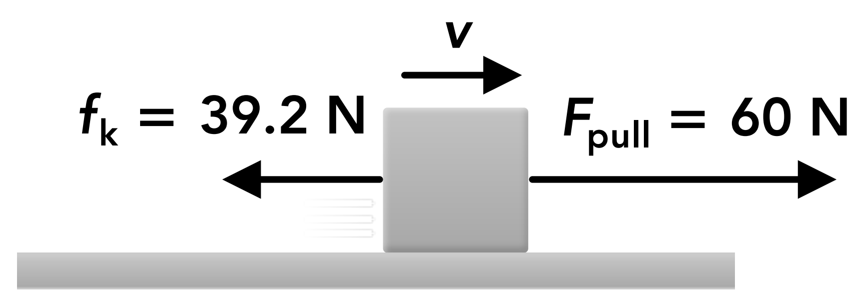
Eventually the applied force equals the maximum static friction force that's possible between the box and the ground.

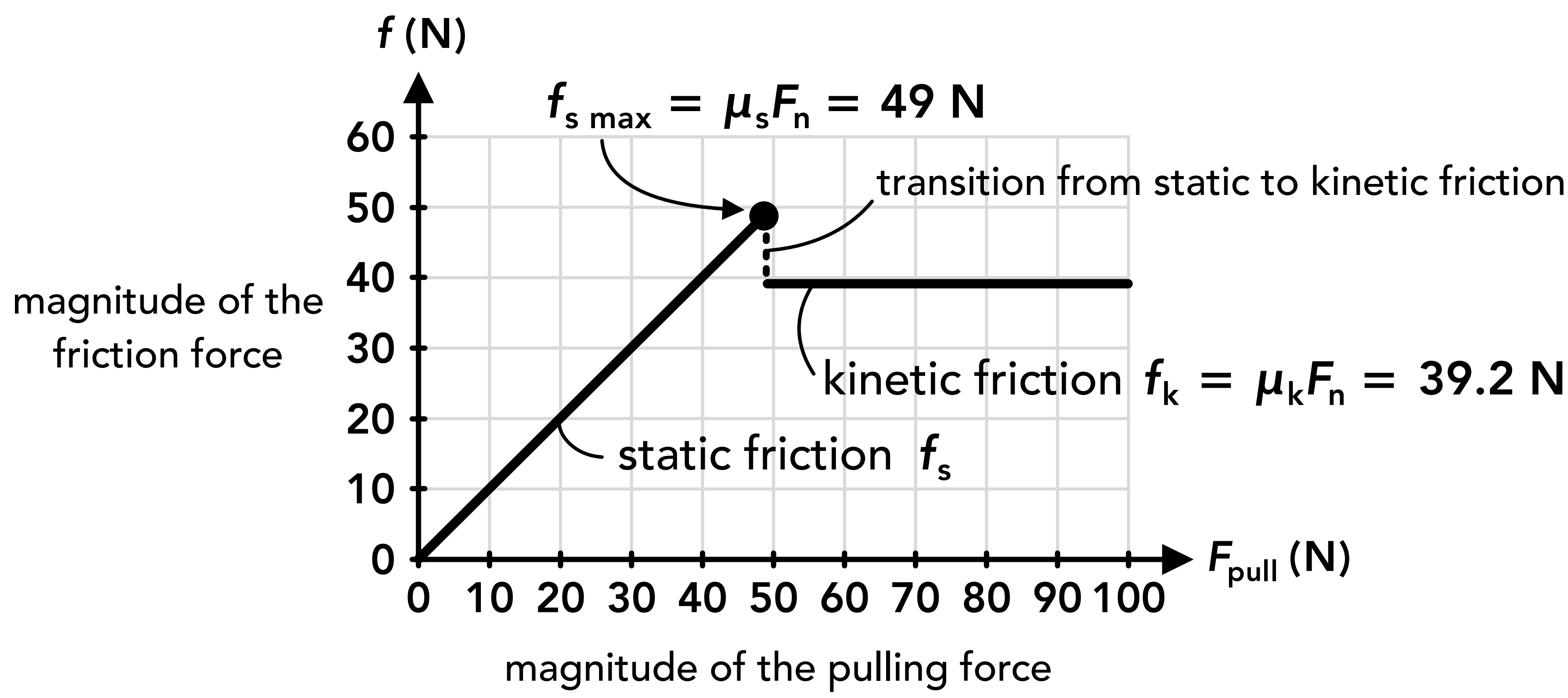


If the applied force is greater than the maximum static friction force, the box will begin to slide and the friction transitions from static friction to kinetic friction.



The kinetic friction force is constant regardless of the applied force or the velocity of the box.





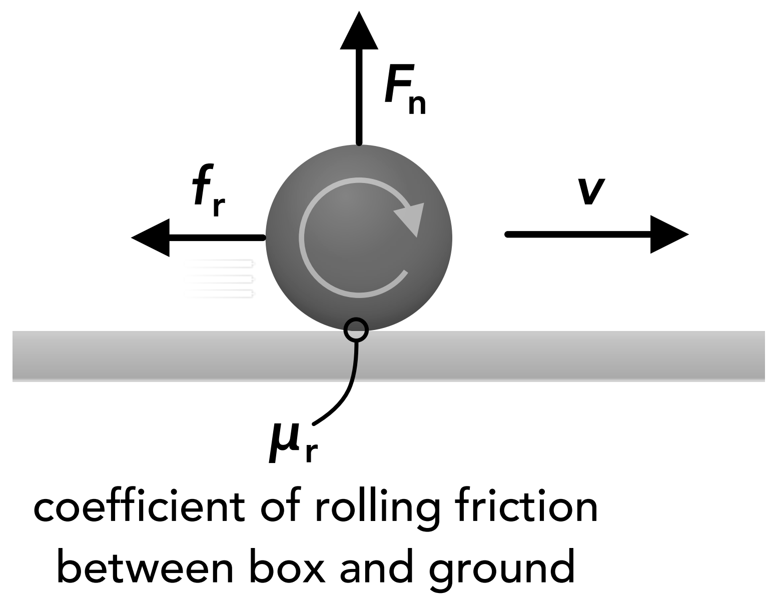
Rolling Friction

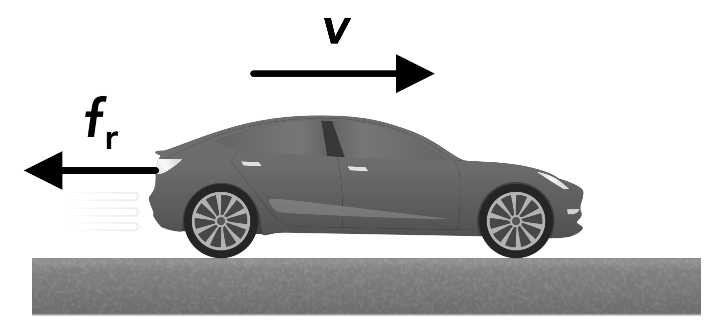
- **Rolling friction** is a type of friction force that acts on an object that is rolling on a surface, also sometimes referred to as **rolling resistance** or **rolling drag**.
- Rolling friction refers to the force acting on the object **while the object is rolling**, not a force that prevents a static object from beginning to roll (unless otherwise stated in a given scenario).
- When an object is rolling along the ground, the surface of the object deforms and interacts with the surface of the ground. This is a complex interaction with many factors, but the overall effect on the rolling object can be treated as a single rolling friction force.

Rolling friction force

$$f_r = \mu_r F_n$$

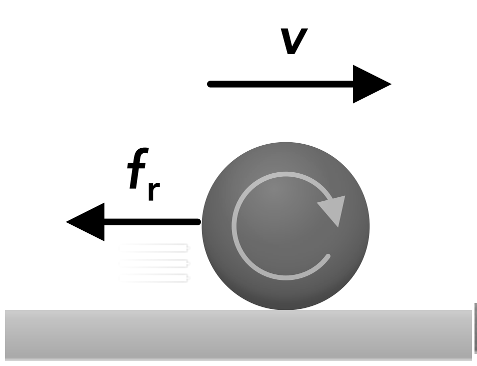
μ_r : coefficient of kinetic friction

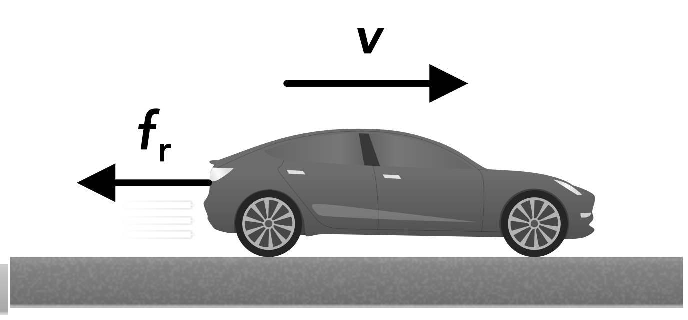




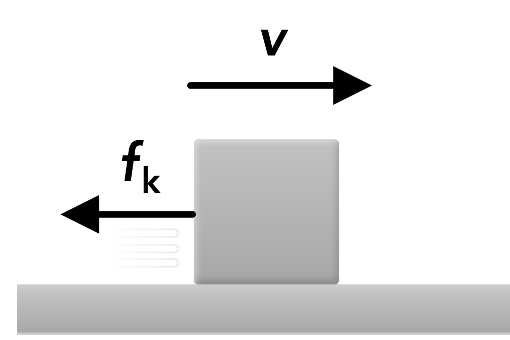
- Rolling friction should be thought of as an "overall friction force" acting a rolling object. This is in contrast to kinetic friction which acts on a sliding object. In most scenarios, only one of these two types of friction should be be used for a moving object.

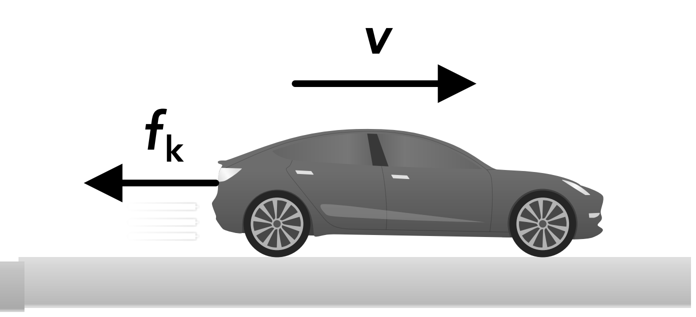
Object or wheels are rolling without slipping, rolling friction is used





Object or wheels are sliding, kinetic friction (sliding friction) is used





car is sliding on ice, wheels are slipping and not turning

SPRING FORCE

Spring Force and Hooke's Law

Variables		SI Unit
F_{sp}	spring force	N
Δx	displacement	m
k	spring constant	$\frac{N}{m}$

- A spring **changes length when a force is applied to both ends**. If the forces pull the ends away from each other the spring gets longer. If the forces push the ends together the spring gets shorter.
- In the real world there are different types of springs with different behaviors, and all materials actually behave similar to springs. But we usually start out by working with "ideal springs".
- An **ideal spring** is…
  - massless: the spring itself has no mass, no inertia and no weight
  - frictionless: there are no friction forces acting on or within the spring itself
  - linearly elastic / follows Hooke's law: the change in length is linearly proportional to the applied force

- **Hooke's Law** states that the magnitude of the force required to stretch or compress a spring by a displacement of Δx is linearly proportional to that the displacement, $F_{sp} = k\Delta x$, where k is the **spring constant** or **stiffness** of the spring.

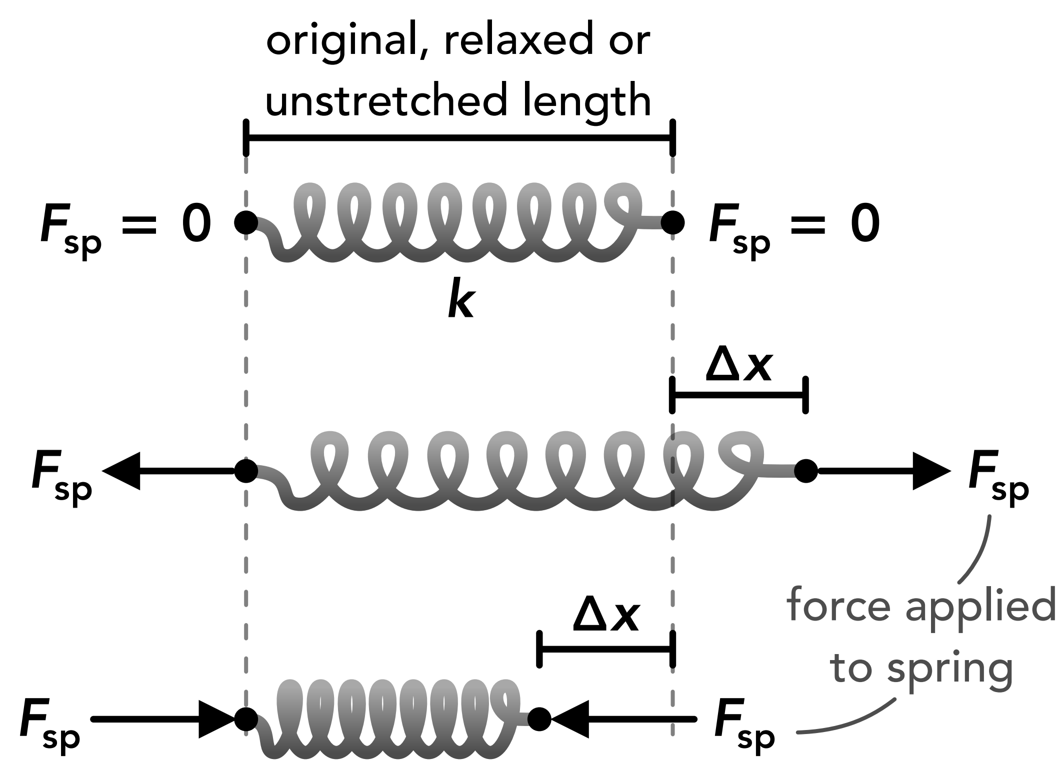
Spring force (Hooke's Law)

$$F_{sp} = k\Delta x$$

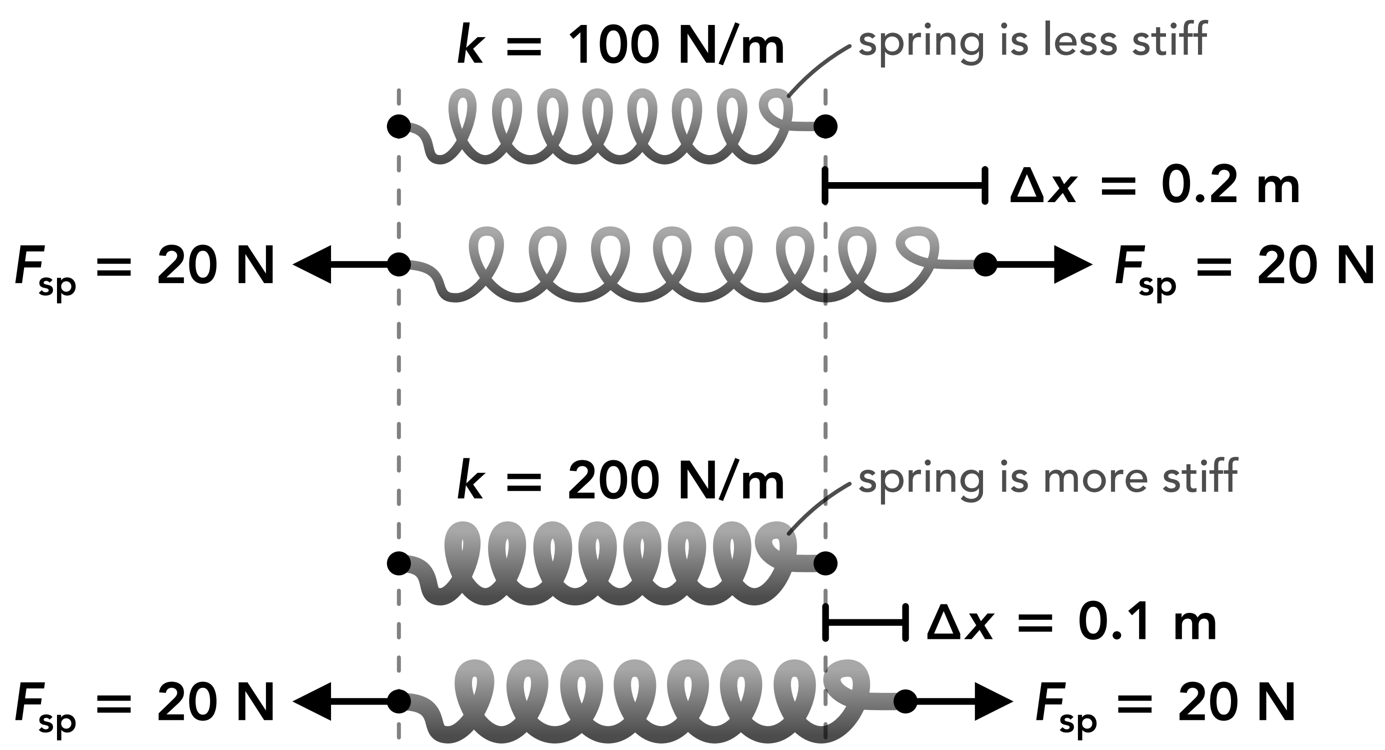
When no force is applied the spring is at its original length, relaxed length or unstretched length

When a tension (pulling) force is applied to both ends the spring gets longer by a change of Δx

When a compression (pushing) force is applied to both ends the spring gets shorter by a change of Δx



- The **spring constant k** is a value that represents the stiffness of a particular spring. A spring that is more stiff has a higher spring constant and requires more force to cause the same displacement as a spring that is less stiff and has a lower spring constant.
- The spring constant has a unit of Newtons/meter (N/m) given by units of force and displacement in Hooke's law.

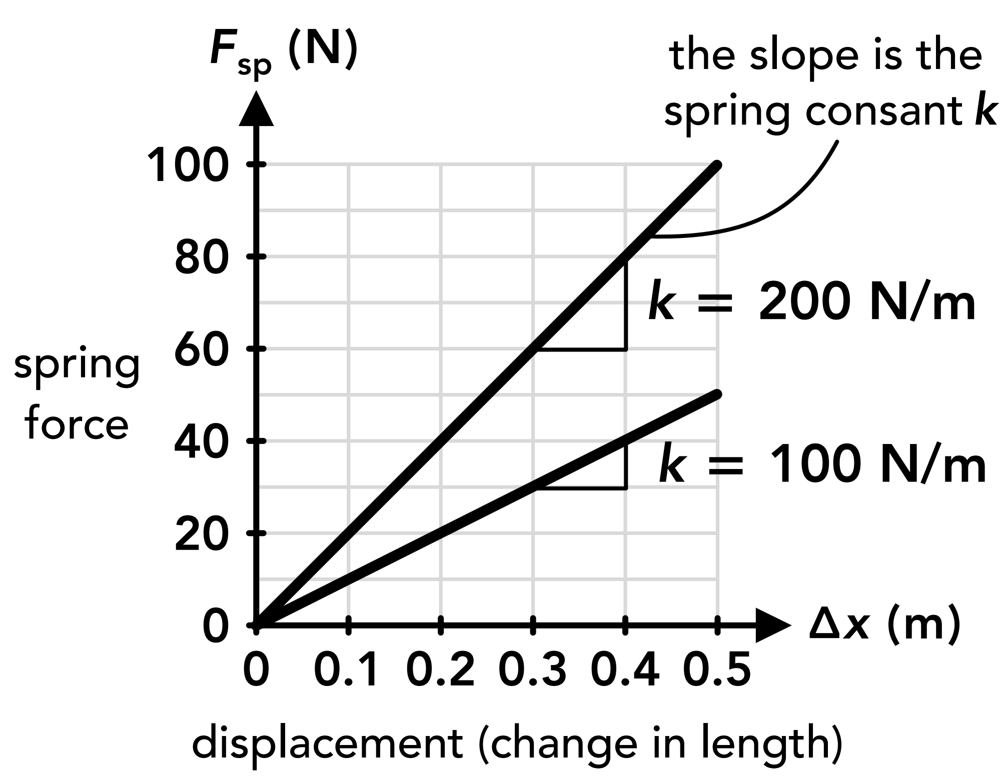


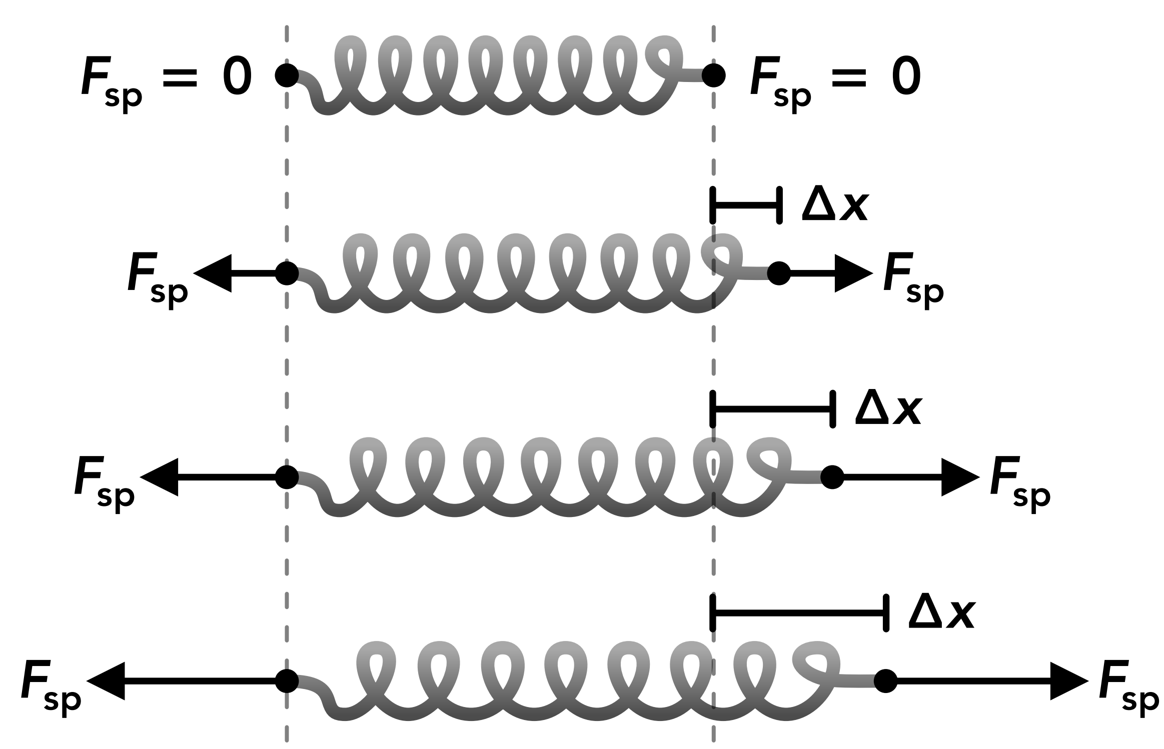
$F_{sp} = k\Delta x$
$(20\text{ N}) = (100\text{ N/m})\Delta x$
$\Delta x = 0.2\text{ m}$

$F_{sp} = k\Delta x$
$(20\text{ N}) = (200\text{ N/m})\Delta x$
$\Delta x = 0.1\text{ m}$

- Since the change in length Δx is linearly proportional to the spring force F_{sp} , a graph of the spring force vs the displacement is a straight line.
- If the spring force is on the vertical axis and the displacement is on the horizontal axis, **the slope of the graph is the spring constant k**. If the axes are flipped the slope is $1/k$.

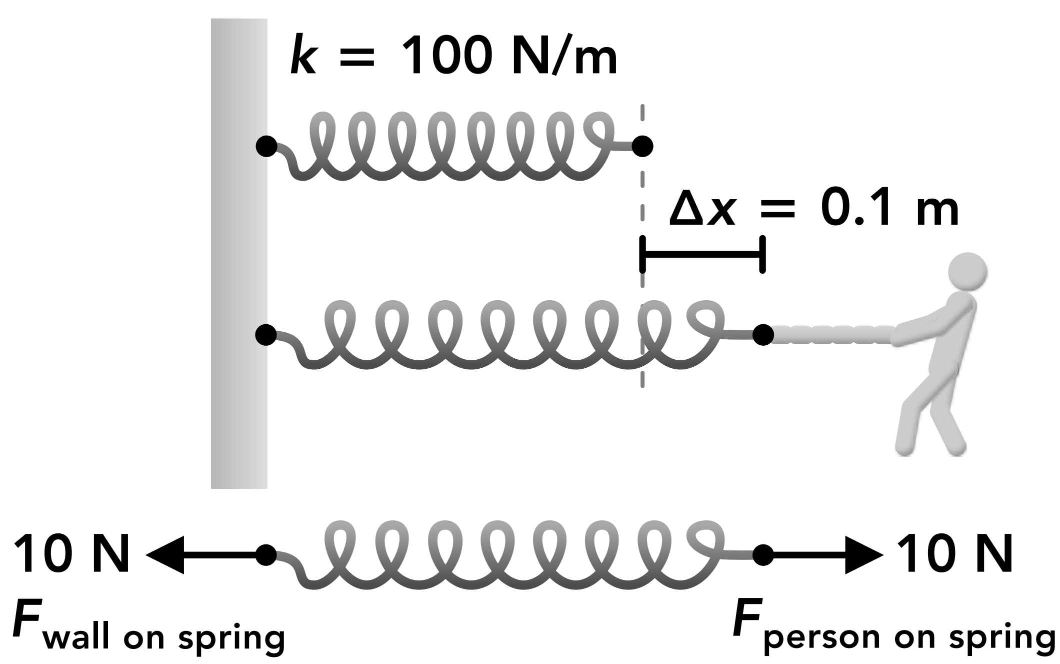
Hooke's Law: $F_{sp} = k\Delta x$



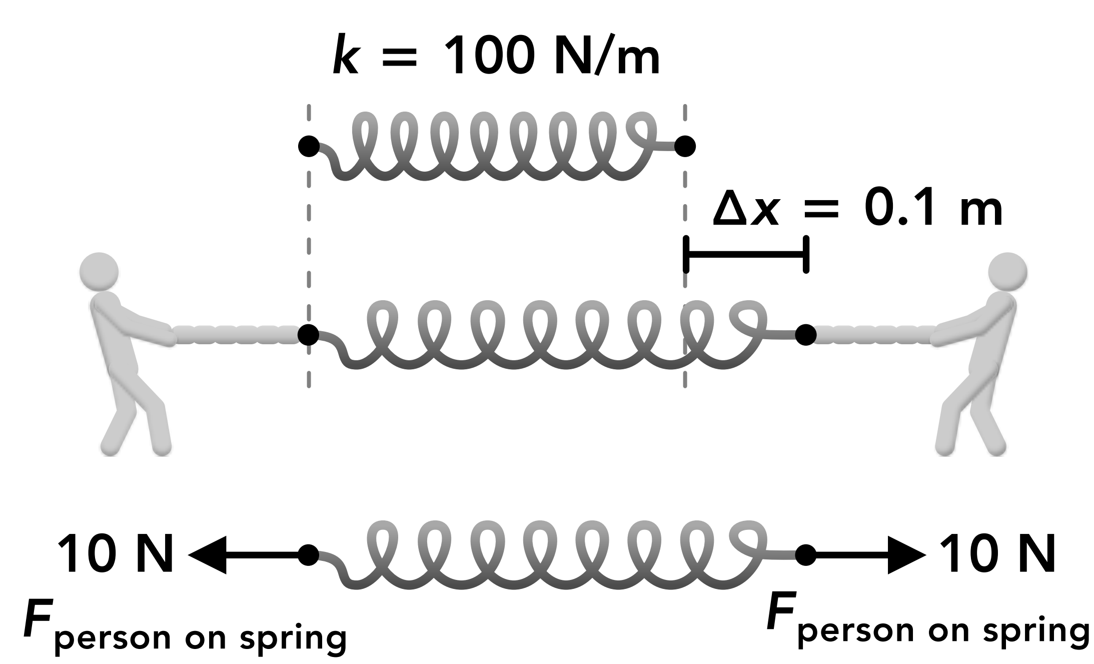


- A source of common confusion is the direction and magnitude of the spring force.
- A "spring force" is not a fundamental type of force like the gravitational force. When we use the term "spring force" we either mean a force exerted on the spring by an object, or the force exerted on an object by the spring.
- First, the forces acting on each end of a spring are equal in magnitude and opposite in direction. Not because they are a pair of equal and opposite forces as described in Newton's 3rd law of motion, but because we're treating the spring as ideal and **we're assuming the net force acting on the spring is zero**. In cases where the the spring is in static equilibrium and not moving (and therefore not accelerating) this must be true according to Newton's 2nd law of motion, $F_{\text{net}} = ma$. Even in cases where one or both ends of the spring are accelerating and the forces acting on the ends are changing, an ideal spring has no mass and **we assume it instantaneously transmits forces from one end to the other**. This is the same thing that happens for an ideal rope when working with tension forces, so you can think of the "spring force" on an object like a tension force acting on the object.
- Even when one end of the spring is fixed to a wall or a non-moving object, the wall still exerts a force on the spring just like if it were being pulled or pushed by a person or some other more "visible" force. If the wall was not exerting this force, the net force would not be zero and the spring would accelerate. Again, this is the same thing that happens with the tension force in a rope.

In both cases the spring is in static equilibrium (not moving) so the net force acting on the spring is zero. The wall exerts a force on the spring just like if it were pulled or pushed by a person.

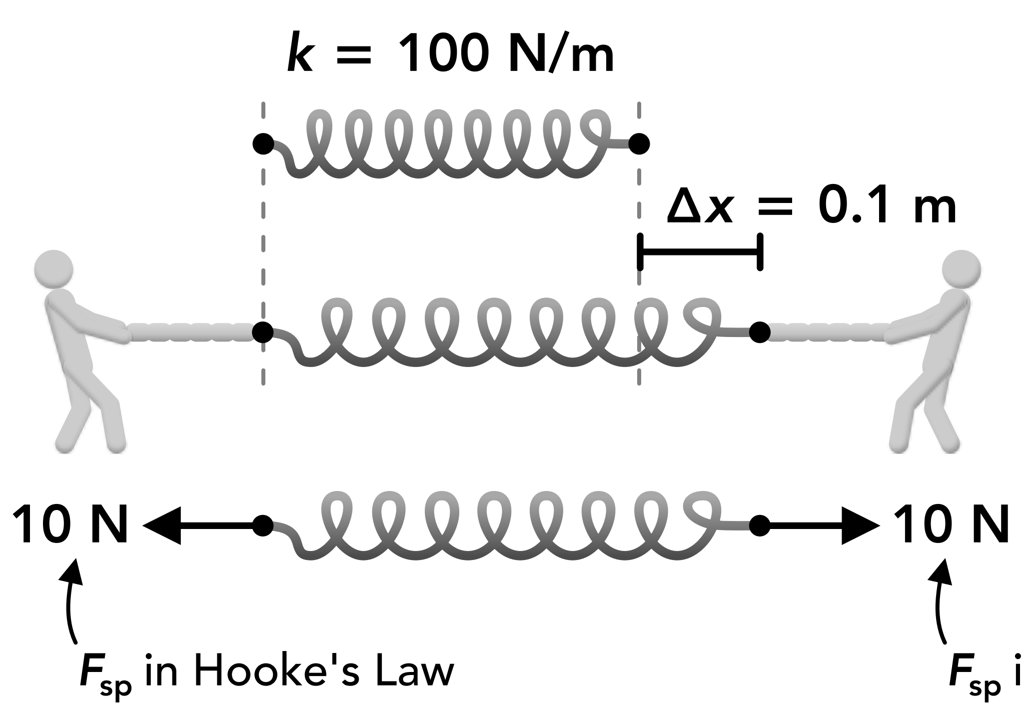


$$\sum F_x = ma_x$$
$$10\text{ N} - 10\text{ N} = m(0\text{ m/s}^2)$$



$$\sum F_x = ma_x$$
$$10\text{ N} - 10\text{ N} = m(0\text{ m/s}^2)$$

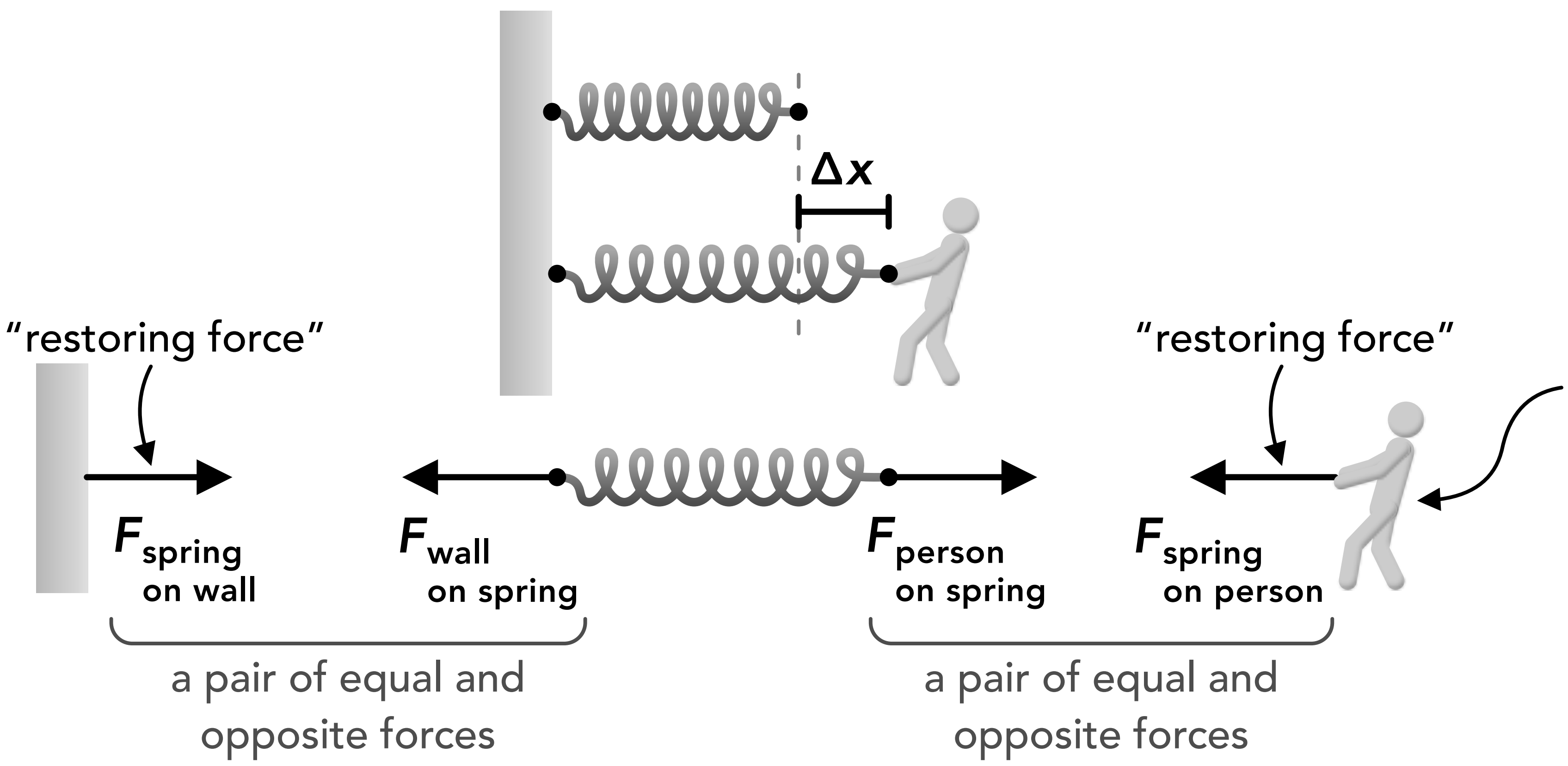
- Second, in the context of Hooke's Law the spring force F_{sp} refers to **the magnitude of the force acting on each end of a spring** (they're the same). We don't double the force or add the forces from each end together.



$$F_{sp} = k\,\Delta x$$
$$(10\text{ N}) = (100\text{ N/m})(0.1\text{ m})$$

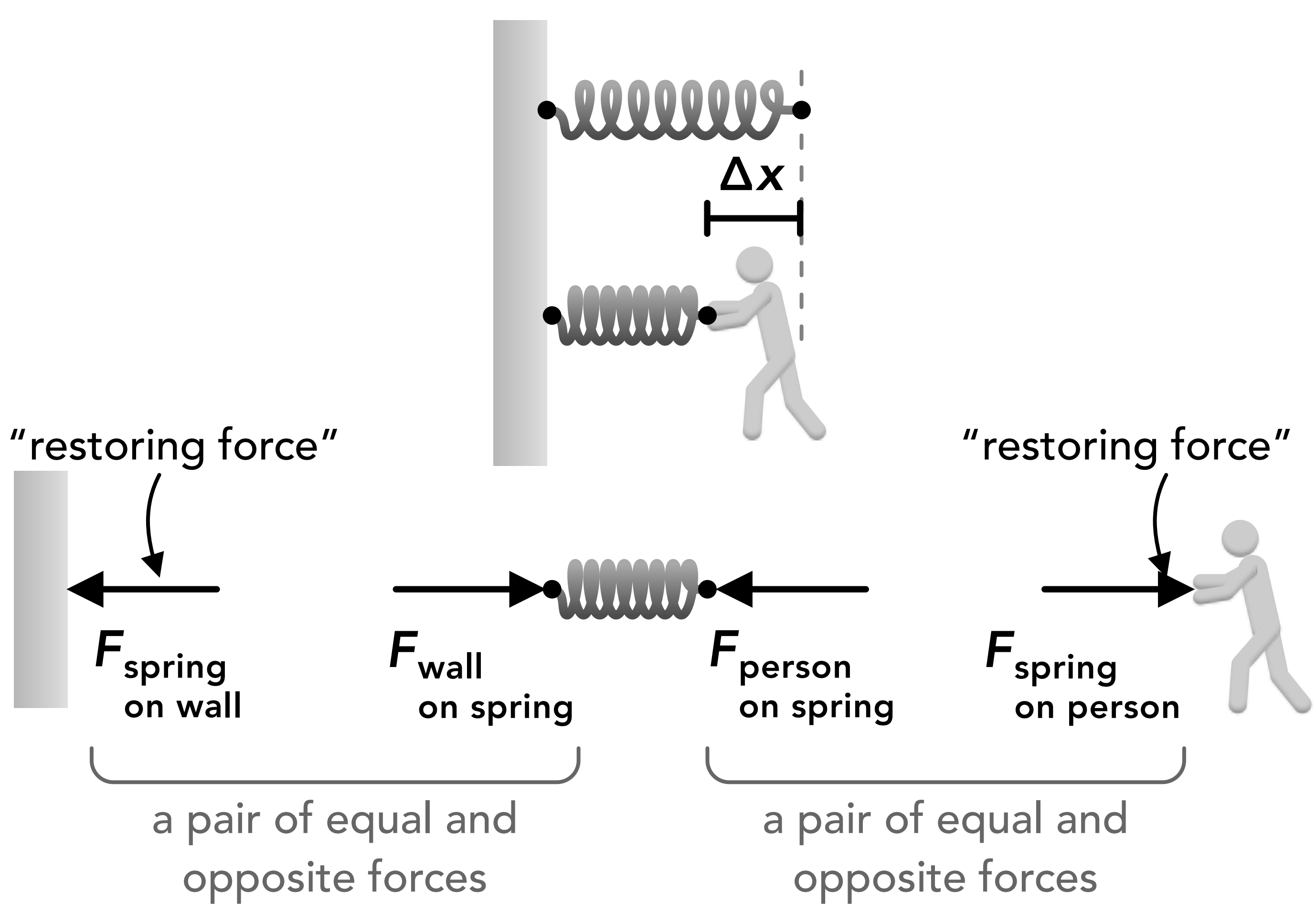
- Third, when we use the term “spring force” we need to be specific about which object the force is exerted on and which object is causing the force. When a spring is attached to an object the spring and the object exert contact forces on each other. The force exerted on the spring by the object is equal and opposite to the force exerted on the object by the spring (these are a pair of forces as described in Newton’s 3rd law of motion).
- When a spring changes length, **it also exerts a force on the objects it’s in contact with** (again this is just from Newton’s 3rd law of motion). The force exerted by the spring on an object is called the **restoring force** because this force is trying to restore the spring to its original length.
- In most cases, we’re focused on an object that is in contact with a spring, not the spring itself. In those scenarios we usually call the force exerted on the object by the spring the “spring force” $\boldsymbol{F}_{sp}$.
- It’s also important to clearly label the forces in a free body diagram so we know what a force is acting on and what is causing the force. Remember, the free body diagram for an object only shows the forces acting on that object.

Free body diagrams of the wall, the spring and the person when the spring is stretched

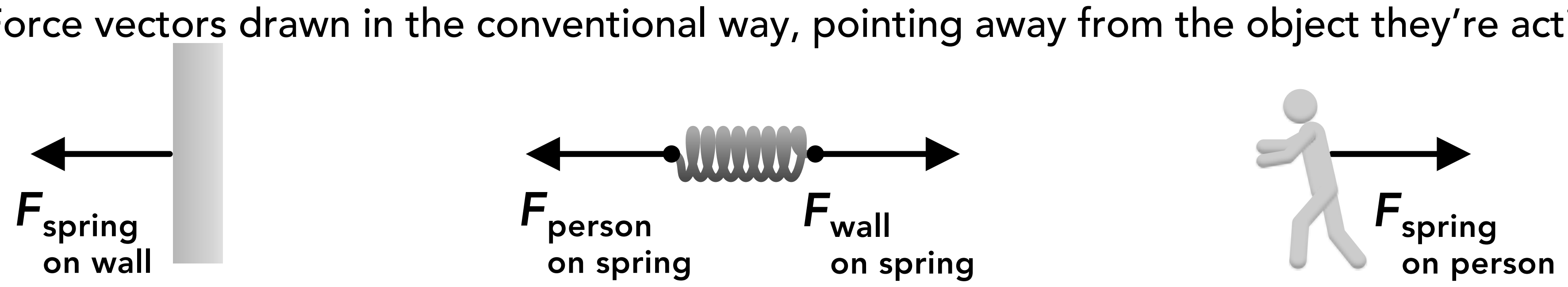


We’re usually focused on the object in contact with the spring, not the spring itself. A free body diagram of the object shows the “restoring force” exerted on the object by the spring, which is equal and opposite to the force exerted on the spring by the object, which is the force in Hooke’s Law.

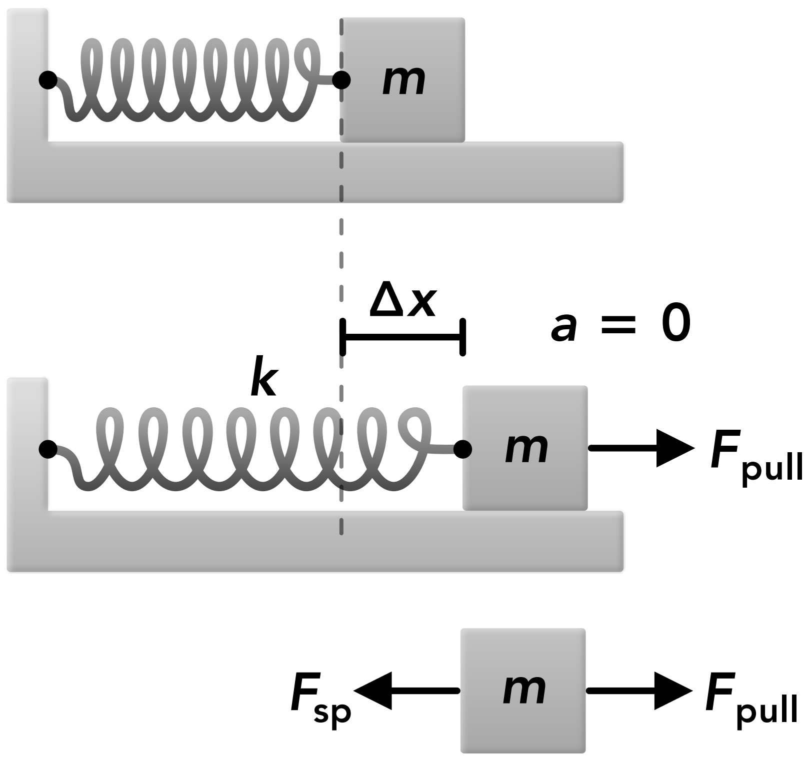
Free body diagrams of the wall, the spring and the person when the spring is compressed



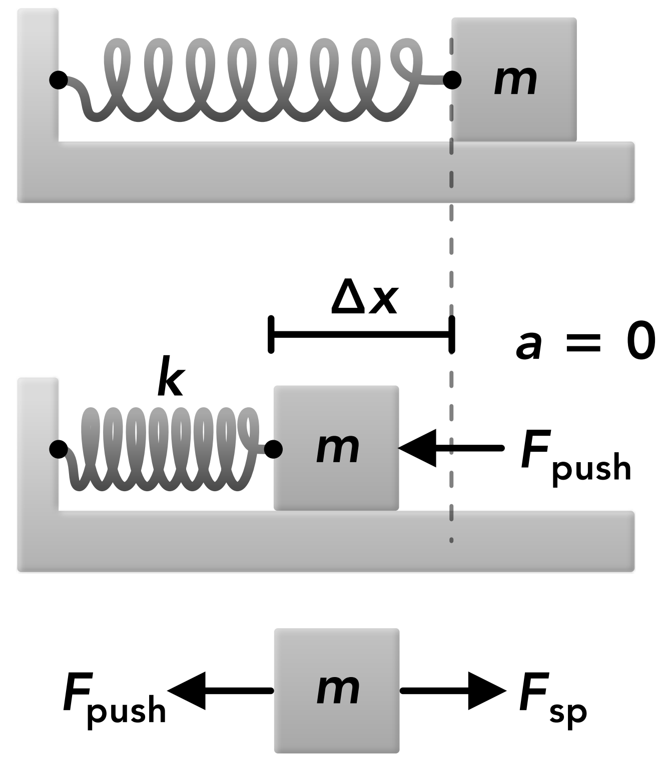
Force vectors drawn in the conventional way, pointing away from the object they’re acting on:



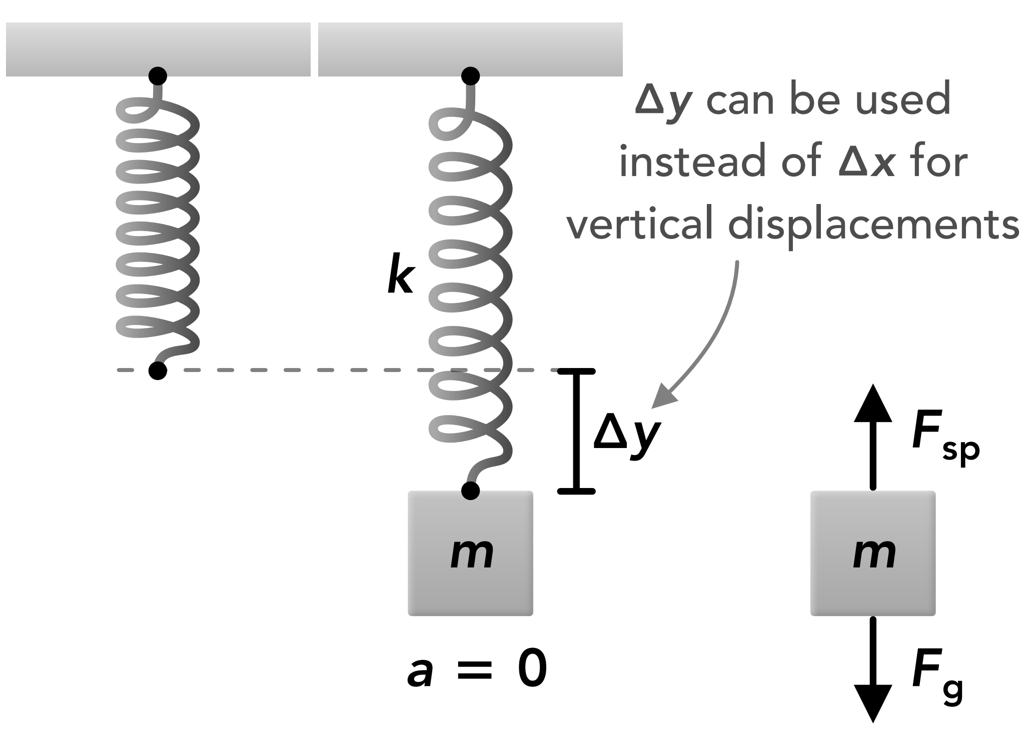
Examples of free body diagrams and Newton's 2nd law involving spring forces



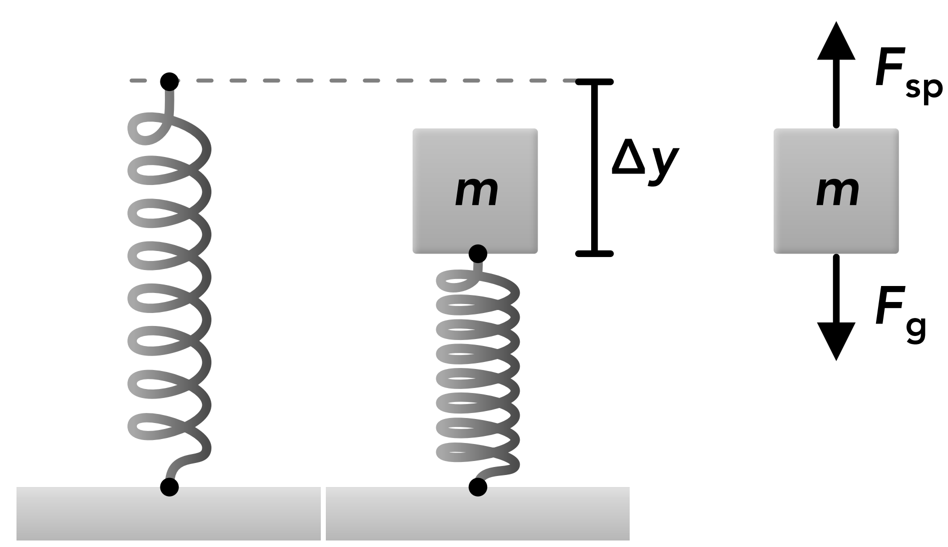
$$\sum F_x = ma_x$$
$$F_{pull} - F_{sp} = m(0)$$
$$F_{sp} = k\,\Delta x \qquad F_{pull} - (k\Delta x) = 0$$



$$\sum F_x = ma_x$$
$$F_{sp} - F_{push} = m(0)$$
$$F_{sp} = k\,\Delta x \qquad (k\Delta x) - F_{push} = 0$$



$$\sum F_y = ma_y$$
$$F_{sp} - F_g = m(0)$$
$$F_{sp} = k\,\Delta y \qquad (k\Delta y) - (mg) = 0$$
$$F_g = mg$$

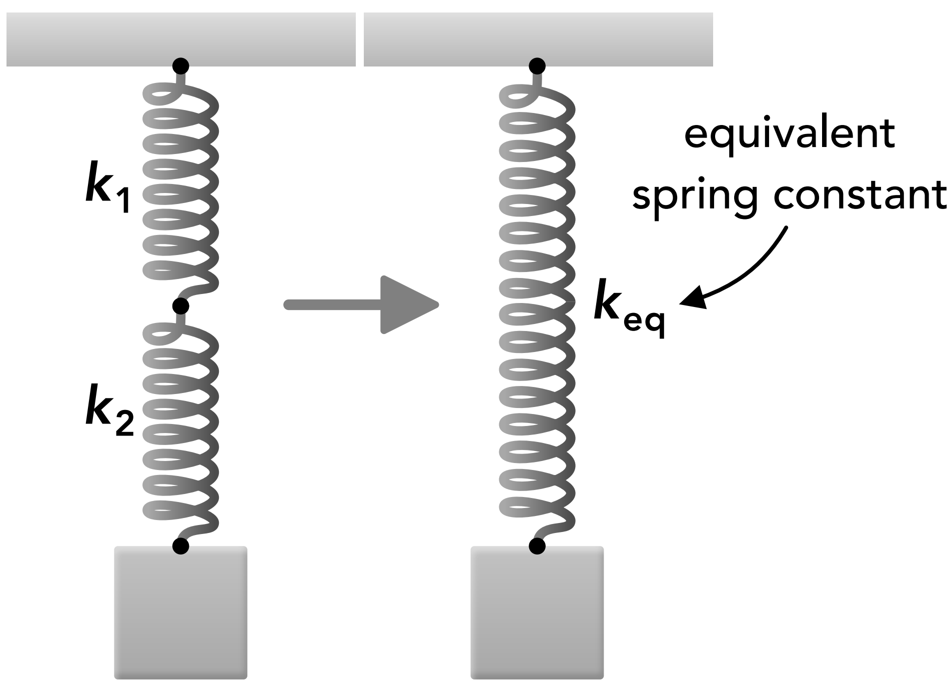


$$\sum F_y = ma_y$$
$$F_{sp} - F_g = m(0)$$
$$F_{sp} = k\,\Delta y \qquad (k\Delta y) - (mg) = 0$$
$$F_g = mg$$

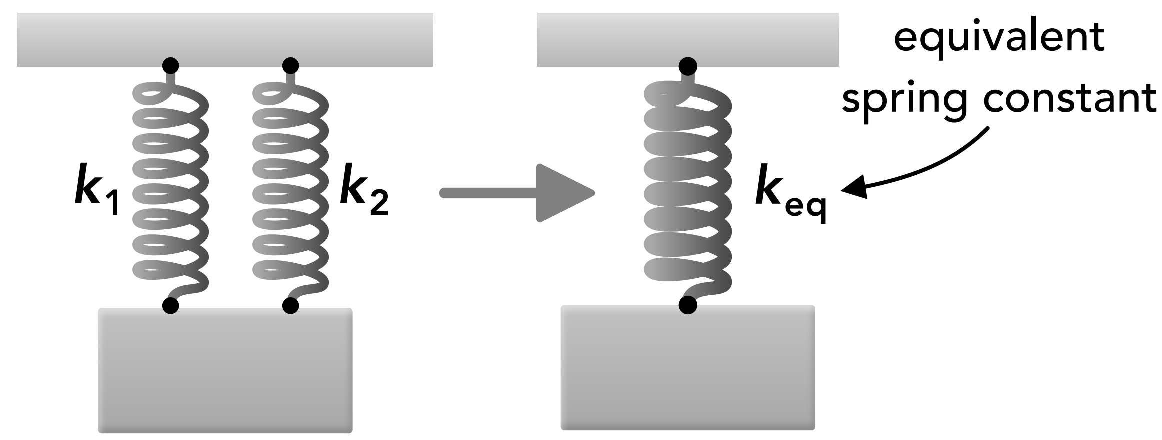
Combining Springs in Series and Parallel

- Multiple springs can be combined together **in series** or **in parallel**.
- Together, the group of springs can be treated as **a single spring with an equivalent spring constant** k_{eq} . The equivalent spring constant is calculated in a different way for springs in series and springs in parallel.

Two springs added in series



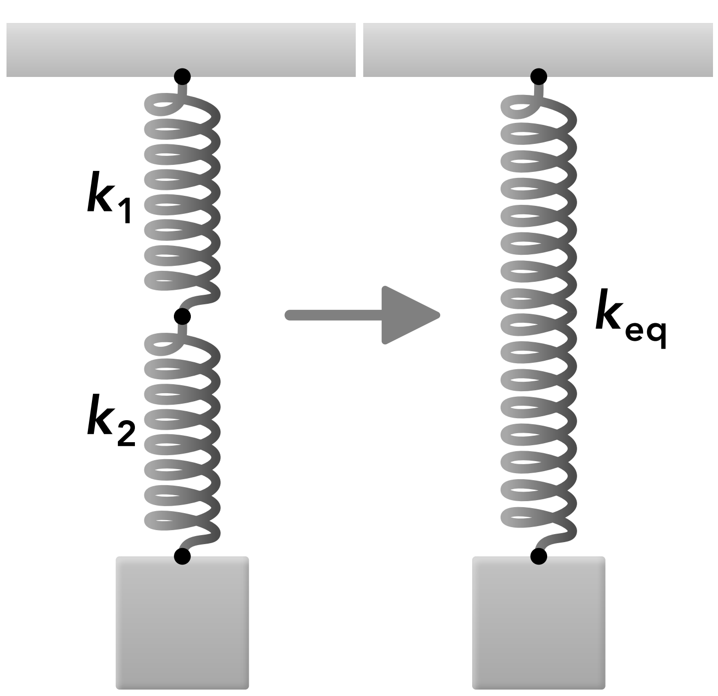
Two springs added in parallel



- Springs added **in series** are connected end-to-end.
- Adding an additional spring in series always **decreases** the equivalent spring constant or stiffness.
- The original lengths of the springs are added together.
- The force applied to the end of a series of springs is the **same force** applied to each individual spring.
- The displacements of each spring are added together.

Equivalent spring constant for springs in series

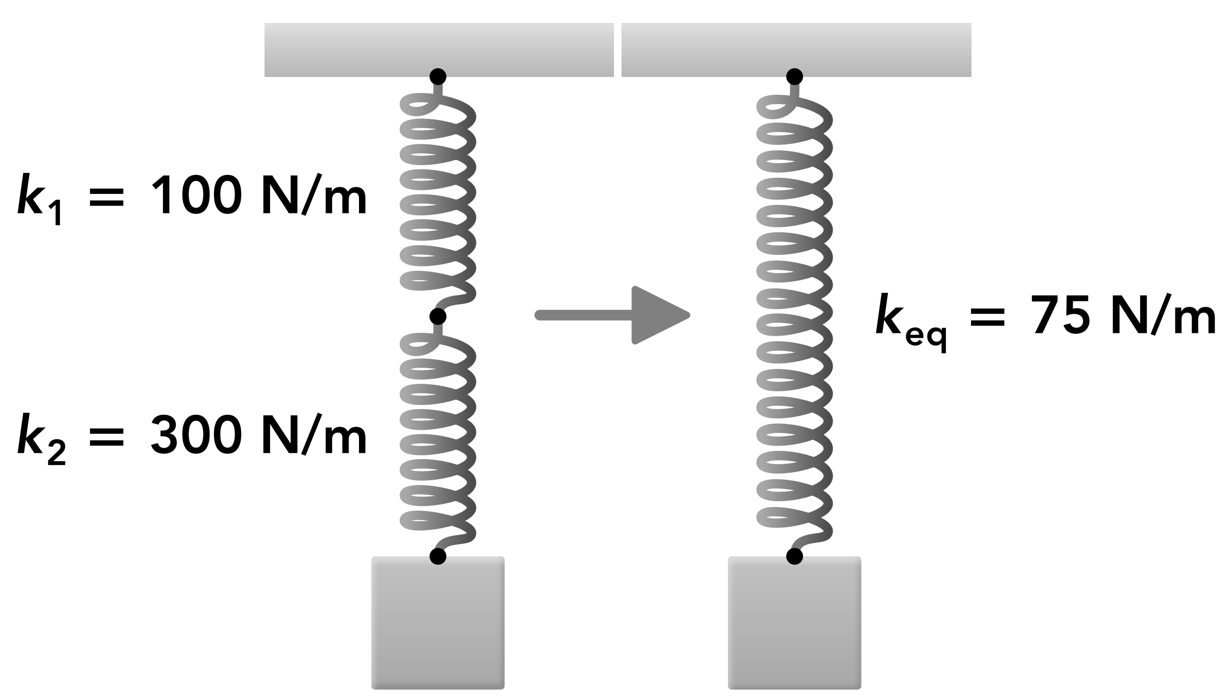
$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots$$



Example:

$$\frac{1}{k_{eq}} = \frac{1}{100} + \frac{1}{300}$$

$$k_{eq} = 75 \text{ N/m}$$



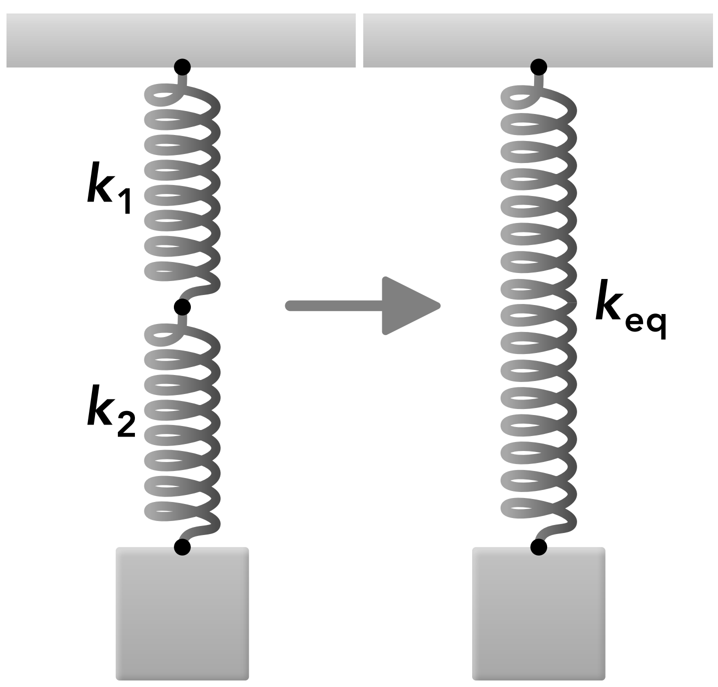
Combining Springs in Series and Parallel

- A group of springs can be added together and treated as **a single spring with an equivalent spring constant k_{eq}**.

- Springs added **in series** are connected end-to-end.
- Adding an additional spring in series always **decreases** the equivalent spring constant or stiffness.
- The original lengths of the springs are added together.
- The force applied to the end of a series of springs is the **same force** applied to each individual spring.
- The displacements of each spring are added together.

Equivalent spring constant for springs in series

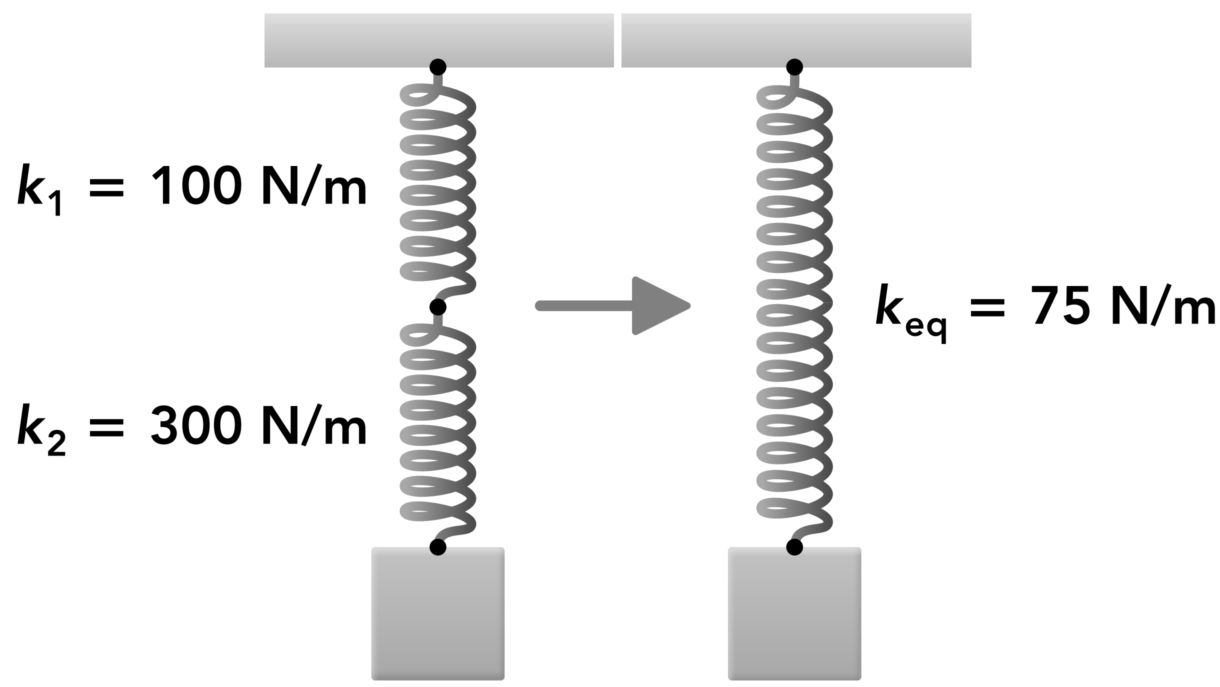
$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots$$



Example:

$$\frac{1}{k_{eq}} = \frac{1}{100} + \frac{1}{300}$$

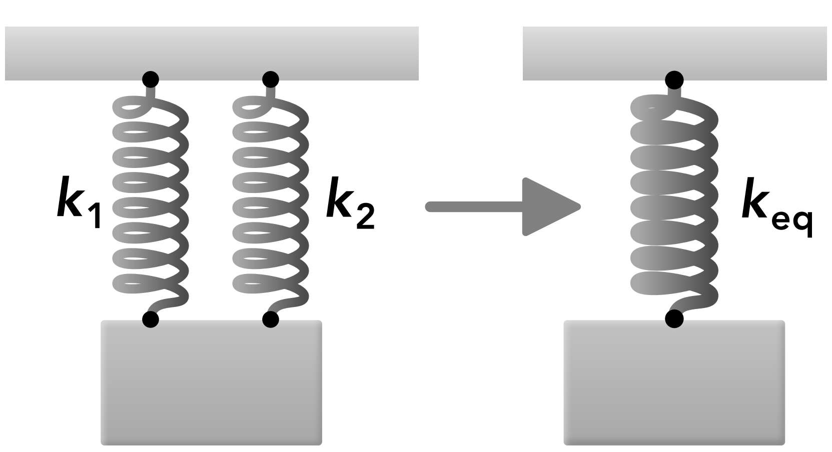
$$k_{eq} = 75 \text{ N/m}$$



- Springs added **in parallel** are all connected to the same two objects or surfaces at each end.
- The equivalent spring constant is just the sum of the individual spring constants.
- Adding an additional spring in parallel always **increases** the equivalent spring constant or stiffness.
- Each spring has the **same displacement** but a **different amount of force**.
- When adding springs in parallel like this, they must have the same original length and we're assuming the object translates (moves linearly) but doesn't rotate, even though there are different forces acting on it at different points.

Equivalent spring constant for springs in parallel

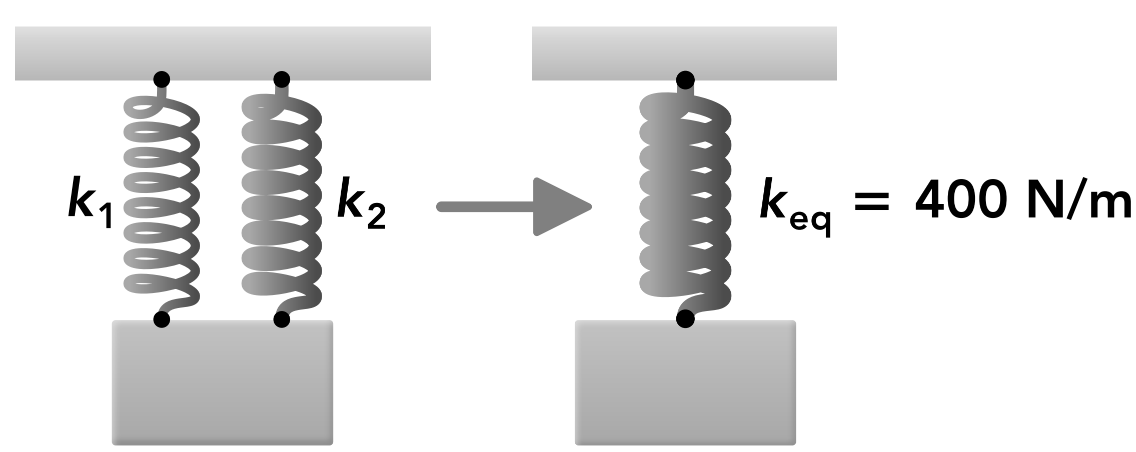
$$k_{eq} = k_1 + k_2 + \ldots$$



Example:

$$k_{eq} = 100 + 300$$

$$k_{eq} = 400 \text{ N/m}$$



$k_1 = 100$ N/m

$k_2 = 300$ N/m

- Groups of springs can be in series and in parallel with other groups of springs. First, the springs within a group are added together (in series or parallel) and then the groups can be added together (in series or parallel).

1. Springs 1 and 3 are added in series to get an equivalent spring constant k_{1+3}
2. Spring "1+3" and spring 2 are added in parallel to get a final equivalent spring constant k_{1+2+3}

$$\frac{1}{k_{1+3}} = \frac{1}{k_1} + \frac{1}{k_3}$$

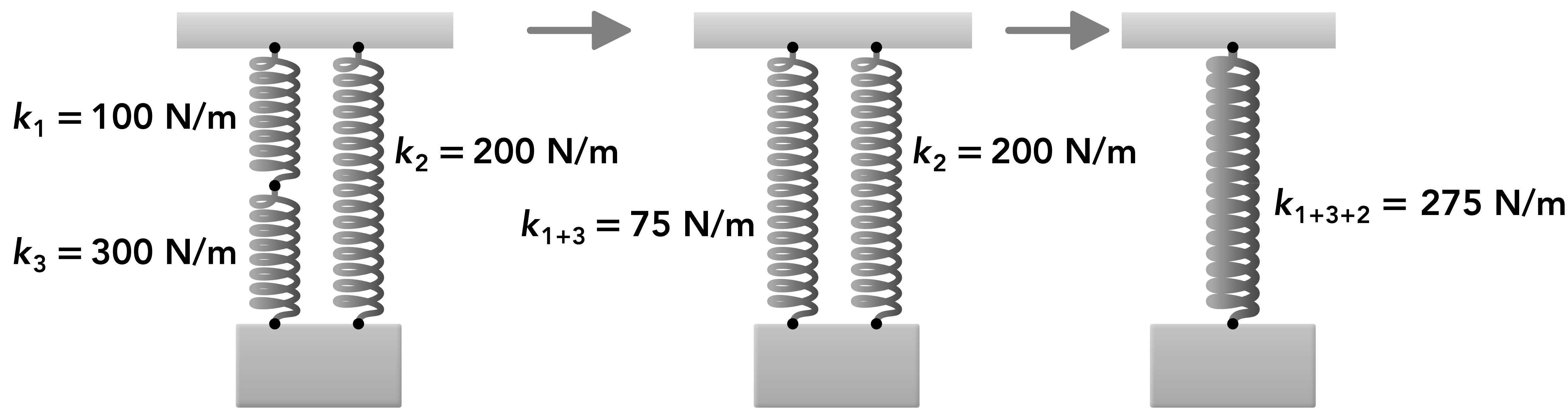
$$\frac{1}{k_{1+3}} = \frac{1}{100} + \frac{1}{300}$$

$$k_{1+3} = 75 \text{ N/m}$$

$$k_{1+3+2} = k_{1+3} + k_2$$

$$k_{1+3+2} = 75 + 200$$

$$k_{1+3+2} = 275 \text{ N/m}$$



1. Springs 1 and 2 are added in parallel to get an equivalent spring constant k_{1+2}
2. Springs 3 and 4 are added in parallel to get an equivalent spring constant k_{3+4}
3. Spring "1+2" and spring "3+4" are added in series to get a final equivalent spring constant $k_{1+2+3+4}$

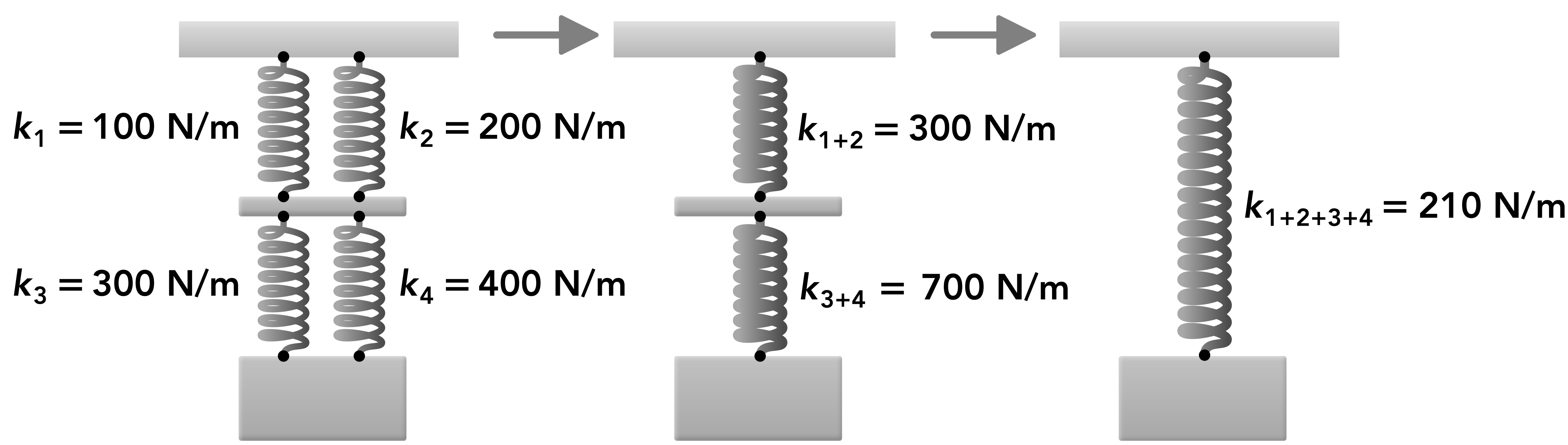
$$k_{1+2} = k_1 + k_2 = 100 + 200 = 300$$

$$k_{3+4} = k_3 + k_4 = 300 + 400 = 700$$

$$\frac{1}{k_{1+2+3+4}} = \frac{1}{k_{1+2}} + \frac{1}{k_{3+4}}$$

$$\frac{1}{k_{1+2+3+4}} = \frac{1}{300} + \frac{1}{700}$$

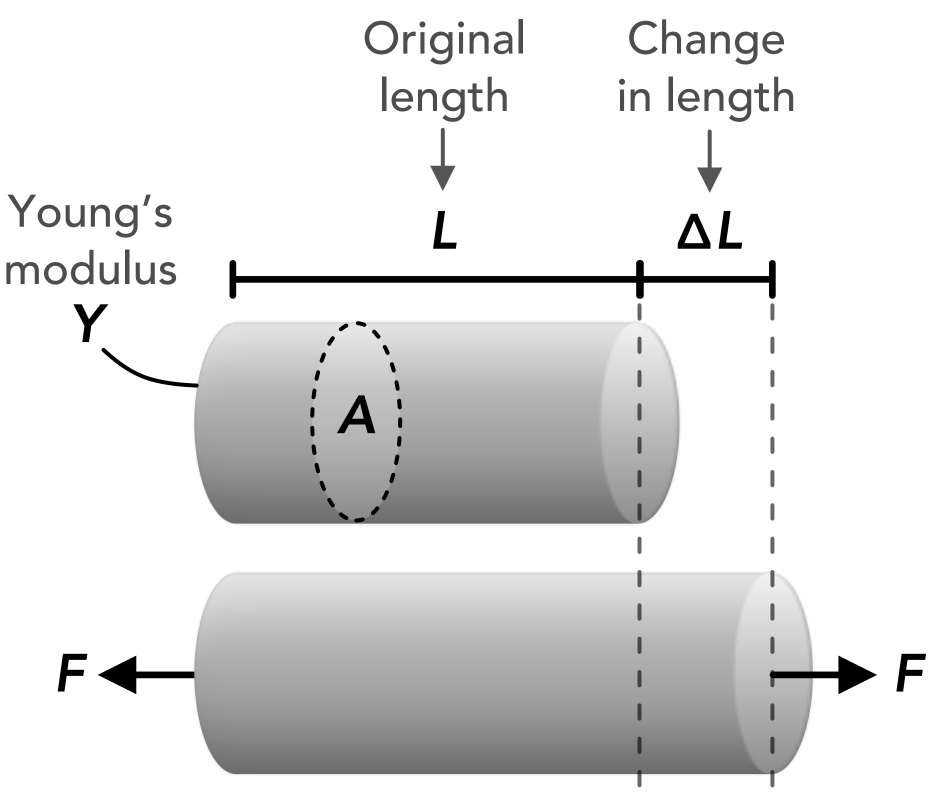
$$k_{1+2+3+4} = 210 \text{ N/m}$$



Elasticity of Materials

- The elastic behavior of materials is complex and depend on many factors such as material properties, the shape and dimensions of the object, the directions of the applied forces, the change in length itself, and more.
- A basic model can be used to describe the elastic behavior of a material in a way that's similar to a spring. Note that this model is only accurate up to a certain amount of strain (percent change in length), after which the material no longer behaves "elastically" and will begin to permanently deform and eventually break.

Variables		SI Unit
F	force	N
k	spring constant	$\frac{\mathrm{N}}{\mathrm{m}}$
Y , E	Young's modulus	$\frac{\mathrm{N}}{\mathrm{m}^2}$
A	cross-sectional area	m^2
L	length	m



"Spring constant" for a material

$$k = \frac{YA}{L}$$

Elastic Force

$$F = \frac{YA}{L}\Delta L$$

Stress

$$\frac{F}{A} = Y\frac{\Delta L}{L}$$

$\frac{\Delta L}{L}$ is called "strain"

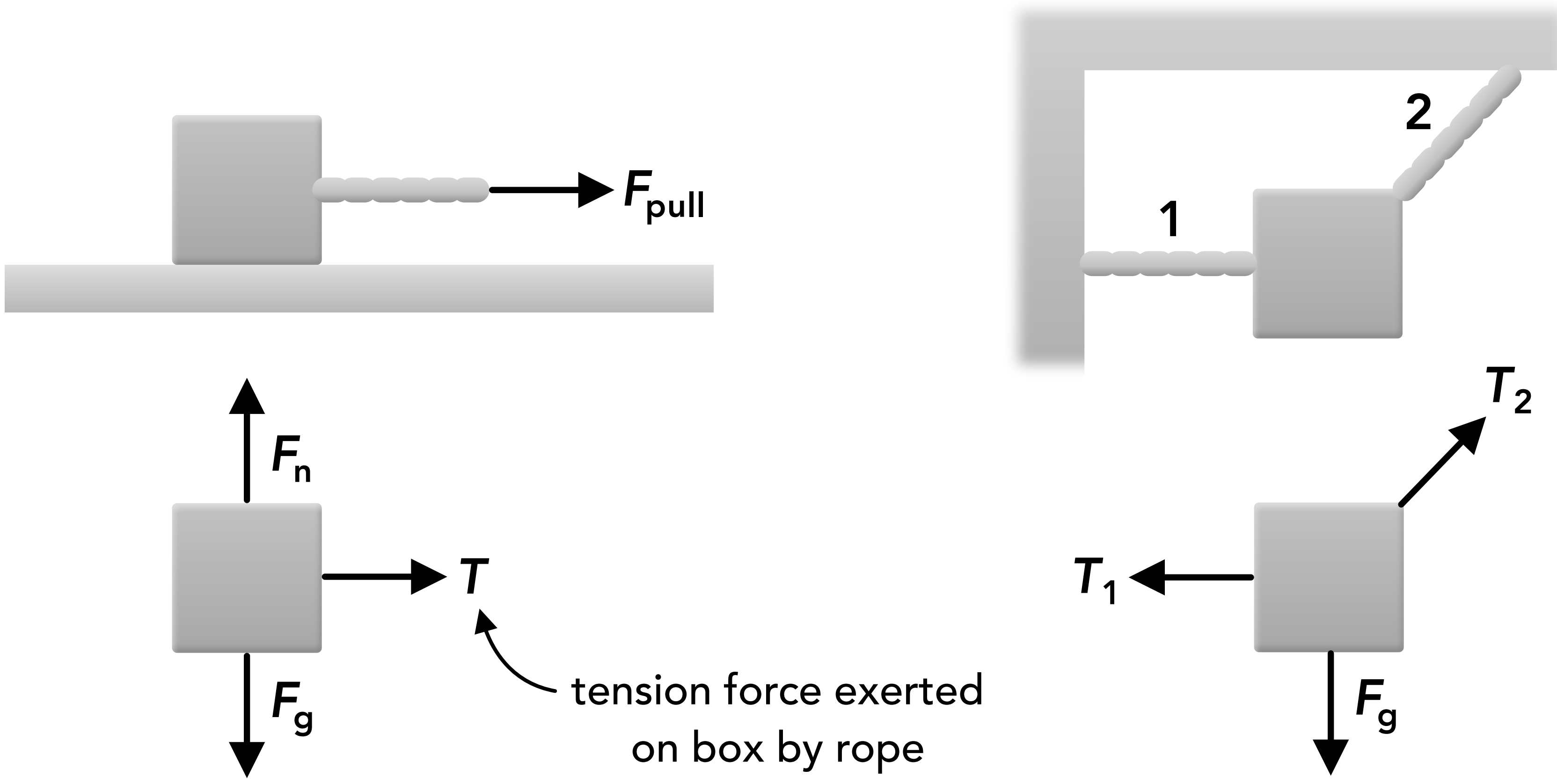
- Young's modulus Y (sometimes referred to as the elastic modulus E) is a property of the material that describes its stiffness. This is a material property that does not depend on the size or shape of the object, and is not the same as a spring constant k. However, an equivalent "spring constant" k can be found using the object's Young's modulus, cross sectional area and original length as shown in the equation above.
- Like a spring, the force applied to the object is proportional to its change in length, now represented as ΔL.
- Because objects are different shapes and sizes, it's often more useful to work with a concept called **stress** which is the amount of force applied per unit of area. It's also more useful to describe the percent change in length, known as **strain** ($\Delta L/L$) instead of the absolute change in length.

TENSION & PULLEY SYSTEMS

Tension

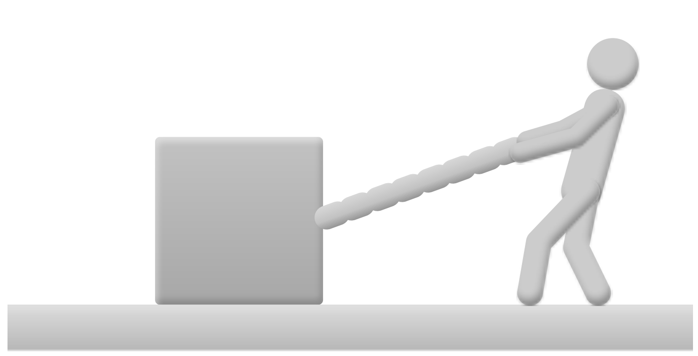
Variables		SI Unit
T	tension force	N

- **Tension** is a **pulling force** that we usually associate with ropes, strings, cables, wires, or other long and thin objects.
- If a rope is attached to an object or surface and the rope is pulled, a tension force arises in the rope and that same tension force is exerted on the objects at both ends of the rope.
- A tension force always acts in the same direction as the rope, and is always a pulling force (not a pushing force).

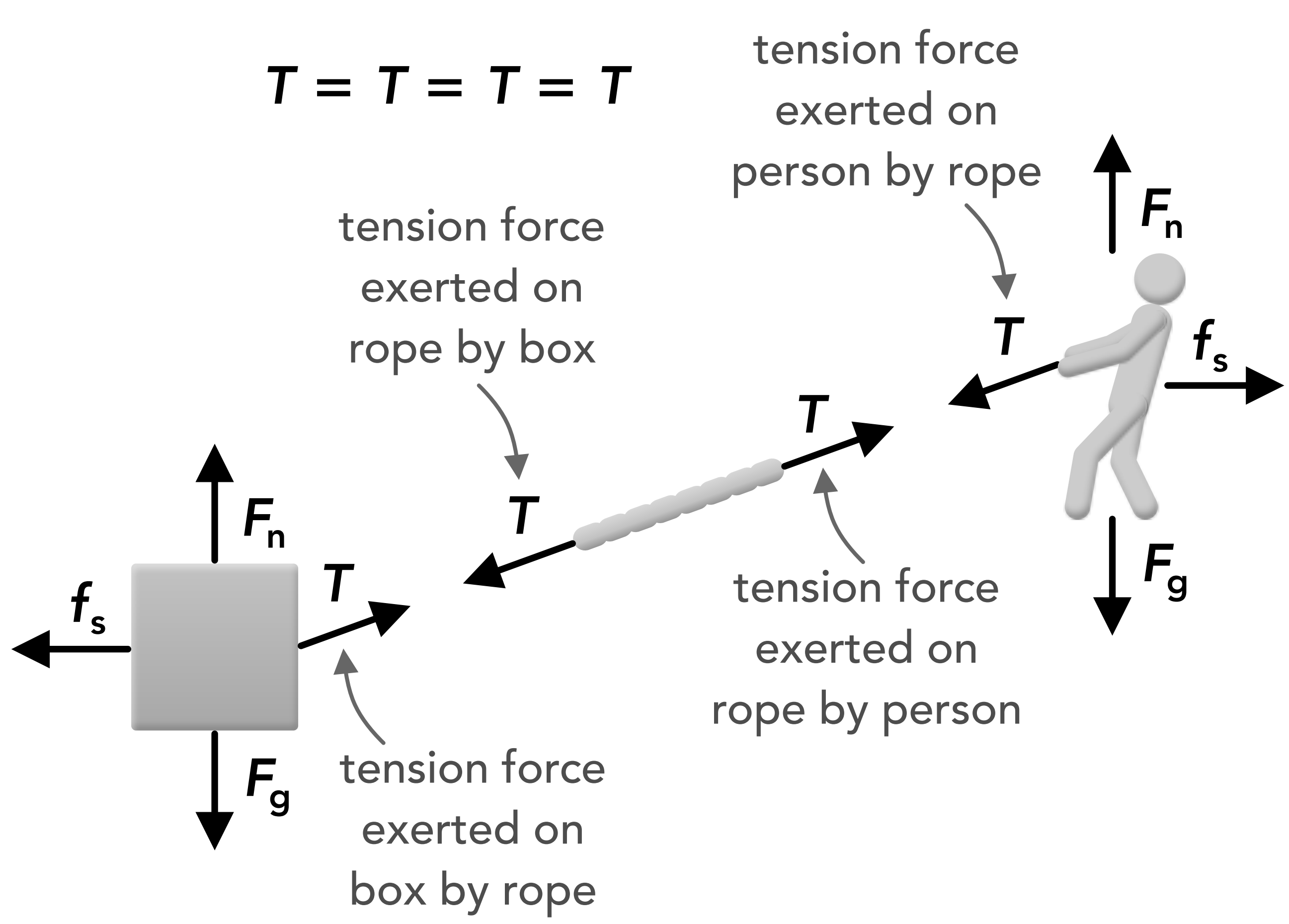


- An **ideal** rope (string, cable, etc.)…
  - is massless: the rope itself has no mass, no inertia and no weight
  - does not change length, regardless of the tension
- We assume that the **tension force is the same at both ends of an ideal rope but acts in opposite directions**. any change in the force at one end is instantly transmitted to the other end. This also applies to ropes passing around ideal pulleys.
- We could think of a rope as a spring with an infinite spring constant (stiffness) which never changes length no matter how much force is applied. Tension force in a rope is like the spring force in a spring.

A person pulling a rope attached to a box



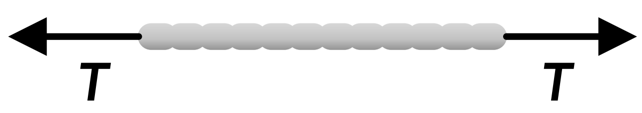
Free body diagrams of the box, rope and person



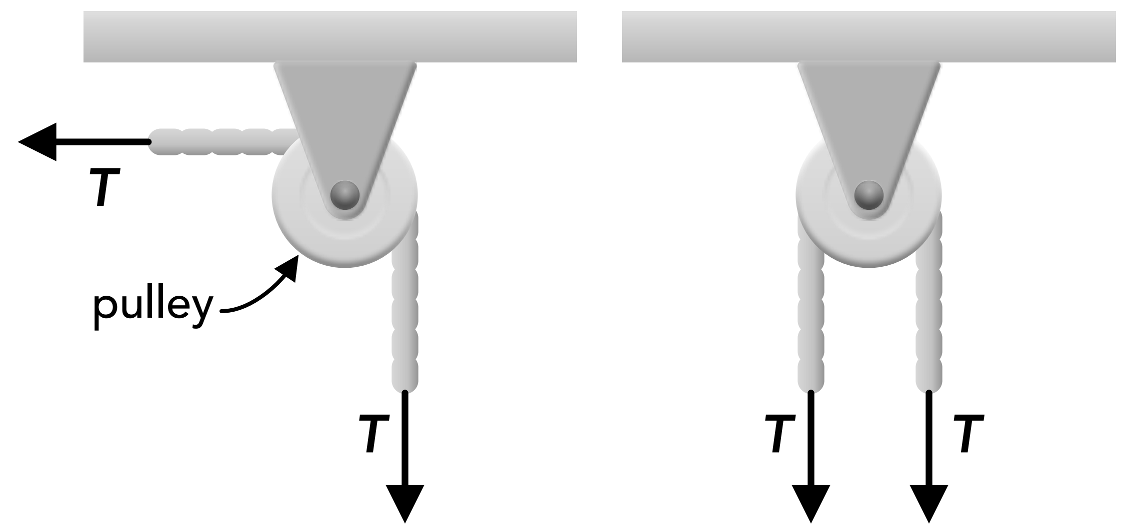
Pulley Systems

- When a rope passes around a pulley, **the rope and the tension force change direction**. The pulley rotates as the rope moves in either direction.

The tension force at the end of a rope acts in the direction of the end of the rope

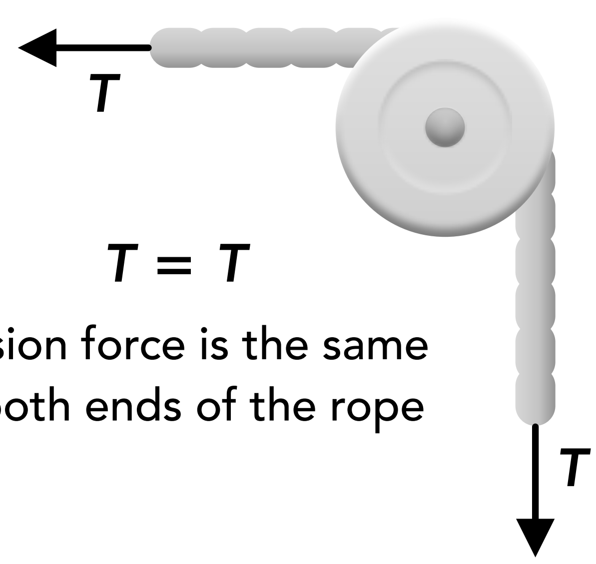


The rope and the tension force change direction around a pulley



- An **ideal** pulley is…
  - massless: the pulley has no mass, no inertia and no weight
  - frictionless: there are no friction forces acting within the pulley and it would rotate freely forever
- When a rope passes around an ideal pulley, **the tension force is still the same at both ends of the rope**.
- When a rope passes around a real or non-ideal pulley that has mass, the pulley has rotational inertia and the torque and rotational dynamics of the pulley need to be considered in addition to the other objects involved.
- If a pulley is not frictionless, the mechanical energy of the system (the ropes and other objects) may be lost as thermal energy due to friction.

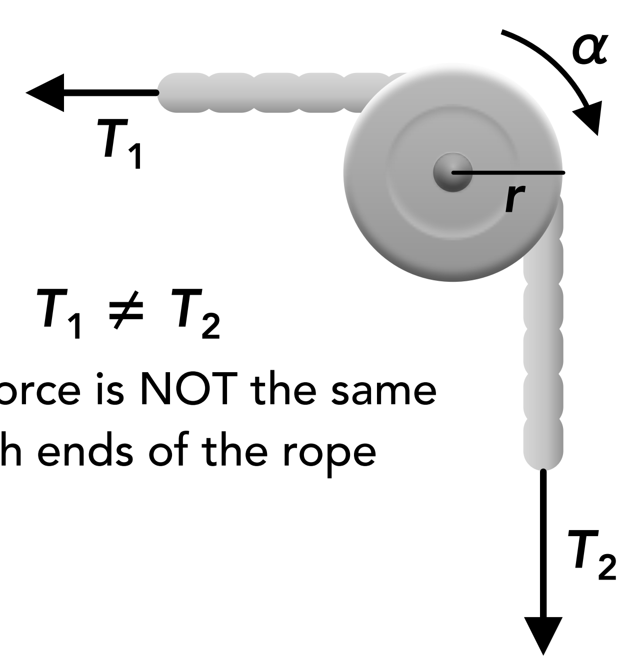
Ideal pulley (massless, frictionless)



$T = T$

tension force is the same at both ends of the rope

Real pulley (with mass)



$T_1 \neq T_2$

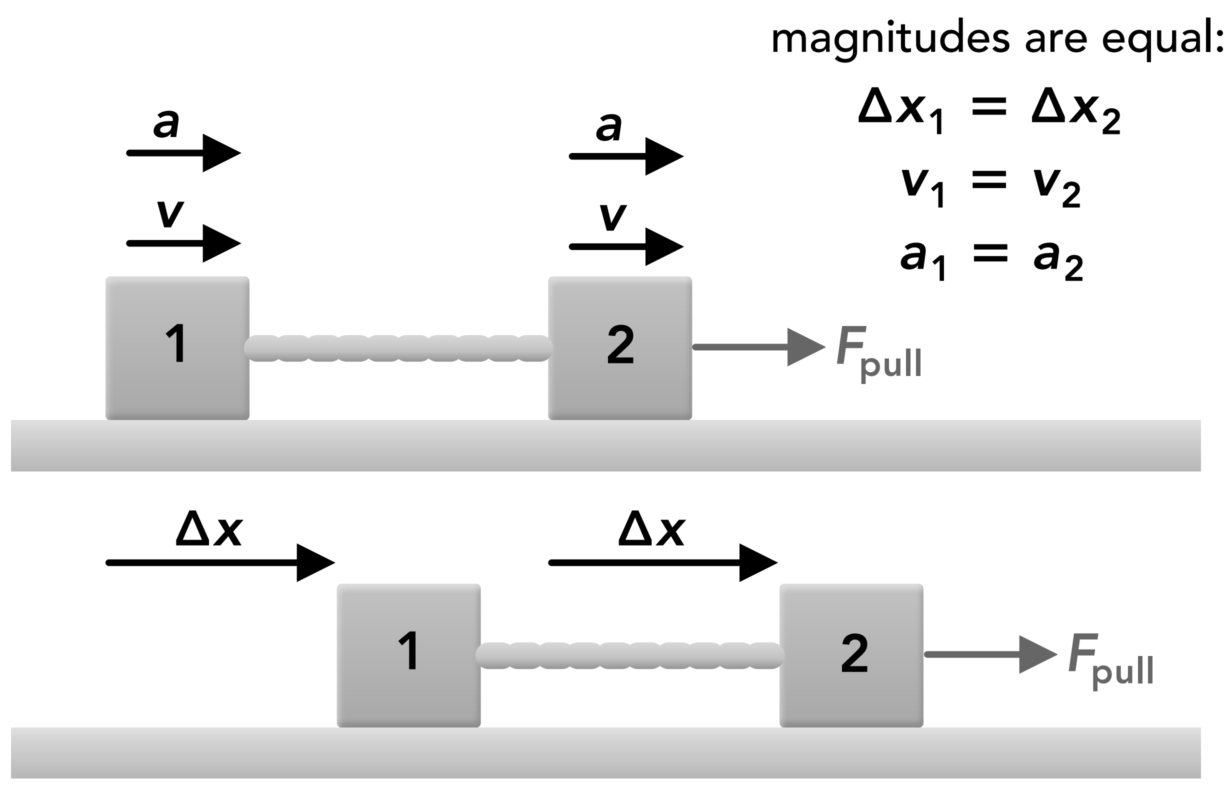
tension force is NOT the same at both ends of the rope

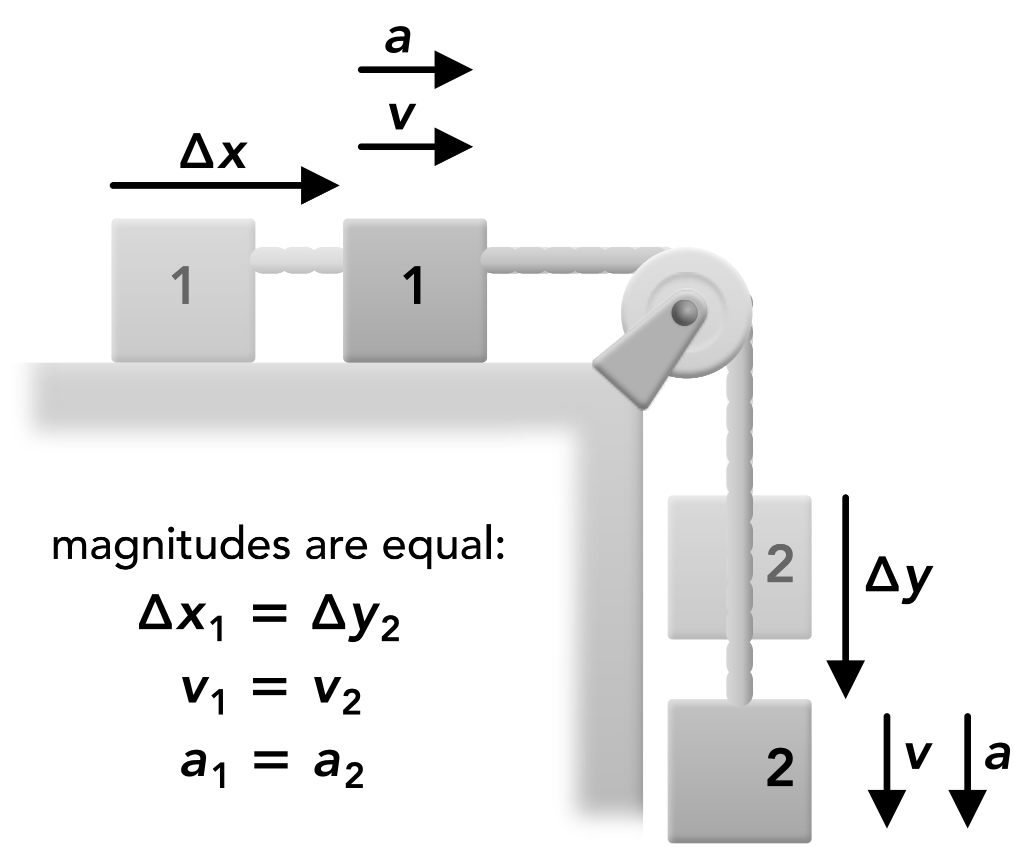
$\tau = rF_{\perp}$
$\tau_{net} = I\alpha$

rotational dynamics of pulley need to be considered

- If the rope is ideal and doesn't change length, **both objects at each end of the rope move together**. During any period of time, both objects must move the same displacement (even if the directions are different) because they're attached to the same rope. If the two objects did not move the same displacement, the rope would have to change length or break.
- Since one object can't move faster than the other, **their displacements, velocities and accelerations have the same magnitude** (but the directions may be different).
- This means we can set the magnitudes of the accelerations equal to each other in order to solve a system of equations that we get from Newton's 2nd law for both obects.

Objects connected by a rope have the same displacement, velocity and acceleration (magnitudes)

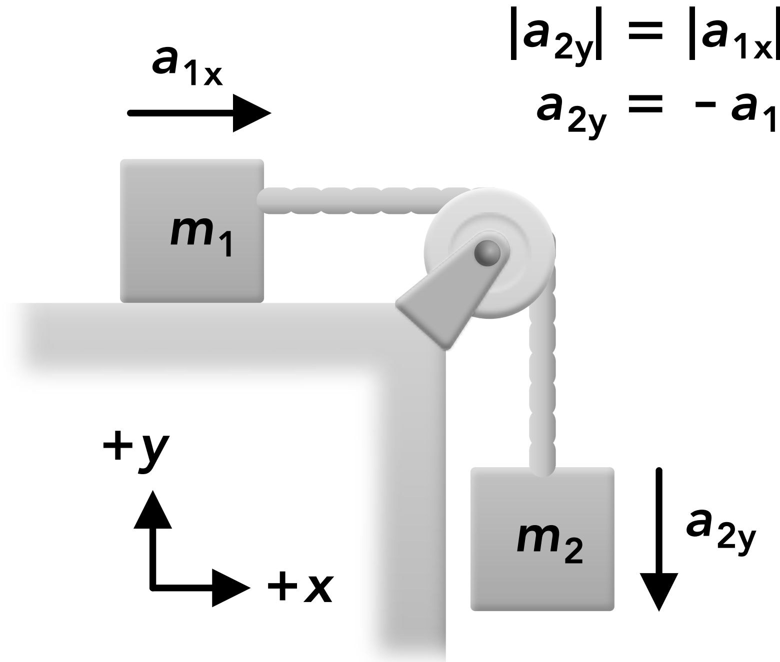




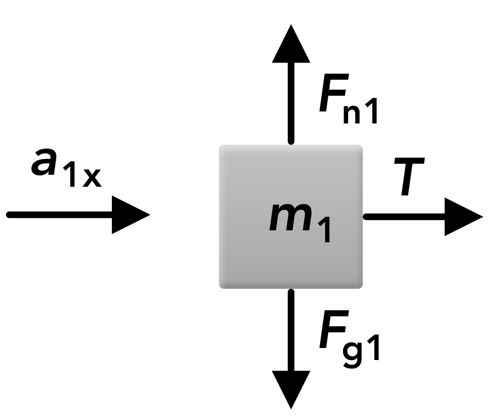
- This means that we can **treat both objects and the rope as a single system (or object)**. If we're trying to find the acceleration, we can draw free body diagrams and apply Newton's 2nd law to each object. Or we can draw a free body diagram and apply Newton's 2nd law to the **system using the total mass of each object**.
- If we do that, **the tension force becomes an internal force and is not included in the free body diagram**. So we can use this method to find the acceleration, but not the tension.
- Instead of using the ***x*** and ***y*** directions for the forces and acceleration, we can use a **new direction that is always parallel to the rope** as it bends around the pulley, and just focus on the positive and negative directions. Think of this as "straightening out" the rope into a "rope axis" while keeping the direction of the forces relative to the rope.

Separate free body diagrams and Newton's 2nd law for each block

$$|a_{2y}| = |a_{1x}|$$
$$a_{2y} = -a_{1x}$$

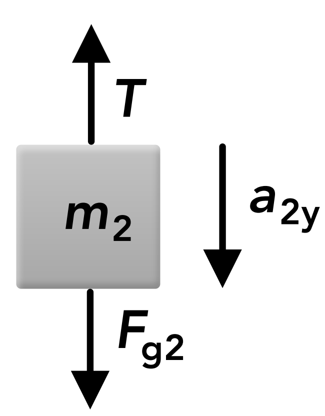


Block 1:



$T = T$

Block 2:



$$\sum F_y = ma_y$$
$$F_{n1} - F_{g1} = m_1(0)$$

$$\sum F_x = ma_x$$
$$T = m_1 a_{1x}$$

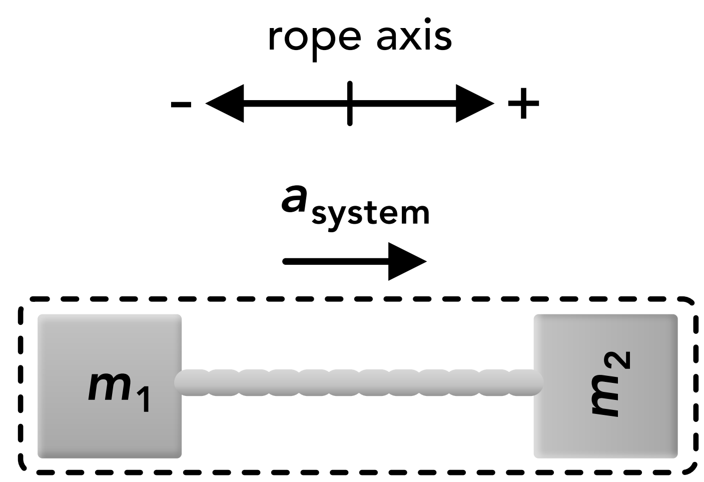
$$\sum F_y = ma_y$$
$$T - F_{g2} = m_2 a_{2y}$$
$$T = m_2 a_{2y} + F_{g2}$$
$$T = m_2 a_{2y} + m_2 g$$

$$T = T$$
$$m_1 a_{1x} = m_2 a_{2y} + m_2 g$$
$$a_{2y} = -a_{1x} \rightarrow m_1 a_{1x} = m_2(-a_{1x}) + m_2 g$$

$$a_{1x} = \frac{m_2 g}{m_1 + m_2}$$

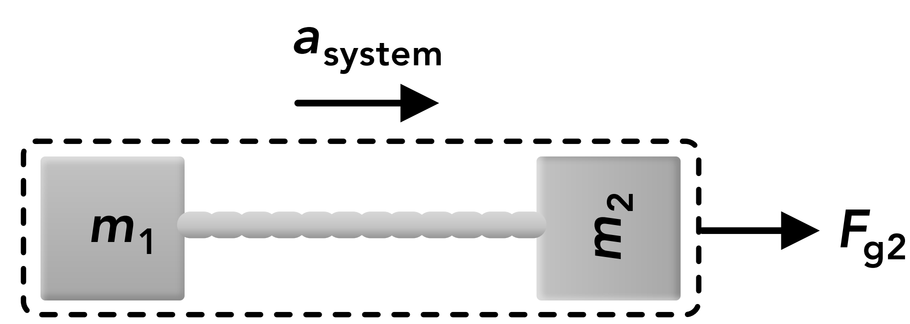
$$a_{2y} = -\frac{m_2 g}{m_1 + m_2}$$

The same blocks and rope are treated as a single system with a total mass, using a shared "rope axis"



$$m_{system} = m_1 + m_2$$

System:



tension is an internal force, not included

net force in the "rope axis" direction:

$$\sum F = m_{system} a_{system}$$
$$F_{g2} = (m_1 + m_2) a_{system}$$
$$m_2 g = (m_1 + m_2) a_{system}$$

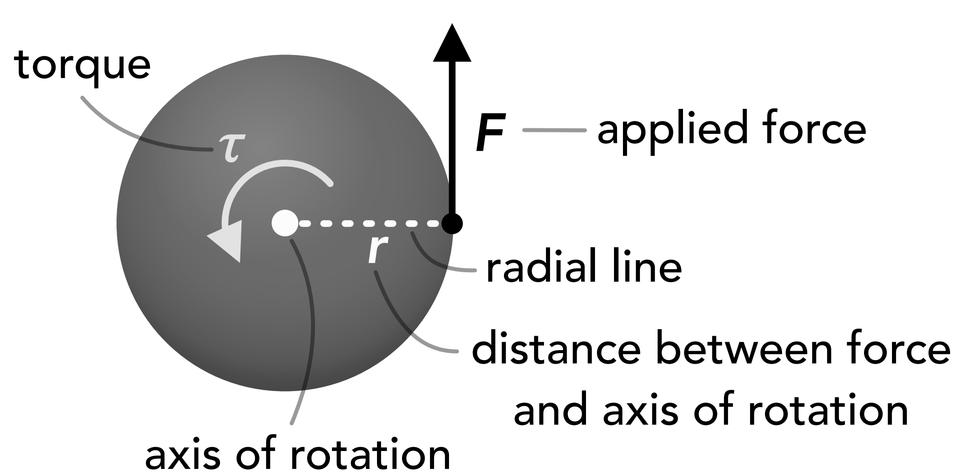
$$a_{system} = \frac{m_2 g}{m_1 + m_2}$$

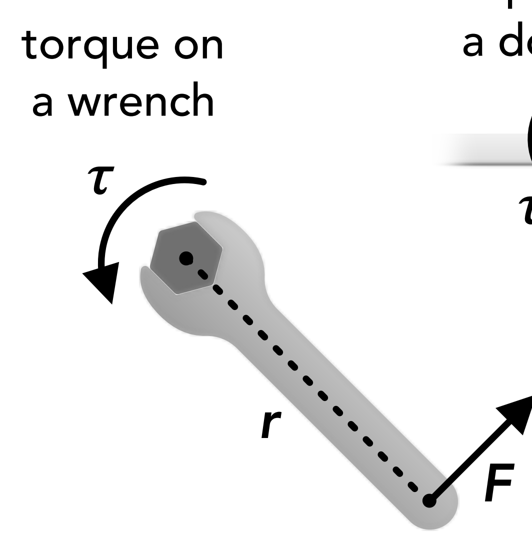
TORQUE

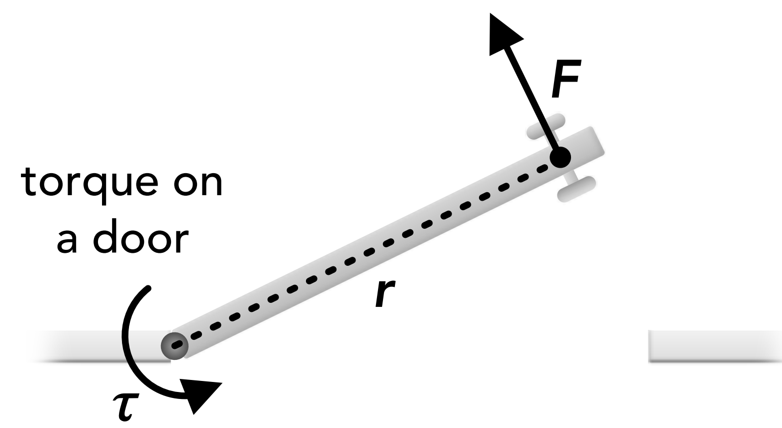
Torque

Variables		SI Unit
τ	torque	N·m
F	force	N
r	distance from rotation axis	m

- In simple terms, **torque** is like a **rotational force**.
- When a force is applied to an object and that force does not point directly at or away from the object's axis of rotation, that force **generates a torque**.
- If an object is forced to rotate around one point or axis, like a wheel rotating about an axle or a door rotating about a hinge, **that is the axis of rotation**. If an object is free to rotate about any axis, its axis of rotation will pass through its **center of mass**.







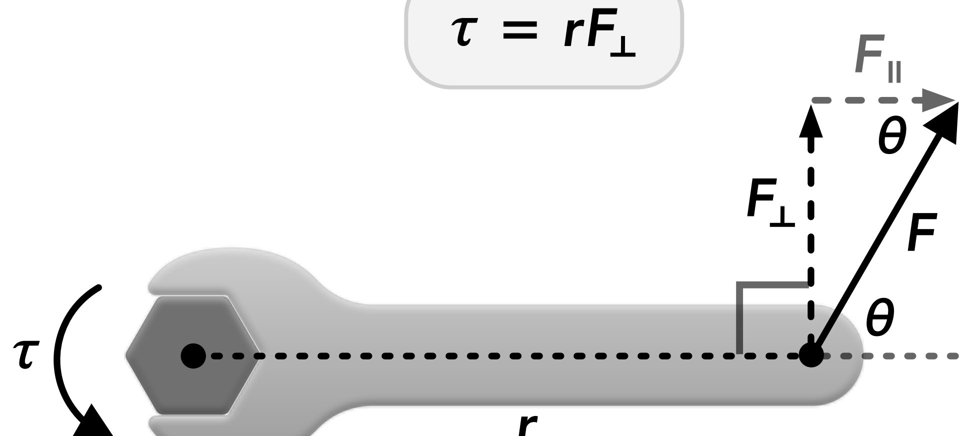
- When you push or pull on a door or a wrench, you're applying a linear force at some distance from the object's axis of rotation and generating a torque on that object which causes it to rotate.

- Torque is represented with the Greek letter τ (tau).
- The SI unit of torque is Newton-meters (N·m) which is given by the equation below.
- Only the component of the force that is **perpendicular** to the radial line between the center of rotation and the point where the force is applied contributes to the torque.
- If the force is not already perpendicular to the radial line, there are two ways to calculate the torque:

Multiply the distance between the axis of rotation and the point where the force is applied (r) times the component of the force vector that is perpendicular to the radial line ($F_\perp$)

Torque

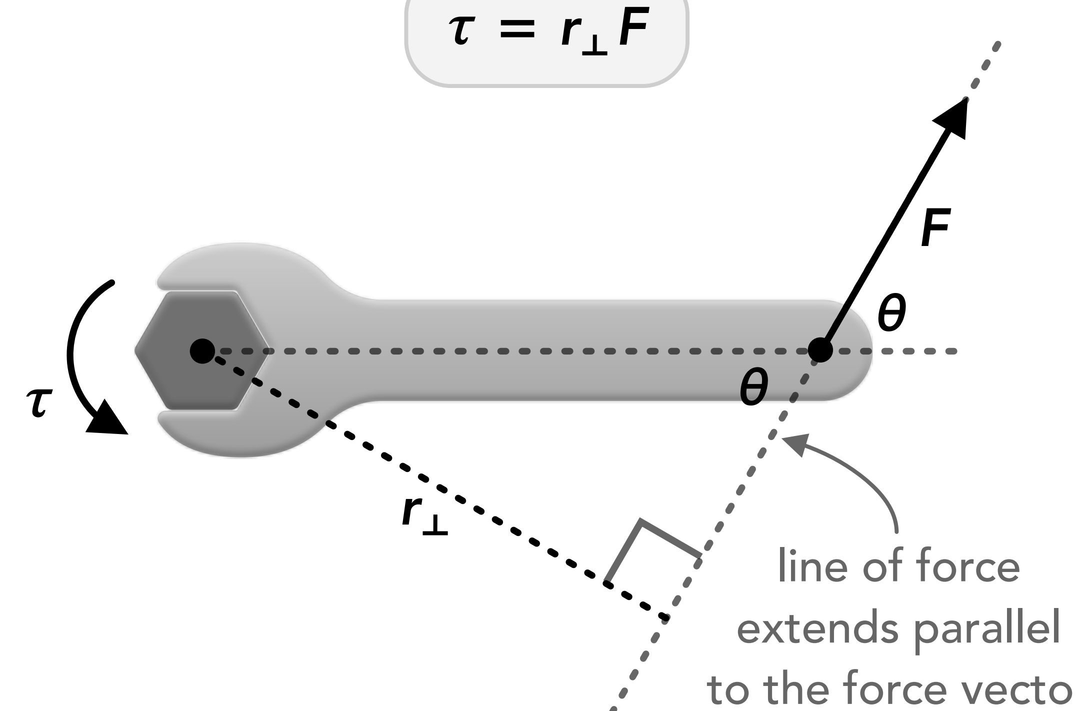
$$\tau = rF_\perp$$



Multiply the perpendiuclar distance between the axis of rotation and the line of force ($r_\perp$) times the force (F)

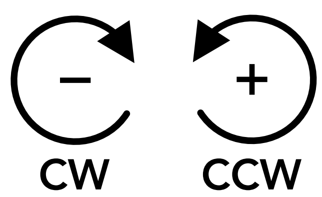
Torque

$$\tau = r_\perp F$$

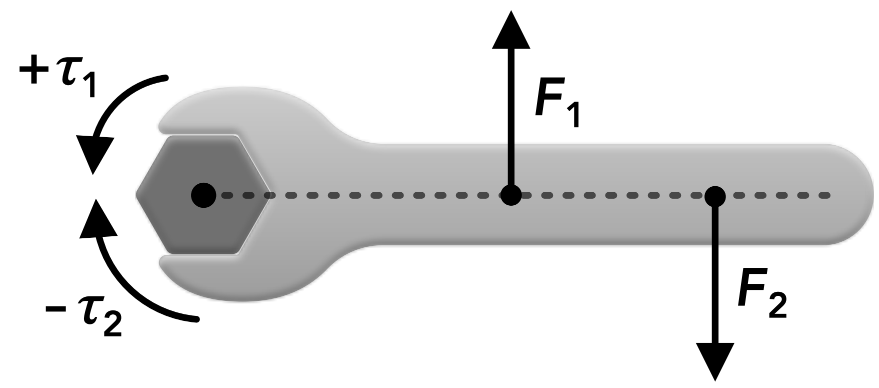


- When looking at the plane of rotation, a torque can either be clockwise (CW) or counterclockwise (CCW).
- **Counterclockwise is the positive direction** using convention, and clockwise is the negative direction, just like in rotational or circular kinematics. This is important when adding several torques to find the net torque.
- The direction of the torque is the direction that the force would cause the object to rotate (CW or CCW).

counterclockwise torque is positive
clockwise torque is negative



F_1 would rotate the wrench counterclockwise so it generates a positive torque

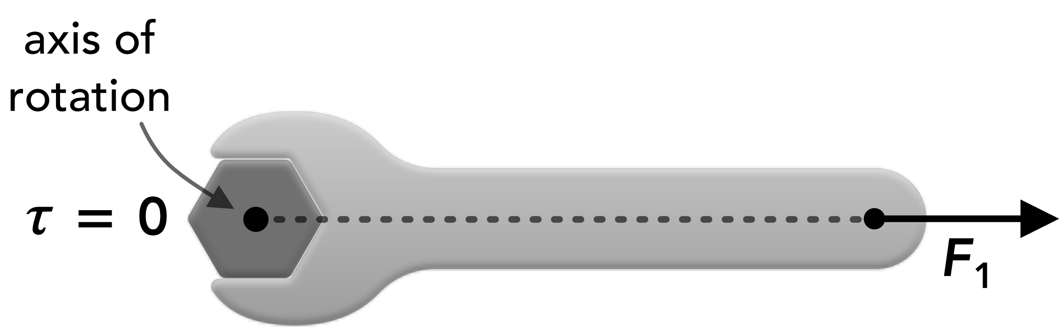


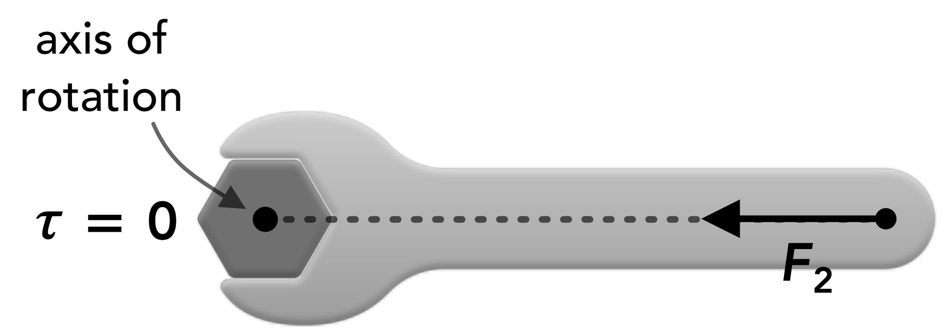
F_2 would rotate the wrench clockwise so it generates a negative torque

---

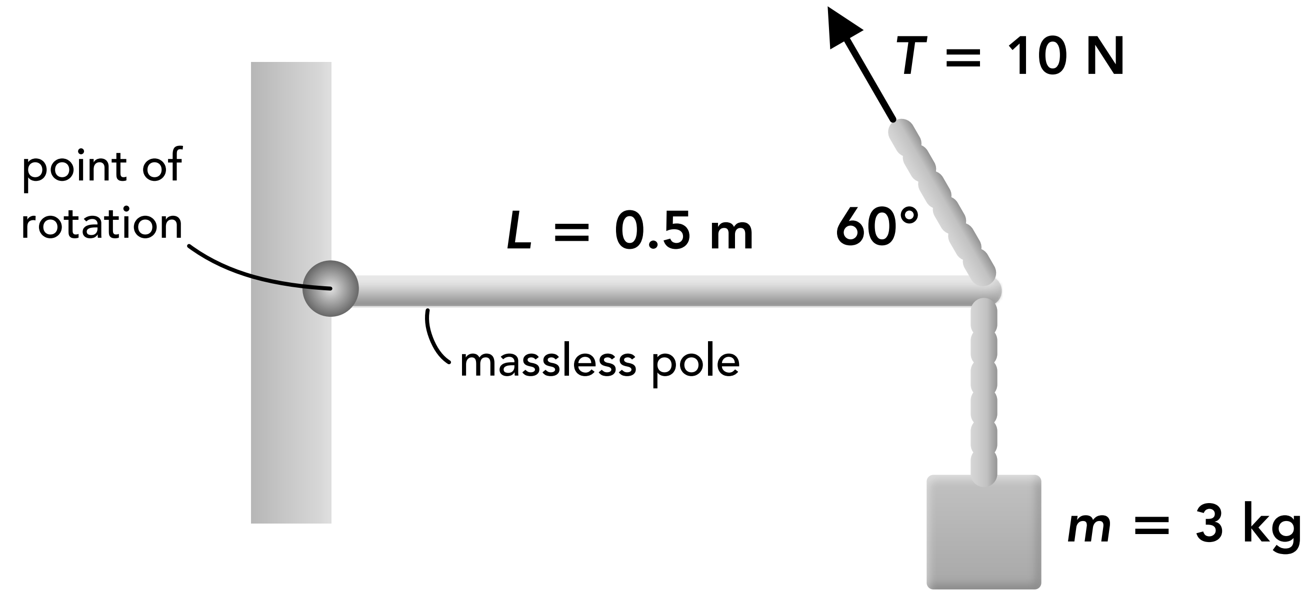
- A force whose line of force passes through the axis of rotation (the force points directly at or away from the axis of rotation) **does not generate a torque** because there is no force component perpendicular to the radial line.

F_1 and F_2 do not generate a torque because they act parallel to the radial line (directly at or away from the axis of rotation)

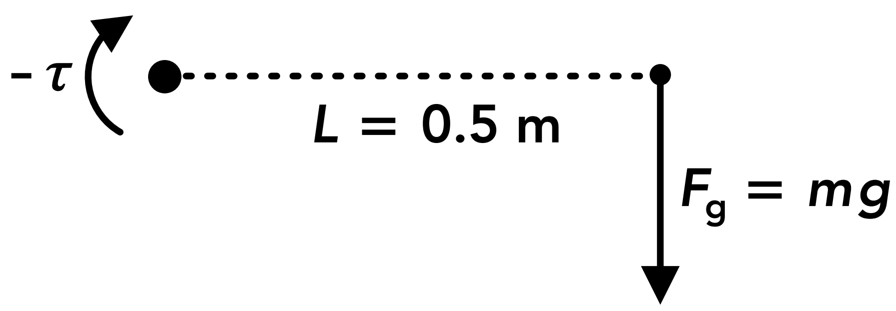




Example: A massless pole is pinned to a wall and is free to rotate about its left end. At the right end of the pole a mass is hanging straight down and a rope pulls the pole up with a tension force at an angle. What are the torques generated by the hanging mass and the tension force about the point of rotation on the left?



Torque from hanging mass:



$\tau = rF_{\perp}$

$\tau = (L)(mg)$

$\tau = (0.5\text{ m})(3\text{ kg})(9.8\text{ m/s}^2)$

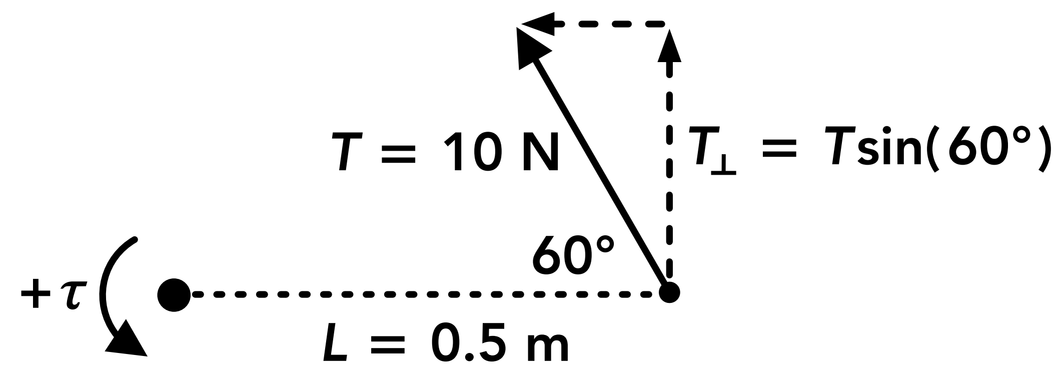
$\tau = 14.7\text{ Nm}$ (magnitude of torque)

↓

$\tau = -14.7\text{ Nm}$

torque is clockwise so it's negative

Torque from upper rope:



$\tau = rF_{\perp}$

$\tau = (L)(T\sin(60°))$

$\tau = (0.5\text{ m})(10\text{ N})\sin(60°)$

$\tau = 4.3\text{ Nm}$ (magnitude of torque)

↓

$\tau = 4.3\text{ Nm}$

torque is counterclockwise so it's positive

ROTATIONAL DYNAMICS

Net Torque and Rotational Dynamics

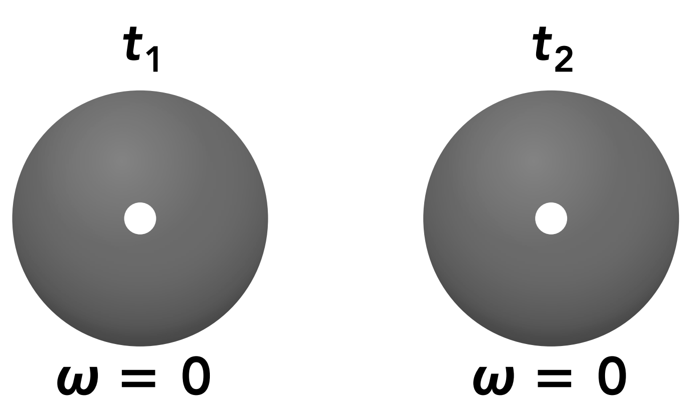
- Newton's laws of motion described how objects move and the relationship between linear forces and linear acceleration. Newton's 1st and 2nd laws of motion can also be applied to torques and rotational motion.
- When working with rotational dynamics, it will help to review the material on rotational kinematics.
- The rotational version of a **force** is a **torque**.
- The rotational version of **acceleration** is **angular acceleration**.
- The rotational version of **mass** is **rotational inertia**, also referred to as the **moment of inertia**.

Variables		SI Unit
τ	torque	$\text{N}\cdot\text{m}$
I	rotational inertia	$\text{kg}\cdot\text{m}^2$
α	angular acceleration	$\frac{\text{rad}}{\text{s}^2}$
m	mass	kg
r	distance from rotation axis	m

- **Newton's 1st law of motion (applied to rotation):** An object at rest (with no angular velocity) will remain at rest and a rotating object will maintain its angular velocity unless there is a net torque acting on the object (the sum of all the torques acting on the object is not zero).
- When we see a rotating or spinning object slow down, there must be a net torque acting on the object caused by forces such as friction or air resistance. In the absence of a net torque a rotating object will rotate forever.
- If an object is not rotating (or if it's rotating at a constant angular velocity) that doesn't mean there are no torques acting on the object, only that the net torque is zero (the torques balance each other in opposite directions).

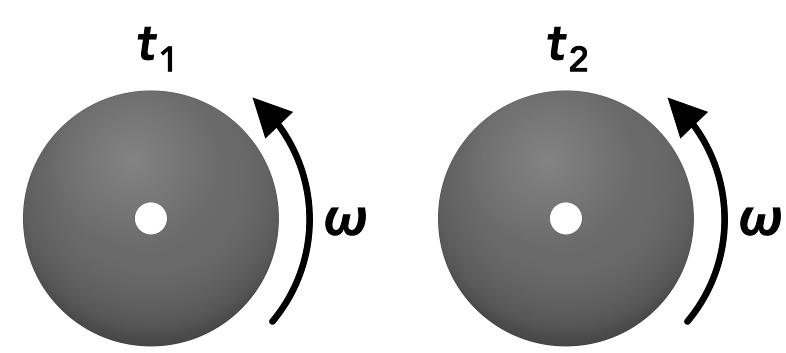
An object at rest (with zero angular velocity) will remain at rest if there is no net torque acting on it

$$\tau_{net} = 0$$
$$\alpha = 0$$



A rotating object will maintain its angular velocity if there is no net torque acting on it

$$\tau_{net} = 0$$
$$\alpha = 0$$

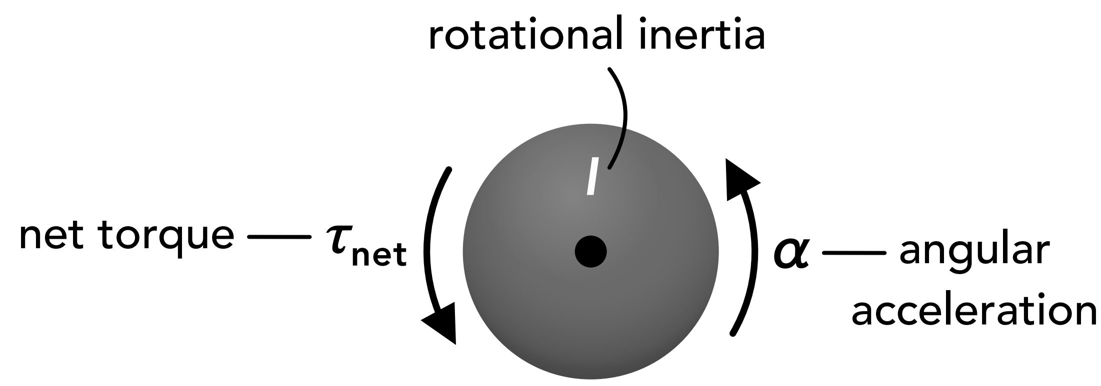


- **Newton's 2nd law of motion (applied to rotation):** A net torque τ_{net} acting on an object with a rotational inertia I will cause an angular acceleration α in the same direction as the net torque, and the net torque is equal to the rotational inertia multiplied by the angular acceleration: $\tau_{net} = I\alpha$
- The **rotational inertia** or the **moment of inertia** is covered in another section, but it's a value that represents the mass of an object and how far that mass is distributed from the axis of rotation.

Newton's 2nd law of motion applied to rotation

$\boxed{\tau_{net} = I\alpha}$ or $\boxed{\sum\tau = I\alpha}$

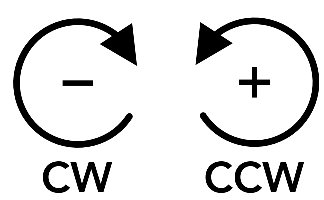
$\sum$: the sum of __

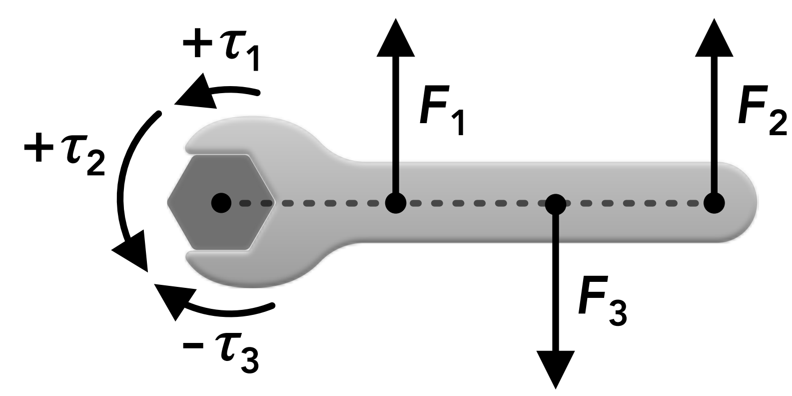


The net torque is the sum of all of the torques acting on an object

$$\tau_{net} = \sum\tau = \tau_1 + \tau_2 - \tau_3$$

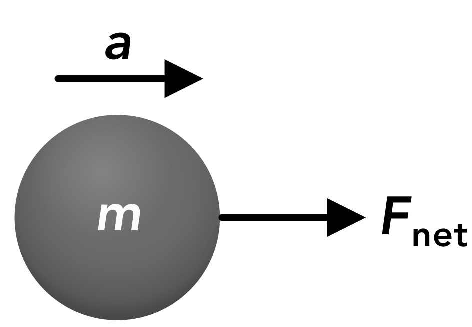
counterclockwise torque is positive
clockwise torque is negative





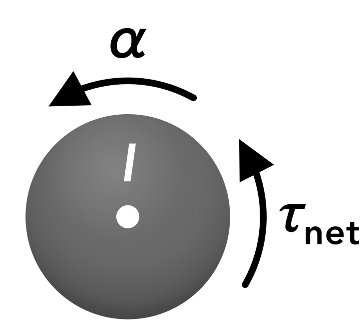
Newton's 2nd law of motion for linear motion and rotational motion

Linear



$\boxed{\sum F = ma}$

Rotational



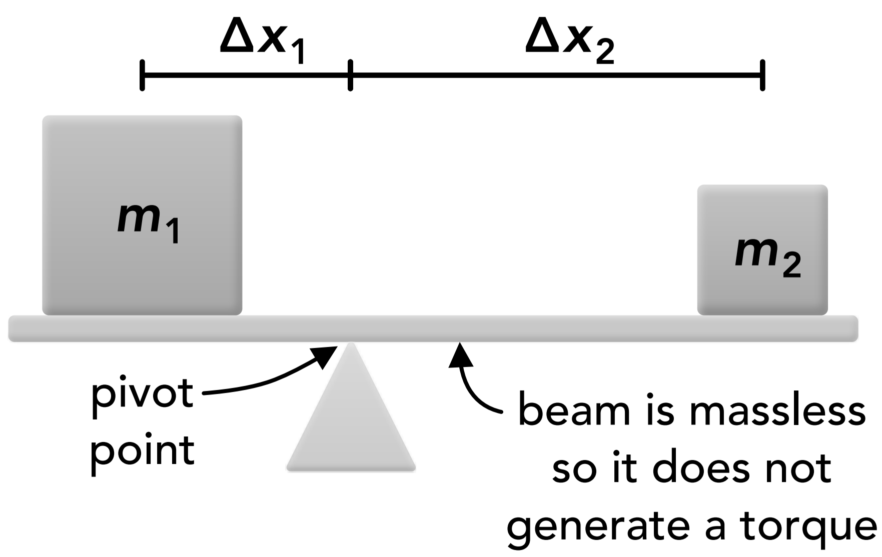
$\boxed{\sum\tau = I\alpha}$

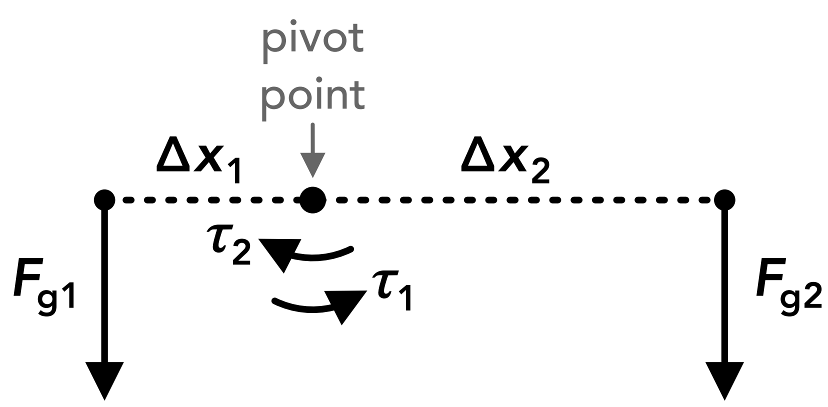
F: force (N)	⟷	τ: torque (N·m)
m: mass (kg)	⟷	I: rotational inertia (kg·m²)
a: acceleration (m/s²)	⟷	α: angular acceleration (rad/s²)

- If the net torque acting on an object or system is zero, the angular acceleration is zero and we say the object or system is in a state of **rotational equilibrium**.
- If an object or system is not rotating (or is rotating at a constant angular velocity), the net torque acting about **any point** on the object or system is zero, not just about the object's pivot point or center of mass. We can use this to analyze the forces and torques acting on an object or system.

Two blocks sit on a massless beam on a pivot point and the system is in rotational equilibrium

$$\alpha = 0$$
$$\tau_{net} = 0$$



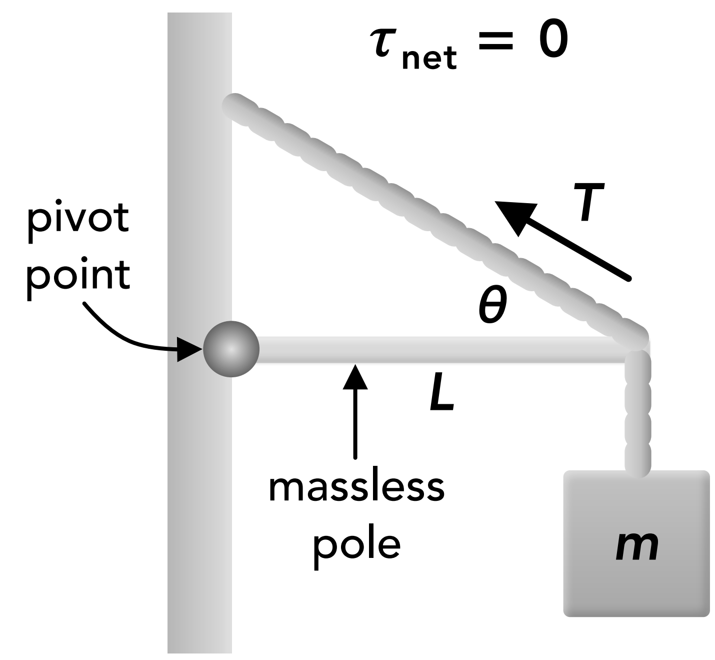


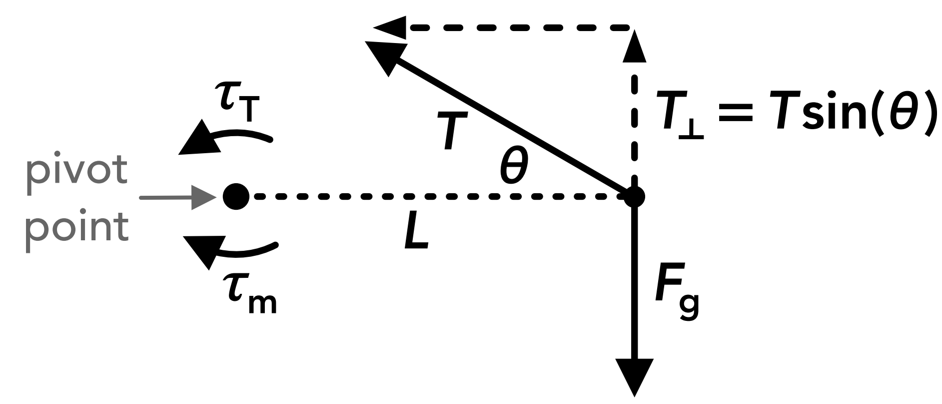
$$\sum \tau = I\alpha$$
$$\tau_1 - \tau_2 = I(0)$$
$$(r_1 F_1) - (r_2 F_2) = 0$$
$$(\Delta x_1 m_1 g) - (\Delta x_2 m_2 g) = 0$$

Any variable can be solved for if the other variables are known

A mass hangs from the end of a massless pole which is supported by an upper rope at an angle, and the system is in rotational equilibrium

$$\alpha = 0$$
$$\tau_{net} = 0$$



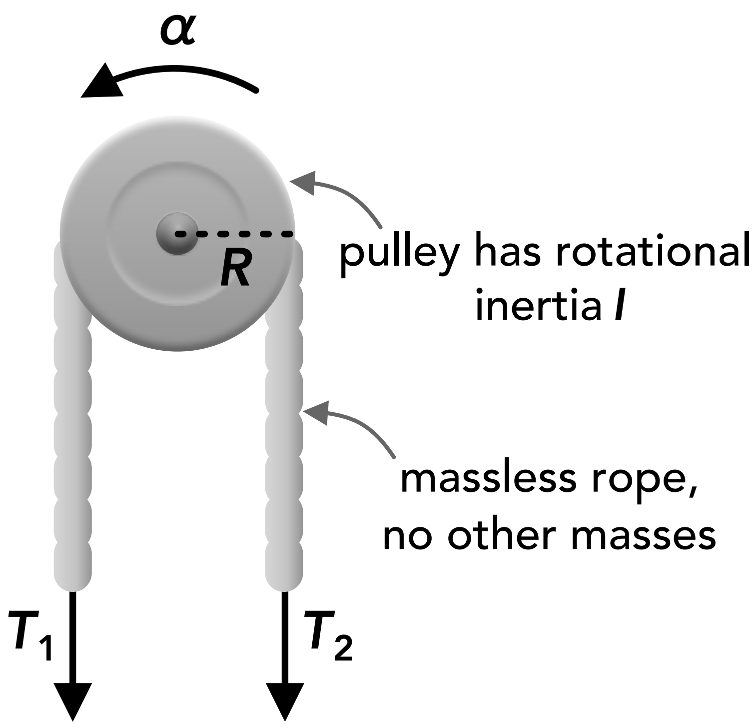


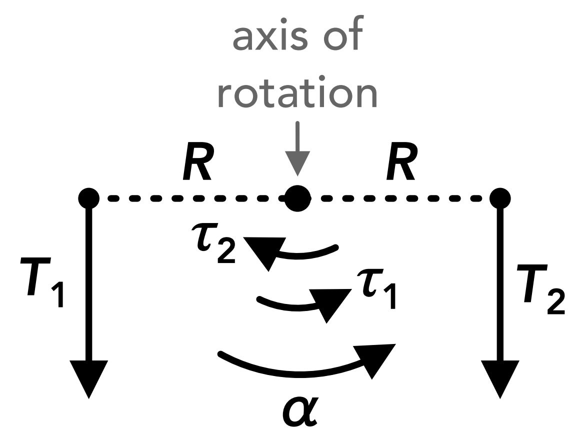
$$\sum \tau = I\alpha$$
$$\tau_T - \tau_m = I(0)$$
$$(r_T F_T) - (r_m F_m) = 0$$
$$(LT\sin(\theta)) - (Lmg) = 0$$

Any variable can be solved for if the other variables are known

- If the net torque acting on an object is not zero, the object is not in rotational equilibrium and it will rotate with an angular acceleration.

A rope is wrapped around a pulley that has mass and rotational inertia, so the tensions in the sections of rope on each side of the pulley are not equal. The ropes are massless and there are no other masses involved. The different tension forces cause the pulley to rotate with an angular acceleration.





$$\sum \tau = I\alpha$$
$$\tau_1 - \tau_2 = I\alpha$$
$$(r_1 F_1) - (r_2 F_2) = I\alpha$$
$$(RT_1) - (RT_2) = I\alpha$$

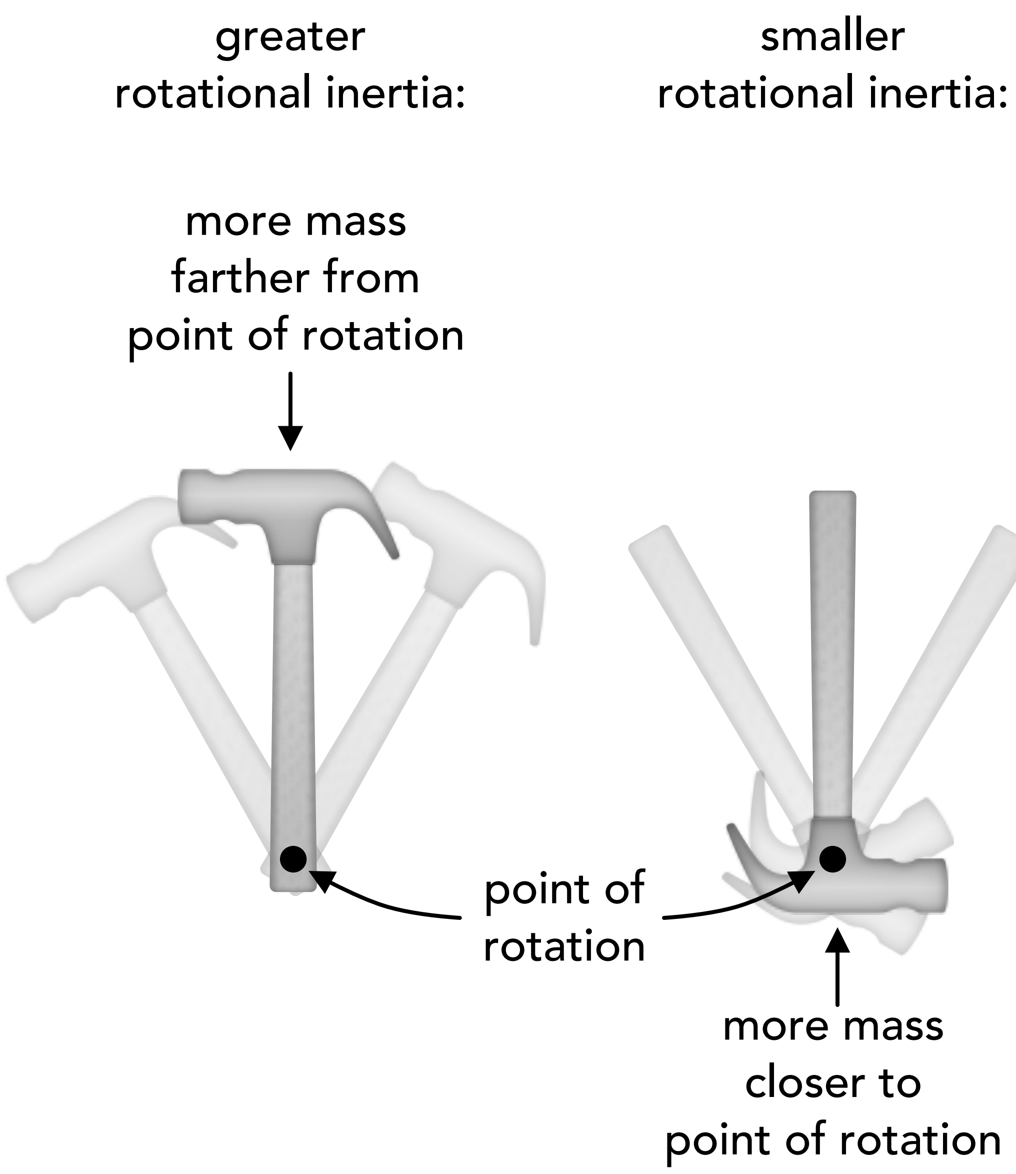
Any variable can be solved for if the other variables are known

Note: If there were other objects attached to the ropes they would also have mass and their own inertia, and the angular acceleration of the pulley would also depend on the linear dynamics of those masses and the rope.

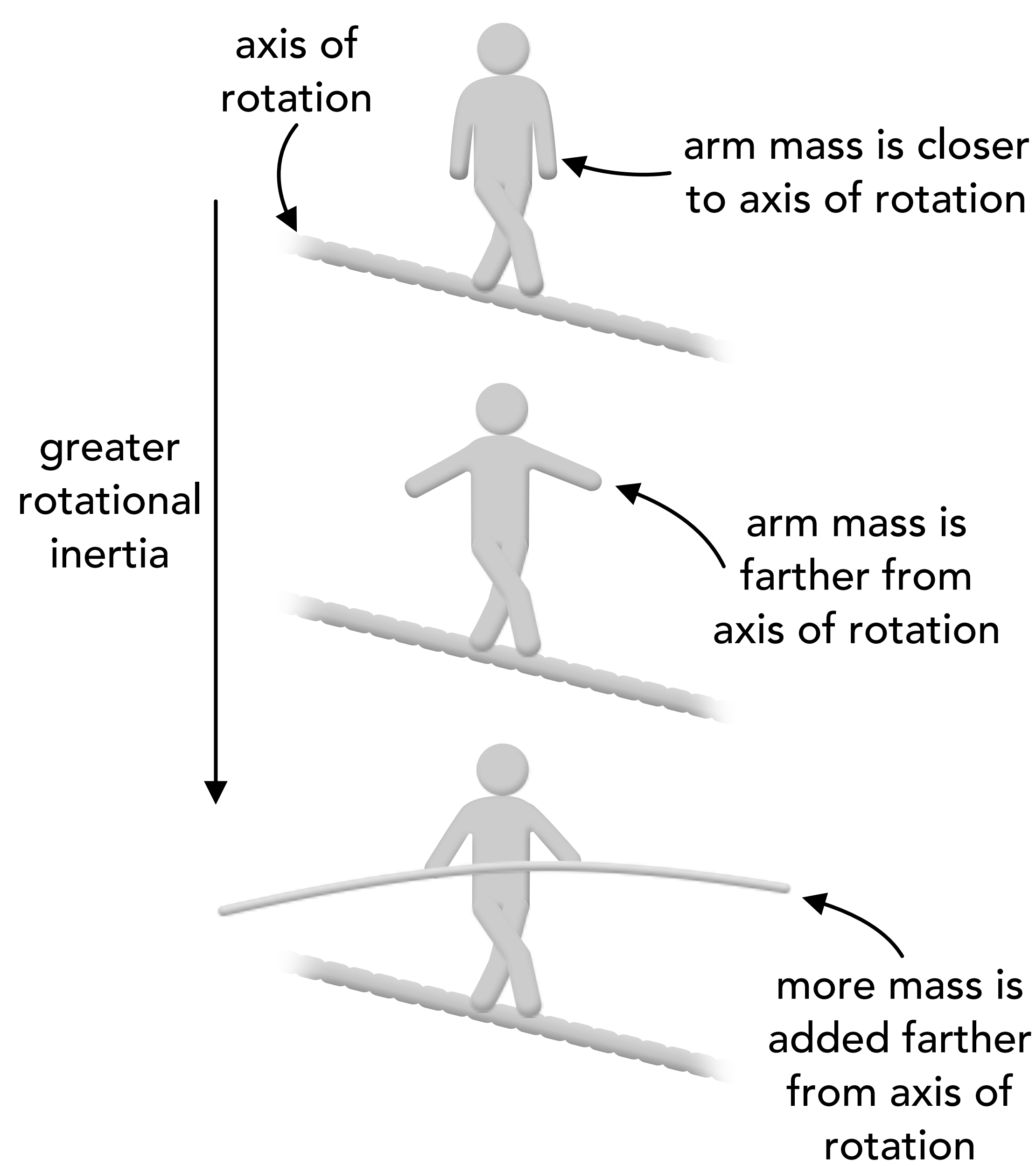
Rotational Inertia (Moment of Inertia)

- An object's **rotational inertia** I, also referred to as the **moment of inertia**, is the object's resistance to angular acceleration. The greater the rotational inertia, the more an object will resist a change to its state of rotation.
- The word "moment" has nothing to do with time and "rotational inertia" may be easier to remember, but the term "moment of inertia" is still widely used.
- The rotational inertia can be thought of as the **position-weighted sum of its mass** or its **mass distribution**.
- The more mass an object has and the farther that mass is distributed from the axis of rotation, the greater the object's rotational inertia.

A hammer is easier to rotate quickly (an angular acceleration) when held and rotated about the end with more mass



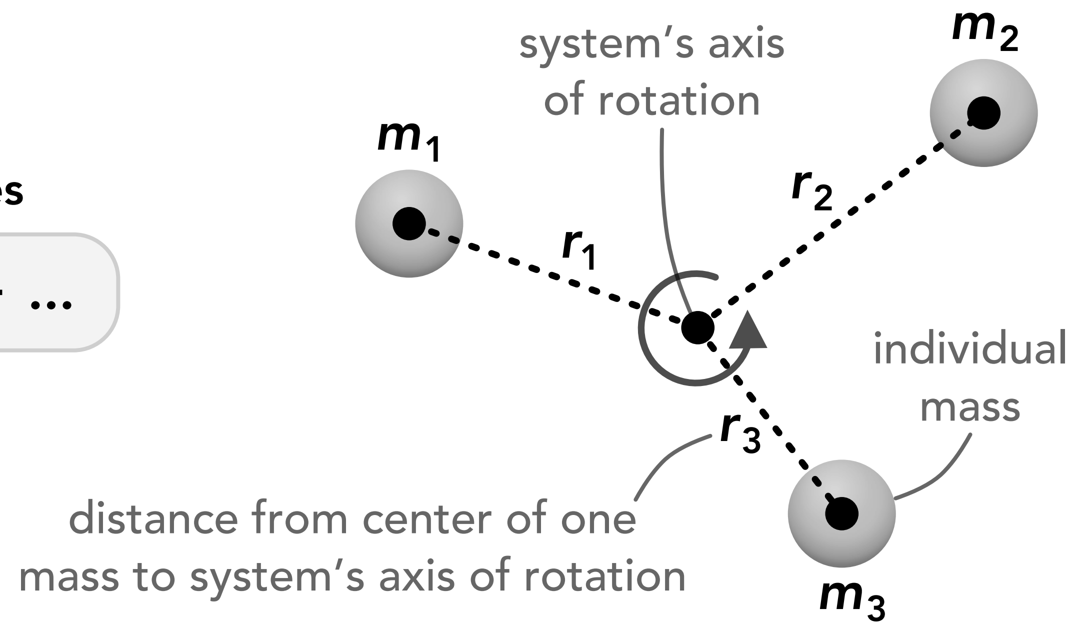
You automatically stick your arms out when trying to balance because it increases your rotational inertia and your resistance to rotation (falling sideways)



- The rotational inertia for a system of point masses (or a group of objects) can be calculated using the equation below, which is the sum of each mass $\boldsymbol{m}$ multiplied by the square of the distance between its own center and the axis of rotation $\boldsymbol{r}^2$.

Rotational inertia for a system of masses

$$I = \sum m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots$$

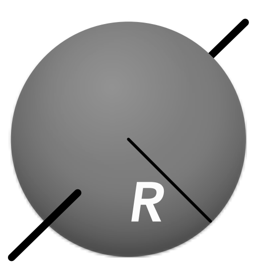


- Any rigid body (an object that does not change shape) can be modeled as a system of many individual point masses or particles (small sections of the object, molecules, atoms or even subatomic particles). If an object has a complex shape the rotational inertia will usually be given if needed.
- Many objects can be modeled as one of the shapes shown below.

Rotational inertia for some common shapes, where $\boldsymbol{m}$ is the total mass of the object, $\boldsymbol{r}$ is the radius, $\boldsymbol{L}$ is the total length of the object, and the axis of rotation is either through the center or one end

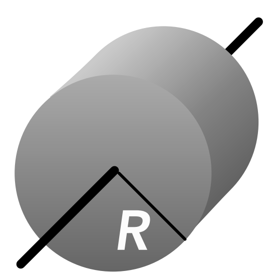
Solid sphere (center)

$I = \frac{2}{5}mR^2$



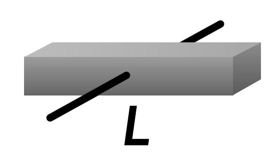
Solid cylinder (center)

$I = \frac{1}{2}mR^2$



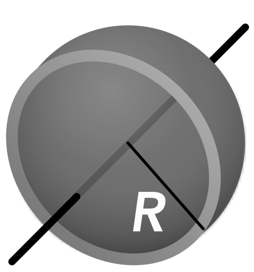
Solid rod (center)

$I = \frac{1}{12}mL^2$



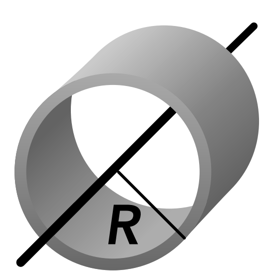
Sphere shell (center)

$I = \frac{2}{3}mR^2$



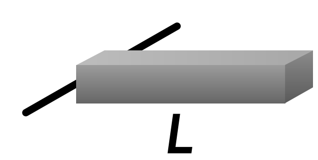
Cylinder shell (center)

$I = mR^2$



Solid rod (end)

$I = \frac{1}{3}mL^2$



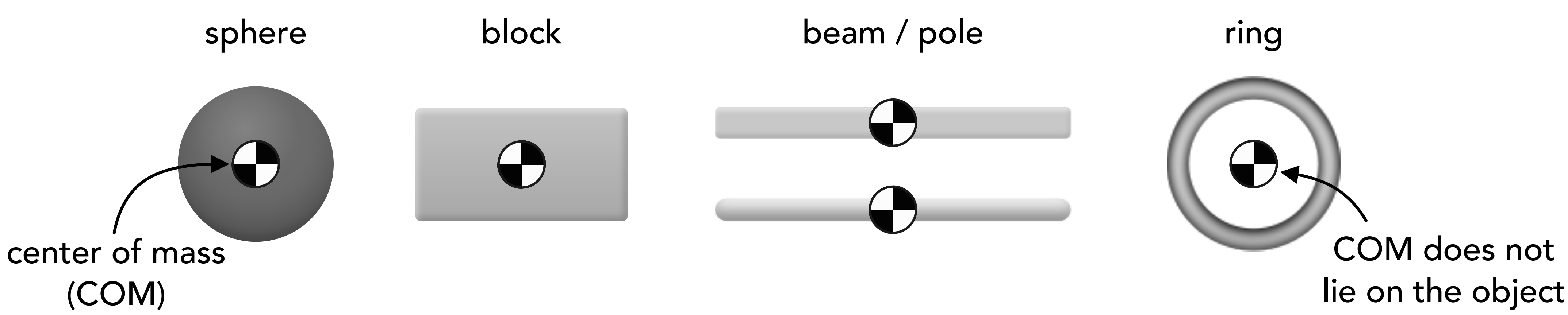
CENTER OF MASS

Center of Mass

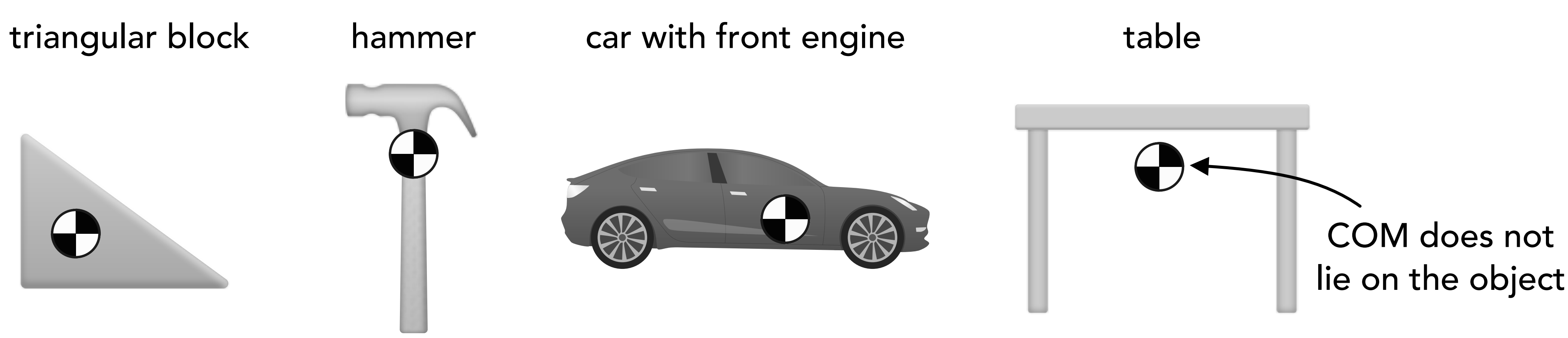
Variables		SI Unit
x	x position	m
y	y position	m
m	mass	kg

- The **center of mass (COM)** of an object or a system of masses is the mass-weighted average position of an object or a system of masses.
- If an object is symmetrical, the center of mass is located at the center of the object (the center of the width, length and height of the object).
- If an object is not symmetrical, the center of mass moves closer to areas of the object with more mass.
- The center of mass of an object or a sytem does not have to lie directly on the object and can be located in the empty space near the physical object(s).

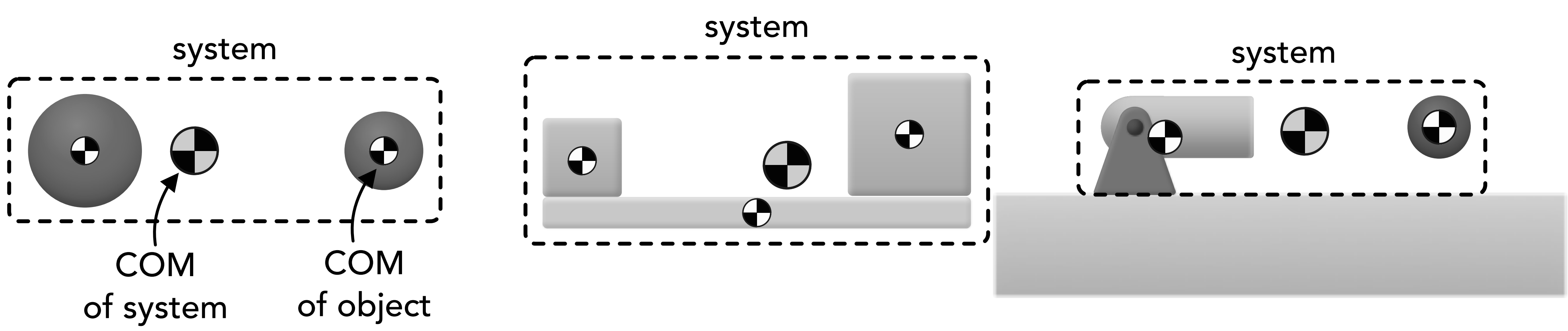
The center of mass of a symmetrical object is located at the center of the object



The center of mass of an asymmetrical object is located closer to areas with more mass

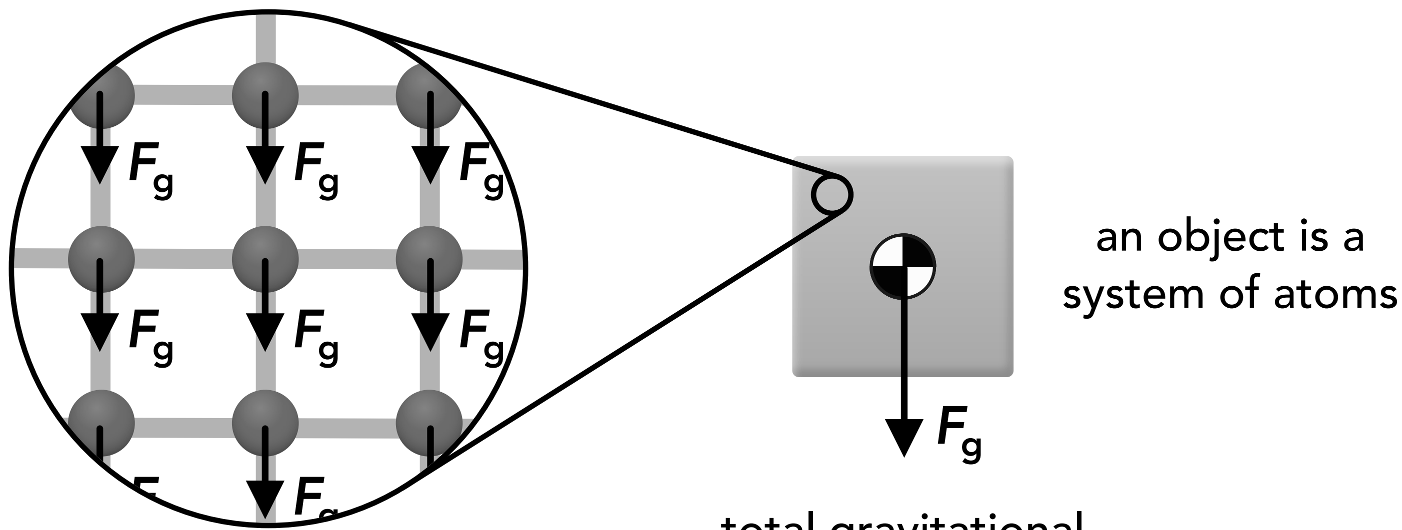


The center of mass of a system depends on the location and mass of each object

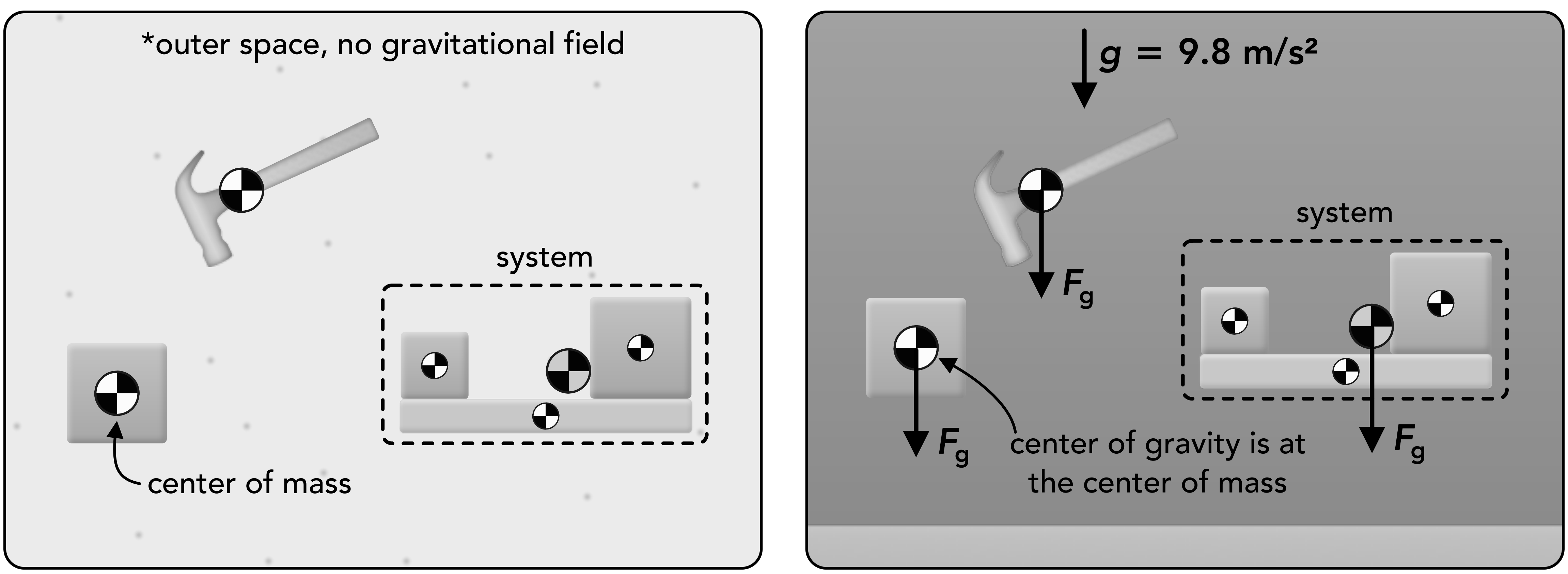


- Objects and systems also have a **center of gravity (COG)**. When an object or system is in a gravitational field, there is a gravitational force acting on every particle in the object or every mass in the system. However, we can treat all of those forces as **a single, total gravitational force acting on a single point - the center of gravity** (this is what we usually do).
- If we assume all of the object or system is within a uniform gravitational field (meaning the acceleration due to gravity g is the same everywhere across the object or system, which is a good approximation) then **the center of mass and the center of gravity are at the same point**.

We can treat the gravitational forces acting on each particle as a single, total gravitational force acting at the object's center of mass



When in a uniform gravitational field, the center of gravity is located at the center of mass



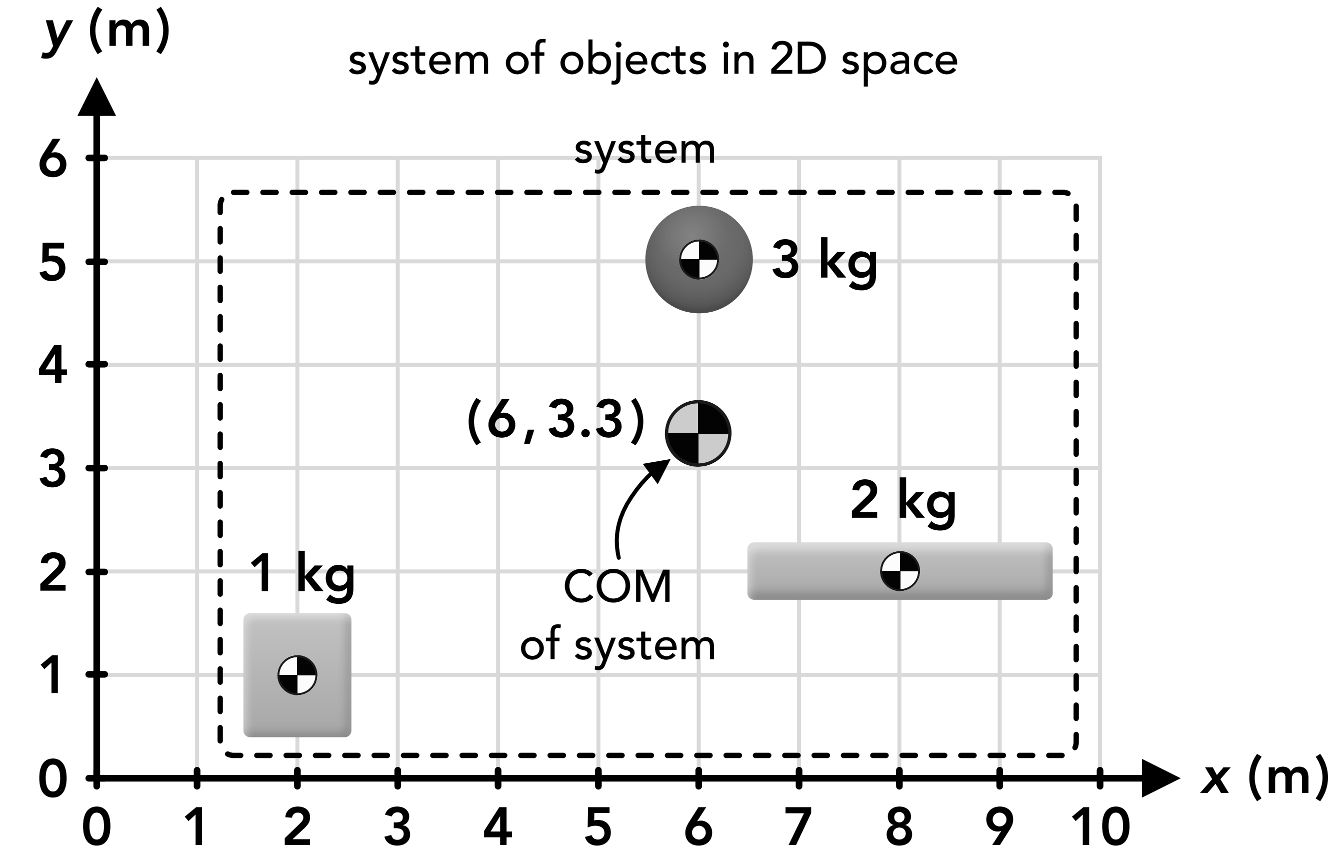
- The center of mass of a system or group of masses can be calculated using the equations below.
- If all of the masses lie on a single line only one coordinate (x) is needed. If the masses are distributed in 2D space then two coordinates (x, y) are needed. Although not covered here, (x, y, z) would be used in 3D space.
- **The coordinates used for each object are the coordinates of that object's own center of mass**, which will be in the middle of the object if the object is symmetrical.
- The origin of the coordinate system is arbitrary. If it's not given you can choose the origin (where x and y are zero).

x coordinate of center of mass of a system

$$x_{COM} = \frac{m_1 x_1 + m_2 x_2 + \ldots}{m_1 + m_2 + \ldots}$$

y coordinate of center of mass of a system

$$y_{COM} = \frac{m_1 y_1 + m_2 y_2 + \ldots}{m_1 + m_2 + \ldots}$$



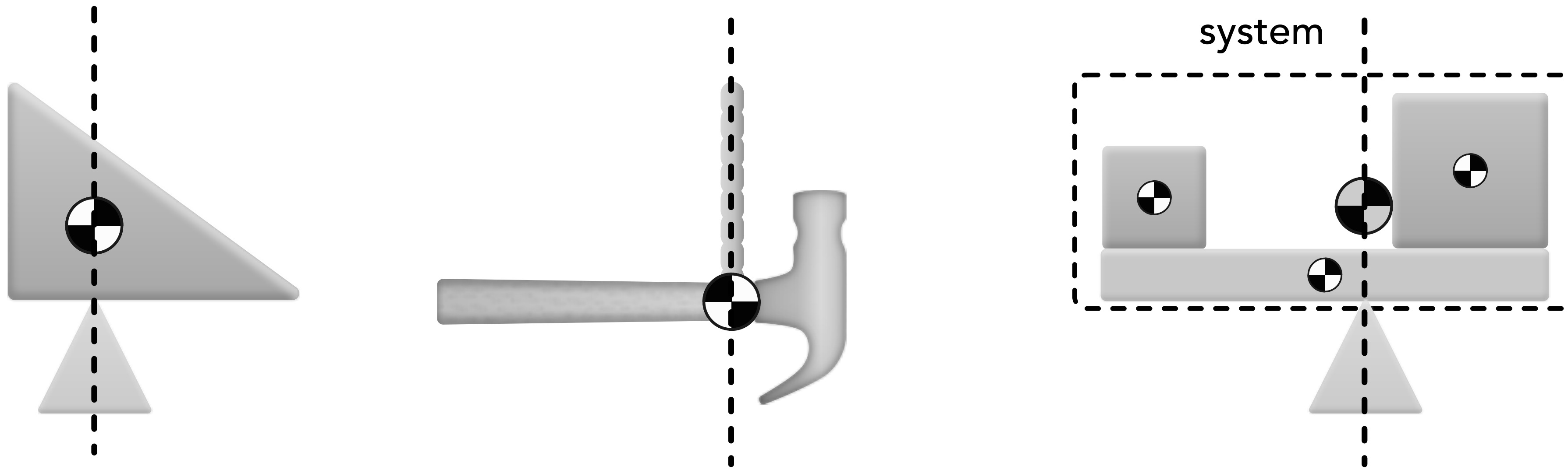
$$x_{COM} = \frac{(1\text{ kg})(2\text{ m}) + (2\text{ kg})(8\text{ m}) + (3\text{ kg})(6\text{ m})}{(1\text{ kg}) + (2\text{ kg}) + (3\text{ kg})} = 6\text{ m}$$

$$y_{COM} = \frac{(1\text{ kg})(1\text{ m}) + (2\text{ kg})(2\text{ m}) + (3\text{ kg})(5\text{ m})}{(1\text{ kg}) + (2\text{ kg}) + (3\text{ kg})} = 3.3\text{ m}$$

COM of system: $(6, 3.3)$

- An object or a system will **balance on its center of mass (center of gravity)** on a pivot point or when suspended from above. Since we can treat the individual gravitational forces acting on each object as a single gravitational force acting on the system's center of mass, that single gravitational force **will not generate a torque if it's directly above or below the point of rotation**.

Objects and systems will be balanced (in rotational equilibrium) when the pivot point or suspension point is in vertical alignment with the center of mass

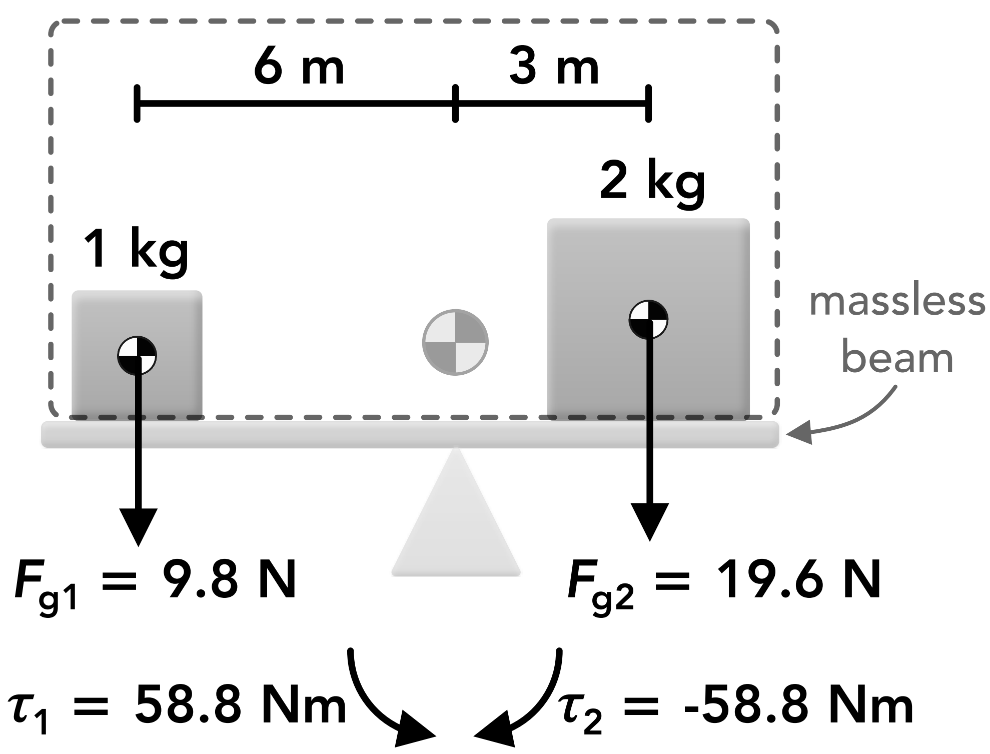


pivot is directly below the object's COM

object is suspended above its COM

pivot is directly below the system's COM

If we look at each object separately, the individual gravitational forces generate individual torques about the pivot point. If the pivot point is directly below the system's center of mass, the net torque will be zero so the system is balanced.



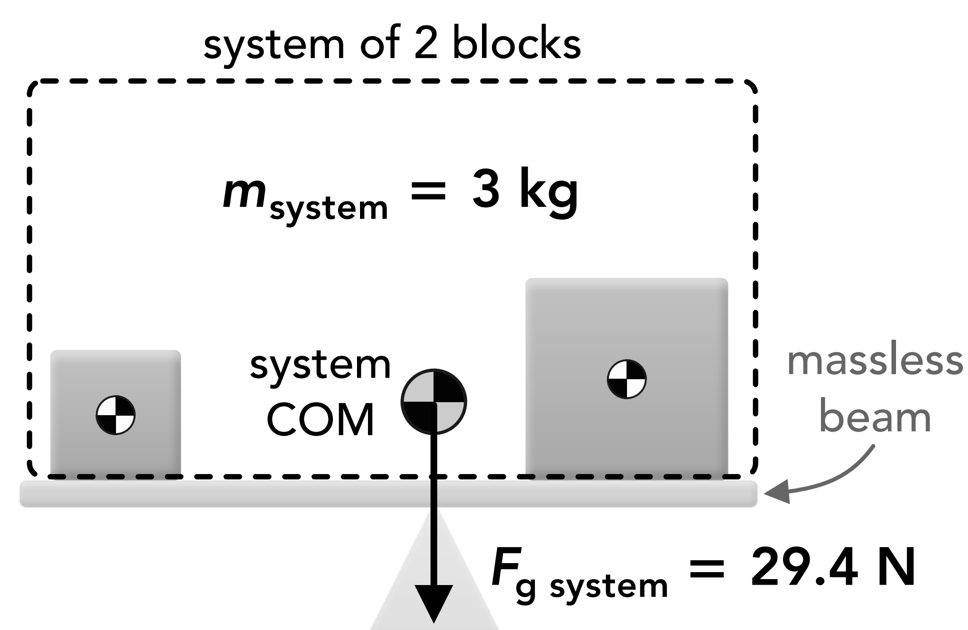
$$\tau_1 = r_1 F_1 = (6\text{ m})(9.8\text{ N})$$
$$\tau_2 = r_2 F_2 = -(3\text{ m})(19.6\text{ N})$$

$$\sum \tau = I\alpha$$
$$(58.8\text{ Nm}) - (58.8\text{ Nm}) = I\alpha$$
$$(0\text{ Nm}) = I(0\text{ rad/s}^2)$$

net torque is zero

angular acceleration is zero

If we treat the objects as a system, a single gravitational force acts at the system's center of mass. If the pivot point is directly below the system's center of mass, that single gravitational force points directly at the point of rotation and does not generate a torque, so the net torque on the system is zero and the system is balanced.



$$\tau_{system} = rF = (0\text{ m})(29.4\text{ N}) = 0\text{ Nm}$$

force is in line with point of rotation, $r = 0$

$$\sum \tau = I\alpha$$
$$(0\text{ Nm}) = I(0\text{ rad/s}^2)$$

net torque is zero

angular acceleration is zero

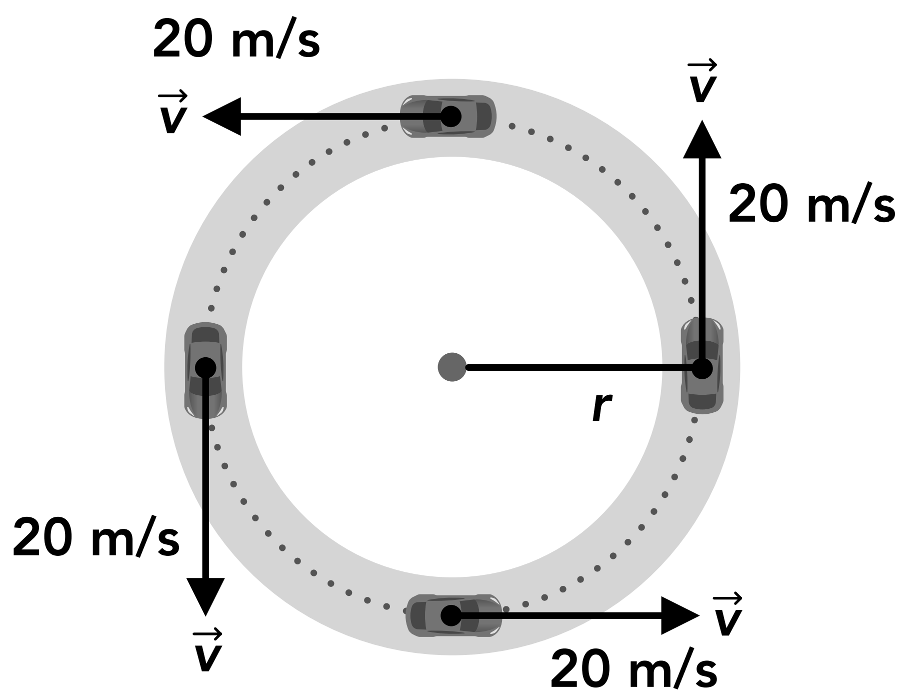
CENTRIPETAL ACCELERATION & FORCE

Uniform Circular Motion

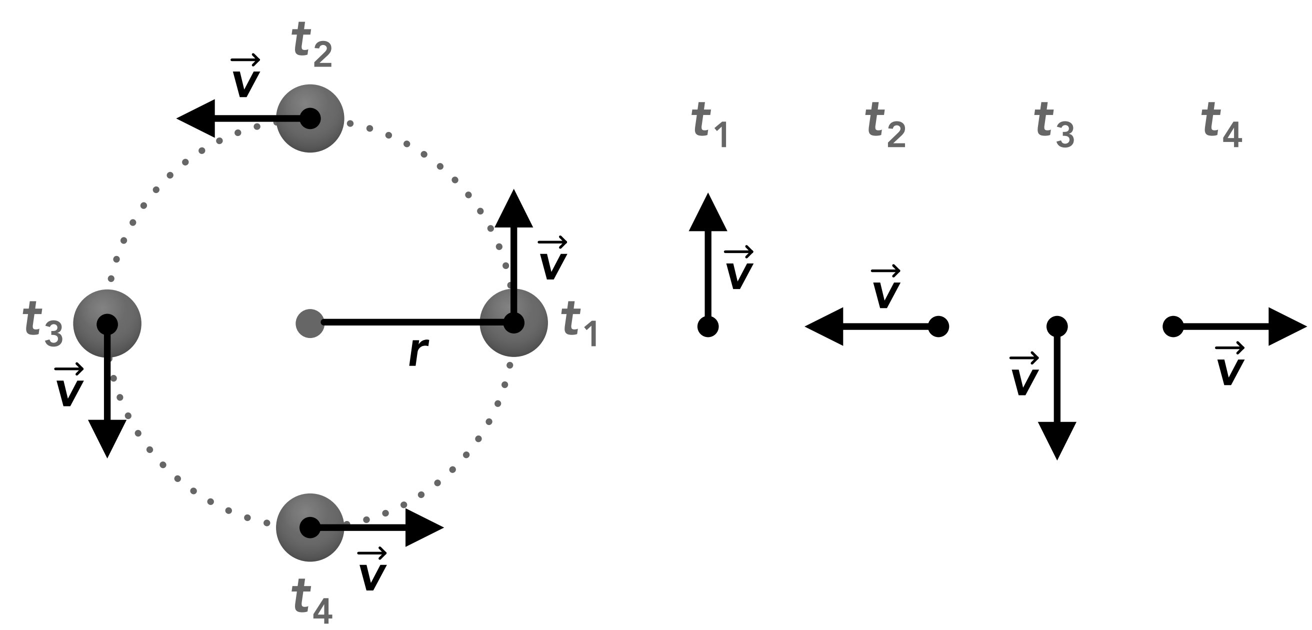
- **Uniform circular motion** is when an object travels in a circular path with a constant speed.
- The direction of the velocity is constantly changing, but the magnitude (speed) stays the same.
- We call the acceleration of the object the **centripetal acceleration** which is covered in another section.
- An object can be in uniform circular motion without completing a full circle.

Variables		SI Unit
v	velocity	$\frac{m}{s}$
r	radius	m
T	period	s
f	frequency	$Hz = \frac{cycles}{s}$
ω	angular velocity	$\frac{rad}{s}$

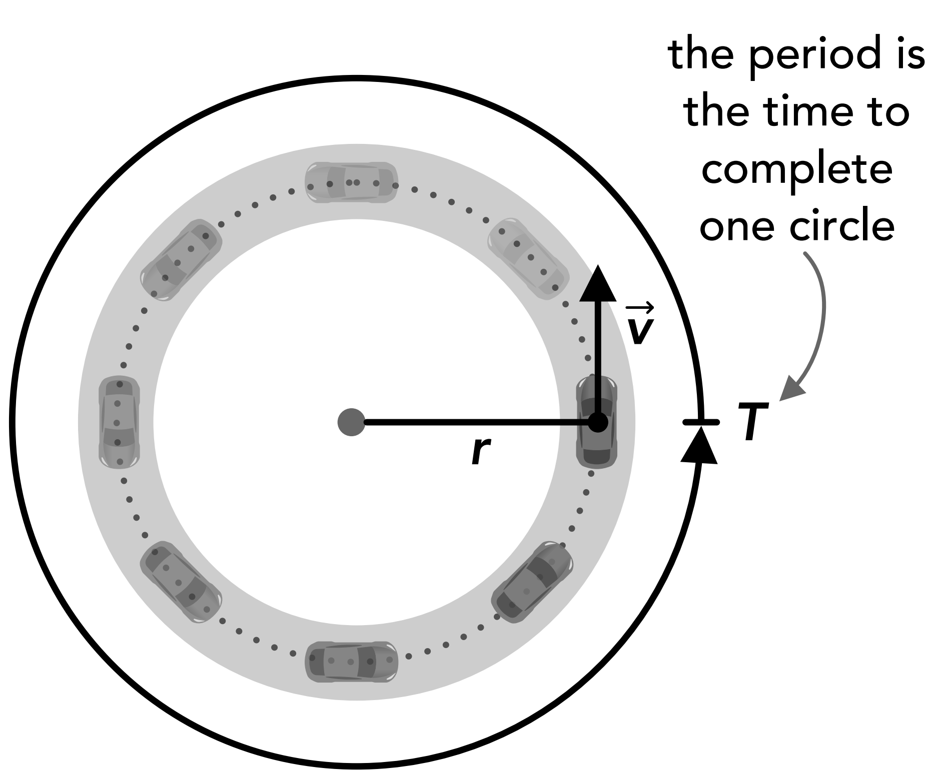
A car in uniform circular motion



The direction of the the velcocity is constantly changing but the magnitude (speed) stays the same



- Sometimes an object in uniform circular motion will repeat several revolutions over and over. In those cases we can describe the motion using period and frequency.
- **Period** (*T*) is the amount of time it takes to complete one circle or revolution. The unit of period is seconds (s).
- **Frequency** (*f*) is the inverse of the period **(1/*T*)** and is the number of circles traveled per second. The unit of frequency is Hertz (Hz) which is cycles/second or 1/s (the numerator has no unit, it's just “something” per second like circles/second, revolutions/second, etc).



Frequency

$$f = \frac{1}{T}$$

f : frequency (Hz, cycles/s)
T : period (s)

an object travels one circumference in one period, so the velocity is related to the period and frequency

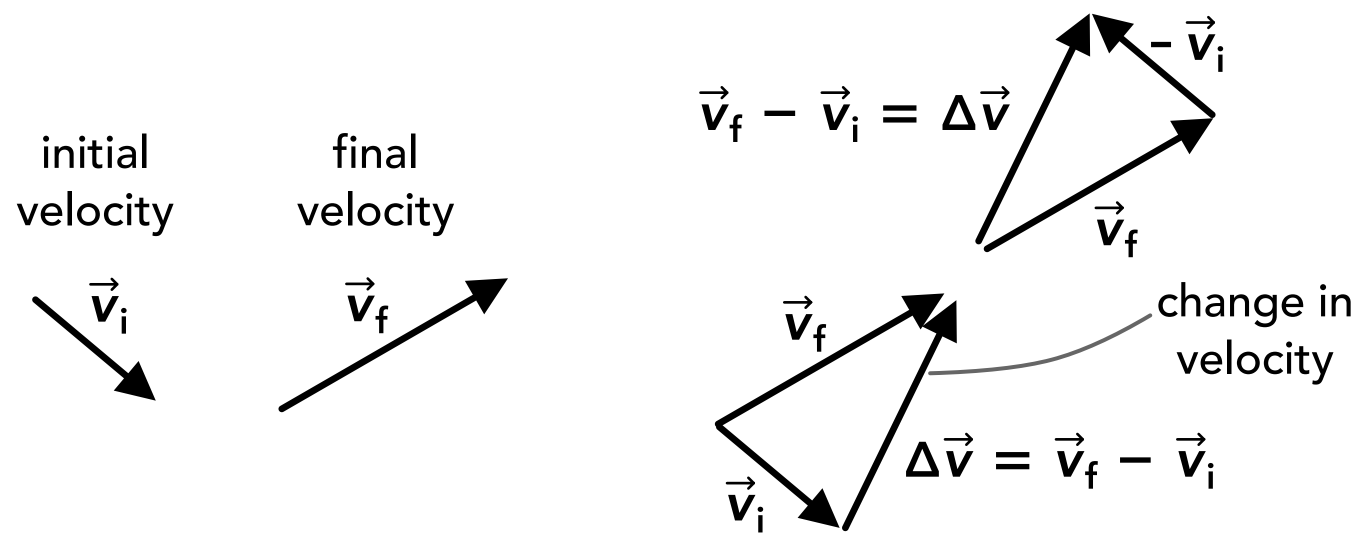
$$v = \frac{2\pi r}{T}$$

$$v = 2\pi r f$$

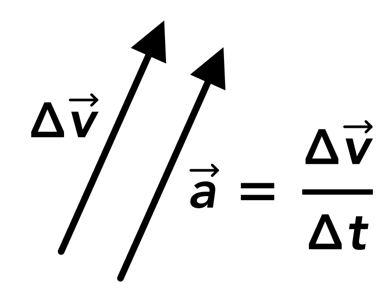
- To understand centripetal acceleration and its related motion, it will help to review velocity and acceleration vectors.
- Acceleration is the change in velocity divided by a period of time.
- Acceleration and velocity are both vector quantities which have a magnitude (value) and a direction, so the acceleration can change the **magnitude** of the velocity (the speed) or the **direction** of the velocity.
- Remember that vectors can be added or subtracted using the tip-to-tail method.

Variables		SI Unit
a_c	centripetal acceleration	$\frac{m}{s^2}$
a	acceleration	$\frac{m}{s^2}$
v	velocity	$\frac{m}{s}$
r	radius	m
t	time	s

The vector representing the change in velocity is the final velocity vector minus the initial velocity vector, which can be found using the tip-to-tail method in one of two ways

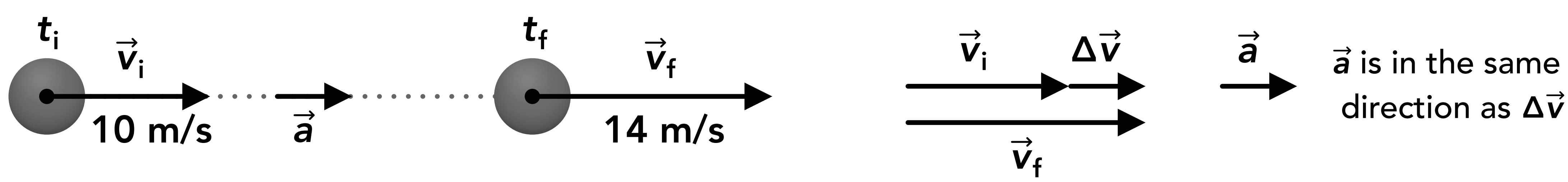


The acceleration vector points in the same direction as the change in velocity vector

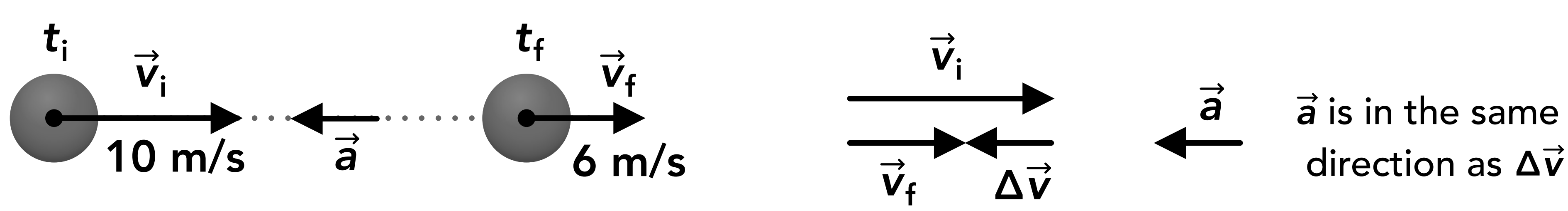


- Let's look at two examples below of an object moving in a straight line. The velocity vectors at two timepoints are shown. What is the direction of the acceleration vector that would cause that change in the velocity vector?

If $\vec{a}$ is parallel to $\vec{v}$ and points in the same direction, the magnitude (speed) of $\vec{v}$ increases and the direction of $\vec{v}$ doesn't change

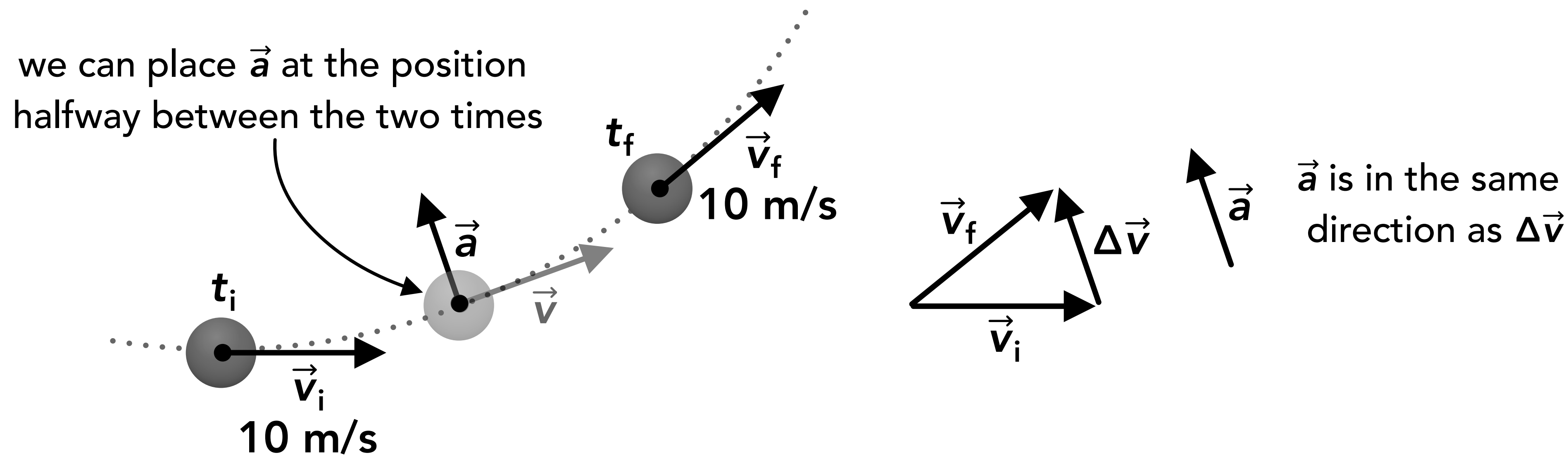


If $\vec{a}$ is parallel to $\vec{v}$ and points in the opposite direction, the magnitude (speed) of $\vec{v}$ decreases and the direction of $\vec{v}$ doesn't change

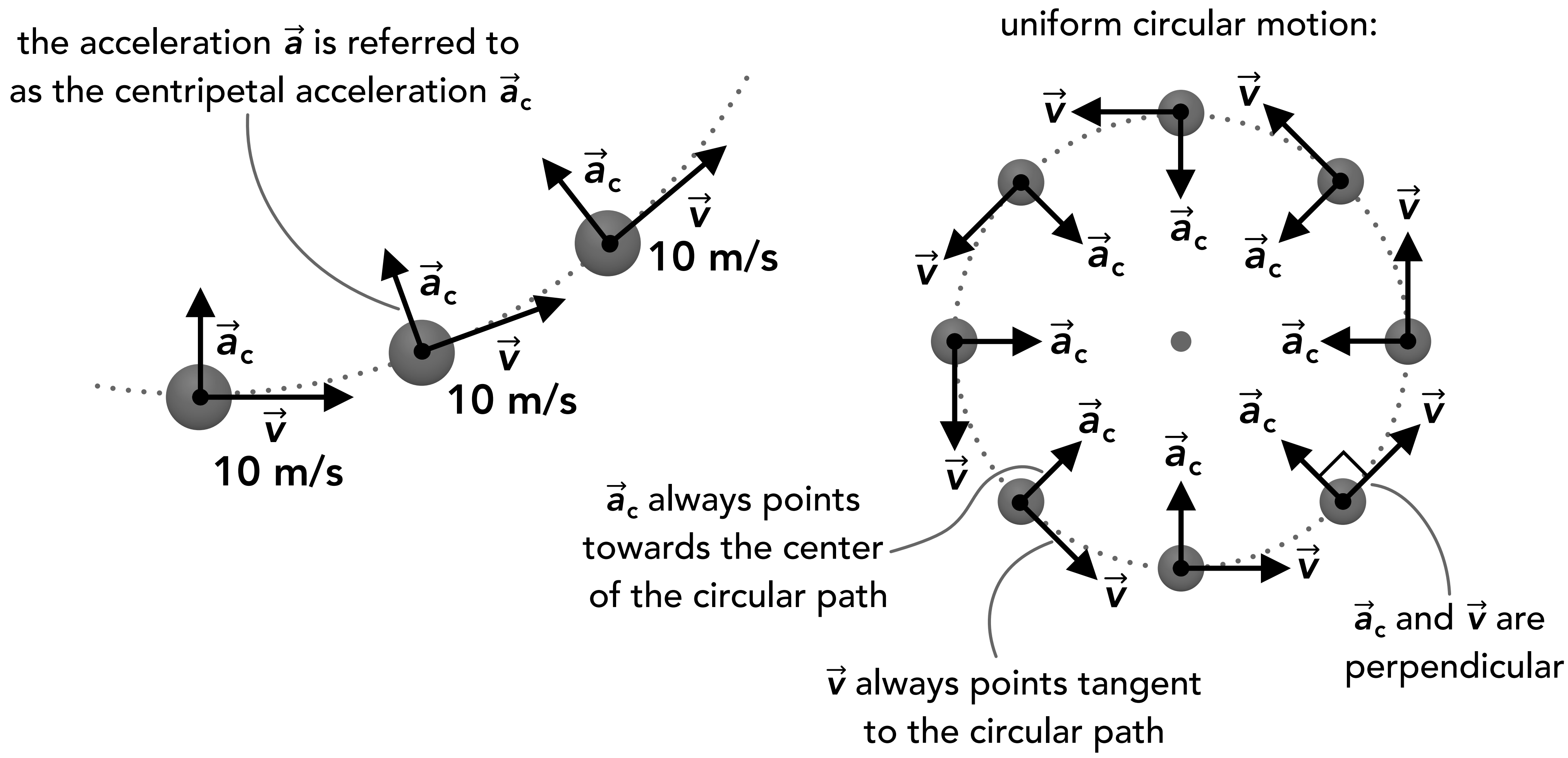


- Now let's look at an object following a curved circular path where the magnitude of the velocity vector (speed) doesn't change. Remember, acceleration can change the direction of the velocity vector, not just the magnitude. What is the direction of the acceleration vector that would cause this change in the velocity vector?

If $\vec{a}$ is not parallel to $\vec{v}$, it causes $\vec{v}$ to change direction and follow a curved path



If $\vec{a}$ is always perpendicular to $\vec{v}$, the direction of $\vec{v}$ changes and the object follows a circular path. The magnitude of $\vec{v}$ doesn't change because no component of $\vec{a}$ points in the same direction as $\vec{v}$



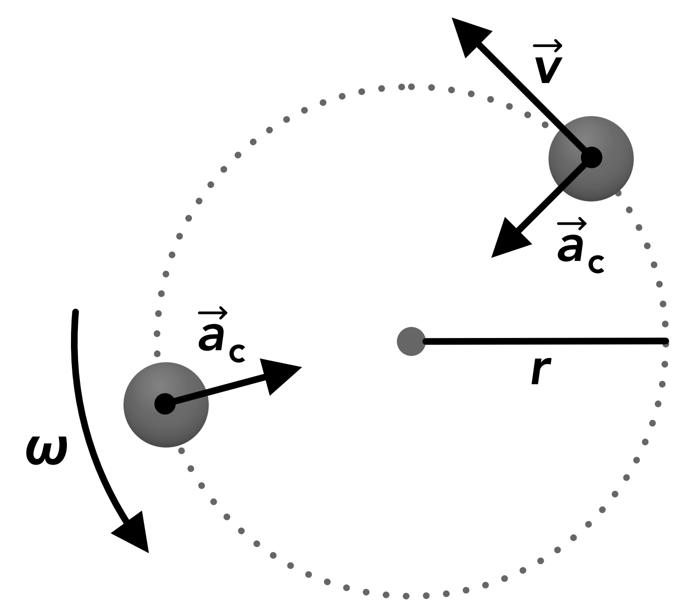
- If the acceleration vector is continuously perpendicular to the velocity vector, the velocity will change direction and end up following a circular path without changing speed. This is uniform circular motion.

- We call the acceleration which results in circular motion the **centripetal acceleration**. "Centripetal" means **acting towards the center** which is the direction of the centripetal acceleration.
- Conceptually, it's important to note that there is nothing special about a centripetal acceleration and this is not some new "type" of acceleration. If an object is moving along a circular path there must be an acceleration that is perpendicular to its velocity and points towards the center of the circle.
- In a physical scenario, the cause of a centripetal acceleration is a **centripetal force** which is any net force that acts towards the center of the circular path. This could be the tension force of a rope, the normal force of a circular track, or any other type of force. Centripetal force is covered in another section.
- The magnitude of the centripetal acceleration is given by the equation below. The greater the speed of the object and the smaller the radius of the circle, the greater the centripetal acceleration that is required to keep the object moving in a circle at that speed.

Centripetal acceleration

$$\vec{a}_c = \frac{v^2}{r} \quad \textbf{(towards center of circle)}$$

v : tangential speed (m/s)
r : radius of circular path (m)



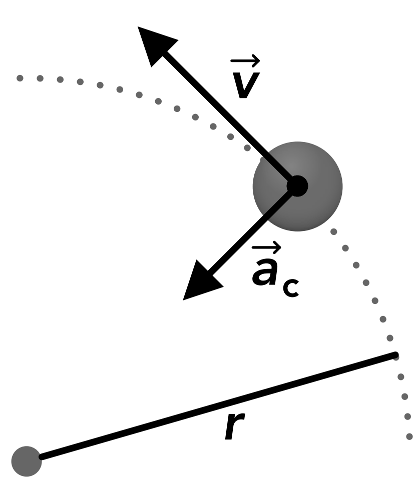
Centripetal acceleration
(other variables substituted for speed)

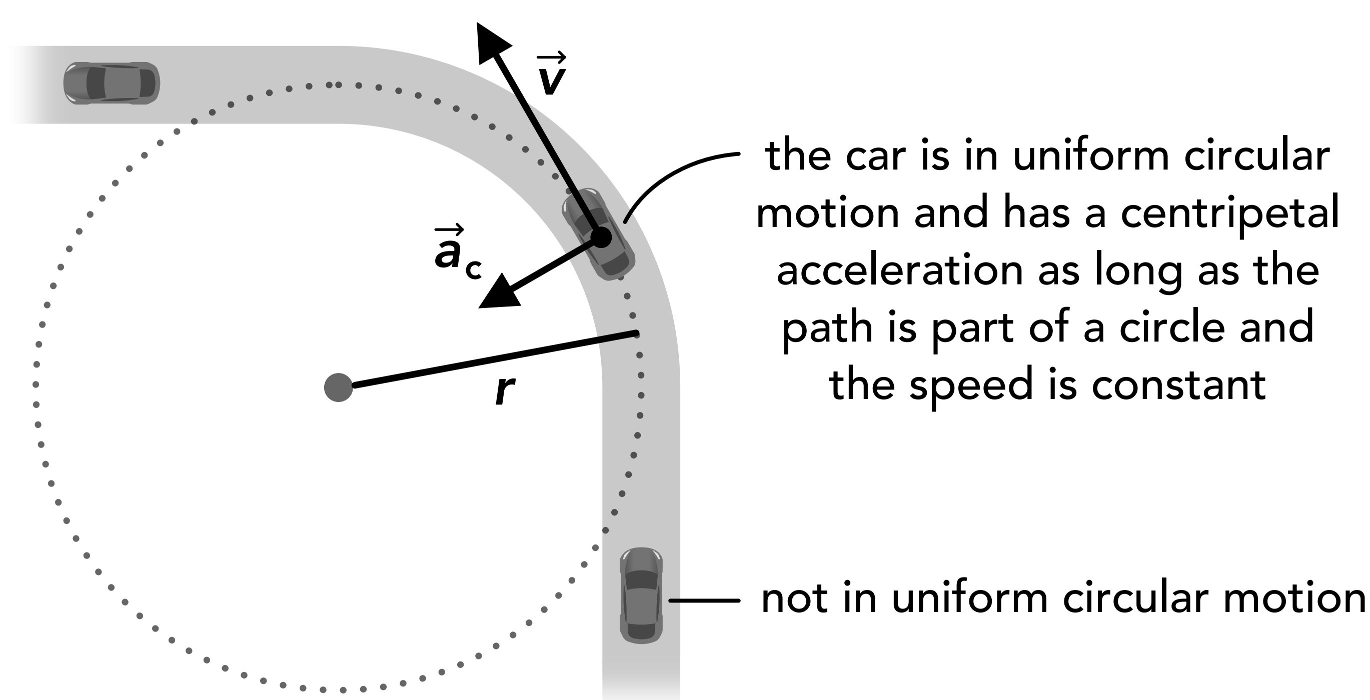
$$a_c = \frac{v^2}{r} = \omega^2 r = (2\pi f)^2 r = \left(\frac{2\pi}{T}\right)^2 r$$

ω : angular speed (rad/s)
f : frequency (Hz = rev/s)
T : period (s)

$$f = \frac{1}{T}$$

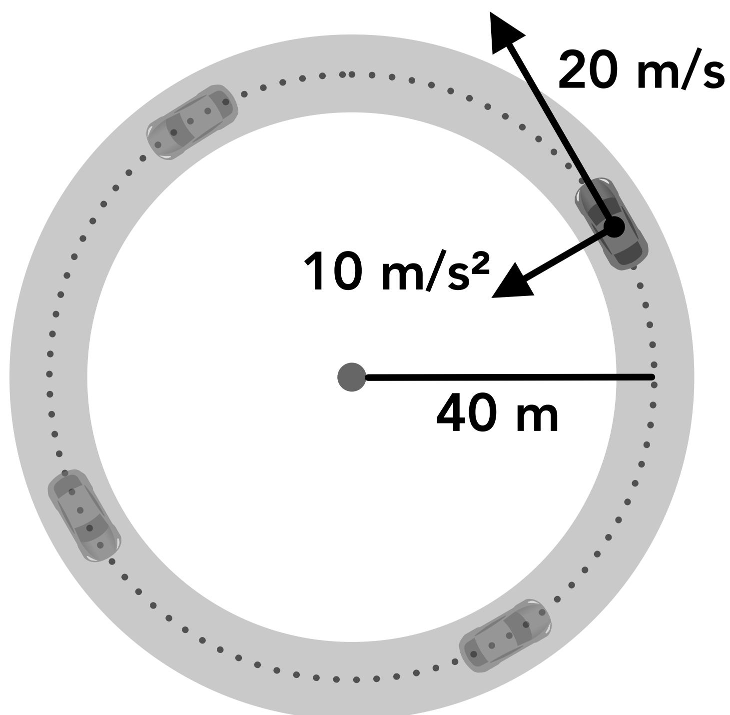
- Even if an object only travels along a segment of a circular path (instead of a full circle) we still consider the object to be in uniform circular motion and the acceleration is still centripetal acceleration.





Calculating centripetal acceleration using different variables for speed

A car is driving around a circle at a constant speed



$$a_c = \frac{v^2}{r} = \frac{(20\text{ m/s})^2}{40\text{ m}} = 10\text{ m/s}^2$$

$$\omega = \frac{v}{r} = \frac{20\text{ m/s}}{40\text{ m}} = 0.5\text{ rad/s}$$

$$a_c = \omega^2 r = 10\text{ m/s}^2$$

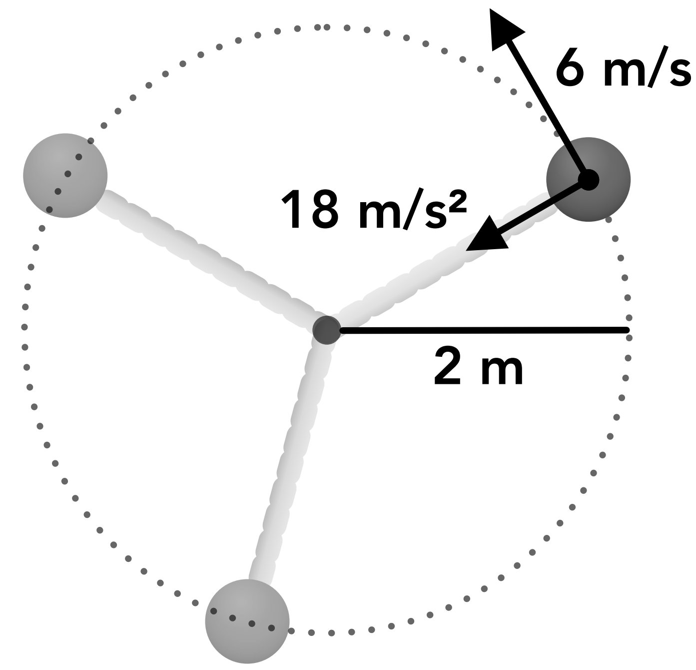
$$T = \frac{2\pi r}{v} = \frac{2\pi(40\text{ m})}{20\text{ m/s}} = 12.57\text{ s}$$

$$a_c = \left(\frac{2\pi}{T}\right)^2 r = 10\text{ m/s}^2$$

$$f = \frac{1}{T} = \frac{1}{12.57\text{ s}} = 0.0796\text{ Hz}$$

$$a_c = (2\pi f)^2 r = 10\text{ m/s}^2$$

An object is tied to a rope and swings around in a circle



$$a_c = \frac{v^2}{r} = \frac{(6\text{ m/s})^2}{2\text{ m}} = 18\text{ m/s}^2$$

$$\omega = \frac{v}{r} = \frac{6\text{ m/s}}{2\text{ m}} = 3\text{ rad/s}$$

$$a_c = \omega^2 r = 18\text{ m/s}^2$$

$$T = \frac{C}{v} = \frac{2\pi(2\text{ m})}{6\text{ m/s}} = 2.1\text{ s}$$

$$a_c = \left(\frac{2\pi}{T}\right)^2 r = 18\text{ m/s}^2$$

$$f = \frac{1}{T} = \frac{1}{2.1\text{ s}} = 0.48\text{ Hz}$$

$$a_c = (2\pi f)^2 r = 18\text{ m/s}^2$$

Centripetal Force

- Remember that if an object is traveling in circular motion then there must be a **centripetal acceleration** (an acceleration vector that points towards the center of the circular path). According to Newton's 2nd law of motion (**$\vec{F}_{net} = m\vec{a}$**) there must be a net force acting on the object in the direction of that acceleration.
- A **centripetal force** is what we call that **net force** acting in the radial direction (towards the center of the circle) which is causing a centripetal acceleration, which results in the circular motion.

Variables		SI Unit
F_c	centripetal force	N
a_c	centripetal acceleration	$\frac{m}{s^2}$
v	velocity	$\frac{m}{s}$
r	radius	m

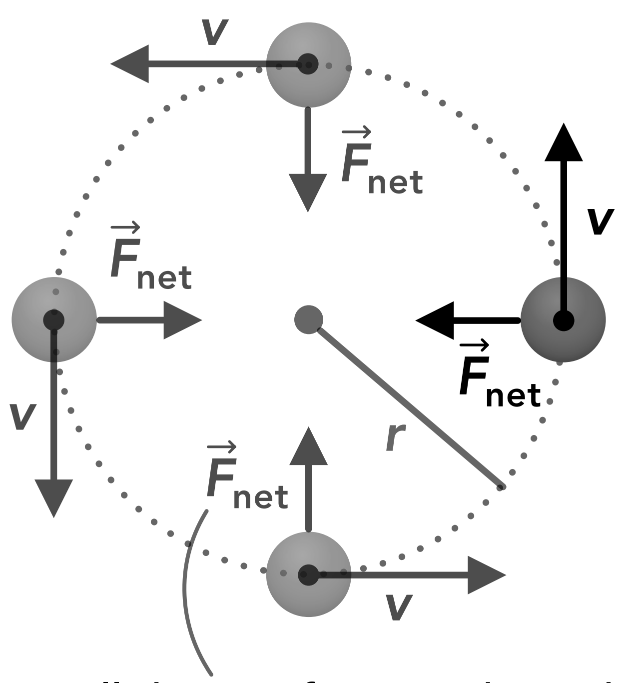
Centripetal force

$$\vec{F}_c = m\frac{v^2}{r} \quad \text{(towards center of circle)}$$

Centripetal acceleration

$$\vec{a}_c = \frac{v^2}{r} \quad \text{(towards center of circle)}$$

$\vec{F}_c = \vec{F}_{net}$

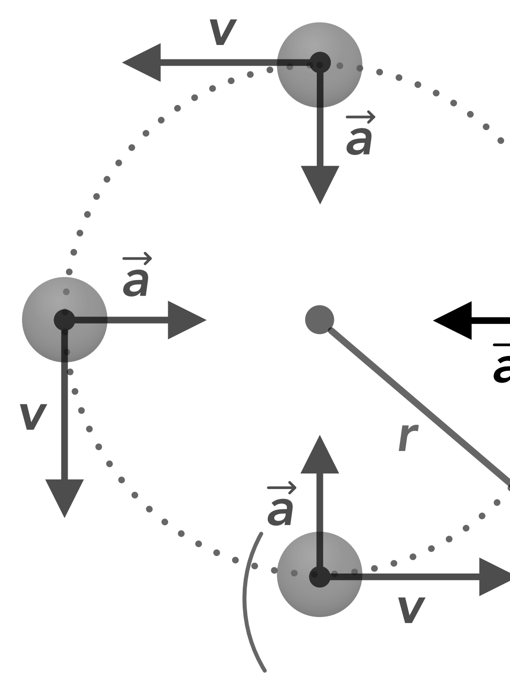


we call this net force in the radial direction the "centripetal force" $\vec{F}_c$

$\vec{F}_c = m\vec{a}_c$

a centripetal force causes a centripetal acceleration

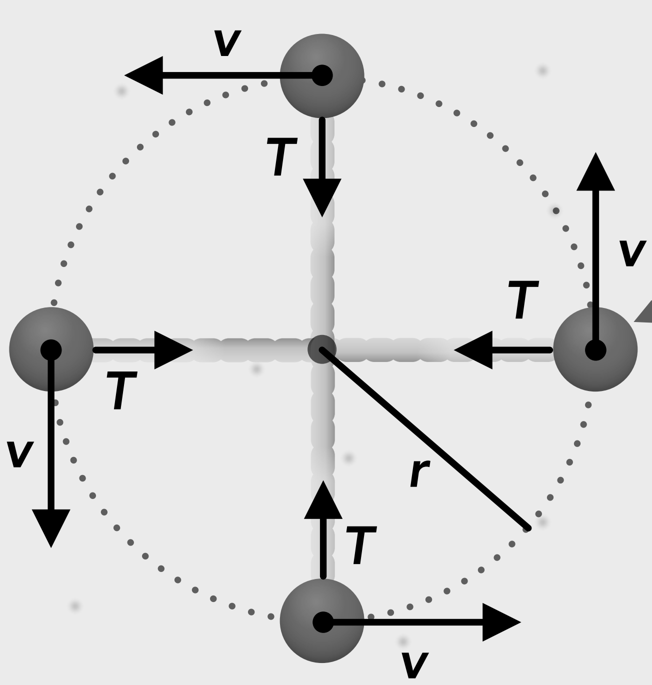
$\vec{a}_c = \vec{a}$



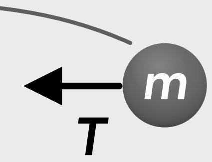
we call this acceleration in the radial direction the "centripetal acceleration" $\vec{a}_c$

- Centripetal force refers to the **net force** acting in the radial direction (towards the center of the circle) which is causing the object to move in circular motion.

A ball attached to a rope swings in uniform circular motion in space (assuming no gravity). The tension force on the ball from the rope is acting as the centripetal force, keeping the ball in circular motion.



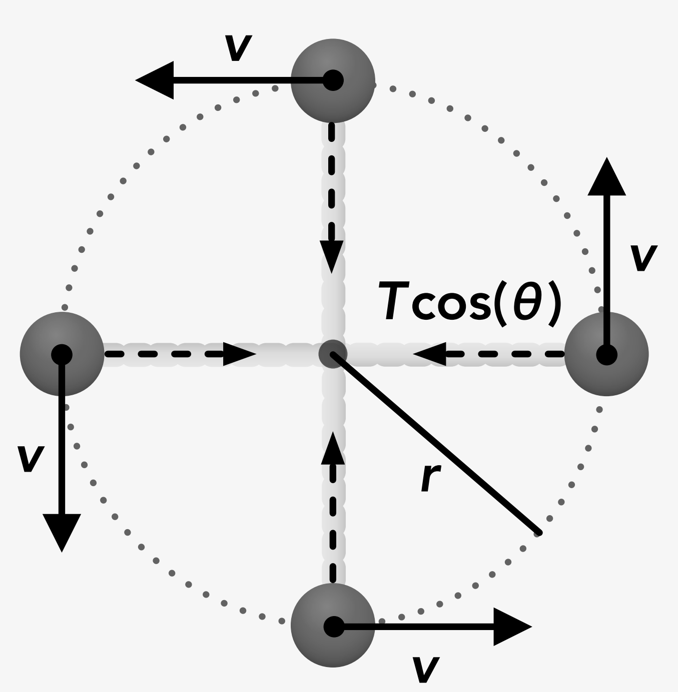
F_c is the same at each point
T is the same at each point



$$T = F_c = m\frac{v^2}{r}$$

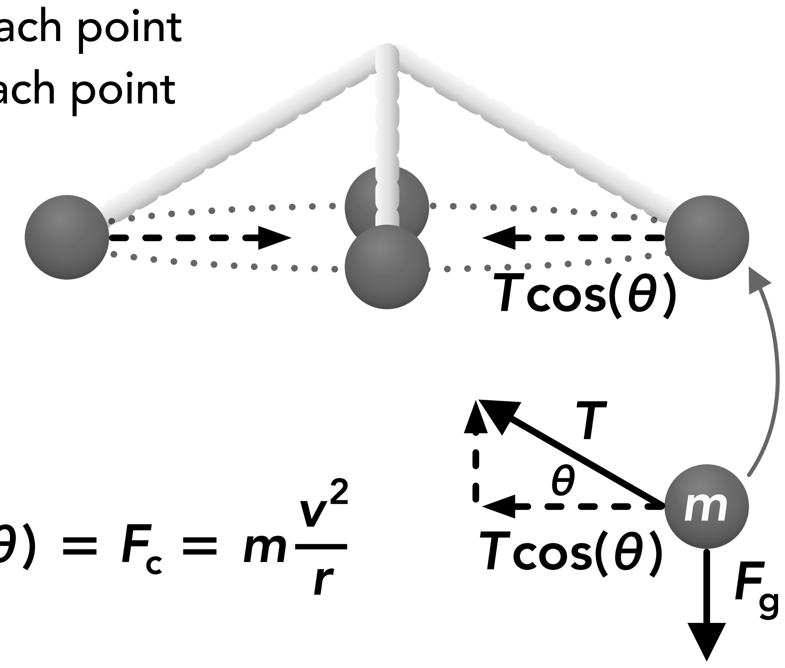
A ball is attached to a rope and swings in uniform circular motion. The circle is horizontal, parallel to the ground, but gravity causes the ball to pull the rope down at an angle. The horizontal component of the tension force, which always points towards the center of the circle, is acting as the centripetal force (not the entire tension force).

top view



side view

F_c is the same at each point
T is the same at each point

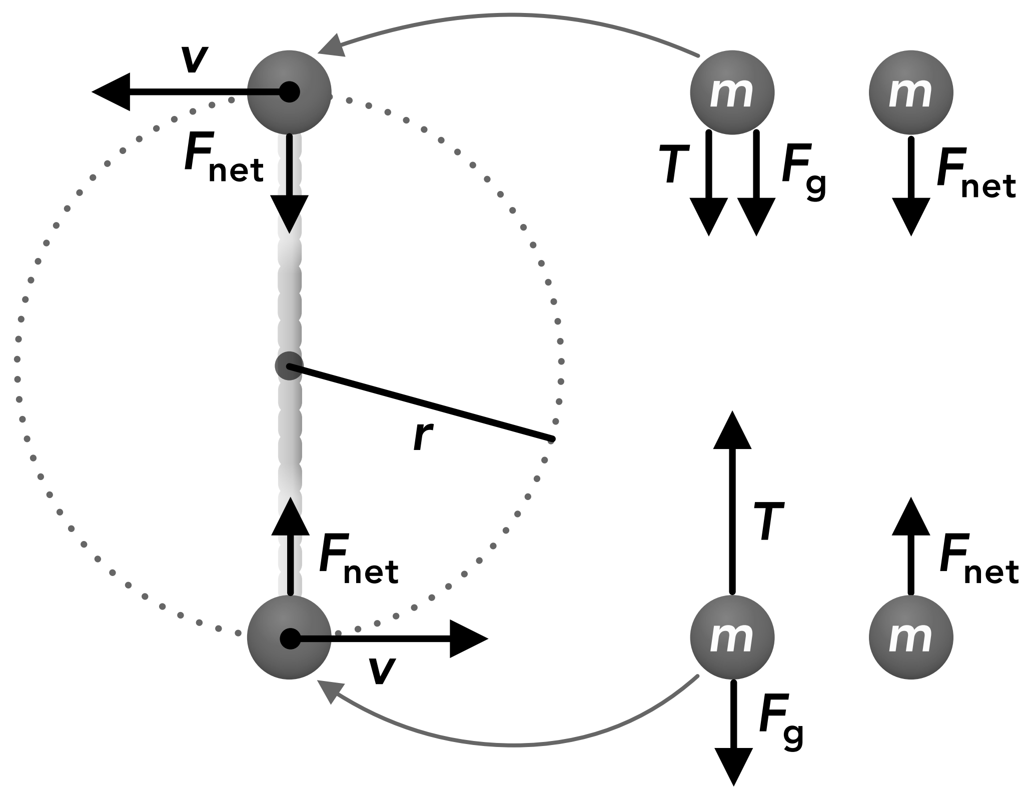


$$T\cos(\theta) = F_c = m\frac{v^2}{r}$$

A ball is attached to a rope and swings in a vertical circle. At each point there is a tension force and a gravitational force acting on the ball. Because the ball is in circular motion, the net force acting on the ball in the radial direction at any time is equal to the centripetal force.

F_c is the same at each point

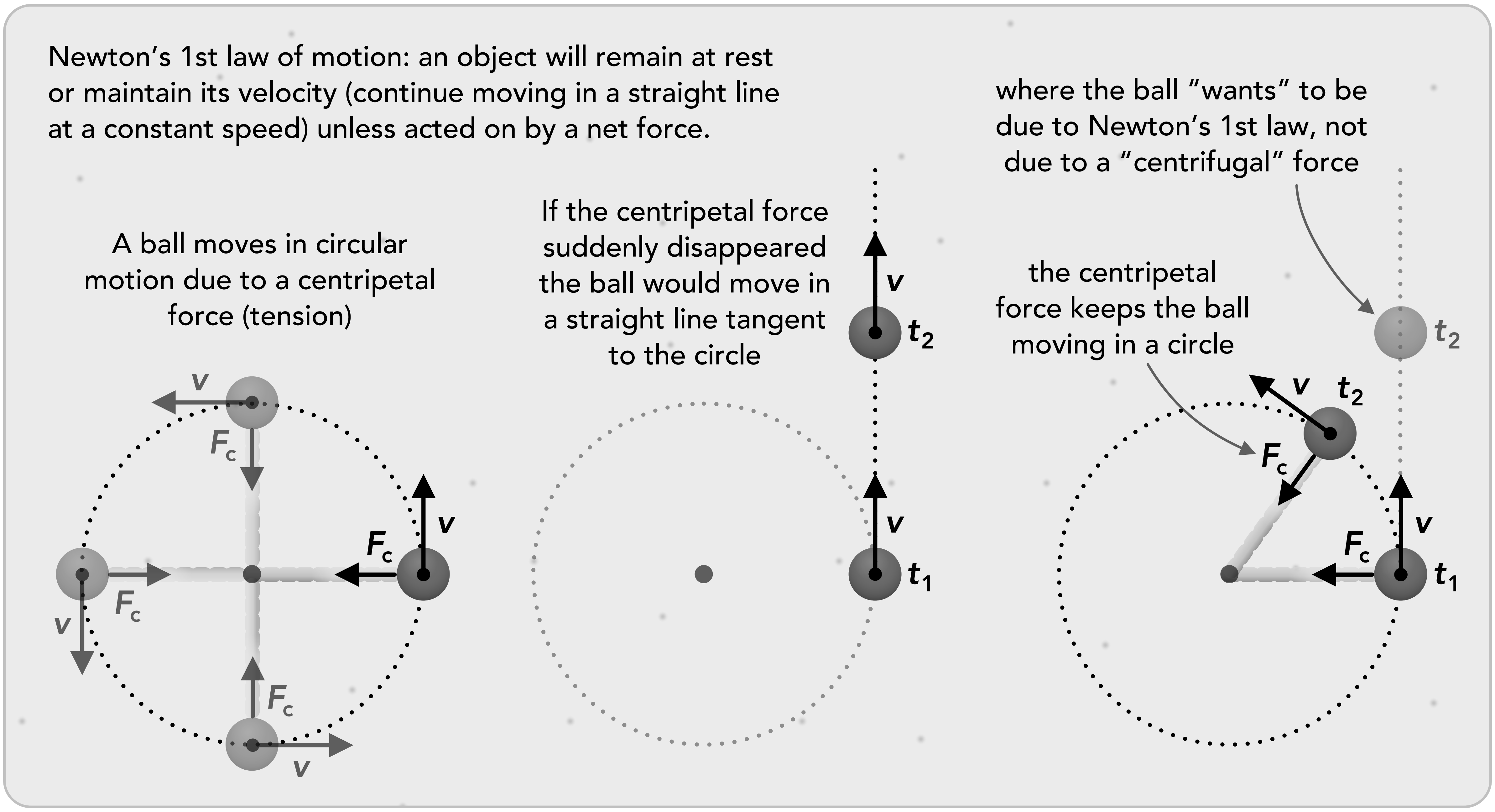
F_g is the same at each point
T changes around the circle



$$F_{net} = F_c$$
$$T + F_g = m\frac{v^2}{r}$$

$$F_{net} = F_c$$
$$T - F_g = m\frac{v^2}{r}$$

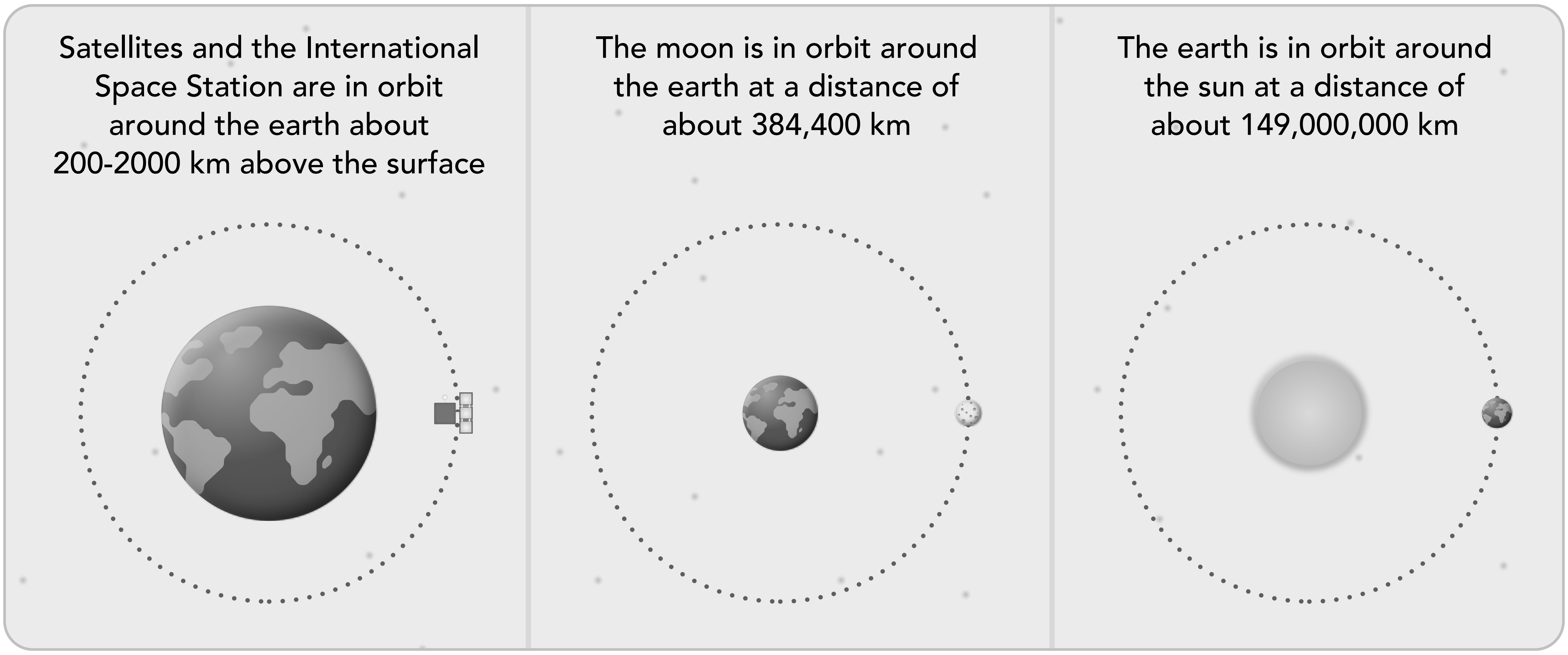
- A common confusion when working with circular motion is the concept of "centrifugal force".
- **Centrifugal force** is a "fictitious force" which is a force that does not actually exist. When the circular motion of an object is viewed in the rotating reference frame (in which the object appears to be stationary) it may appear that a force is pushing or pulling the object **away from the center of the circle**, which we call a centrifugal force. This imaginary force only arises because of the rotating reference frame.
- The only real force acting on the object is an **inwards centripetal force**, not an outwards centrifugal force.
- According to Newton's 1st law of motion an object will maintain its velocity (continue moving in a straight line at a constant speed) unless acted on by a net force. In circular motion we call that net force the centripetal force. If that centripetal force suddenly disappeared the object would travel in a straight line tangent to the circle.



ORBITAL MOTION

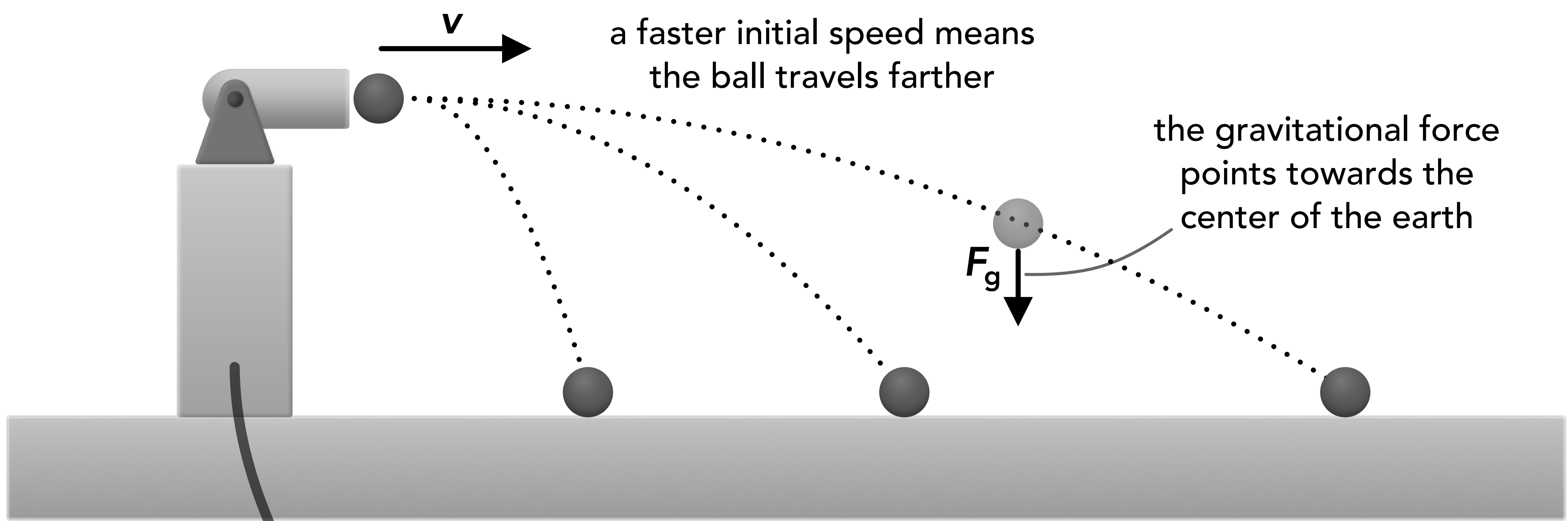
Orbital Motion

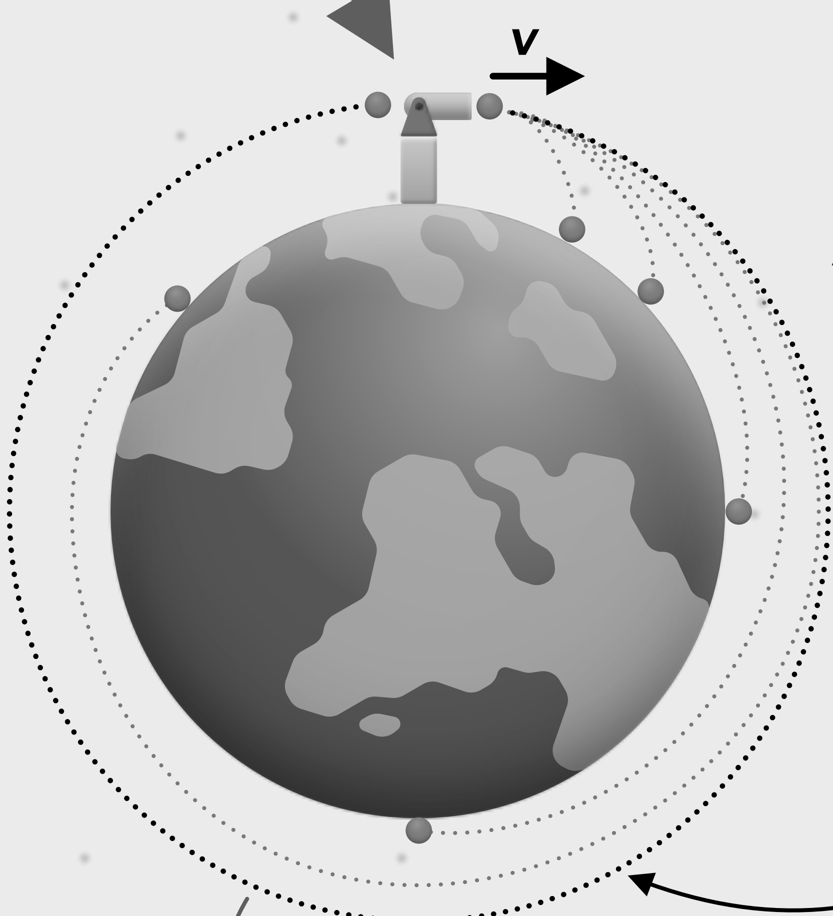
- **Orbital motion** is when an object follows a circular or elliptical path (an "orbit") around another object, where the only force acting on the object is gravity.
- Technically the two objects are both in orbit around a common point called a "barycenter" but we'll focus on a simplified version for now, where one of the objects has much more mass than the other and the barycenter is approximately at the center of the larger object.



- An object in orbital motion around a planet is actually in **projectile motion** or **free fall**.
- The only force acting on the object is the **gravitational force** from the planet which always points towards the center of the planet. It may seem like some force is required to keep the object moving, but we know from Newton's 1st law of motion that an object in motion will continue moving on its own unless a net force is applied to stop it from moving.
- We're only going to focus on the continuous orbital motion itself, not the cause of the initial velocity that started the orbital motion or changes to the orbital motion.

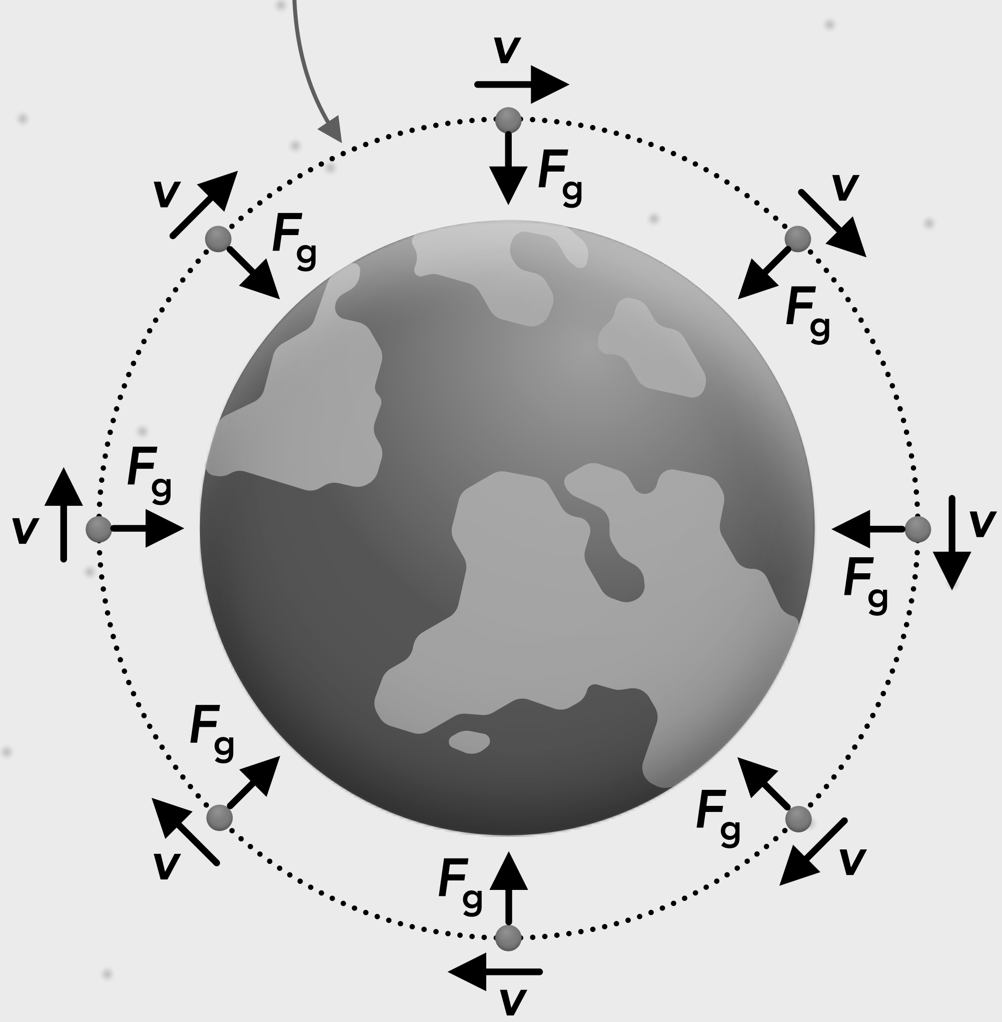
In projectile motion (or free fall) the only force acting on the object is the gravitational force (ignoring air resistance)





As the speed and the range increase to a larger scale, the curvature of the earth becomes relevant and the ground begins to “fall away” below the path of the ball.

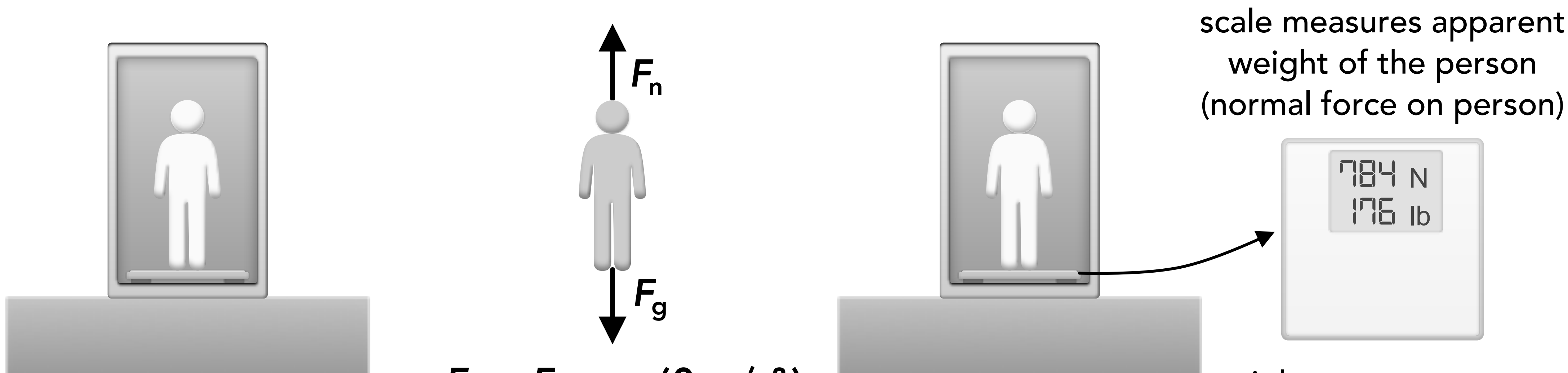
If the initial speed is fast enough (~ 7,900 m/s) the ball will never hit the ground, the projectile motion completely circles the earth and the ball ends up where it started. At this velocity the ball is in orbital motion around the earth.



Once the ball is in orbit its speed is constant and the only force acting on it is the gravitational force (assuming no air resistance). The gravitational force is perpendicular to the velocity and changes its direction, but no force is required for the ball to keep moving.

- Since an object in orbital motion is in projectile motion or free fall, and the only force acting on it is gravity, **the object has no apparent weight** and it experiences "weightlessness".
- Gravity is still acting on an object in orbital motion. For example, the International Space Station (ISS) is in orbit around the earth at an altitude of about 400 km. At that distance from the earth, the acceleration due to gravity g is still about 8.7 m/s² or 89% of the acceleration due to gravity at the surface of the earth.
- An astronaut in the ISS orbiting the earth will feel like they're falling, because they are. It's the same thing as being in an elevator that is in free fall where you appear to be weightless. The difference is that an object in orbital motion is also moving sideways very fast, and the direction of the gravitational force keeps rotating.

Your apparent weight is the normal force supporting you from below, which is equal to the actual weight if the net force and acceleration are zero



person:

$m = 80$ kg

$F_g = mg = 784$ N

$F_n - F_g = m(0 \text{ m/s}^2)$

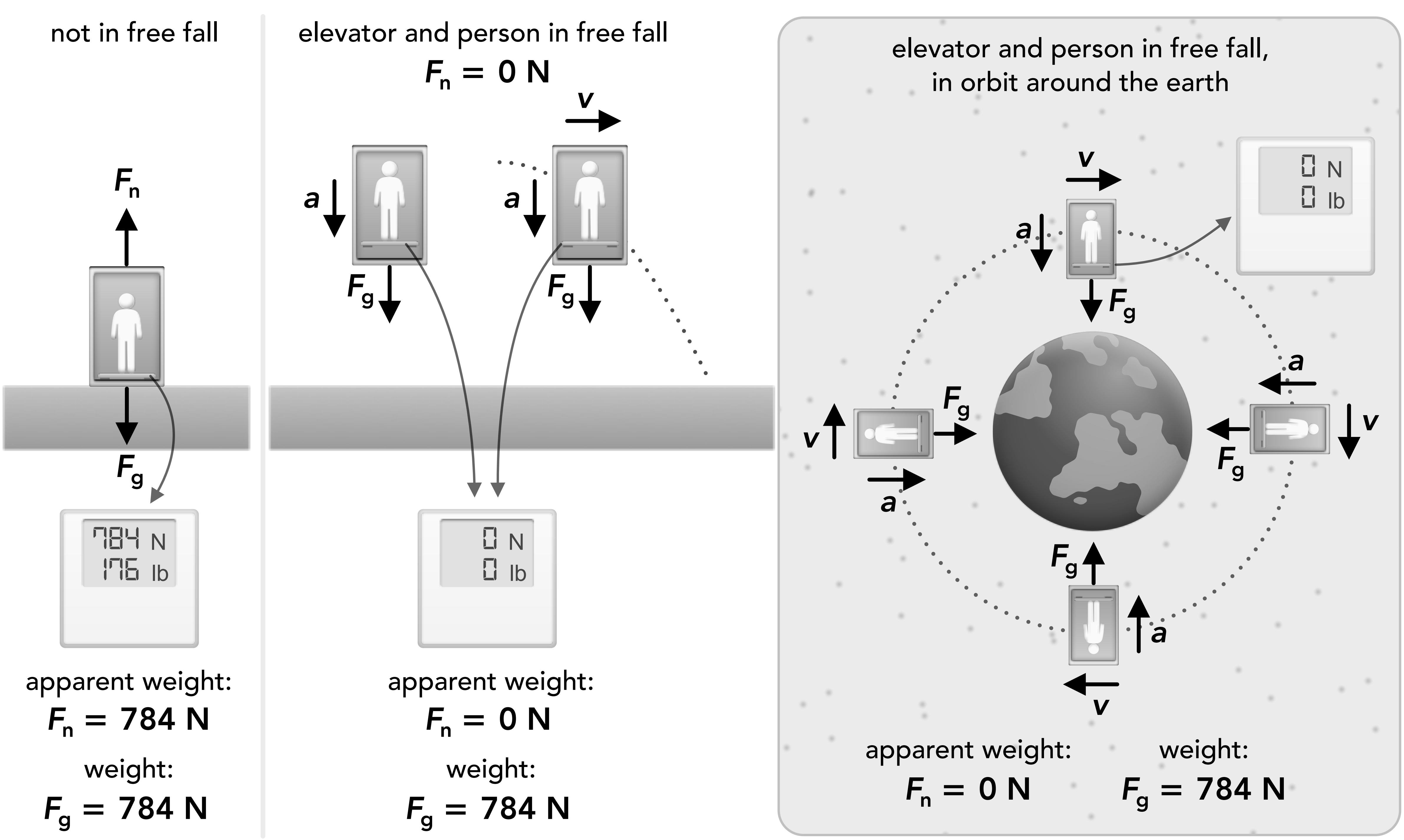
$F_n = F_g$

$F_n = 784$ N

$F_g = 784$ N ← weight

$F_n = 784$ N ← apparent weight

When you're in free fall and the only force acting on you is the gravitational force, you have no apparent weight and you experience "weightlessness", even if you have some velocity. This is the case for an object in orbital motion.

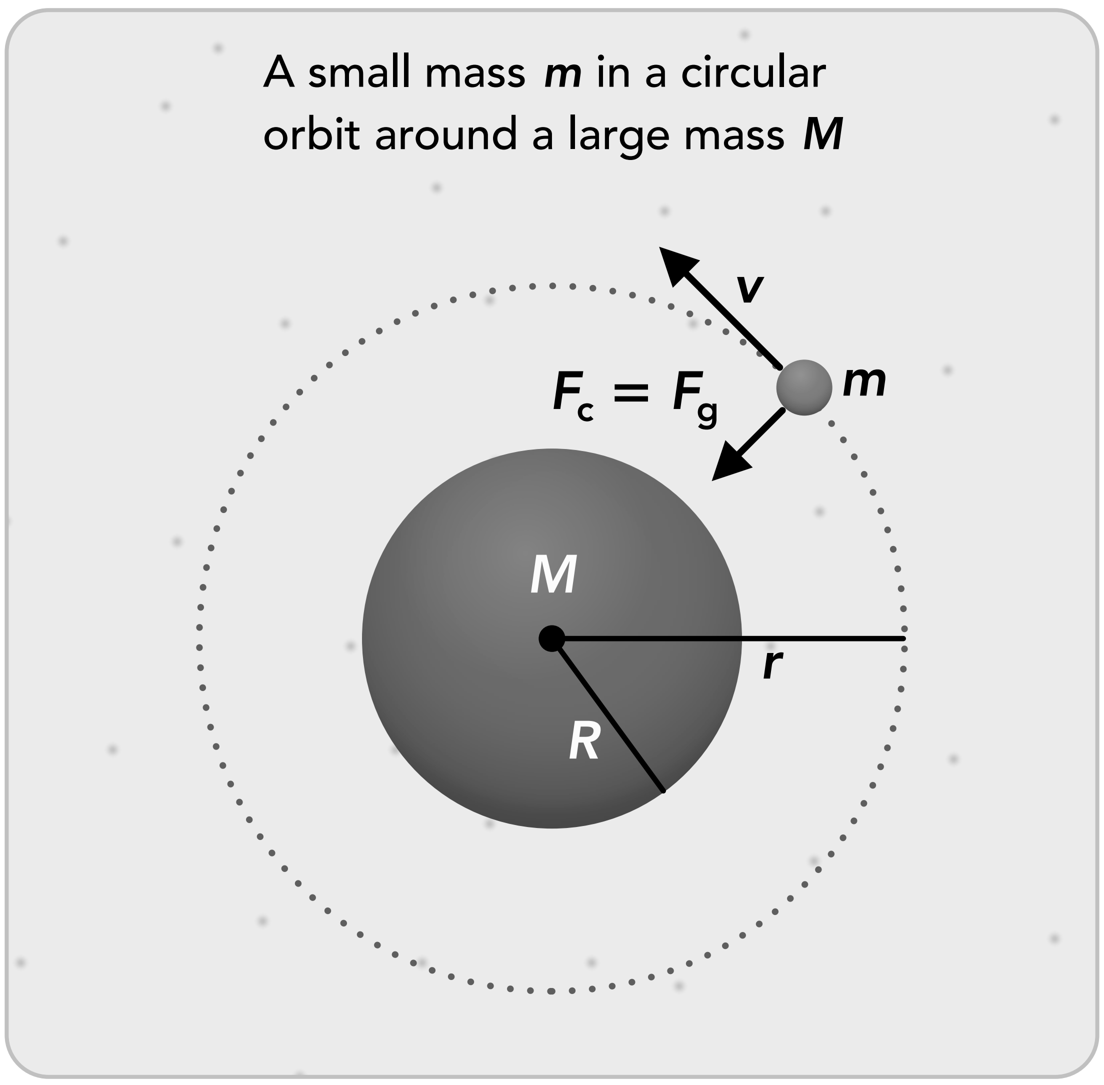


Circular Orbits

Constants		Unit	Name
G	6.67×10^{-11}	$\frac{m^3}{kg \cdot s^2}$	gravitational constant

Variables		SI Unit
M	planet mass	kg
m	object mass	kg
R	planet radius	m
r	orbital radius	m
v	orbital speed	$\frac{m}{s}$
T	orbital period	s
F_g	gravitational force	N
F_c	centripetal force	N

- The path of an object in orbit around a planet can be a circular orbit or an elliptical orbit. The same laws which are covered in the elliptical orbit section also apply to circular orbits.
- An object in a **circular orbit** is in **uniform circular motion** around the planet. Remember that an object in circular motion must have a centripetal force acting on it which always points towards the center of the circle.
- For circular orbits, **the centripetal force is the gravitational force** and the centripetal acceleration is the gravitational acceleration at that distance from the center of the planet. This means we can equate some concepts from uniform circular motion and Newton's law of universal gravitation.



The centripetal force for a circular orbit is the gravitational force acting on the small mass so we can combine circular motion and gravitation

$$F_c = F_g$$

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$v^2 = \frac{GM}{r} \qquad T = \frac{2\pi r}{v}$$

Orbital velocity

$$v = \sqrt{\frac{GM}{r}}$$

Orbital period

$$T = 2\pi\sqrt{\frac{r^3}{GM}}$$

- Using the equation above for the orbital velocity we can derive an equation for the kinetic energy K of an object in a circular orbit.
- The gravitational potential energy U_g of the object in orbit is just the gravitational potential energy of the two-mass system, regardless of the motion of either mass (so this is not specific to orbital motion).
- The total energy of an object in a circular orbit is the sum of kinetic energy and the potential energy.

Variables		SI Unit
E	total energy	J
K	kinetic enerrgy	J
U_g	potential energy	J

Kinetic energy of object in a circular orbit

$$K = \frac{1}{2}mv^2 = \frac{GMm}{2r}$$

Gravitational potential energy of two-mass system

$$U_g = -\frac{GMm}{r}$$

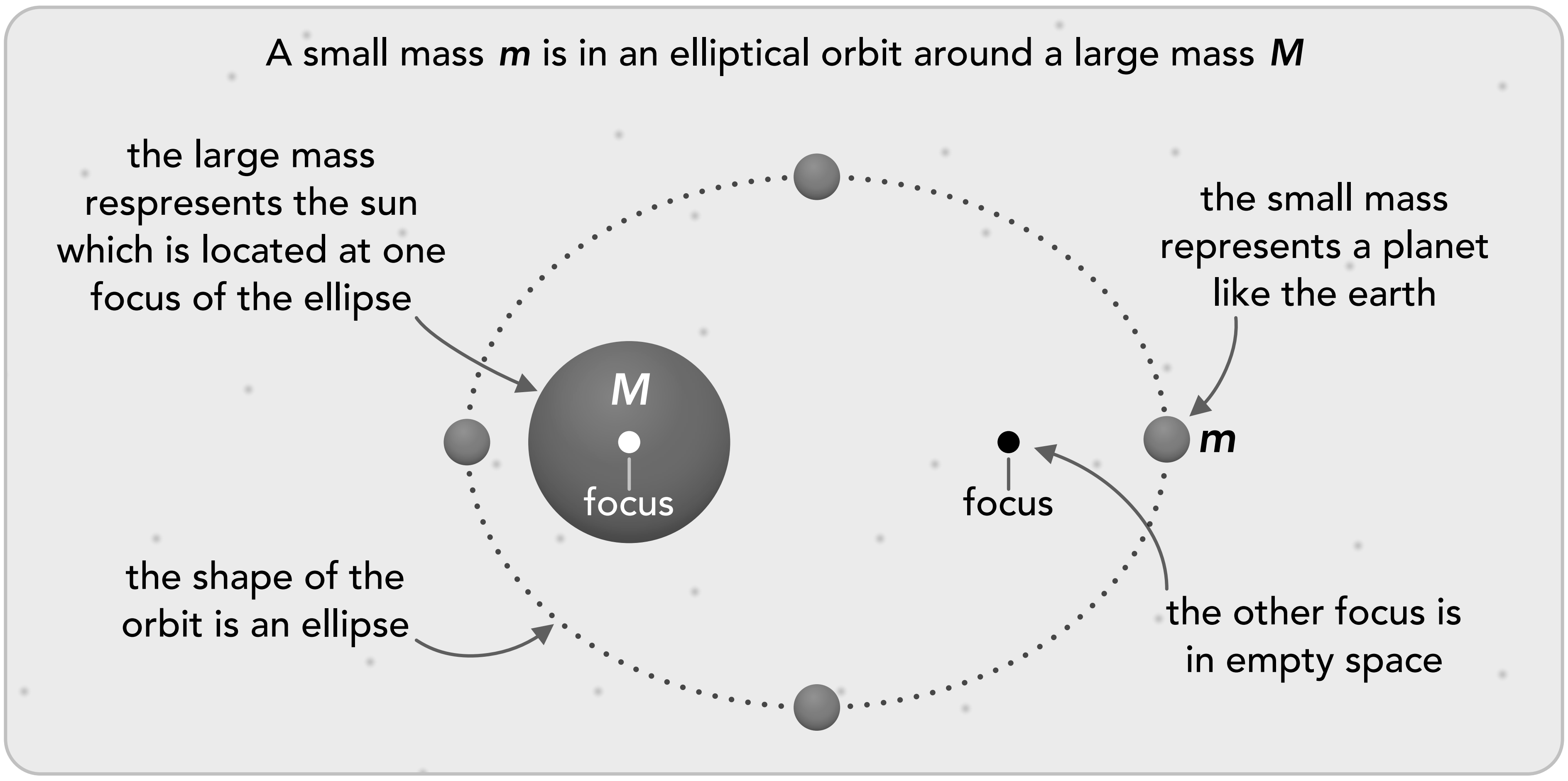
Total energy of object in a circular orbit

$$E = K + U_g = -\frac{GMm}{2r}$$

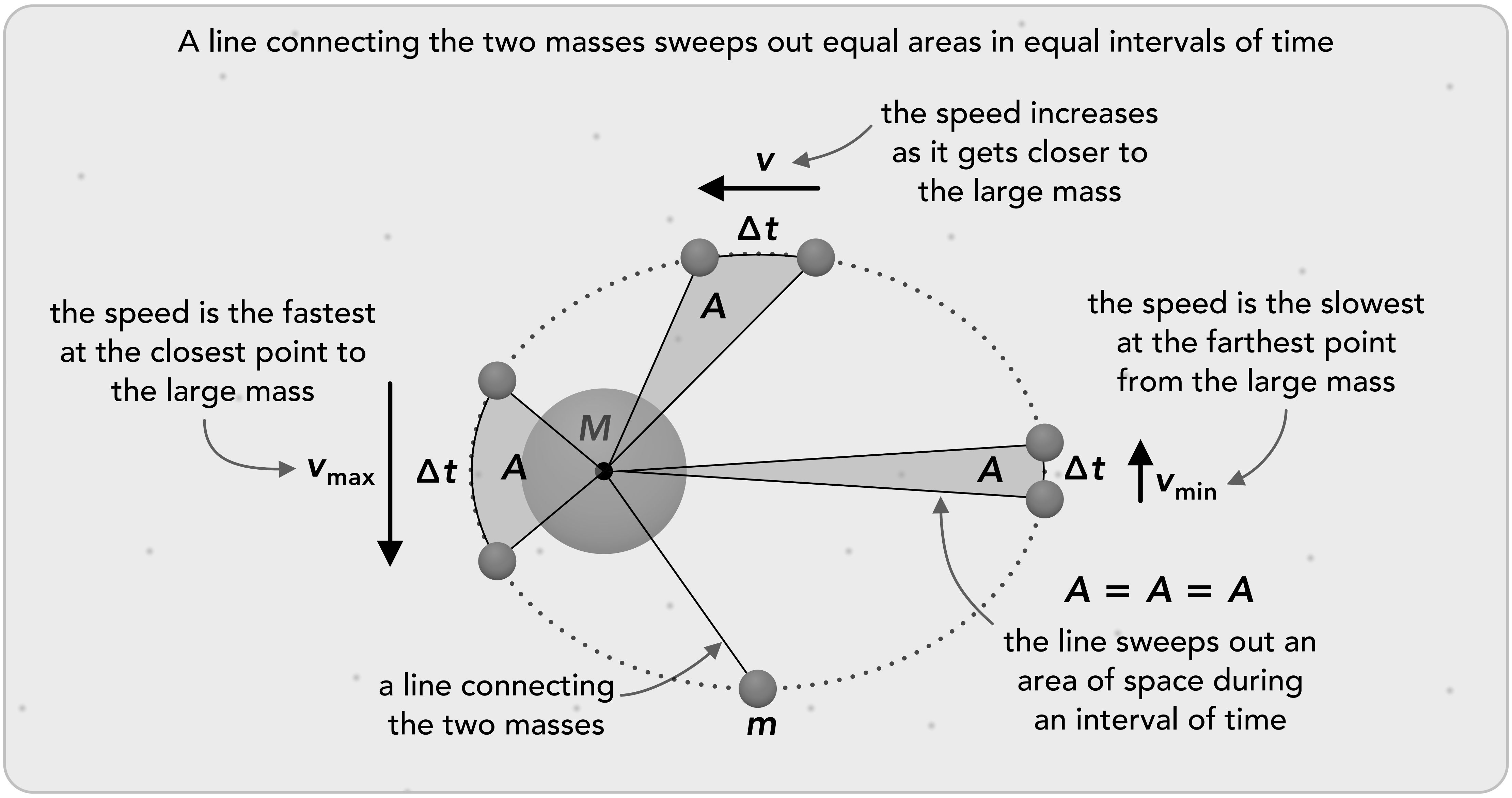
Elliptical Orbits

- Most real orbits are **elliptical orbits** which means the path is an ellipse instead of a perfect circle. A circular orbit is a special case of an elliptical orbit where the eccentricity is zero and the two focal points are at the center. The laws governing elliptical orbits also apply to circular orbits.
- In the early 1600's Johannes Kepler described the orbits of the planets around the sun. **Kepler's laws of planetary motion** are given below.

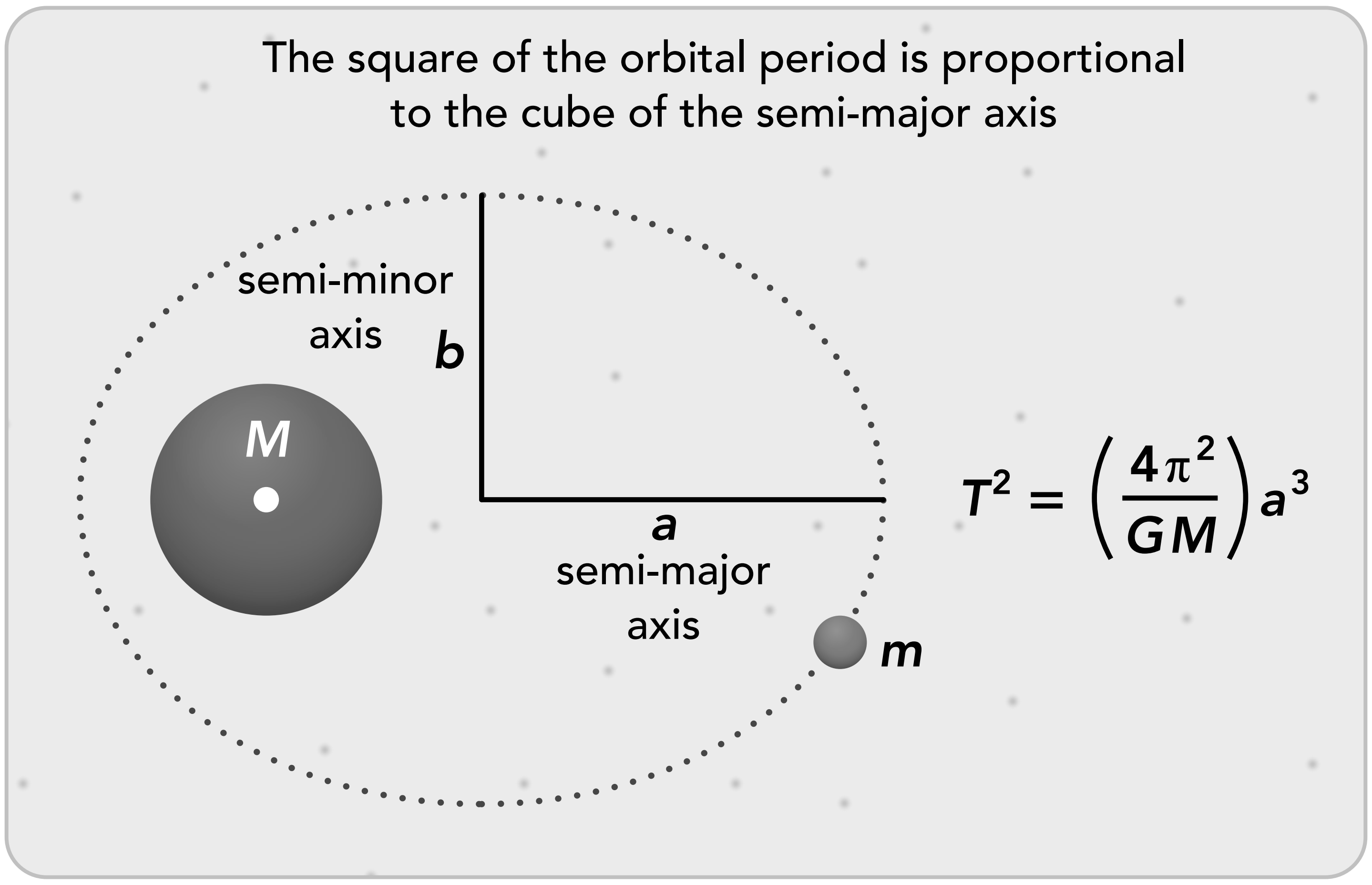
- **Law 1**: The orbit of a planet is an ellipse with the sun at one of the two foci.
- An ellipse has two foci or focal points. If one of the masses (the sun) is much larger than the other (a planet) then the center of the larger mass aligns with one focus of the ellipse according to Kepler's 1st law.



- **Law 2**: A line connecting the planet and the sun sweeps out equal areas during equal intervals of time.
- Imagine a line connecting the two masses which follows the orbital motion. During any 1 second interval (or maybe 1 month on a planetary scale) that imaginary line will sweep out or cover the same amount of area, regardless of where the planet is in the orbit. This law relates to the orbital speed and the orbital period.
- **The planet (small mass) will move faster when it's closer to the sun (large mass), and slower when it's farther away from the sun**. From Newton's law of gravitation, the gravitational force between the two masses is stronger when they are closer together. Also, because the path is elliptical and not circular, the gravitational force on the small mass is not perpendicular to its velocity (except at the left and right ends of the orbit shown). This means the gravitational force will have a component that's parallel to the velocity and it will cause the small mass to accelerate and its speed will change throughout the orbit.



- **Law 3**: The square of a planet's orbital period is proportional to the cube of the semi-major axis of its orbit.
- Unlike a circle which has a single radius, an ellipse has a semi-major axis and a semi-minor axis which are the longer "radius" and the shorter "radius" (the longest and shortest distances from the center to the perimeter).
- The equation for the period of an elliptical orbit is similar to the period of a circular orbit, but the semi-major axis is used instead of the radius.



$$T^2 = \left(\frac{4\pi^2}{GM}\right)a^3$$

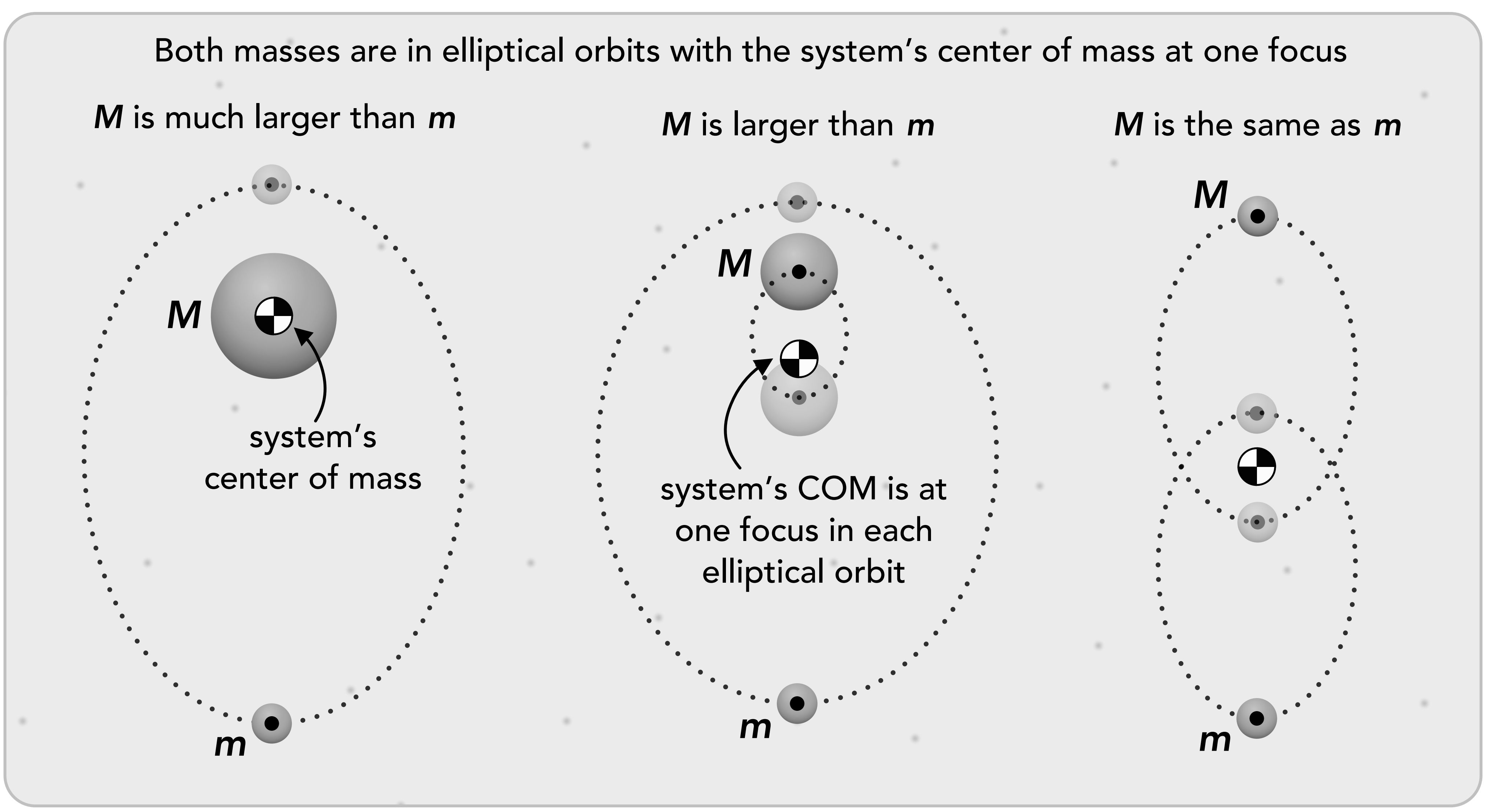
Orbital period for elliptical orbit

$$T = 2\pi\sqrt{\frac{a^3}{G(M+m)}}$$

Orbital period for elliptical orbit (assuming M is much larger than m)

$$T = 2\pi\sqrt{\frac{a^3}{GM}}$$

- Kepler's laws describe the elliptical orbit of a small mass around a large mass. In reality, **both masses orbit the shared center of mass of the system (called the barycenter) in elliptical orbits**. That center of mass is located at one focus of each elliptical orbit.
- When one mass is much larger than the other (as in the case of the sun and the earth) the system's center of mass is within the larger mass and is very close its center. So it's a fair approximation that a focus of the elliptical orbit of the smaller mass is located at the center of the larger mass, instead of at the system's center of mass.
- As the two masses become more similar in size, the system's center of mass moves towards the middle of the two masses. If the masses are equal, the center of mass is directly between them and they appear to orbit each other.



TYPES OF ENERGY

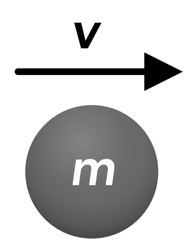
Kinetic Energy

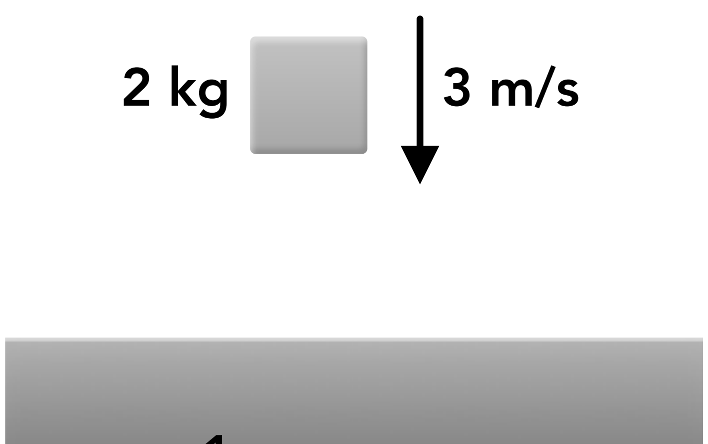
- **Kinetic energy** is the energy of an object or system due to its motion. There are two types: translational kinetic energy and rotational kinetic energy. Both of these are types of **mechanical energy**.
- Kinetic energy is a **scalar quantity** (not a vector quantity) so it's always positive, it does not have a direction and it does not depend on the direction of the object's or system's velocity.
- **Translational kinetic energy** or linear kinetic energy (often just referred to as kinetic energy) is the energy of an object or system due to its linear motion and depends on its mass and linear speed.

Variables		SI Unit
K	kinetic energy	J
m	mass	kg
v	speed	$\frac{m}{s}$

Kinetic energy

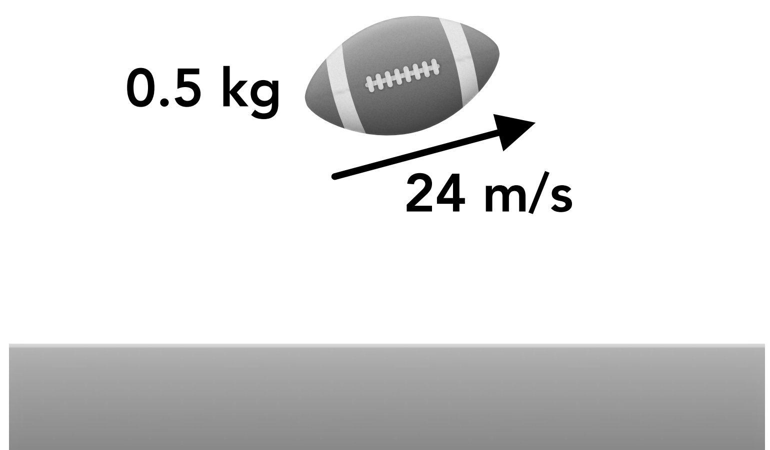
$$K = \frac{1}{2}mv^2$$





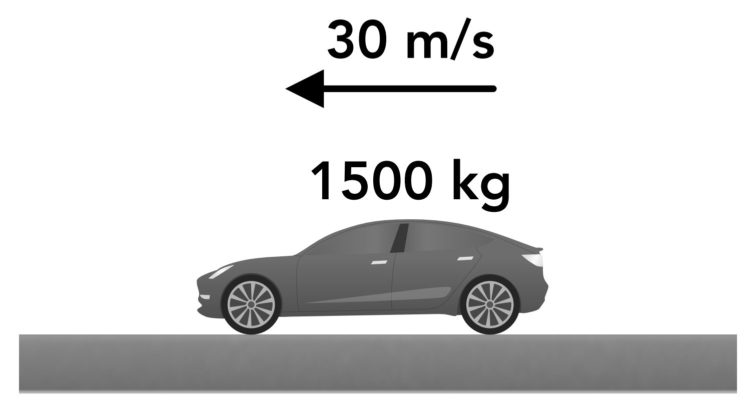
$$K = \frac{1}{2}(2\text{ kg})(3\text{ m/s})^2$$

$$K = 9\text{ J}$$



$$K = \frac{1}{2}(0.5\text{ kg})(24\text{ m/s})^2$$

$$K = 144\text{ J}$$



$$K = \frac{1}{2}(1500\text{ kg})(30\text{ m/s})^2$$

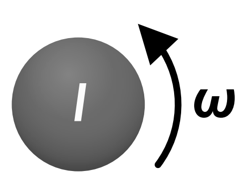
$$K = 675{,}000\text{ J}$$

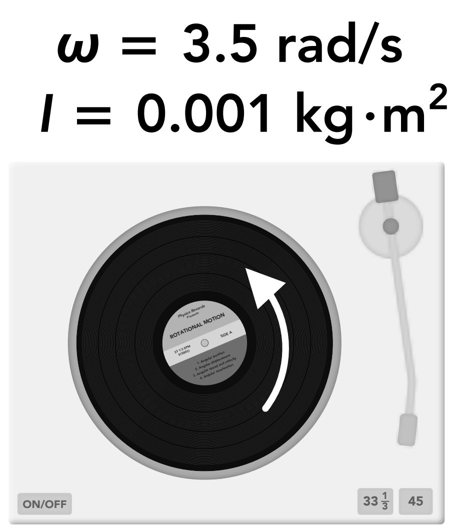
- **Rotational kinetic energy** is the energy of an object or system due to its rotational motion and depends on its rotational inertia and angular speed.

Variables		SI Unit
K_{rot}	rotational kinetic energy	J
I	rotational inertia	$kg \cdot m^2$
ω	angular speed	$\frac{rad}{s}$

Rotational kinetic energy

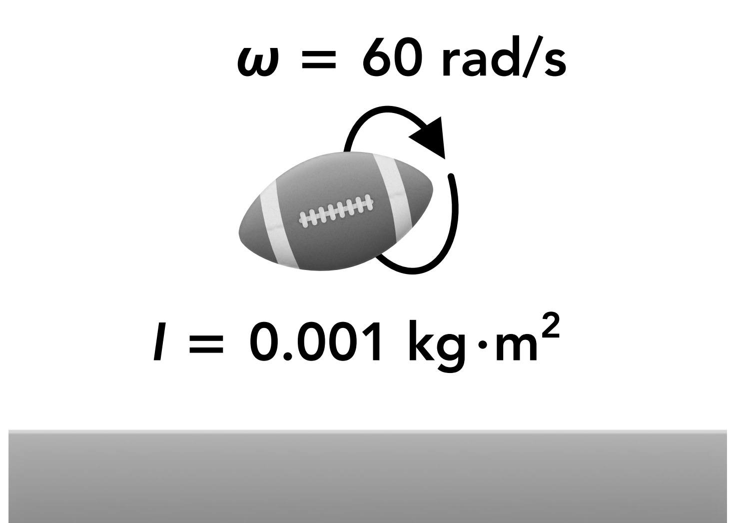
$$K_{rot} = \frac{1}{2}I\omega^2$$





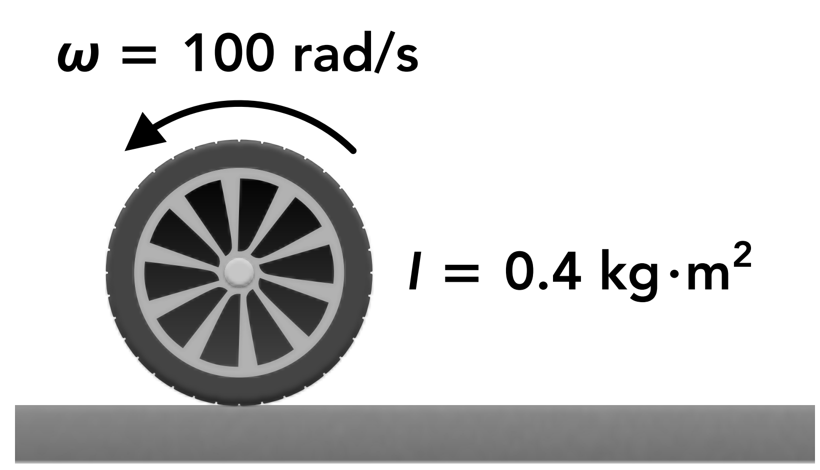
$$K_{rot} = \frac{1}{2}(0.001)(3.5)^2$$

$$K_{rot} = 0.006\text{ J}$$



$$K_{rot} = \frac{1}{2}(0.001)(60)^2$$

$$K_{rot} = 1.8\text{ J}$$



$$K_{rot} = \frac{1}{2}(0.4)(100)^2$$

$$K_{rot} = 2{,}000\text{ J}$$

Gravitational Potential Energy

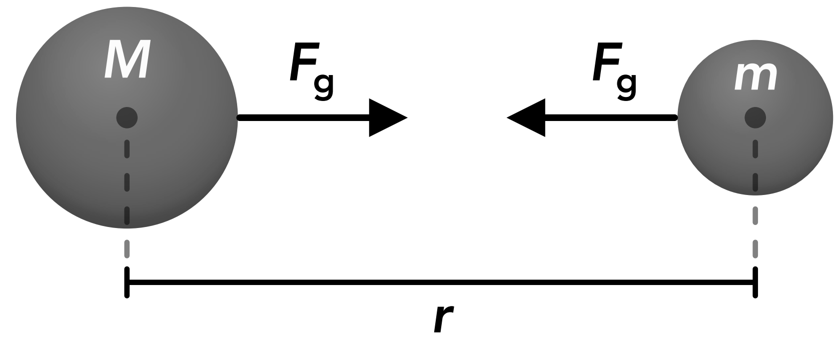
- **Gravitational potential energy** is the energy of a system of two masses due to the gravitational force pulling them together. The two masses are usually an object and the earth. This is a type of **mechanical energy**.
- It's important to remember that gravitational potential energy is a property of **a system** of two masses. A single object can't have gravitational potential energy on its own, although it is very common to say that it does. When you see "the potential energy of the ball", replace that with "the potential energy of the ball-earth system".
- There are usually two equations that are used to calculate gravitational potential energy. One is used for planet-sized distances and one is used for changes in height near the surface of a planet. These are actually the same equation, the second one is derived from the first using approximations.
- The SI unit of gravitational potential energy is a joule (J), the same as all types of energy.

Constants		Unit	Name
G	6.67×10^{-11}	$\frac{m^3}{kg \cdot s^2}$	gravitational constant

Variables		SI Unit
U_g	gravitational potential energy	J
M	planet mass	kg
m	object mass	kg
r	distance between centers	m
y	height	m
g	gravitational acceleration	$\frac{m}{s^2}$

- The gravitational potential energy of a two-mass system is derived from the gravitational force given by Newton's law of universal gravitation which is shown below. Note that when the two masses are an infinite distance apart the gravitational force between them approaches zero.

$$F_g = \frac{GMm}{r^2} \qquad F_g = 0 \text{ at } r = \infty$$

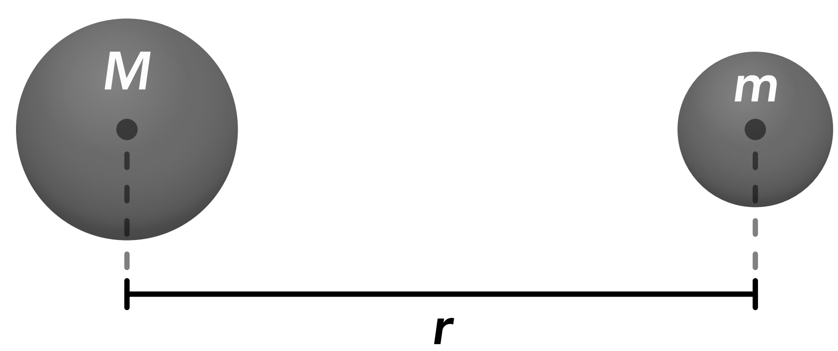


- The gravitational potential energy of a two-mass system exists because of the gravitational force between them. If the gravitational force is zero when the objects are an infinite distance apart, **the gravitational potential energy is also zero when the objects are an infinite distance apart**.
- The gravitational force between the two masses is attractive and each mass "wants" to move towards the other, so **energy must be added to the system (work must be done on the system) in order to move them apart** and increase the distance r between them.
- For those two reasons, the gravitational potential energy of a two-mass sytem is negative instead of positive. An increase in r must increase U_g (which changes from a bigger negative value to a smaller negative value, a positive increase) and U_g must be zero when $r = \infty$.

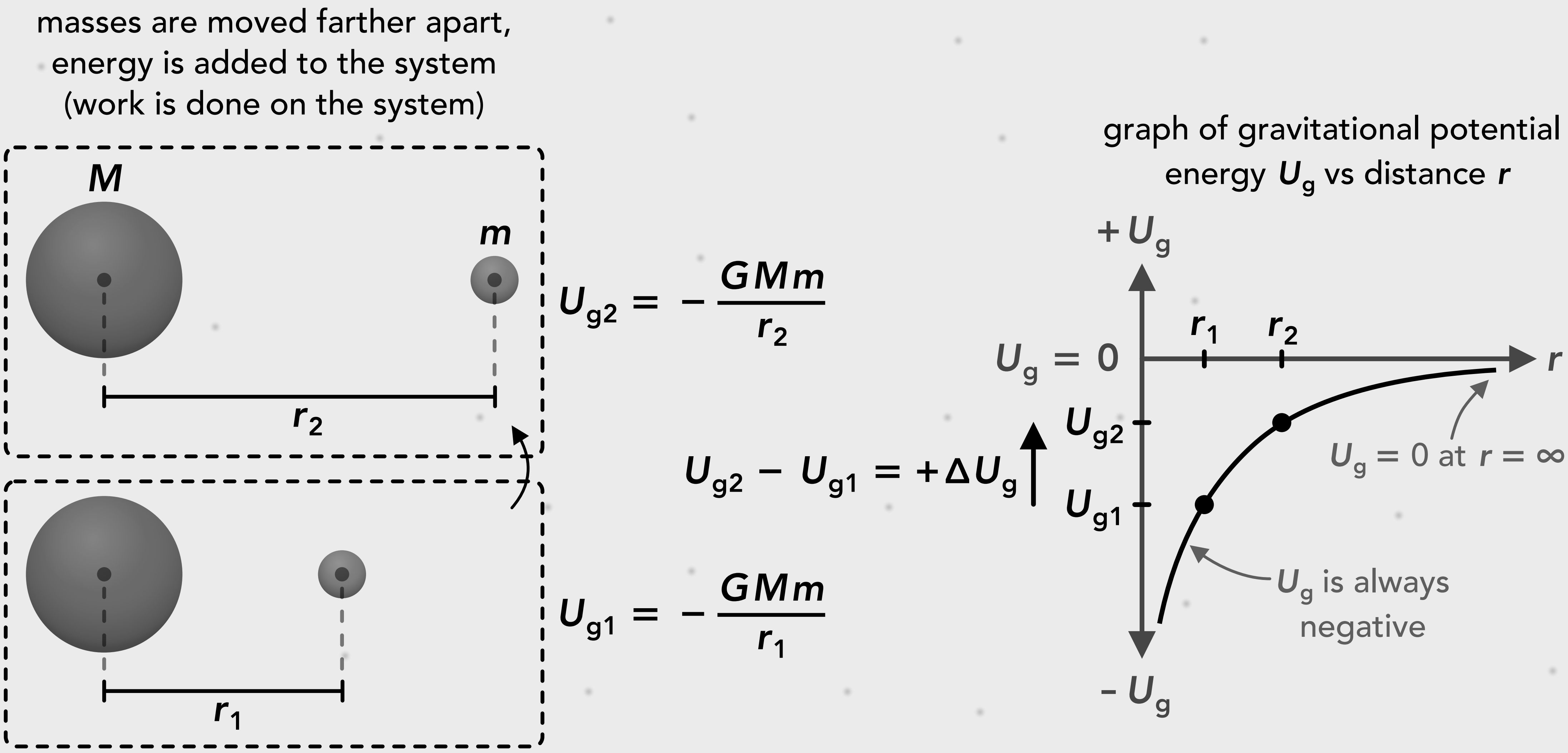
Gravitational potential energy of a two-mass system

$$\boxed{U_g = -\frac{GMm}{r}}$$

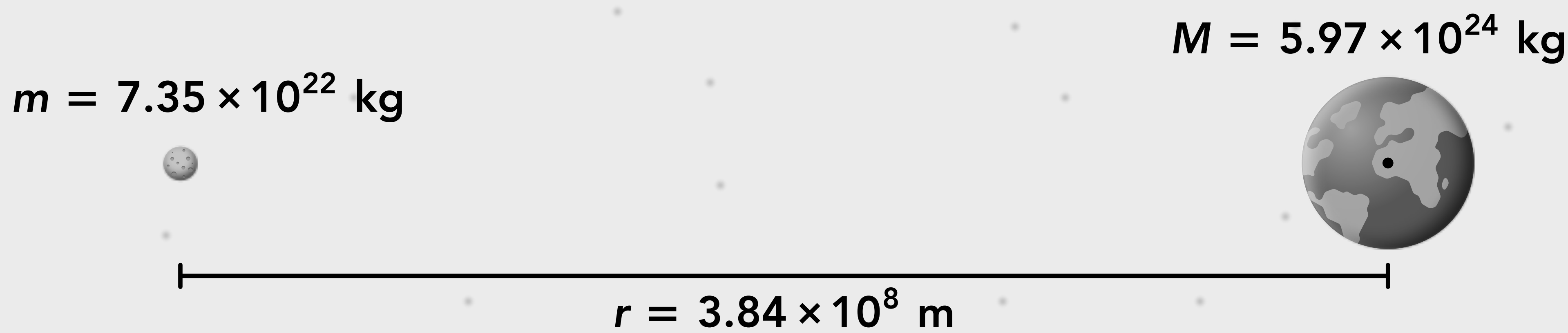
$$U_g = 0 \text{ at } r = \infty$$



The change in the gravitational potential energy of a two-mass system is positive as the distance between the masses increases



Example: Gravitational potential energy of the earth-moon system

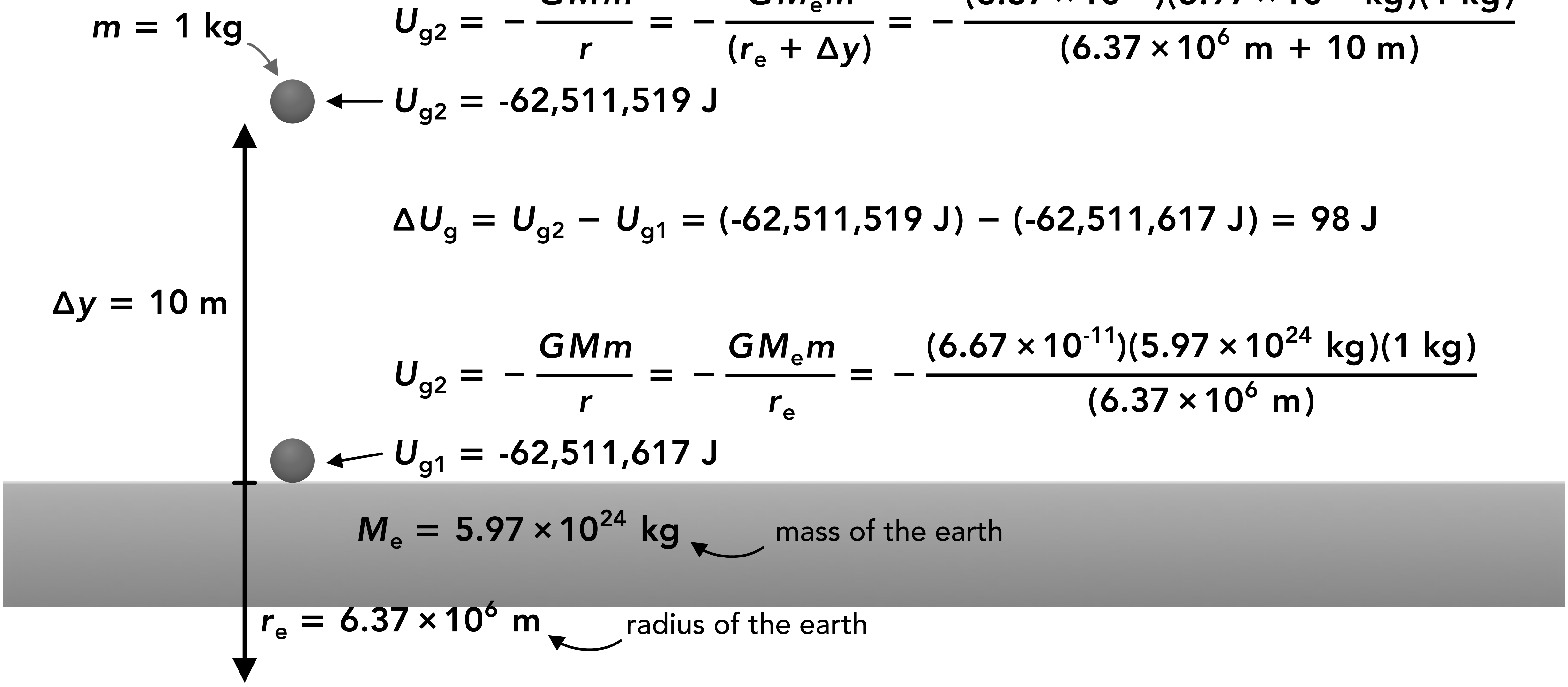


$$U_g = -\frac{GMm}{r} = -\frac{(6.67\times10^{-11})(5.97\times10^{24}\ \text{kg})(7.35\times10^{22}\ \text{kg})}{(3.84\times10^{8}\ \text{m})} = -7.62\times10^{28}\ \text{J}$$

- What if we're working with an object at a relatively small height above the ground and we want to know the gravitational potential energy of the object-earth system?
- The equation given above still represents the gravitational potential energy of the object-earth system, but we're going to get very large (negative) values for the potential energy. We'll also find that it's more useful to focus on the **change** or the **difference** in the potential energy between two heights.
- Below is an example of the gravitational potential energy of a ball-earth system at two different heights, then how we can simplify the equation for relatively small changes in height near the surface of the earth.

Calculating the change in gravitational potential energy of the ball-earth system between two different heights, using the original equation for gravitational potential energy

U_g: gravitational potential energy of **ball-earth system**



$$U_{g2} = -\frac{GMm}{r} = -\frac{GM_e m}{(r_e + \Delta y)} = -\frac{(6.67 \times 10^{-11})(5.97 \times 10^{24}\ \text{kg})(1\ \text{kg})}{(6.37 \times 10^{6}\ \text{m} + 10\ \text{m})}$$

$$U_{g2} = \text{-62,511,519 J}$$

$$\Delta U_g = U_{g2} - U_{g1} = (\text{-62,511,519 J}) - (\text{-62,511,617 J}) = 98\ \text{J}$$

$$U_{g2} = -\frac{GMm}{r} = -\frac{GM_e m}{r_e} = -\frac{(6.67 \times 10^{-11})(5.97 \times 10^{24}\ \text{kg})(1\ \text{kg})}{(6.37 \times 10^{6}\ \text{m})}$$

$$U_{g1} = \text{-62,511,617 J}$$

We can simplify the equation for the change in gravitational potential energy using some approximations when working with relatively small changes in height above the surface of the earth

M_e : mass of the earth
r_e : radius of the earth
m : object mass
Δy : object's change in height

$$\Delta U_g = U_{g2} - U_{g1}$$

simplify

$$\Delta U_g = \left(\frac{-GM_e m}{(r_e + \Delta y)}\right) - \left(\frac{-GM_e m}{r_e}\right)$$

$$\Delta U_g = \frac{GM_e m}{r_e} - \frac{GM_e m}{(r_e + \Delta y)}$$

$$\Delta U_g = GM_e m\left(\frac{1}{r_e} - \frac{1}{(r_e + \Delta y)}\right)$$

$$\Delta U_g = GM_e m\left(\frac{(r_e + \Delta y)}{r_e(r_e + \Delta y)} - \frac{r_e}{r_e(r_e + \Delta y)}\right)$$

$$\Delta U_g = GM_e m \frac{\Delta y}{r_e(r_e + \Delta y)}$$

if the object's change in height is much less than the radius of the earth, the radius plus the change in height is approximately equal to the radius

$$\Delta y \ll r_e$$

$$(r_e + \Delta y) \approx r_e$$

$$\Delta U_g = GM_e m \frac{\Delta y}{r_e^2}$$

$$\Delta U_g = \frac{GM_e}{r_e^2} m\Delta y$$

if we assume the acceleration due to gravity is constant for this change in height

$$g = \frac{GM_e}{r_e^2}$$

$$\Delta U_g = gm\Delta y$$

Calculating the change in gravitational potential energy of the ball-earth system between two different heights, using the simplified equation for the change in gravitational potential energy

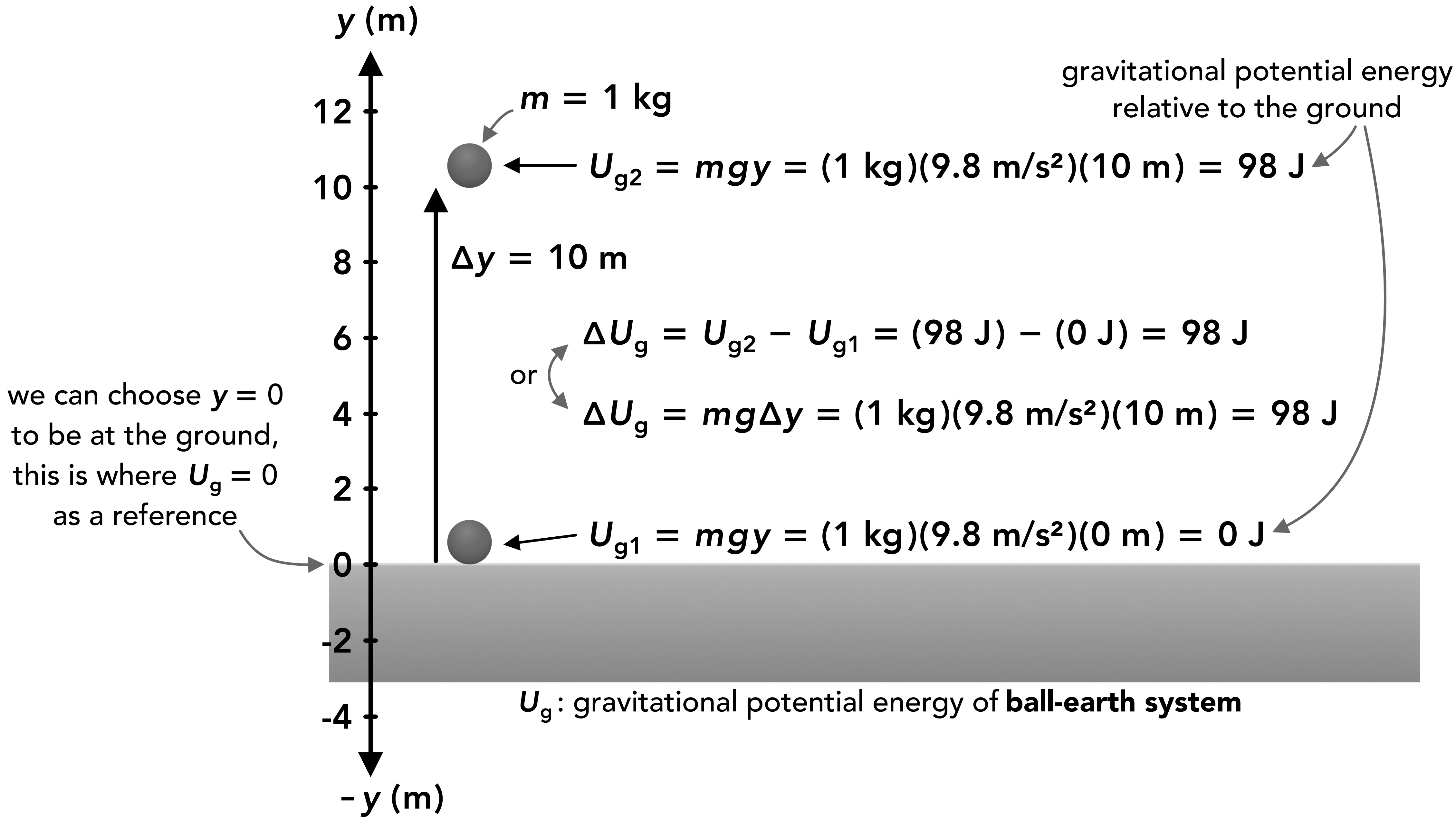
Change in gravitational potential energy of an object-earth system

$$\Delta U_g = mg\Delta y$$

Gravitational potential energy of an object-earth system *relative to a reference point

$$U_g = mgy \qquad U_g = 0 \text{ at } y = 0$$

these mean the same thing, they're just two different ways to represent changes in the gravitational potential energy



- Above we use the simplified equation for the change in gravitational potential energy for the ball-earth system. The change in height is small relative to the radius of the earth, and the acceleration due to gravity is constant. **We get the same value for the change in potential energy as before: 98 J**. This is the change in the potential energy of the system regardless of which equation we use.
- Instead of calculating the change in potential energy **we can establish a reference point where $y = 0$ and $U_g = 0$**. Then we can calculate the potential energy when the object is at different heights **relative to that reference point**.
- Remember that this value is not the actual, absolute gravitational potential energy of the sytem, **this is just a different way to represent changes in the gravitational potential energy**, which will be helpful when using the conservation of energy and work.
- Also, notice that the change in potential energy only depends on the change in height, **it does not depend on the path that the object takes between the two heights**.

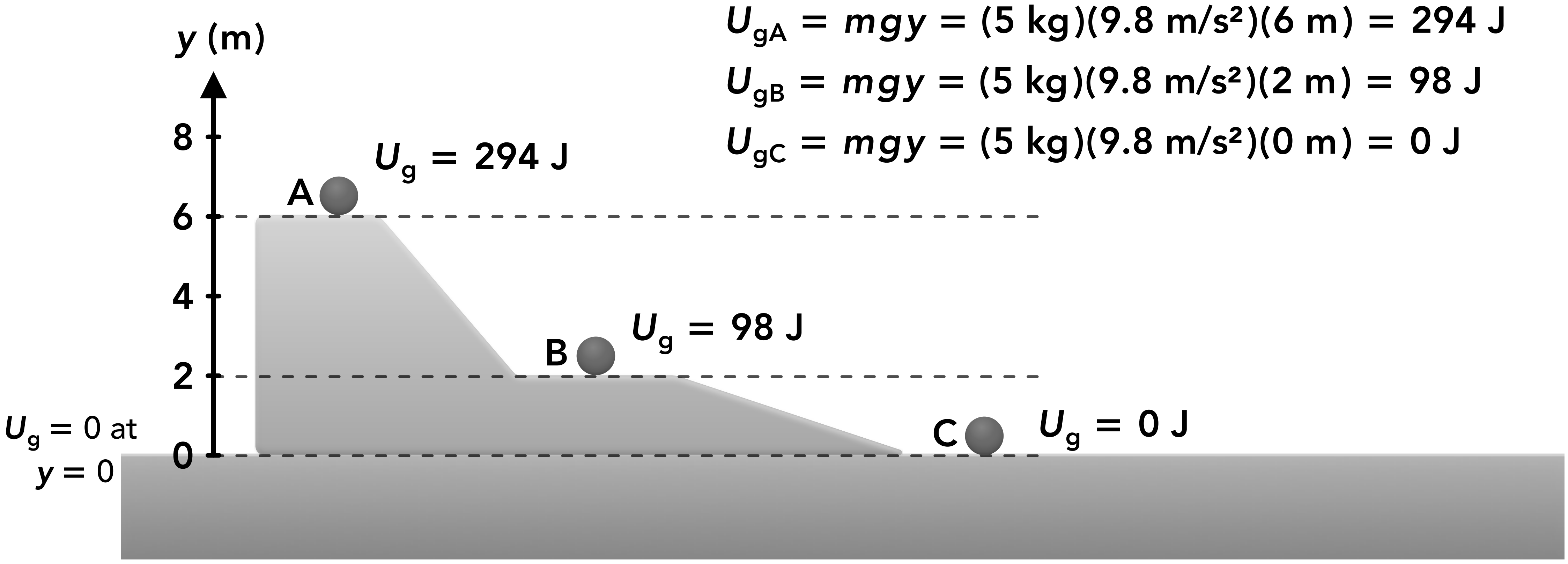
Example: A 5 kg ball rolls down a ramp with multiple sections and reaches the ground. What is the ball's gravitational potential energy (technically the energy of the ball-earth system) at points A, B and C if:

the reference point (y = 0) is at the ground

$U_{gA} = mgy = (5\text{ kg})(9.8\text{ m/s}^2)(6\text{ m}) = 294\text{ J}$

$U_{gB} = mgy = (5\text{ kg})(9.8\text{ m/s}^2)(2\text{ m}) = 98\text{ J}$

$U_{gC} = mgy = (5\text{ kg})(9.8\text{ m/s}^2)(0\text{ m}) = 0\text{ J}$

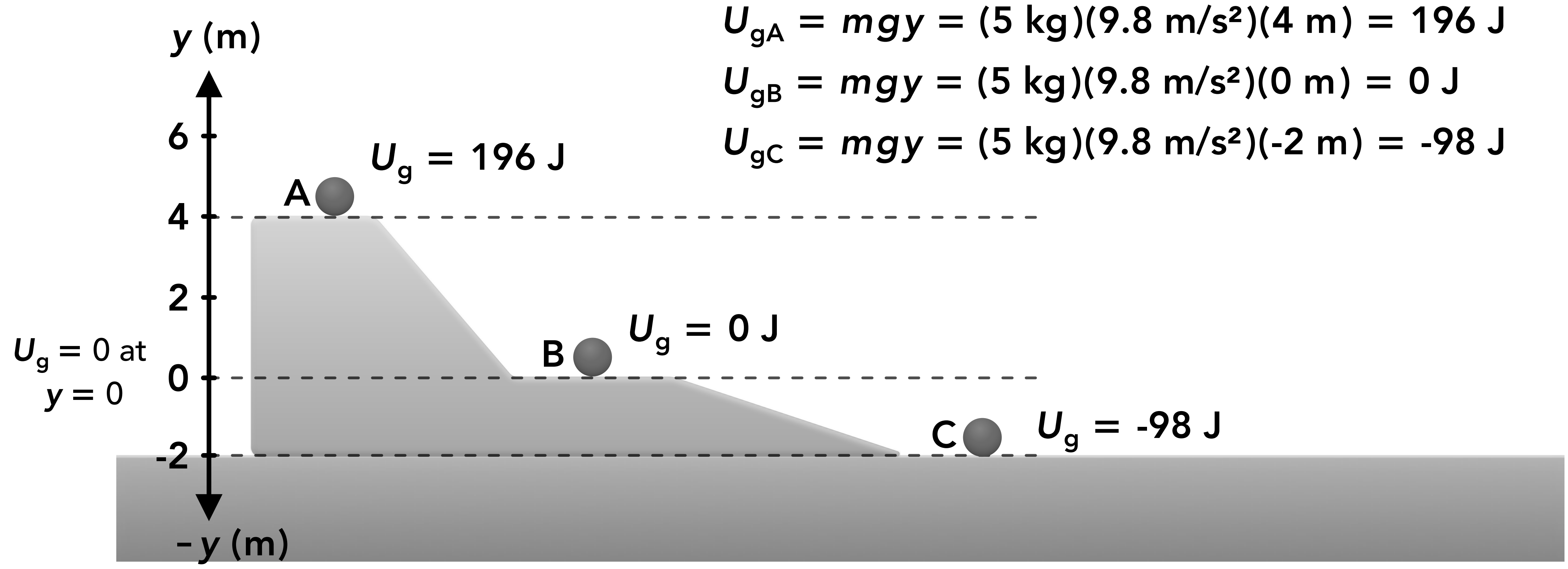


the reference point (y = 0) is at the height of point B

$U_{gA} = mgy = (5\text{ kg})(9.8\text{ m/s}^2)(4\text{ m}) = 196\text{ J}$

$U_{gB} = mgy = (5\text{ kg})(9.8\text{ m/s}^2)(0\text{ m}) = 0\text{ J}$

$U_{gC} = mgy = (5\text{ kg})(9.8\text{ m/s}^2)(-2\text{ m}) = -98\text{ J}$

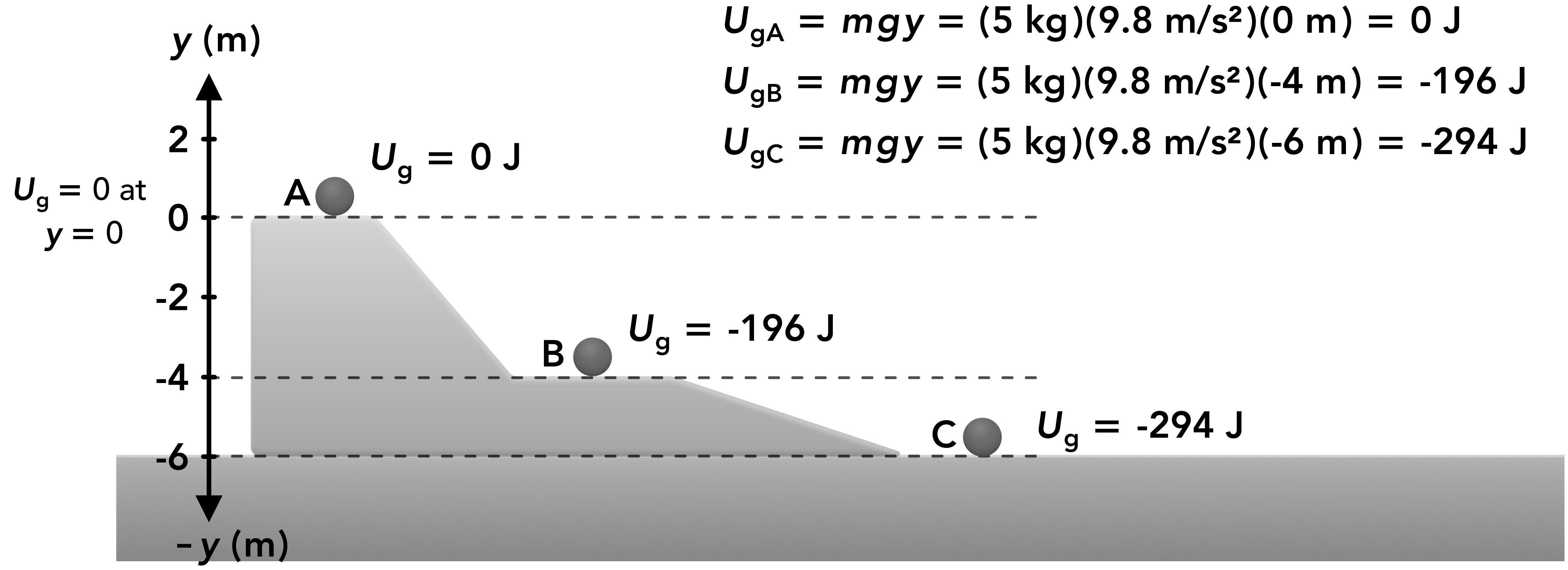


the reference point (y = 0) is at the height of point A

$U_{gA} = mgy = (5\text{ kg})(9.8\text{ m/s}^2)(0\text{ m}) = 0\text{ J}$

$U_{gB} = mgy = (5\text{ kg})(9.8\text{ m/s}^2)(-4\text{ m}) = -196\text{ J}$

$U_{gC} = mgy = (5\text{ kg})(9.8\text{ m/s}^2)(-6\text{ m}) = -294\text{ J}$



Other Types of Energy

- Kinetic energy, gravitational potential energy and spring potential energy are all types of **mechanical energy**.
- There are other types of energy that are categorized as **non-mechanical energy** such as thermal energy, sound energy, light energy, chemical energy and electrical energy.

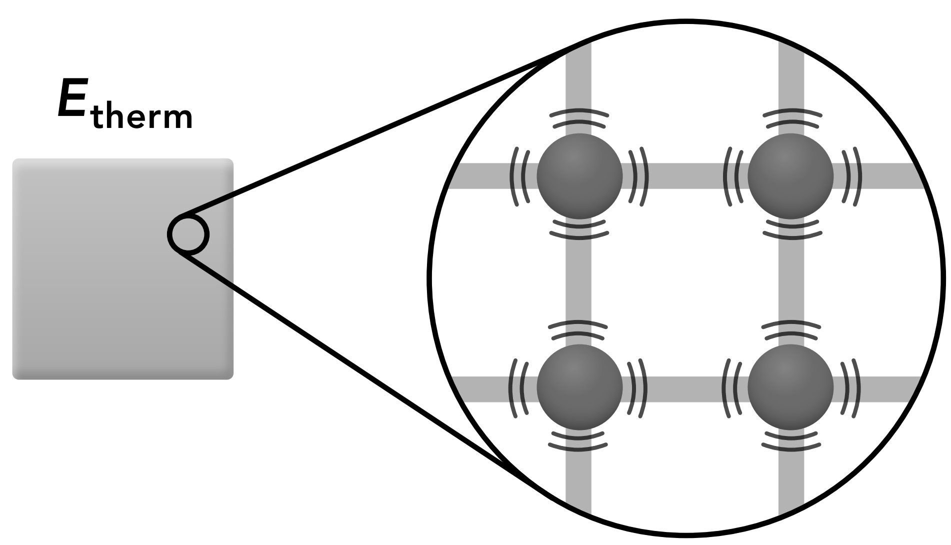
Mechanical energy

- K - Kinetic energy
- U_g - Gravitational potential energy
- U_{sp} - Spring (elastic) potential energy

Non-mechanical energy

- E_{therm} - Thermal energy
- E_{sound} - Sound energy
- E_{light} - Light energy
- E_{chem} - Chemical energy
- E_{elec} - Electrical energy

- **Thermal energy** is the energy of an object or system due to the vibrations of the atoms in the material.
- The thermal energy of an object is really the total kinetic energy of all of its atoms, and an object's temperature is the average kinetic energy of all of its atoms.
- The atoms in an object with a higher temperaure are vibrating (translating and rotating) more than the atoms in an object with a lower temperature.
- When there is kinetic friction between two objects, some of the kinetic energy of the moving object is converted into the increased thermal energy of both objects, raising their temperatures.



the thermal energy and temperature of the block is due to the vibrational kinetic energy of its atoms

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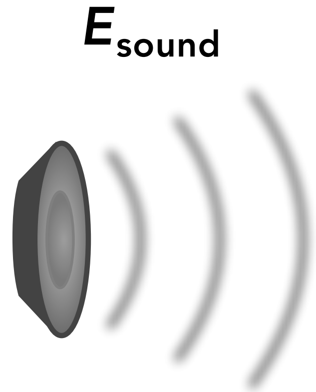
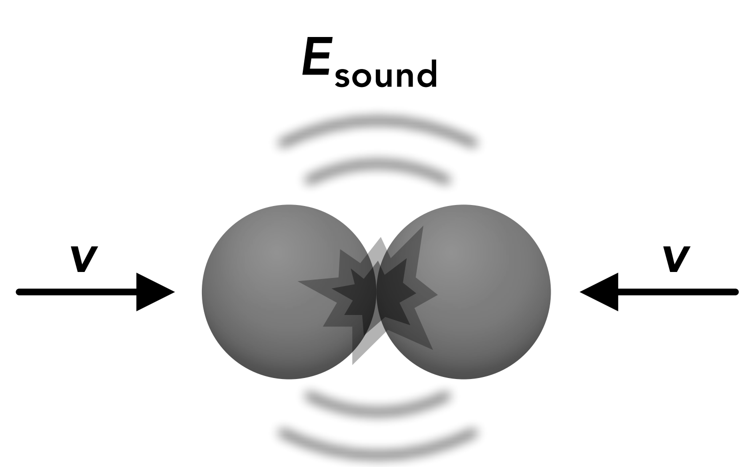
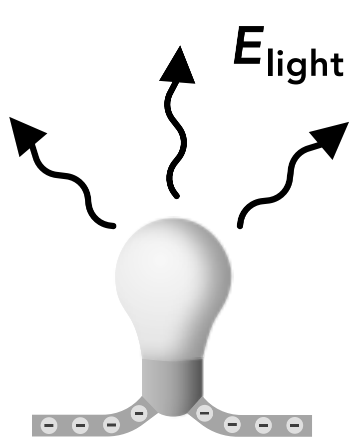
- **Sound energy** is the energy of sound waves traveling through the air or another medium.
- Sound energy is a combination of pressure energy (a form of potential energy) and the kinetic energy of the molecules as they move or vibrate.
- During a collision of two objects, some of the kinetic energy is converted into sound energy, creating the sound that you hear from the collision.

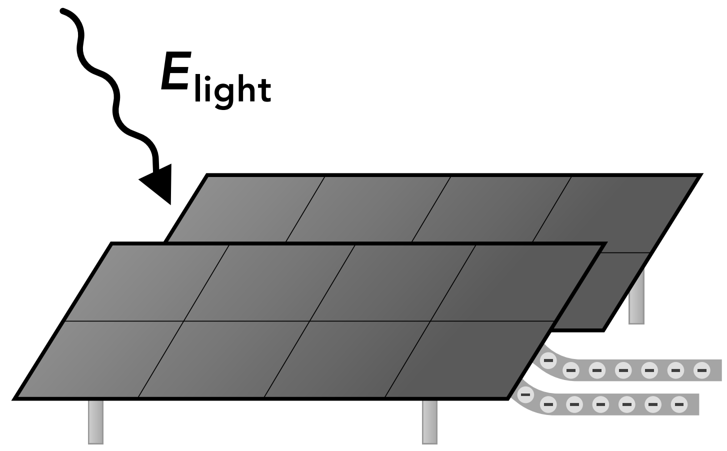




some of the kinetic energy during is a collision is transformed into sound energy

- **Light energy** is the energy of light waves, which is just a form of electromagnetic wave energy.
- The electrical energy in the filament of a lightbulb is converted into light energy (and thermal energy) which is the light emitted from the bulb.
- The light energy from the sun can be converted into electrical energy using solar panels.

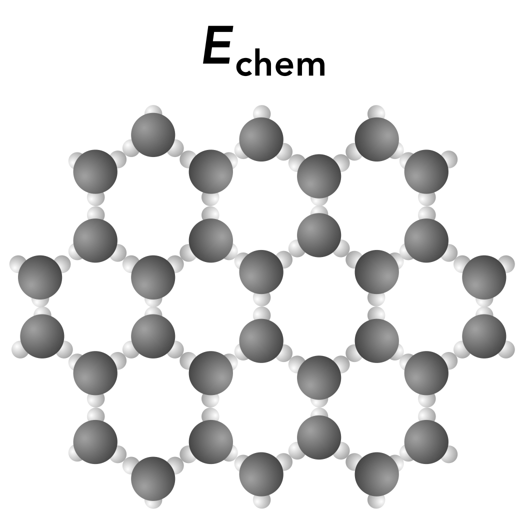




energy is converted between electrical energy and light energy in a light bulb and a solar panel

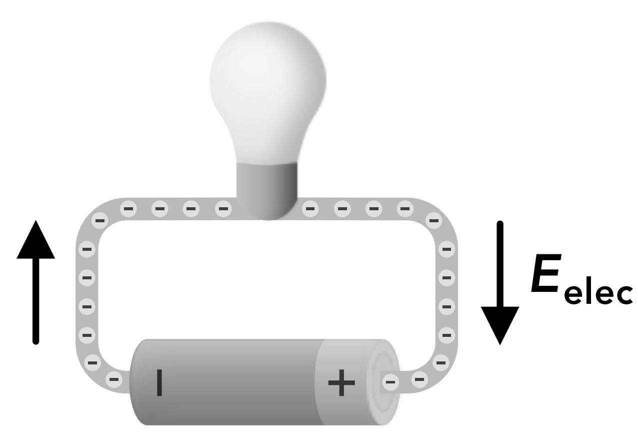
- **Chemical energy** is the energy stored in the chemical bonds in a material.
- Energy is required to form chemical bonds, and chemical energy is released and converted into other forms when those chemical bonds are broken or change in some way during a chemical reaction.
- The chemical energy in the food we eat is converted into other forms of energy that we can use.
- The chemical energy in a battery is converted into electrical energy, which can be converted into other types of energy like light, sound, heat or mechanical energy.

chemical energy is stored in chemical bonds



during chemical reactions the energy is converted into light, thermal, electrical or other forms of energy

- **Electrical energy** is the energy due to the movement of electrons.
- The chemical energy in a battery is converted into electrical energy when connected to a circuit, which can be converted into other types of energy like light, heat, sound or mechanical energy.



the chemical energy in a battery is converted into electrical energy, which is converted into light and thermal energy in the lightbulb

ENERGY, WORK & POWER

The Law of Conservation of Energy

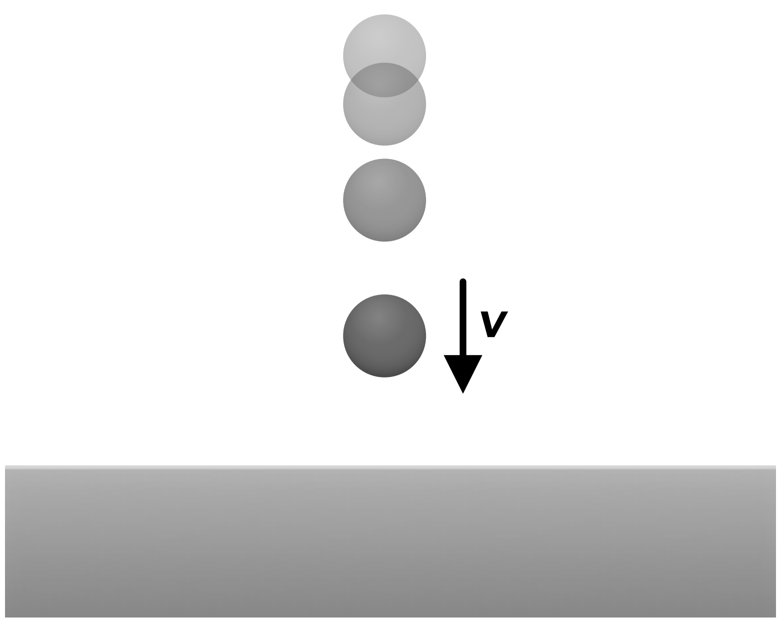
- **The law of conservation of energy**: the total amount of energy in the universe or an isolated system is conserved (it's constant and doesn't change over time).
- There are many different types of energy (kinetic energy, gravitational potential energy, etc.) and energy can be converted or transformed between those different types, but it cannot be created or destroyed.

Variables		SI Unit
E	energy	J
K	kinetic energy	J
U_g	gravitational potential energy	J
U_{sp}	spring potential energy	J

- Energy can be converted or transformed from one type of energy to any other type of energy. In the real world there are some conversions which are more common and less likely to happen in the reverse direction.

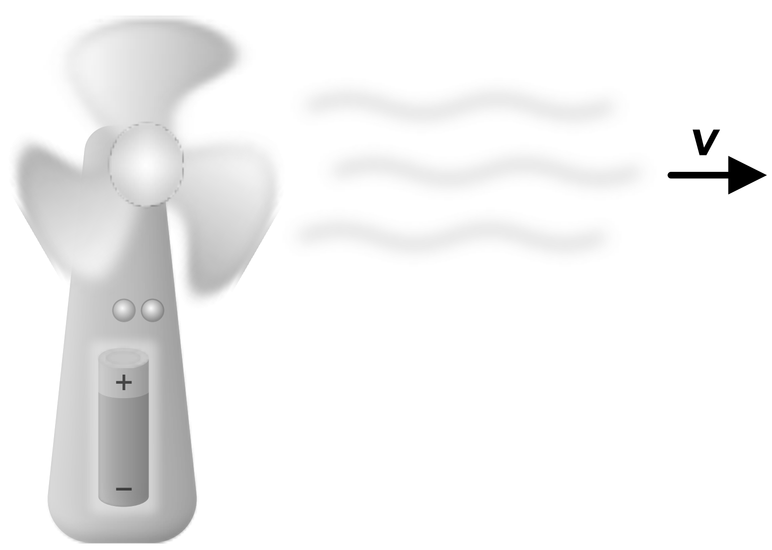
Gravitational potential energy is converted into kinetic energy as a ball falls towards the earth

$$U_g \rightarrow K$$

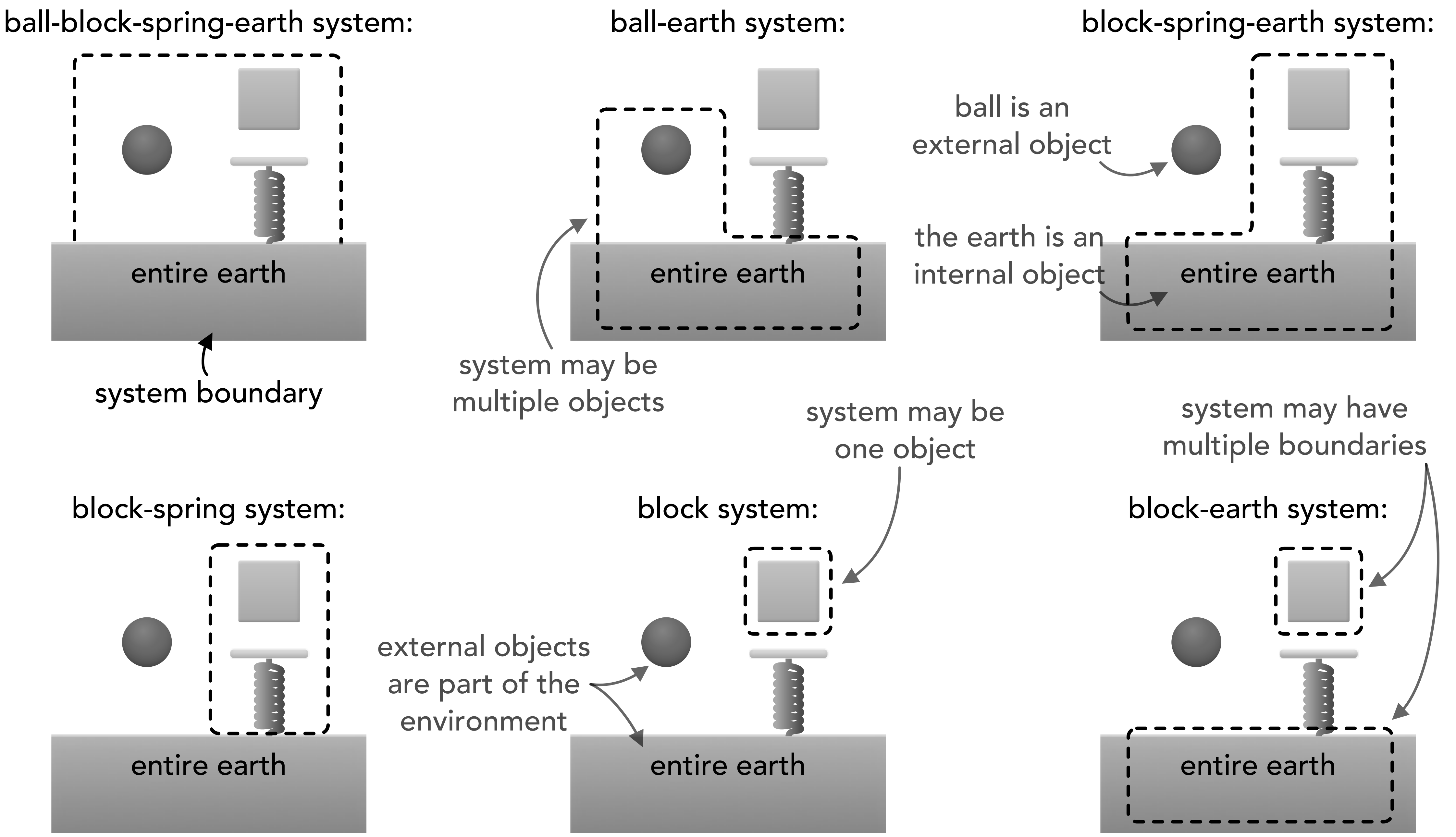


Chemical energy in a fan battery is converted into electrical energy, which is converted into rotational kinetic energy, translational kinetic energy, sound energy, light energy and thermal energy

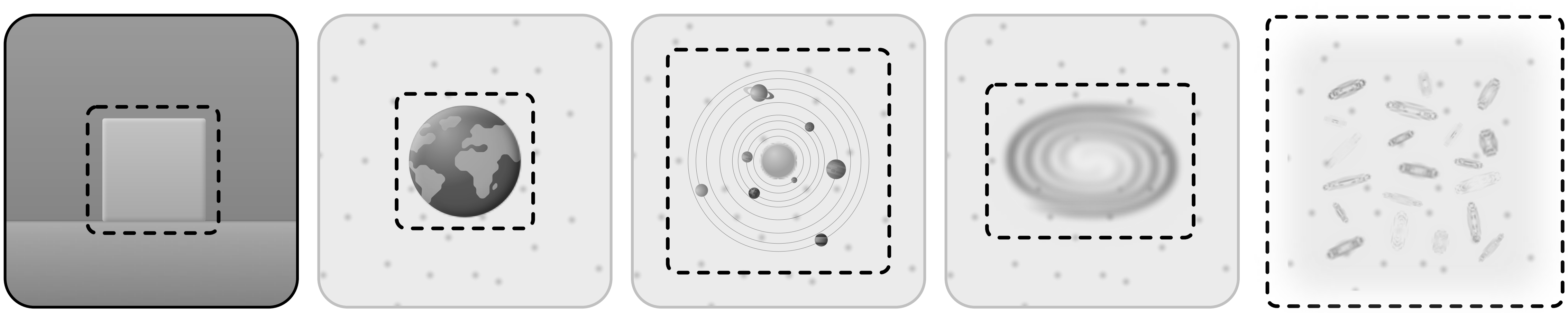
$$E_{chem} \rightarrow E_{elec} \rightarrow K_{rot} + K + E_{sound} + E_{light} + E_{therm}$$



- A **system** can be thought of as a selected group of objects which is separated from its **environment** by a chosen **boundary line**. There are no predefined or existing systems, a system is just what we choose it to be based on the objects that we're studying or the problem we're solving.
- Once a boundary line is drawn and a system is chosen, everything in the universe (energy, forces, objects) are considered either inside the system (internal) or outside the system (external) at any one moment in time. Energy and objects may move into or out of the system depending on the situation.
- Objects do not have to be in contact with each other to both be in the system. Mutliple separate boundary lines may be drawn to define the system so that only specific objects are included.



A system can be any size, from one object to the entire universe



- Energy and objects may move into or out of a system across the system boundary line.
- An **isolated system** is defined as a system where **energy does not enter or leave the system**. This is just a definition. If energy does enter or leave the system, it is not an isolated system.
- According to the law of conservation of energy, **the total amount of energy within an isolated system is conserved over time**. Energy within the system can be converted back and forth between different types, but the **total** amount of energy stays the same in an isolated system.

Conservation of energy (universe and isolated systems)

$$\boxed{\Delta E_{\text{total}} = 0 \quad , \quad E_{\text{total i}} = E_{\text{total f}}}$$

Isolated system: no energy moves into or out of the system, no work is done on or by the system, and no net external forces are acting on the system

---

If no energy enters or leaves a system (to or from the environment) that system is an "isolated system"

isolated system

$$E_{\text{total}} = K + U_g + U_{sp} + E_{\text{therm}} + E_{\text{light}} + E_{\text{sound}} + E_{\text{chem}} + E_{\text{elec}} + \dots$$

$$\Delta E_{\text{total}} = 0$$

$$K \leftrightarrow U_g \leftrightarrow U_{sp} \leftrightarrow E_{\text{therm}}$$
$$E_{\text{light}} \leftrightarrow E_{\text{sound}} \leftrightarrow E_{\text{chem}} \leftrightarrow E_{\text{elec}}$$

environment

$$E_{\text{total}} = K + U_g + U_{sp} + E_{\text{therm}} + E_{\text{light}} + E_{\text{sound}} + E_{\text{chem}} + E_{\text{elec}} + \dots$$

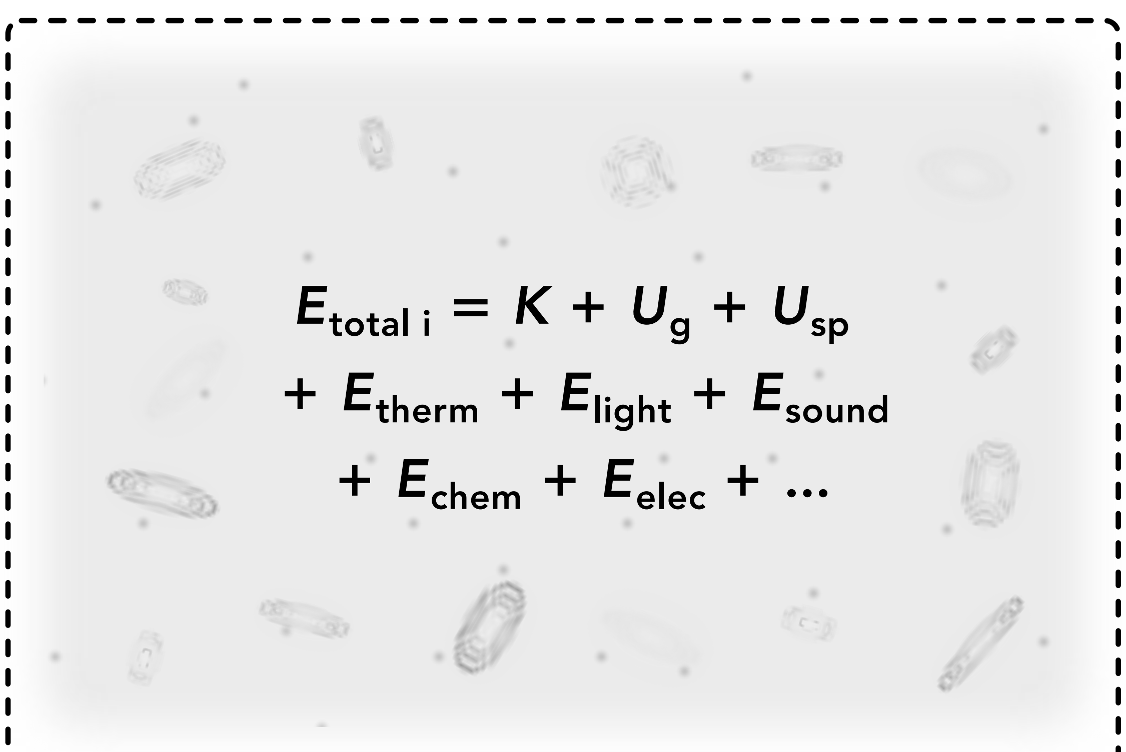
$$\Delta E_{\text{total}} = 0$$

$$K \leftrightarrow U_g \leftrightarrow U_{sp} \leftrightarrow E_{\text{therm}}$$
$$E_{\text{light}} \leftrightarrow E_{\text{sound}} \leftrightarrow E_{\text{chem}} \leftrightarrow E_{\text{elec}}$$

- The universe is considered an isolated system because there is nothing "outside" the universe and no external environment for energy to be transferred to or from.
- If you add up the total amount of each type of energy in the universe at one moment in time, that total will be the same at a different moment in time. The amounts of each type of energy may change, but the total amount of energy stays the same.

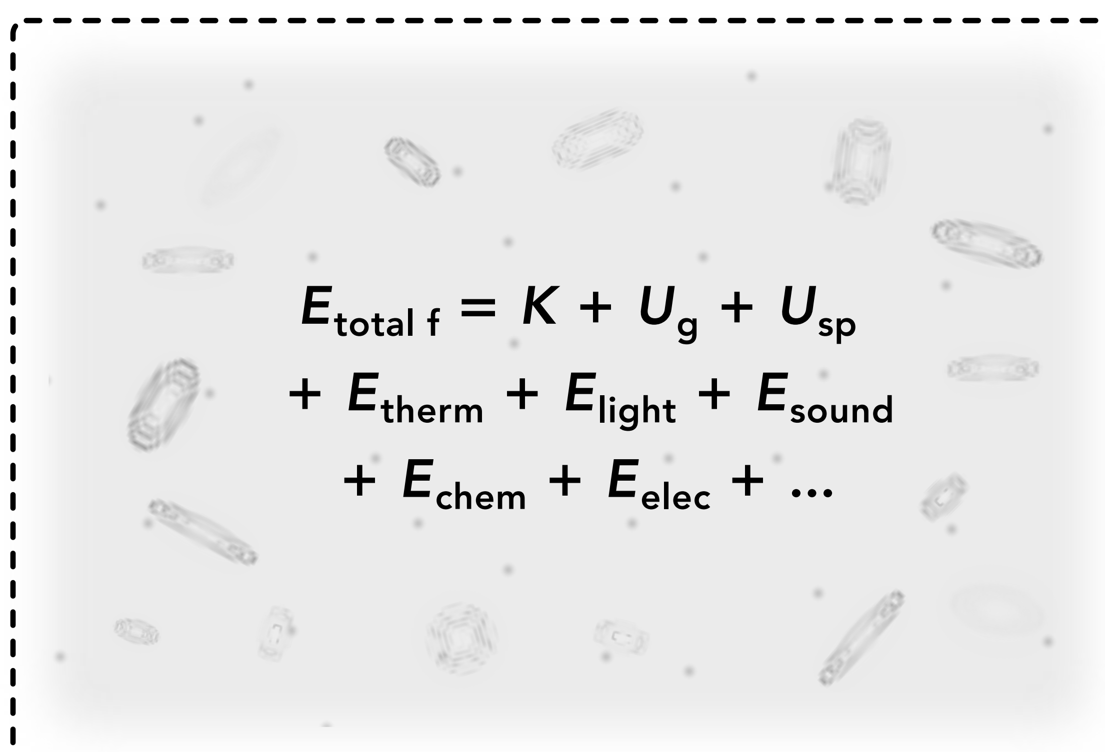
The total amount of energy in the universe stays the same over time

the universe at time t_i



$$E_{\text{total i}} = K + U_g + U_{sp} + E_{\text{therm}} + E_{\text{light}} + E_{\text{sound}} + E_{\text{chem}} + E_{\text{elec}} + \dots$$

the universe at time t_f



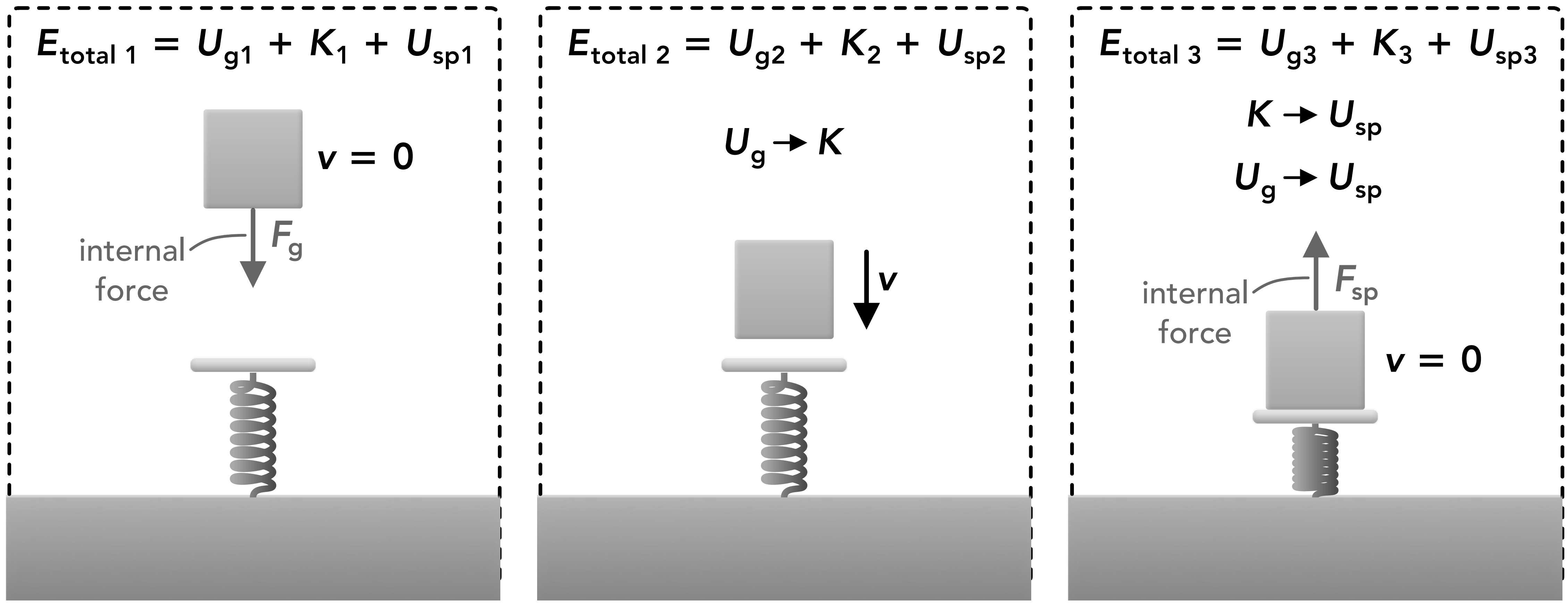
$$E_{\text{total f}} = K + U_g + U_{sp} + E_{\text{therm}} + E_{\text{light}} + E_{\text{sound}} + E_{\text{chem}} + E_{\text{elec}} + \dots$$

$$E_{\text{total i}} = E_{\text{total f}}$$

The total amount of energy in an isolated system stays the same over time

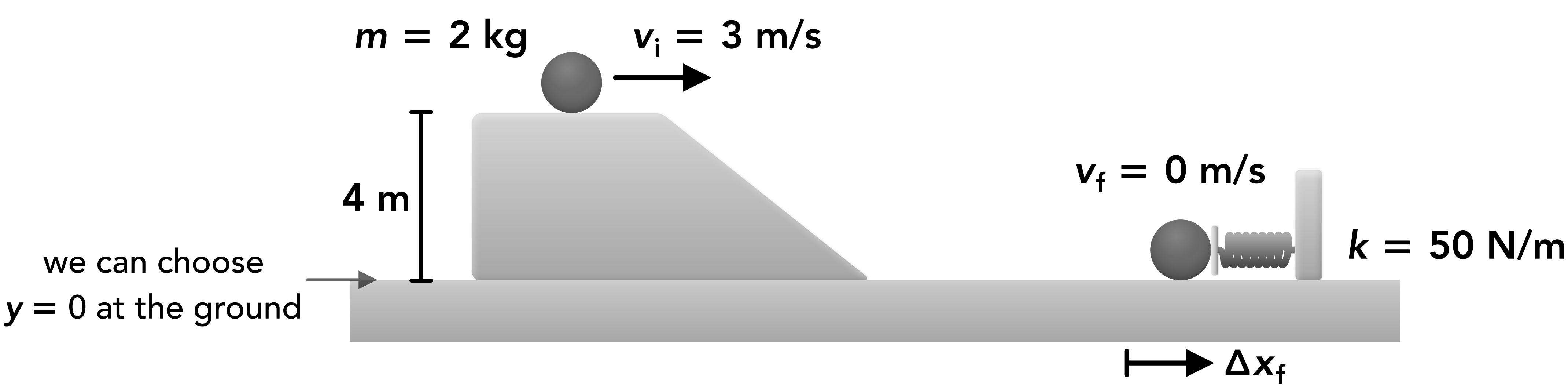
In the isolated system below, the block, the spring and the earth are all internal objects. The gravitational force between the block and the earth is an internal force which converts energy within the system. The spring force between the block and the spring is also is an internal force which converts energy within the sytem. The gravitational force and the spring force are NOT external forces and do NOT do work on the system.

block-spring-earth system at time t_1	block-spring-earth system at time t_2	block-spring-earth system at time t_3



t_1	t_2	t_3
$U_{g1} = 10$ J	$U_{g2} = 5$ J	$U_{g3} = 3$ J
$K_1 = 0$ J	$K_2 = 5$ J	$K_3 = 0$ J
$U_{sp1} = 0$ J	$U_{sp2} = 0$ J	$U_{sp3} = 7$ J
$E_{total\ 1} = 10$ J	$E_{total\ 2} = 10$ J	$E_{total\ 3} = 10$ J

Example: A 2 kg ball is 4 m above the ground with a speed of 3 m/s. It rolls down a ramp and contacts a spring with a spring constant of 50 N/m. The ball compresses the spring and momentarily comes to a stop. At that moment, how much is the spring compressed? Assume there is negligible friction.



*We're assuming that the ball-spring-earth system is an isolated system so the total energy in the system is conserved over time. The final total energy is equal to the initial total energy:

there is kinetic energy, gravitational potential energy and spring potential energy involved in this scenario

$$E_{\text{total i}} = E_{\text{total f}}$$

$$K_i + U_{g\,i} + U_{sp\,i} = K_f + U_{g\,f} + U_{sp\,f}$$

$$\frac{1}{2}mv_i^2 + mgy_i + \frac{1}{2}k\Delta x_i^2 = \frac{1}{2}mv_f^2 + mgy_f + \frac{1}{2}k\Delta x_f^2$$

$K = \frac{1}{2}mv^2$ $\qquad \frac{1}{2}(2\text{ kg})(3\text{ m/s})^2$ $\qquad \frac{1}{2}(2\text{ kg})(0\text{ m/s})^2$

$U_g = mgy$ $\qquad + (2\text{ kg})(9.8\text{ m/s}^2)(4\text{ m})$ $\qquad + (2\text{ kg})(9.8\text{ m/s}^2)(0\text{ m})$

$U_{sp} = \frac{1}{2}k\Delta x^2$ $\qquad + \frac{1}{2}(50\text{ N/m})(0\text{ m})^2$ $\qquad + \frac{1}{2}(50\text{ N/m})\Delta x_f^2$

$$87.4 = 25\Delta x_f^2$$

$$1.87\text{ m} = \Delta x_f$$

Work

- The law of conservation of energy says the total amount of energy within an isolated system stays the same over time. But if the system is not isolated energy can move into or out of the sytem, so the energy within the system is not conserved (but the total amount of energy in the universe is conserved).
- **Work** is the transfer of energy into or out of a system. This happens when external forces are applied over some displacement.
- The SI unit of work is a joule (J), the unit of energy. 1 J = 1 N·m.

Variables		SI Unit
W	work	$\mathrm{J = N \cdot m}$
E	energy	J
F	force	N
d	displacement	m

Work

$$\boxed{\Delta E_{\text{system}} = W}$$

- If energy is transferred **into** the system (from the environment) we say work is done "on" the system. The total amount of energy inside the system **increases**. The total amount of energy in the environment decreases.
- If energy is transferred **out of** the system (to the environment) we say work is done "by" the system. The total amount of energy inside the system **decreases**. The total amount of energy in the environment increases.

work done on the system by the environment is positive and increases the energy in the system

system (not isolated)

$$E_{\text{total}} = K + U_g + U_{sp} + E_{\text{therm}} + E_{\text{light}} + E_{\text{sound}} + E_{\text{chem}} + E_{\text{elec}} + \ldots$$

$$\Delta E_{\text{total}} = W$$

$$K \leftrightarrow U_g \leftrightarrow U_{sp} \leftrightarrow E_{\text{therm}}$$
$$E_{\text{light}} \leftrightarrow E_{\text{sound}} \leftrightarrow E_{\text{chem}} \leftrightarrow E_{\text{elec}}$$

$+W \leftarrow$

$\rightarrow -W$

environment

$$E_{\text{total}} = K + U_g + U_{sp} + E_{\text{therm}} + E_{\text{light}} + E_{\text{sound}} + E_{\text{chem}} + E_{\text{elec}} + \ldots$$

$$\Delta E_{\text{total}} = -W$$

$$K \leftrightarrow U_g \leftrightarrow U_{sp} \leftrightarrow E_{\text{therm}}$$
$$E_{\text{light}} \leftrightarrow E_{\text{sound}} \leftrightarrow E_{\text{chem}} \leftrightarrow E_{\text{elec}}$$

work done by the system on the environment is negative and decreases the energy in the system

---

Work done on the system is positive and increases the system's energy

system (not isolated): $E_{\text{total}} = 10\ \text{J}$

environment: $E_{\text{total}} = \ldots\ \text{J}$

system (not isolated):

$E_{\text{total}} = 14\ \text{J}$

$\Delta E_{\text{total}} = W = +4\ \text{J}$

$4\ \text{J} \leftarrow$

environment:

$E_{\text{total}} = \ldots - 4\ \text{J}$

$\Delta E_{\text{total}} = -4\ \text{J}$

Work done by the system is negative and decreases the system's energy

system (not isolated): $E_{\text{total}} = 10\ \text{J}$

environment: $E_{\text{total}} = \ldots\ \text{J}$

system (not isolated):

$E_{\text{total}} = 2\ \text{J}$

$\Delta E_{\text{total}} = W = -8\ \text{J}$

$\rightarrow 8\ \text{J}$

environment:

$E_{\text{total}} = \ldots + 8\ \text{J}$

$\Delta E_{\text{total}} = +8\ \text{J}$

- The result of work is a change in the amount of energy within a system. But work is done on or by a system due to external forces when the system moves, and work can also be calculated as the displacement of the system multiplied by the component of the external force that is parallel to that displacement.

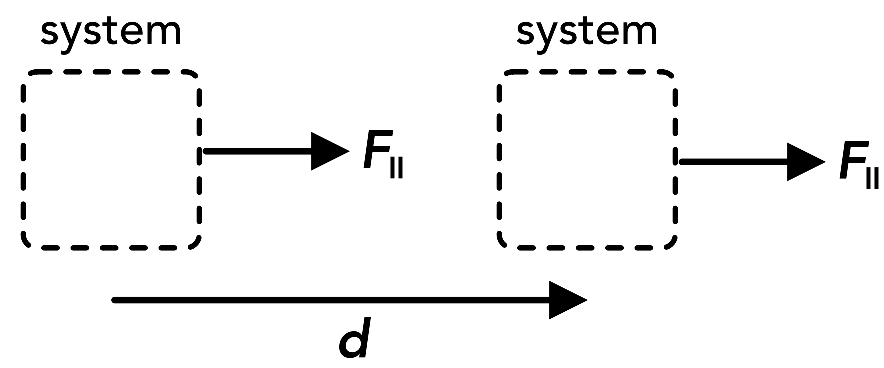
Work

$$W = F_{\parallel} d$$

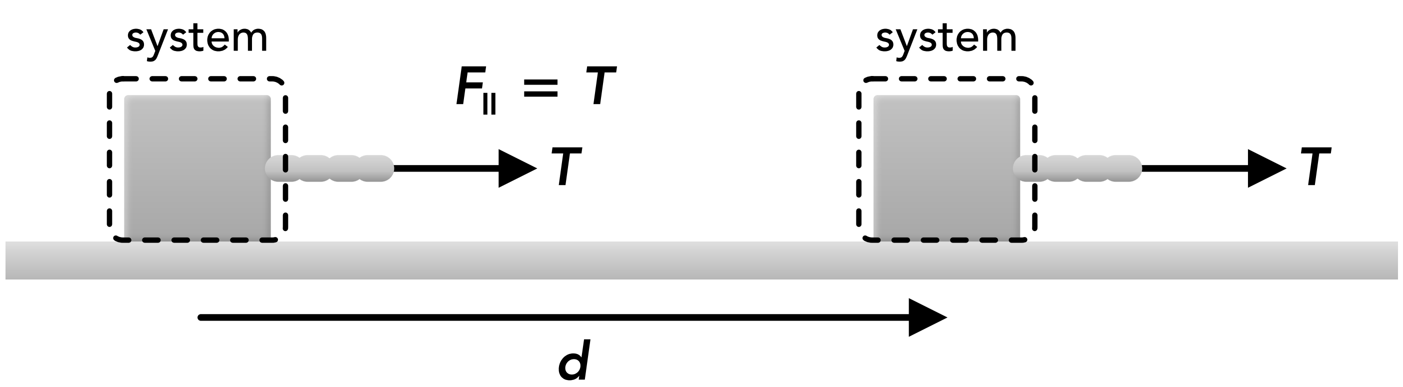
$F_{\parallel}$: component of force parallel to d

*F is an external force

d : displacement of the system

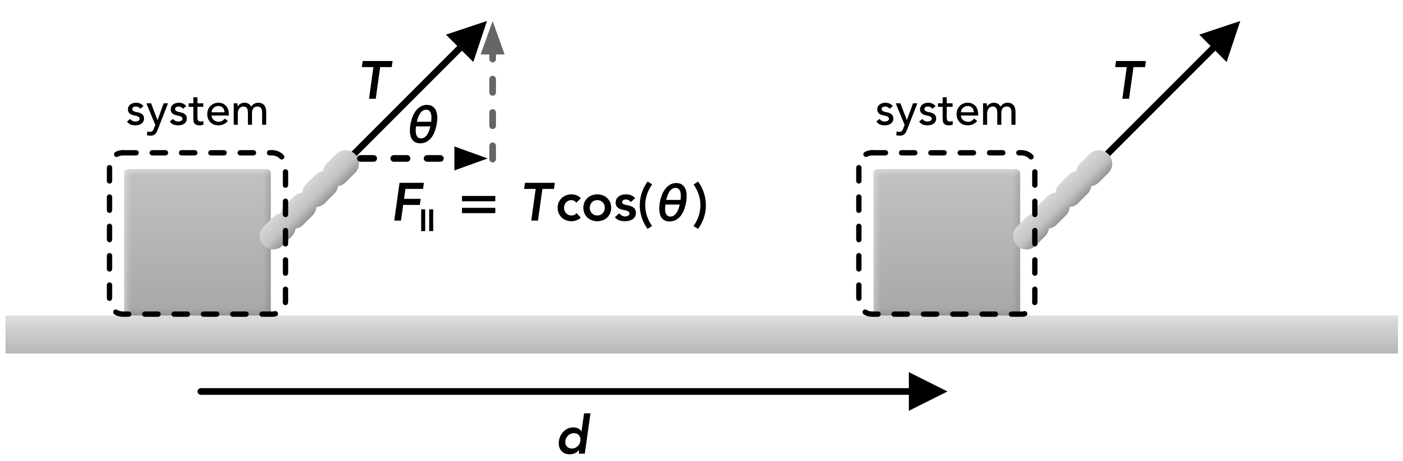


Only the force component parallel to the displacement does work on the system, and the sign of the work (positive or negative) depends on if the force component and displacement are in the same direction or opposite directions



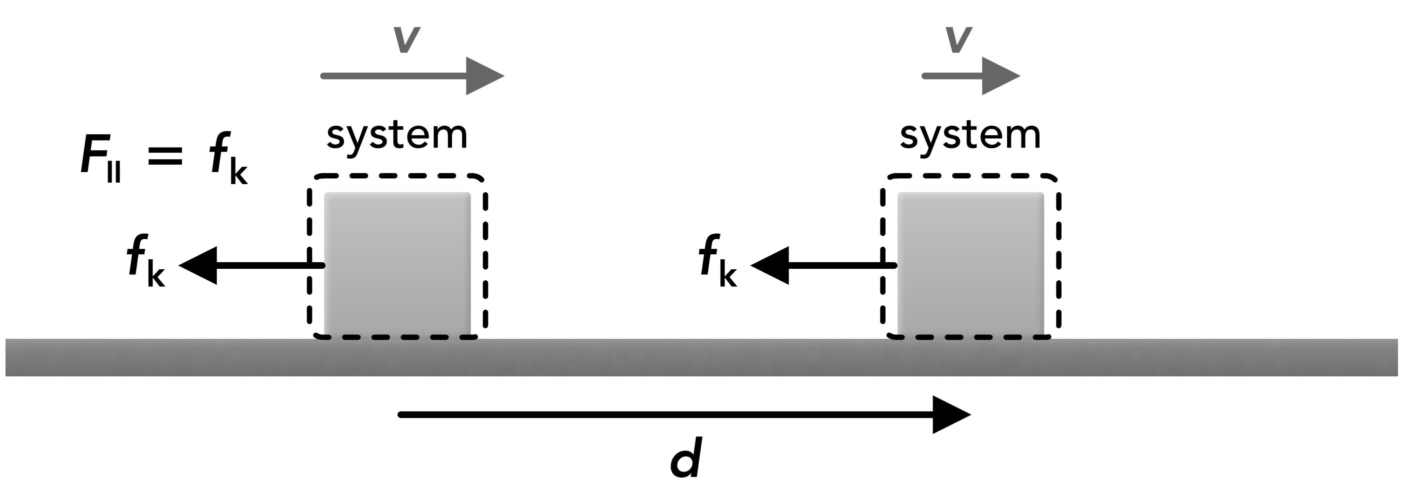
$F_{\parallel}$ and d are in the same direction, work is positive

$$W = F_{\parallel} d = Fd$$



$F_{\parallel}$ and d are in the same direction, work is positive

$$W = F_{\parallel} d = T\cos(\theta)\, d$$



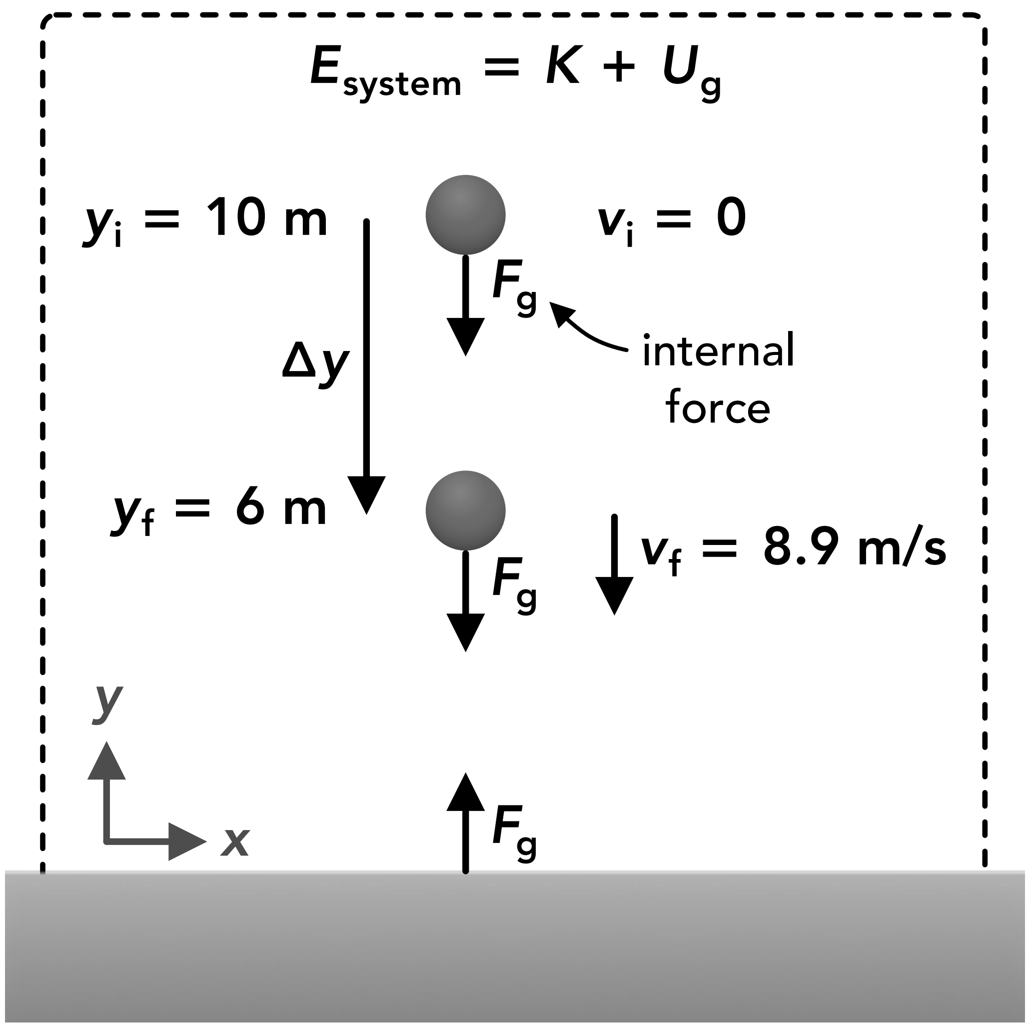
$F_{\parallel}$ and d are in opposite directions, work is negative

$$W = F_{\parallel} d = -f_k d$$

- **Internal forces** are any forces acting between objects within a system (internal objects). Internal forces can convert energy between different types within the system, but they do not change the total energy in the system.
- **External forces** are any forces acting between an object outside of a system (external object) and an object inside the system (internal object).
- **Work is done on or by a system due to external forces**. Internal forces do not do work on a system, they only only convert energy between different types within a system.

If the earth is included in the system, the gravitational force between the ball and the earth is an internal force which converts gravitational potential energy into kinetic energy within the system but it does not do work on the ball.

ball-earth system (isolated)



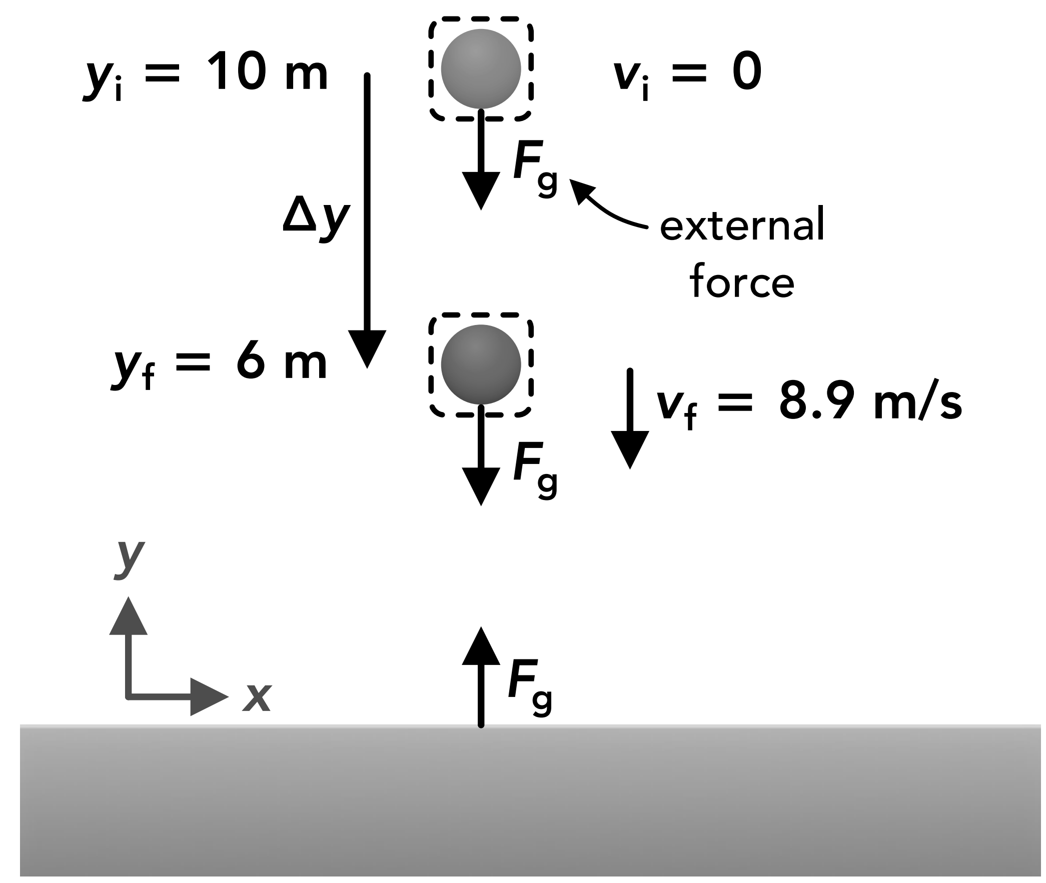
$$\Delta E_{system} = 0$$
$$E_{system\ i} = E_{system\ f}$$
$$K_i + U_{gi} = K_f + U_{gf}$$
$$K_f - K_i = -(U_{gf} - U_{gi})$$
$$\Delta K = -\Delta U_g$$
$$\Delta K = -(mg(6\text{ m} - 10\text{ m}))$$
$$\Delta K = mg(4\text{ m})$$

the change in the kinetic energy of the ball-earth system is equal to the negative of the change in gravitational potential energy

If the earth is not included in the system, the gravitational force acting on the ball is an external force which does work on the ball system, changing its kinetic energy. The ball system does not have gravitational potential energy because the earth is not included in the system.

ball system (not isolated)

$$E_{system} = K$$



$$\Delta E_{system} = W$$
$$\Delta K = F_{\parallel} d$$
$$\Delta K = (-mg)(6\text{ m} - 10\text{ m})$$
$$\Delta K = mg(4\text{ m})$$

the change in the kinetic energy of the ball system is equal to the work done on the system

Power

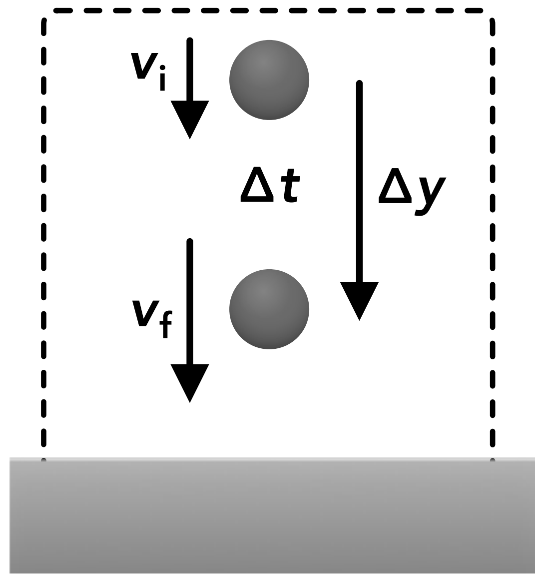
- **Power** is the rate of energy converted or transferred over time.
- The SI unit for power is a watt (W) which should not be confused with the variable for work (***W***). 1 watt is equal to 1 joule/second (J/s).
- There are several equations used to calculate power. In each case, "power" means an amount of energy per unit of time.

Variables		SI Unit
P	power	$\mathrm{W} = \frac{\mathrm{J}}{\mathrm{s}}$
E	energy	J
W	work	J
F	force	N
v	velocity	$\frac{\mathrm{m}}{\mathrm{s}}$

If energy is being converted from one type of energy to another, the power is the change in one type of energy divided by the period of time

Power

$$P = \frac{\Delta E}{\Delta t}$$



Example:

$$P = \frac{\Delta K}{\Delta t} = \frac{-\Delta U_g}{\Delta t}$$

If energy is transferred into or out of a system through work, the power is the amount of work done divided by the period of time, which is also equal to the parallel force component multiplied by the velocity of the system

Power

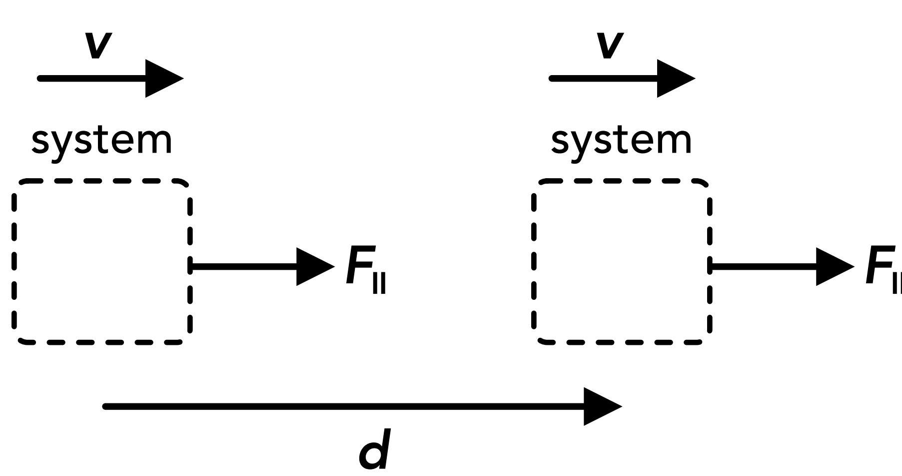
$$P = \frac{W}{\Delta t} = F_{\parallel} v$$

$F_{\parallel}$: component of force parallel to v

*F is an external force

v : velocity of the system

$$P = \frac{W}{\Delta t} = \frac{F_{\parallel} d}{\Delta t} = F_{\parallel} v$$



MOMENTUM & IMPULSE

Momentum

Variables		SI Unit
p	momentum	$\frac{kg \cdot m}{s}$
m	mass	kg
v	velocity	$\frac{m}{s}$

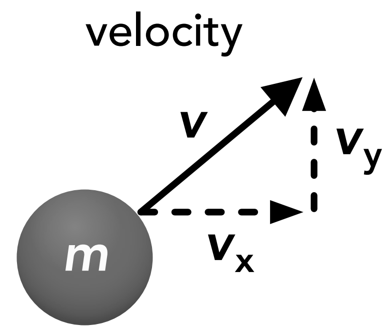
- The **linear momentum** of an object or system is its mass times its velocity. The word "momentum" by itself usually refers to linear momentum.
- Momentum is a **vector** so it has a magnitude and a direction.
- The momentum vector is in the same direction as the velocity vector.
- Linear momentum may seem similar to linear kinetic energy but momentum is a vector while kinetic energy is a scalar quantity, and the equations are different.

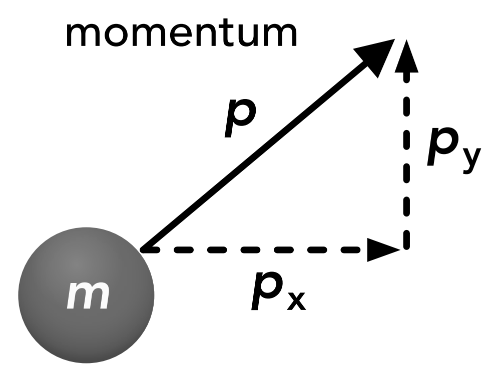
Momentum

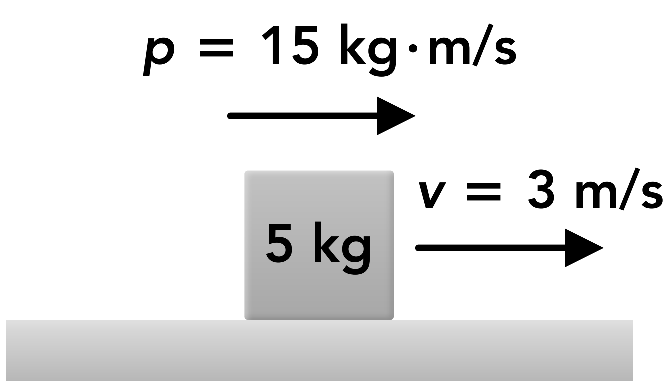
$\vec{p} = m\vec{v}$

Momentum vector components

$p_x = mv_x$ $\quad$ $p_y = mv_y$

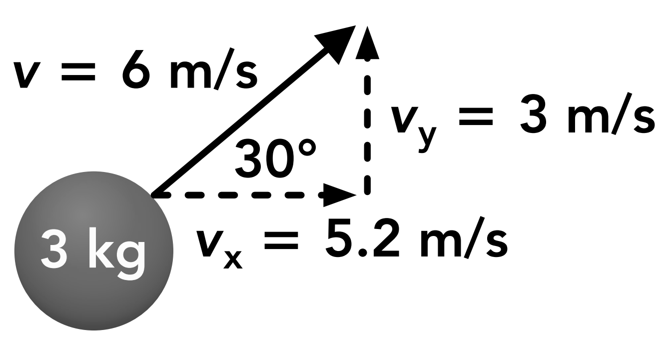


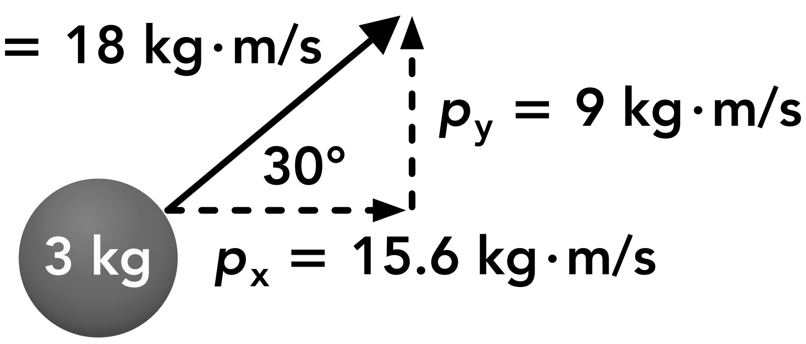




$p = (5 \text{ kg})(3 \text{ m/s})$

$p = 15 \text{ kg}\cdot\text{m/s}$





$$p = (3 \text{ kg})(6 \text{ m/s}) = 18 \text{ kg}\cdot\text{m/s}$$

$v_x = (6 \text{ m/s})\cos(30°)$ $\quad$ $p_x = (18 \text{ kg}\cdot\text{m/s})\cos(30°)$

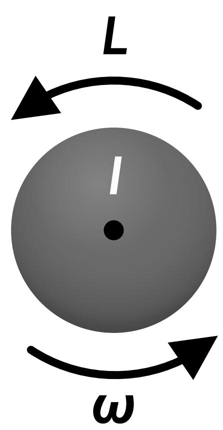
$v_y = (6 \text{ m/s})\sin(30°)$ $\quad$ $p_y = (18 \text{ kg}\cdot\text{m/s})\sin(30°)$

- The **angular momentum** of an object or system is its rotational inertia times its angular velocity.
- Angular momentum is a **vector** so it has a magnitude and a direction.
- The angular momentum vector is in the same direction as the angular velocity, either clockwise (CW) or counterclockwise (CCW).
- Angular momentum may seem similar to rotational kinetic energy but angular momentum is a vector while rotational kinetic energy is a scalar, and the equations are different.

Variables		SI Unit
L	angular momentum	$\frac{kg \cdot m^2}{s}$
I	rotational inertia	$kg \cdot m^2$
ω	angular velocity	$\frac{rad}{s}$

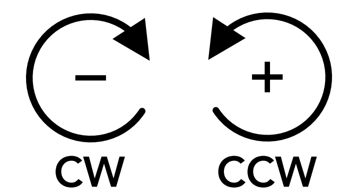
Angular momentum

$$L = I\omega$$

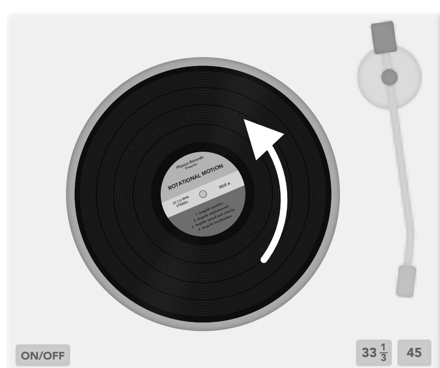


angular momentum is in the same direction as the angular velocity

counterclockwise direction is positive
clockwise direction is negative

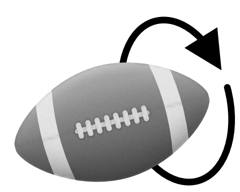


$\omega = 3.5$ rad/s
$I = 0.001\ kg \cdot m^2$



$L = (0.001)(3.5)$
$L = 0.0035\ kg \cdot m^2/s$

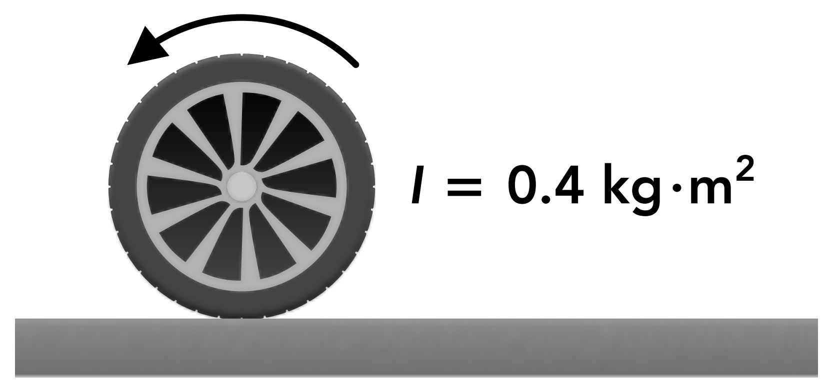
$\omega = 60$ rad/s



$I = 0.001\ kg \cdot m^2$

$L = (0.001)(60)$
$L = 0.06\ kg \cdot m^2/s$

$\omega = 100$ rad/s



$I = 0.4\ kg \cdot m^2$

$L = (0.4)(100)$
$L = 40\ kg \cdot m^2/s$

Impulse

- **Impulse** is the change in momentum of an object or system, which is caused by an external force applied over a period of time.
- When a force is applied to an object or system, Newton's 2nd law of motion ($\vec{F}_{net} = m\vec{a}$) says that force causes an acceleration. Acceleration is a change in velocity over time, so that force changes the object's or system's velocity, which means it also changes the momentum (which depends on the velocity).
- Impulse is a **vector** so it has a magnitude and a direction. The impulse has the same direction as the applied force.

Variables		SI Unit
J	impulse	$\frac{kg \cdot m}{s} = N \cdot s$
p	momentum	$\frac{kg \cdot m}{s}$
F	force	N
t	time	s

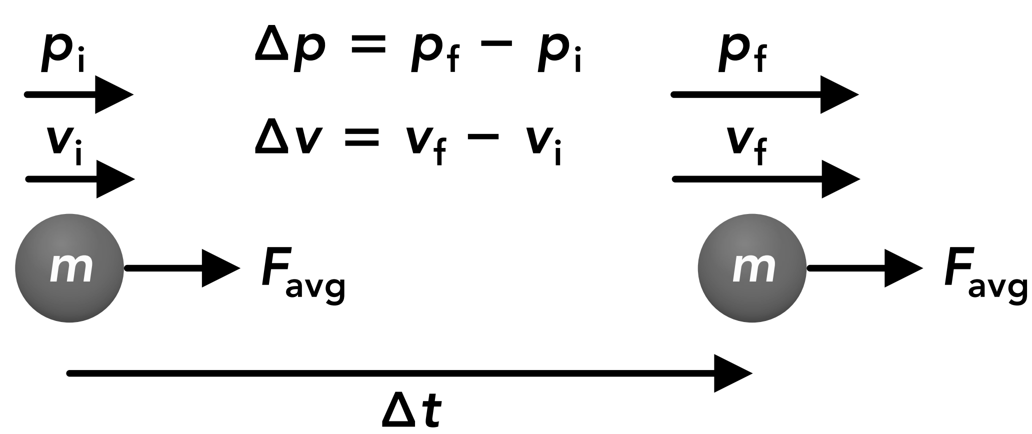
Impulse

$$\boxed{\vec{J} = \Delta\vec{p} = \vec{F}_{avg}\Delta t}$$

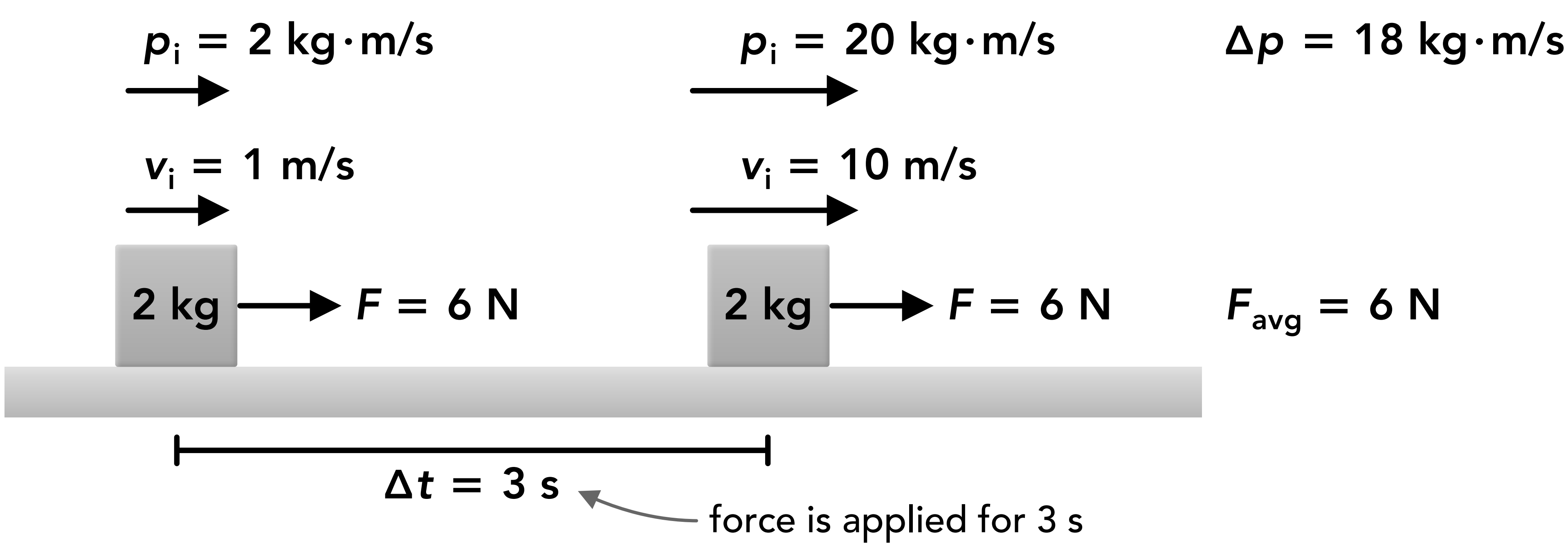
F_{avg} : average force over time

If we assume m is constant:

$$F_{avg}\Delta t = m\Delta v$$

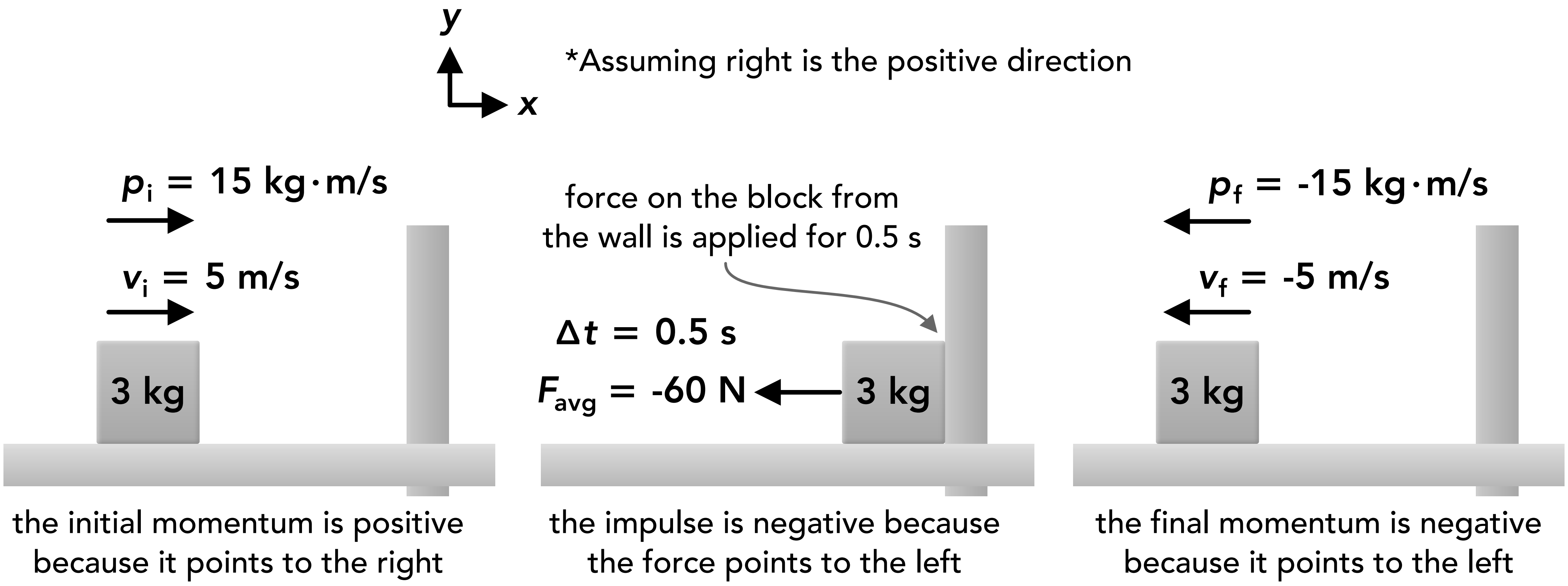


Example: A constant force is applied to a block sliding on a frictionless surface. The force is in the same direction as the initial velocity and momentum so the impulse from the force increases the block's momentum and velocity.



$$J = \Delta p = F_{avg}\Delta t$$
$$J = p_f - p_i = F_{avg}\Delta t$$
$$J = (2\text{ kg})(10\text{ m/s}) - (2\text{ kg})(1\text{ m/s}) = (6\text{ N})(3\text{ s})$$
$$J = 18\text{ kg}\cdot\text{m/s} = 18\text{ N}\cdot\text{s or kg}\cdot\text{m/s}$$

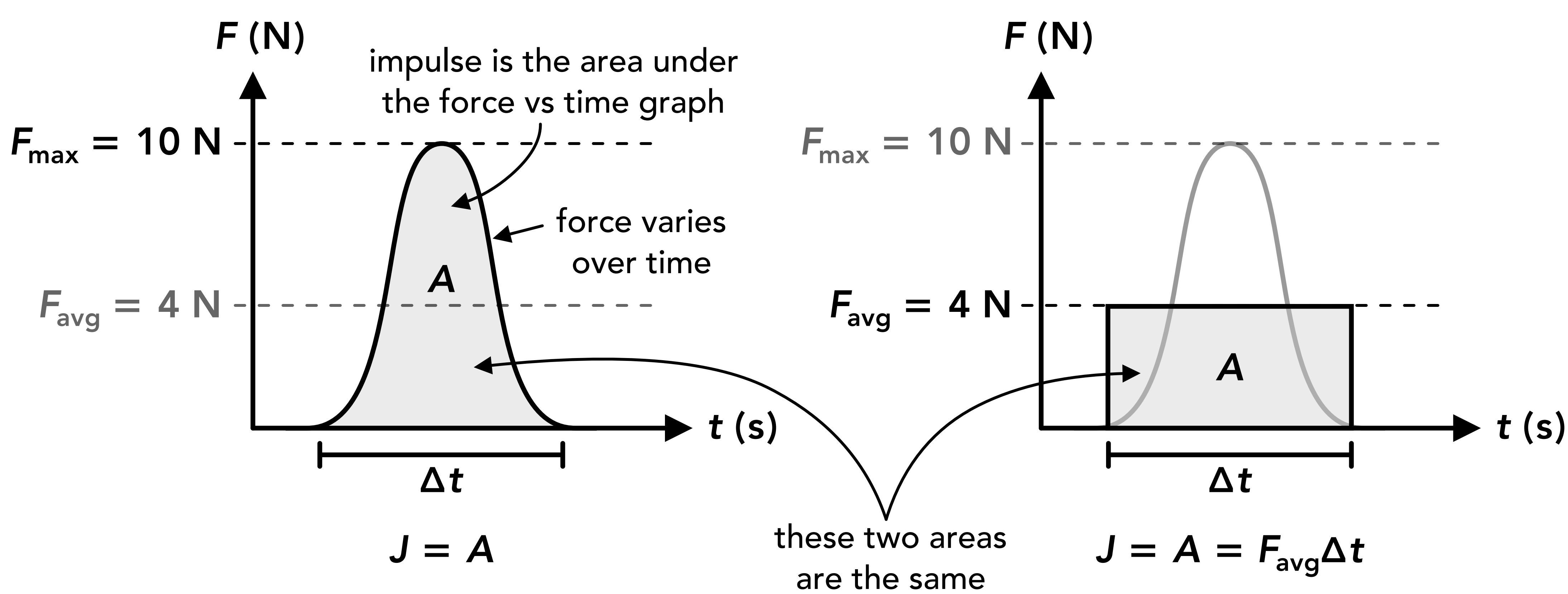
Example: A block slides on a frictionless surface and bounces off a wall, which applies a force to the block for a short period of time. The force is in the opposite direction as the initial velocity and momentum so the impulse from the force decreases and reverses the block's momentum and velocity.



$$J = \Delta p = F_{avg}\Delta t$$
$$J = p_f - p_i = F_{avg}\Delta t$$
$$J = (3\text{ kg})(-5\text{ m/s}) - (3\text{ kg})(5\text{ m/s}) = (-60\text{ N})(0.5\text{ s})$$
$$J = -30\text{ kg}\cdot\text{m/s} = -30\text{ N}\cdot\text{s or kg}\cdot\text{m/s}$$

- The force applied to the object or system can vary in magnitude over time. This is often the case in the real world.
- F_{avg} is the average magnitude of the force over the period of time that it's acting on the object.
- If we have a graph of the force applied vs time, **the impulse is the area under the curve** (the area between the graphed line and the horizontal axis). In calculus, that would be the integral of the force over time. But if we know the value of the average force or if the graph is a rectangle or a triangle, we can find the area using geometry.

The area under the curve of the force vs time graph is equal to the area under the curve of the average force during that same interval, and the areas are equal to the impulse



- The concept of impulse also applies to rotational dynamics, although there's no word for "rotational impulse".
- When a torque is applied to an object or system, that torque causes an angular acceleration so the torque changes the object's angular velocity and its angular momentum (which depends on the angular velocity).

Variables		SI Unit
τ	torque	$N \cdot m$
L	angular momentum	$\frac{kg \cdot m^2}{s}$
I	rotational inertia	$kg \cdot m^2$
ω	angular velocity	$\frac{rad}{s}$

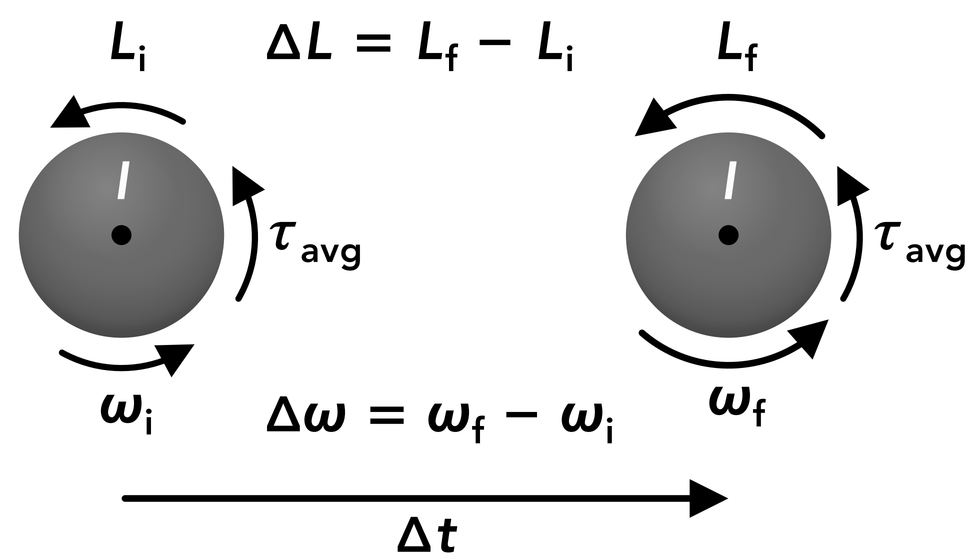
Rotational impulse

$$\boxed{\Delta L = \tau_{avg} \Delta t}$$

τ_{avg} : average torque over time

If we assume I is constant:

$$\tau_{avg}\Delta t = I\Delta\omega$$



CONSERVATION OF MOMENTUM

Law of Conservation of Momentum

- When a scenario involves multiple objects we can consider them as a system.
- If there is **no external force** acting on the system we call it an **isolated sytem**.
- **The law of conservation of momentum**: the total momentum within an isolated is conserved (it's constant over time) regardless of the internal interactions between the objects in the system. The momentum of each object is not conserved, only the total momentum of the system.
- This does not mean that the total momentum within a system is zero (although it can be), it means the change in momentum is zero between two times.
- If there is a net external force on the system, the system is not isolated and an impulse is exerted on the objects in the system, changing the total momentum.

Variables		SI Unit
p	momentum	$\frac{kg \cdot m}{s}$
m	mass	kg
v	velocity	$\frac{m}{s}$
J	impulse	$\frac{kg \cdot m}{s}$
F	force	N
t	time	s

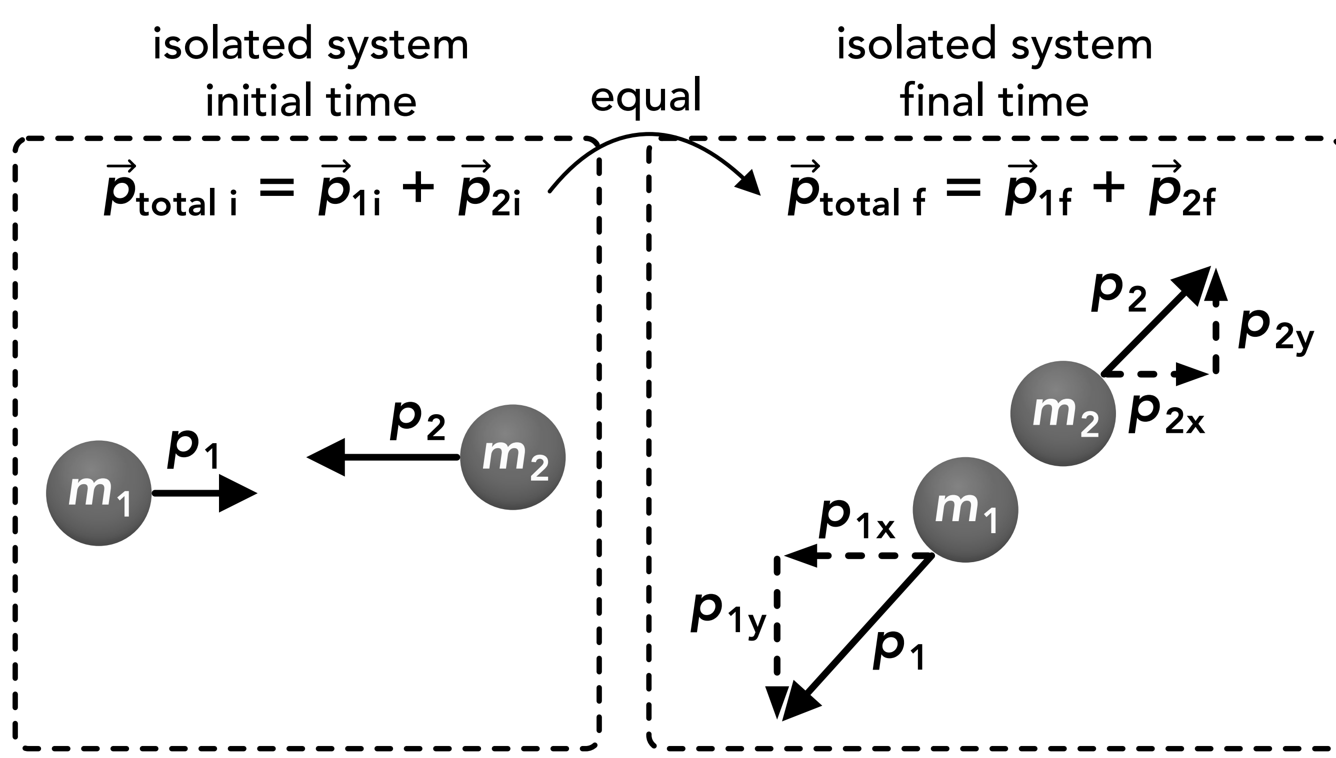
Law of conservation of momentum (universe and isolated systems)

$$\boxed{\Delta\vec{p}_{total} = 0 \quad , \quad \vec{p}_{total\ i} = \vec{p}_{total\ f}}$$

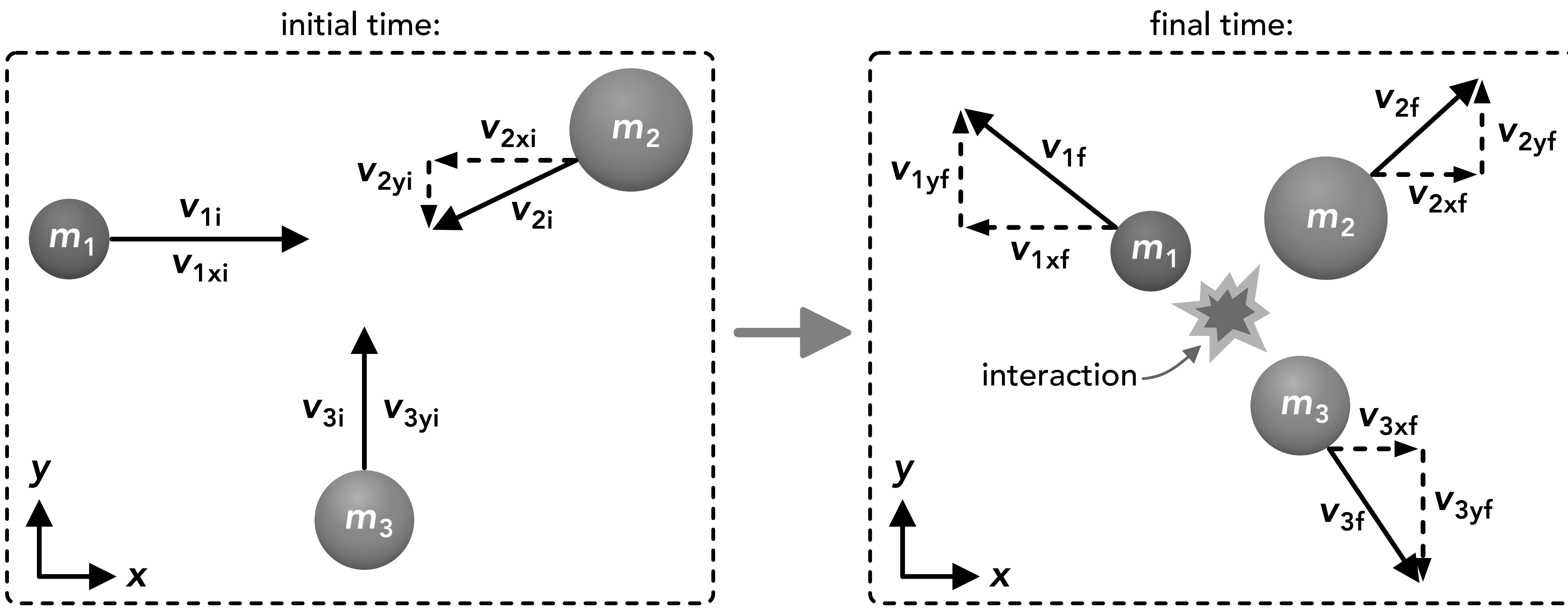
$$\Delta p_{x\ total} = 0 \quad , \quad p_{xi\ total} = p_{xf\ total}$$

$$\Delta p_{y\ total} = 0 \quad , \quad p_{yi\ total} = p_{yf\ total}$$

Isolated system: there is no net external force acting on the system



Momentum is a vector so momentum is conserved in the *x* and *y* directions



$$\vec{p}_{\text{total i}} = \vec{p}_{\text{total f}}$$

x **momentum components:**

$$p_{xi\ \text{total}} = p_{xf\ \text{total}}$$

the initial *x* momentum components for each object

the final *x* momentum components for each object

$$p_{1xi} + p_{2xi} + p_{3xi} = p_{1xf} + p_{2xf} + p_{3xf}$$

$$m_1 v_{1xi} + m_2 v_{2xi} + m_3 v_{3xi} = m_1 v_{1xf} + m_2 v_{2xf} + m_3 v_{3xf}$$

y **momentum components:**

$$p_{yi\ \text{total}} = p_{yf\ \text{total}}$$

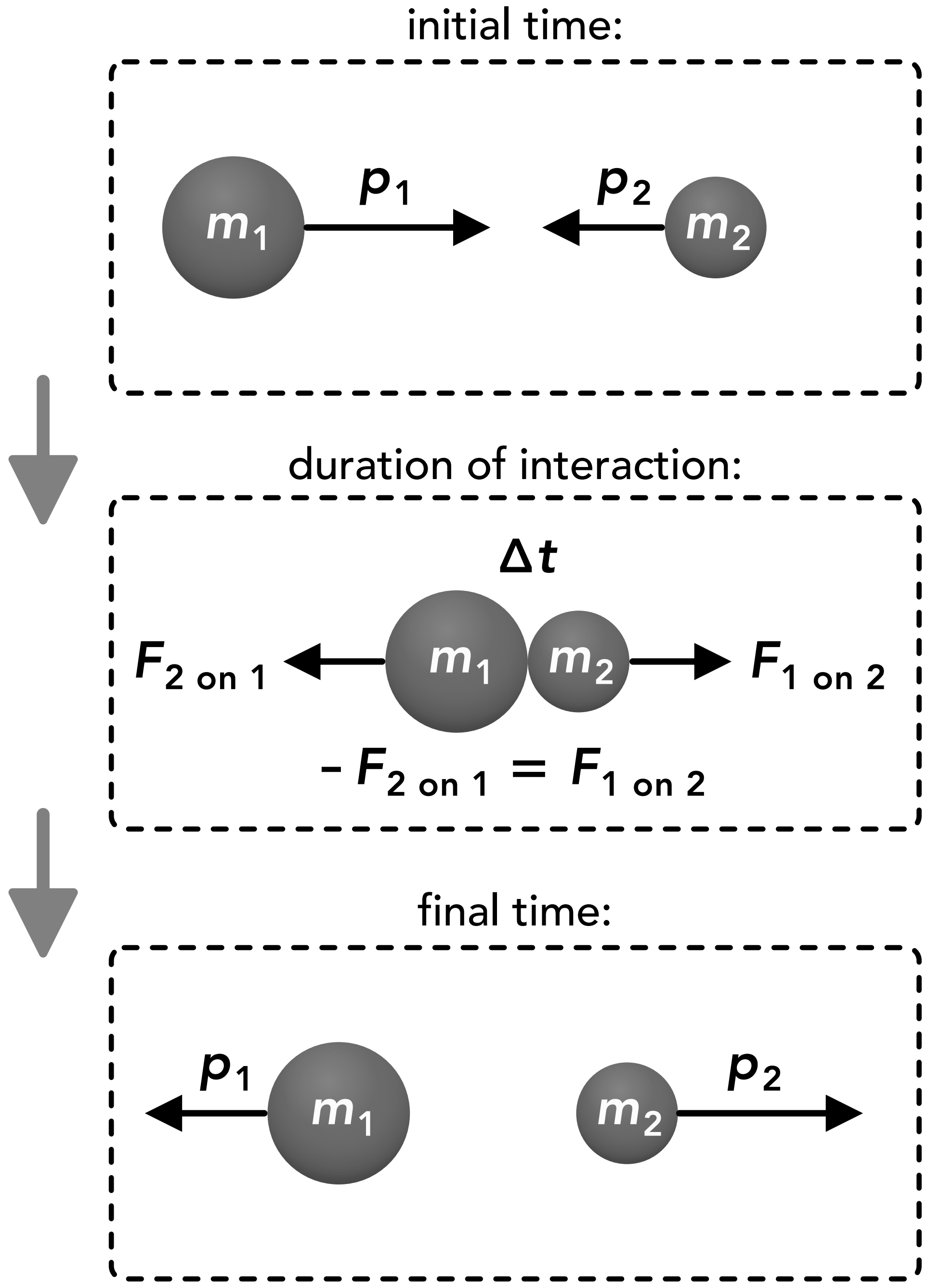
the initial *y* momentum components for each object

the final *y* momentum components for each object

$$p_{1yi} + p_{2yi} + p_{3yi} = p_{1yf} + p_{2yf} + p_{3yf}$$

$$m_1 v_{1yi} + m_2 v_{2yi} + m_3 v_{3yi} = m_1 v_{1yf} + m_2 v_{2yf} + m_3 v_{3yf}$$

- When two objects **inside** a system interact, the forces they exert on each other (internal forces) are a pair of forces which are equal in magnitude and opposite in direction (Newton's 3rd law of motion).
- If the force acting on one object has the same magnitude and is exerted for the same duration as the force on the other object, the impulse exerted on each object is the same but in opposite directions. This means the net impulse on the pair of objects is zero, so **the total change in momentum is zero**.



$$\vec{p}_{\text{total i}} = \vec{p}_{1\text{i}} + \vec{p}_{2\text{i}}$$

the impulse exerted on each object is the same (but in opposite directions)

$$J_1 = -F_{2\text{ on }1}\Delta t$$
$$J_2 = F_{1\text{ on }2}\Delta t$$
$$J_{\text{total}} = J_1 + J_2 = 0 \longrightarrow \Delta p_{\text{total}} = 0$$

the total change in momentum of the two objects is zero

$$\vec{p}_{\text{total f}} = \vec{p}_{1\text{f}} + \vec{p}_{2\text{f}}$$

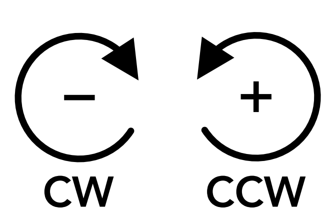
- **The law of conservation of angular momentum**: the total angular momentum of an isolated system is conserved (it's constant over time) regardless of the interactions within the system.
- For rotation, an **isolated system** means there is **no net external torque** acting on the system.
- Angular momentum is a **vector** which has the same direction as the angular velocity (either clockwise or counterclockwise).

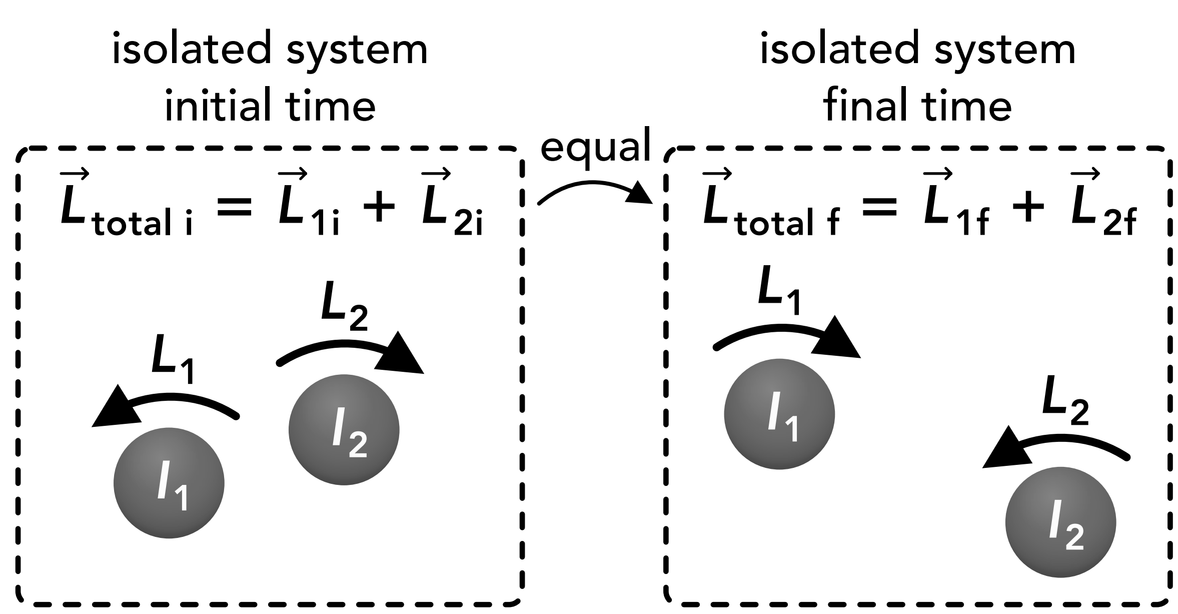
Variables		SI Unit
L	angular momentum	$\frac{\text{kg}\cdot\text{m}^2}{\text{s}}$
I	moment of inertia	$\text{kg}\cdot\text{m}^2$
ω	angular velocity	$\frac{\text{rad}}{\text{s}}$

Law of conservation of angular momentum (universe and isolated systems)

$$\Delta\vec{L}_{\text{total}} = 0 \quad , \quad \vec{L}_{\text{total i}} = \vec{L}_{\text{total f}}$$

Isolated system: there is no net external torque acting on the system

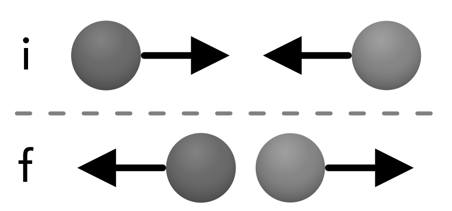
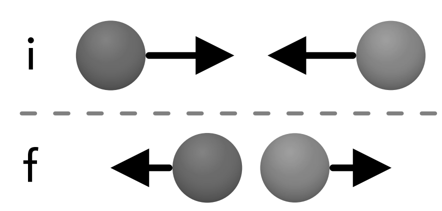
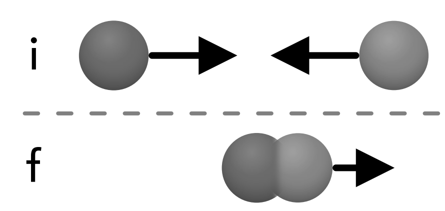
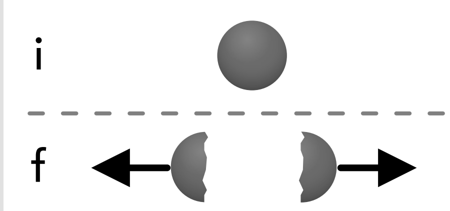




Types of Collisions & Events

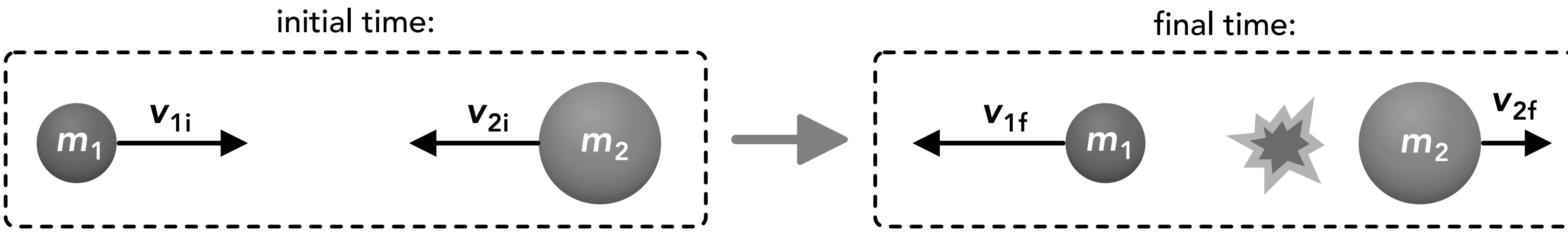
Variables		SI Unit
p	**momentum**	$\frac{kg \cdot m}{s}$
K	**kinetic energy**	J

- There are several types of “events” which can be studied using the law of conservation of momentum such as collisions and explosions.
- If the system is defined so that all of the relevant objects are included within the system and there are no external forces acting on the system, then **the total momentum in the system is conserved** and we can use that law to analyze the individual objects before and after the event.
- For an elastic collision the total kinetic energy of the system is also conserved.

Elastic collision (perfectly elastic)	Inelastic collision (partially elastic)	Perfectly inelastic collision	Explosion
			
momentum is conserved $\vec{p}_{\text{total i}} = \vec{p}_{\text{total f}}$	momentum is conserved $\vec{p}_{\text{total i}} = \vec{p}_{\text{total f}}$	momentum is conserved $\vec{p}_{\text{total i}} = \vec{p}_{\text{total f}}$	momentum is conserved $\vec{p}_{\text{total i}} = \vec{p}_{\text{total f}}$
kinetic energy is conserved $K_{\text{total i}} = K_{\text{total f}}$	kinetic energy is NOT conserved $K_{\text{total i}} \neq K_{\text{total f}}$	kinetic energy is NOT conserved $K_{\text{total i}} \neq K_{\text{total f}}$	kinetic energy is NOT conserved $K_{\text{total i}} \neq K_{\text{total f}}$

Elastic collision (perfectly elastic collision)

- An **elastic collision** (a perfectly elastic collision) is when two or more objects collide with each other and then move away from each other (or in the same direction at different speeds).
- **The total kinetic energy of the system is conserved** in a perfectly elastic collision.



$$\vec{p}_{\text{total i}} = \vec{p}_{\text{total f}}$$

$$p_{1i} + p_{2i} = p_{1f} + p_{2f}$$

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \longleftarrow \text{(equation 1)}$$

$$K_{\text{total i}} = K_{\text{total f}}$$

$$K_{1i} + K_{2i} = K_{1f} + K_{2f}$$

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \longleftarrow \text{(equation 2)}$$

system of two equations can be used to solve for unknown variables

---

By combining the two equations above (conservation of momentum and conservation of kinetic energy) using substitution and some algebra, we get an equation that we can use when two objects collide elastically and we don't know either of the final velocities. This can then be used with the conservation of momentum as a simpler set of two equations to solve.

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \longleftarrow \text{(equation 1)}$$

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

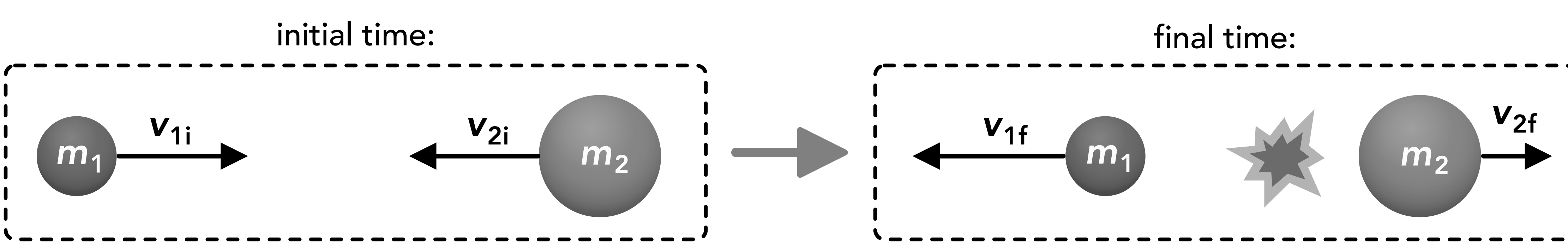
$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

$$\longrightarrow v_{1i} + v_{1f} = v_{2i} + v_{2f} \longleftarrow \text{(equation 3)}$$

if both final velocities are unknown
(elastic collision only)

Inelastic collision (partially elastic collision)

- An **inelastic collision** (a partially elastic collision) is when two or more objects collide with each other and then move away from each other (or in the same direction at different speeds).
- **The total kinetic energy of the system is not conserved** in an inelastic collision. Some of the initial kinetic energy is converted into thermal energy, sound energy, light energy or energy that deforms the objects.



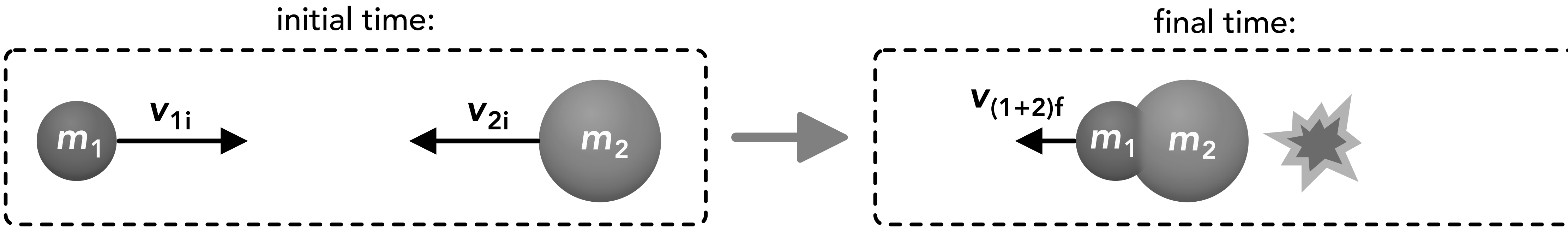
$$\vec{p}_{\text{total i}} = \vec{p}_{\text{total f}}$$

$$p_{1i} + p_{2i} = p_{1f} + p_{2f}$$

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

Perfectly inelastic collision

- A **perfectly inelastic collision** is when two or more objects collide with each other and stick together, so the objects move together with the **same final velocity** after the collision and can be treated as a single object.
- **The total kinetic energy of the system is not conserved** in a perfectly inelastic collision. Some or all of the initial kinetic energy is converted into thermal energy, sound energy, light energy or energy that deforms the objects.

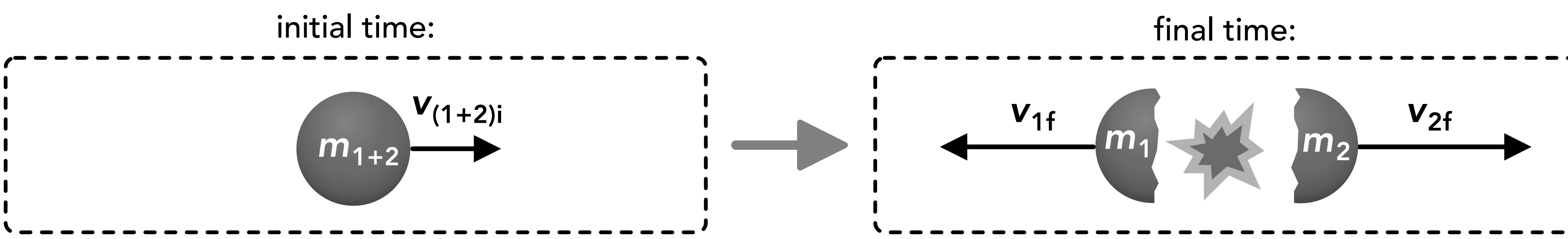


$$\vec{p}_{\text{total i}} = \vec{p}_{\text{total f}}$$
$$p_{1i} + p_{2i} = p_{(1+2)f}$$
$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_{(1+2)f}$$

the two objects stick together and move as one object

Explosion

- An **explosion** is when one object breaks apart into smaller pieces or a group of objects start together and then move away from each other. The object or group of objects may have some velocity before the explosion.
- **The total kinetic energy of the system is not conserved** in an explosion. Since kinetic energy is a scalar quantity and not a vector quantity it does not depend on direction and all of the kinetic energies will be positive. We can imagine an event where the initial kinetic energy is zero but there are final kinetic energies after the explosion.



$$\vec{p}_{\text{total i}} = \vec{p}_{\text{total f}}$$
$$p_{(1+2)i} = p_{1f} + p_{2f}$$
$$(m_1 + m_2) v_{(1+2)i} = m_1 v_{1f} + m_2 v_{2f}$$

the pieces or group start together as one object

SIMPLE HARMONIC MOTION

Simple Harmonic Motion

Variables		SI Unit
T	period	s
f	frequency	$\text{Hz} = \frac{\text{cycles}}{\text{s}}$
A	amplitude	m
x, y	position	m
v	velocity	$\frac{\text{m}}{\text{s}}$
a	acceleration	$\frac{\text{m}}{\text{s}^2}$

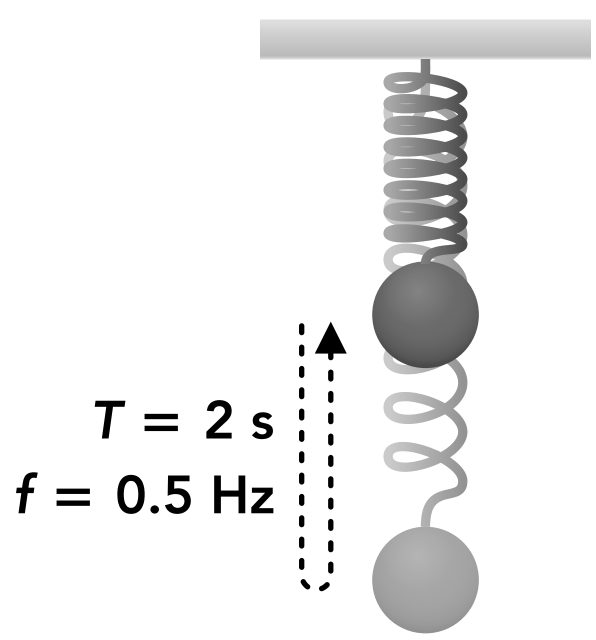
- We know that a net force acting on an object causes the object to accelerate (Newton's laws of motion). In many cases that motion is in one direction, but there are cases where the direction of the net force alternates back and forth repeatedly which causes the object to oscillate.
- **Periodic motion** is any motion that repeats in equal intervals of time. This is a broad category which could include things like a person on a swing, a person in a rocking chair, a heartbeat or pulse, waves crashing on the shore, the tides alternating between high and low, and any example of uniform circular motion like wheels rotating or planets orbiting the sun.
- **Simple harmonic motion** specifically refers to the periodic motion of an object that occurs due to a **restoring force** which is proportional to the distance of the object from its equilibrium position. The most common examples are a mass attached to a spring and a simple pendulum.
- The **period** is the duration of one oscillation or cycle, how long it takes the object to return to its original position.
- The **frequency** is the number of oscillations or cycles per second, which is the inverse of the period.

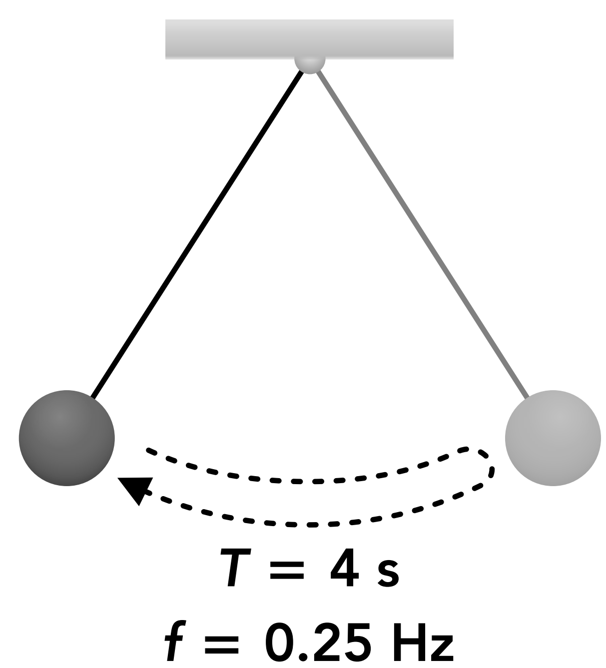
Objects in simple harmonic motion (and other periodic motion) repeat their motion with the same period and frequency

Frequency

$$f = \frac{1}{T}$$

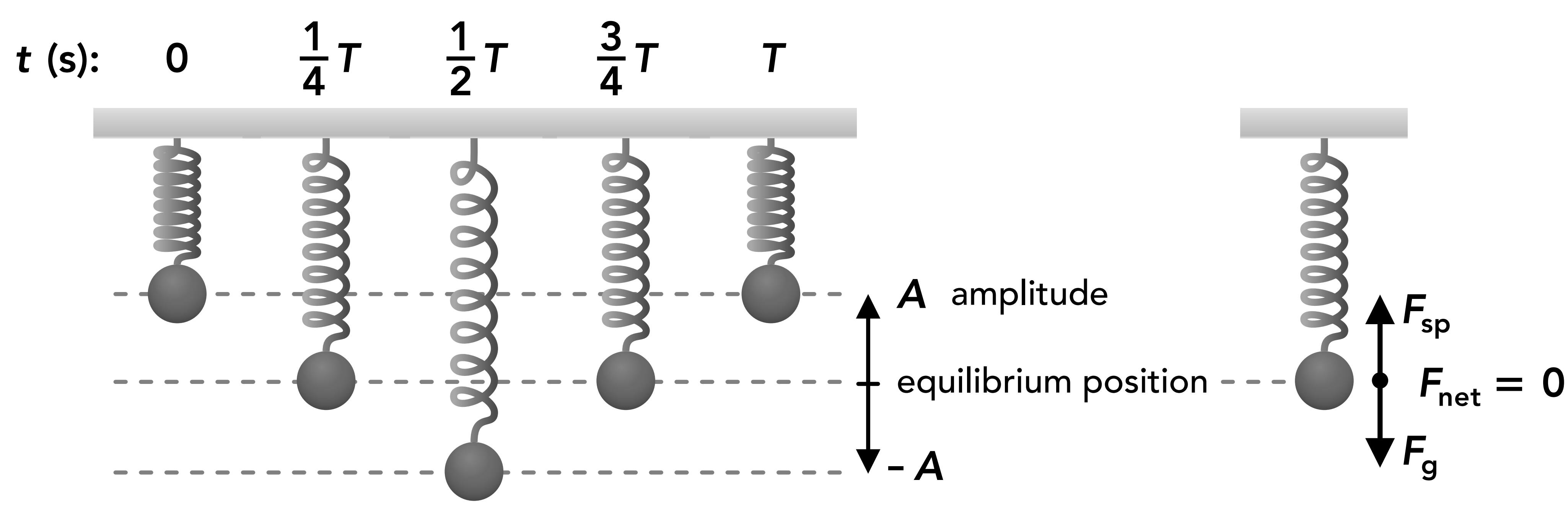
f : frequency (Hz, cycles/s)
T : period (s)

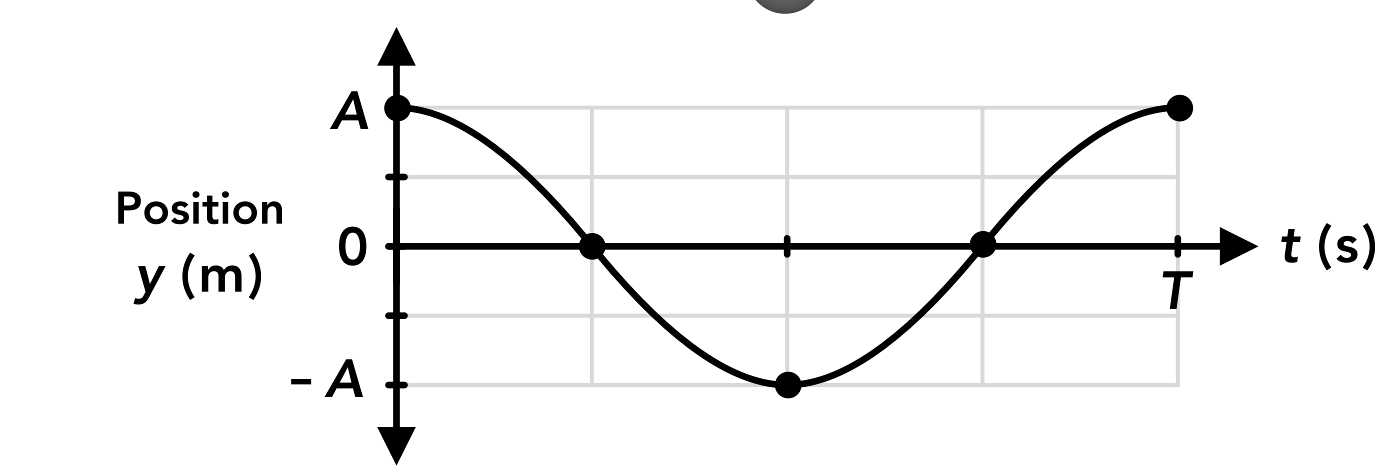




- The **equilibrium position** of a simple harmonic motion is the position where **the net force on the object is zero** and the acceleration is zero. If the object is placed in the equilibrium position and released it will not move. For a horizontal mass-spring system this is the position where the spring is at its original length and the spring force is zero. For a vertical mass-spring system this is the stretched length where the upwards spring force is equal to the downwards gravitational force. For a pendulum this is lowest position where the pendulum is vertical.
- The **amplitude** of a simple harmonic motion is the distance between the equilibrium position and one end of the oscillation. The distance between both ends of the oscillation is twice the amplitude.
- If we graph the position of the object over time the graph is **sinusoidal** (it's a sine wave, with some phase shift).
- For any motion, the value of the velocity graph is the slope of the postion graph at any point in time, and the value of the acceleration graph is the slope of the velocity graph at any point in time. This results in all three graphs being sinusoidal as seen below.
- We can describe the position, velocity and acceleration over time using the wave equations below which depend on the amplitude, frequency and time.

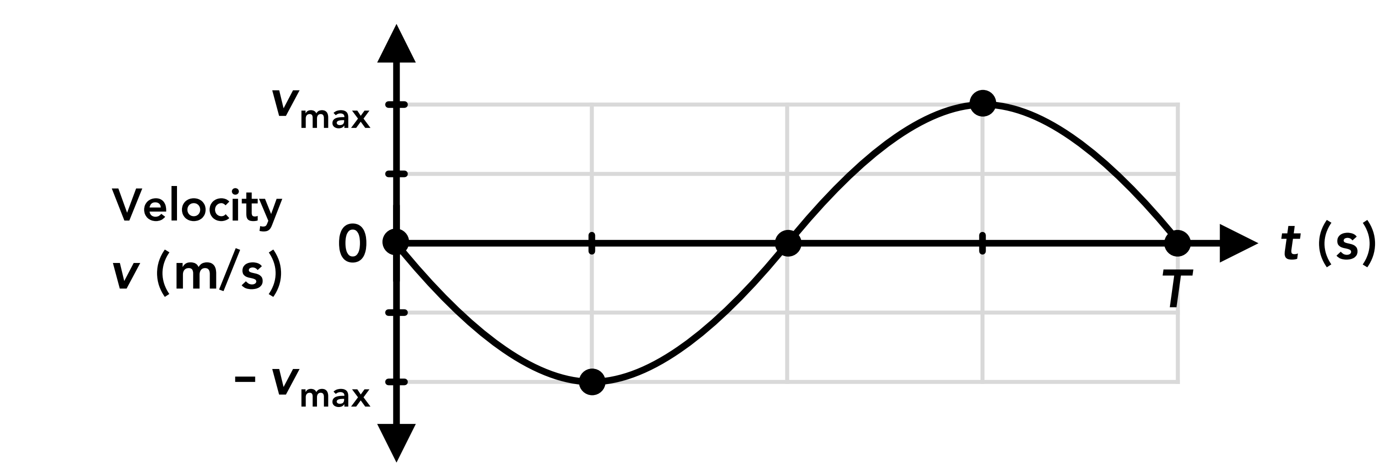
Graphs of the position, velocity and acceleration of a mass-spring system for one oscillation (one period)





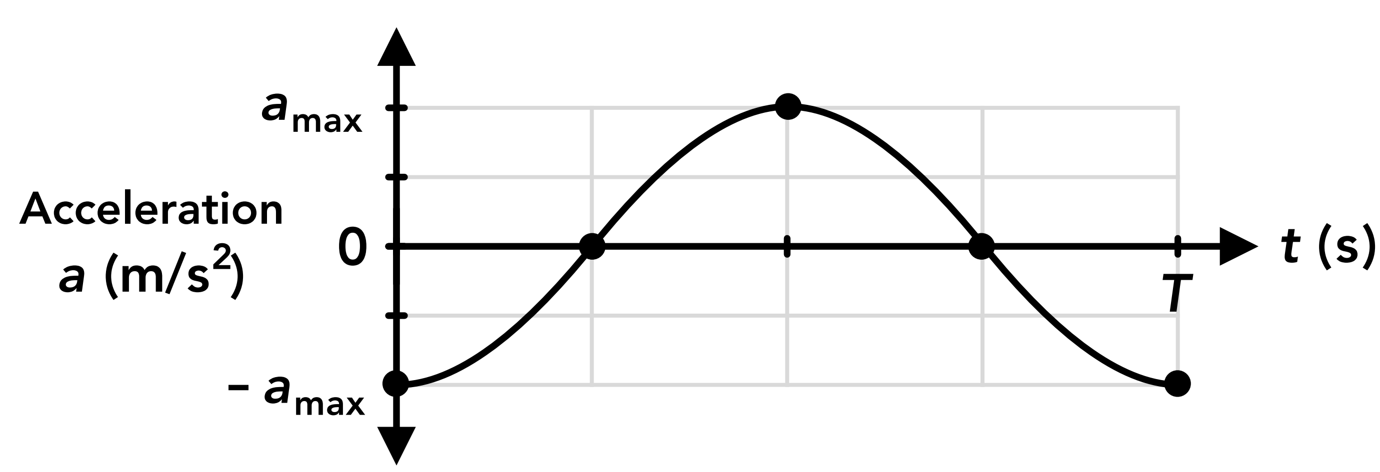
$$x(t) = A\cos(2\pi ft)$$

$$x_{max} = A$$



$$v_x(t) = -v_{max}\sin(2\pi ft)$$

$$v_{max} = 2\pi fA$$



$$a_x(t) = -a_{max}\cos(2\pi ft)$$

$$a_{max} = (2\pi f)^2 A$$

Mass-Spring Systems

- One example of simple harmonic motion is a mass attached to a spring oscillating horizontally or vertically.
- For a horizontal mass-spring system the spring force acts as the restoring force.
- For a vertical mass-spring system a combination of the spring force and the gravitational force act as the restoring force.
- We're going to assume there are no friction or drag forces so the system does not lose any energy and continues oscillating forever. This is called the "undamped" case.
- The period of a mass-spring oscillation depends on the mass of the object and the spring constant (we assume the spring is massless). Notice that it does not depend on the amplitude.

Variables		SI Unit
T	period	s
f	frequency	$\text{Hz} = \frac{\text{cycles}}{\text{s}}$
A	amplitude	m
m	mass	kg
k	spring constant	$\frac{\text{N}}{\text{s}}$
U_{sp}	spring potential energy	J
K	kinetic energy	J

Period of a mass-spring oscillation

$$T_{sp} = 2\pi\sqrt{\frac{m}{k}}$$

Frequency of a mass-spring oscillation

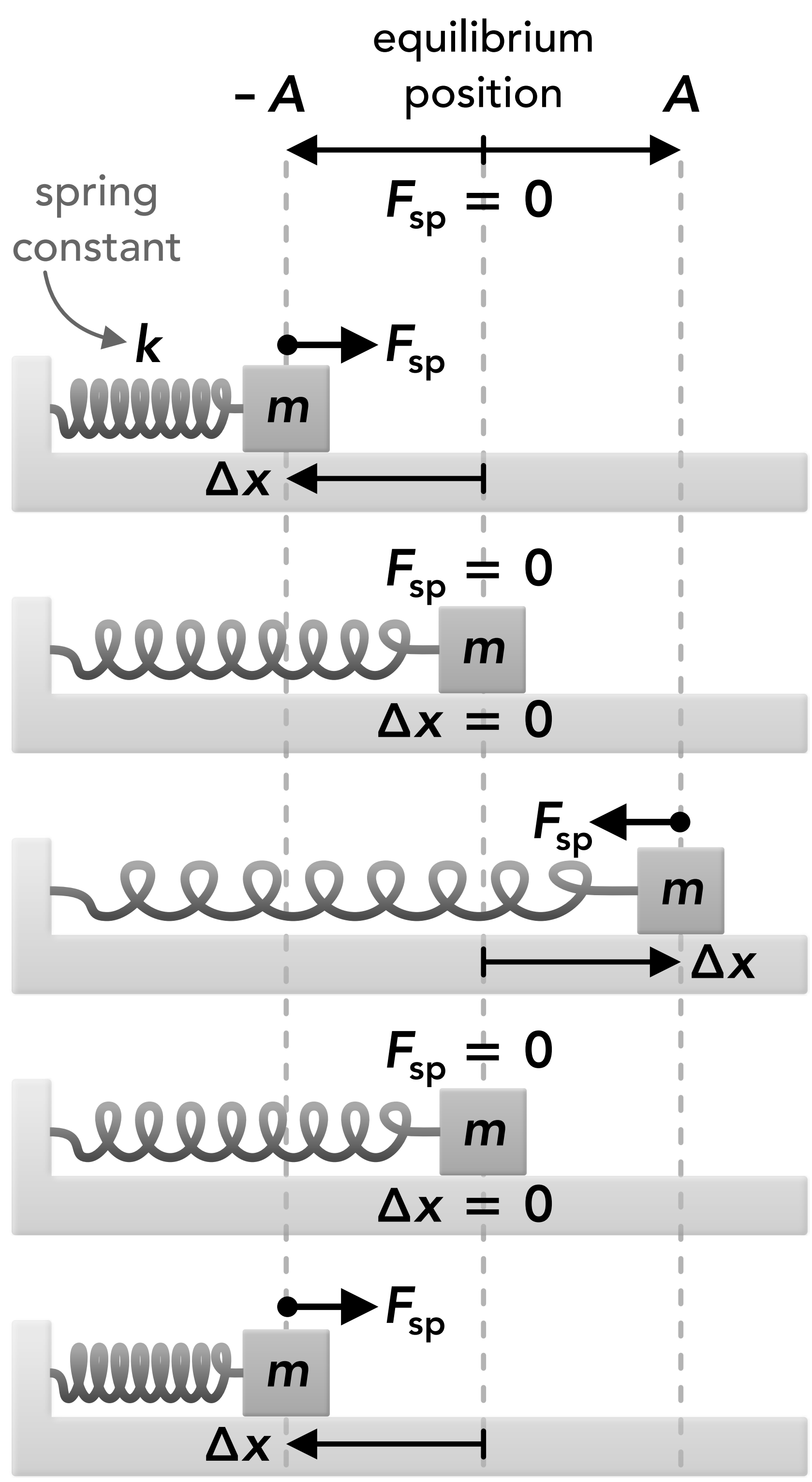
$$f_{sp} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Maximum velocity of a mass-spring oscillation

$$v_{max} = A\sqrt{\frac{k}{m}}$$

- For a **horizontal** mass-spring system the equilibrium position is at the unstretched spring position where the displacement is zero and the spring force is zero.
- The restoring force is always the spring force acting on the mass which alternates directions when the spring switches between being stretched or compressed. The spring force is at its maximum magnitude when the object is at the maximum displacement from the equilibrium position.
- The velocity of the mass is at its maximum magnitude at the equilibrium position and zero at the maximum displacement when the object is momentarily at rest while it reverses direction.
- The acceleration of the mass depends on the spring force and is at its maximum magnitude at the maximum displacement and zero at the equilibrium position.
- The spring potential energy depends on the displacement and is at its maximum value at the maximum displacement and zero at the equilibrium position.
- The kinetic energy depends on the velocity of the mass and is at its maxium value at the equilibrium position and zero at the maximum displacement.

Horizontal mass-spring system (assuming no friction)

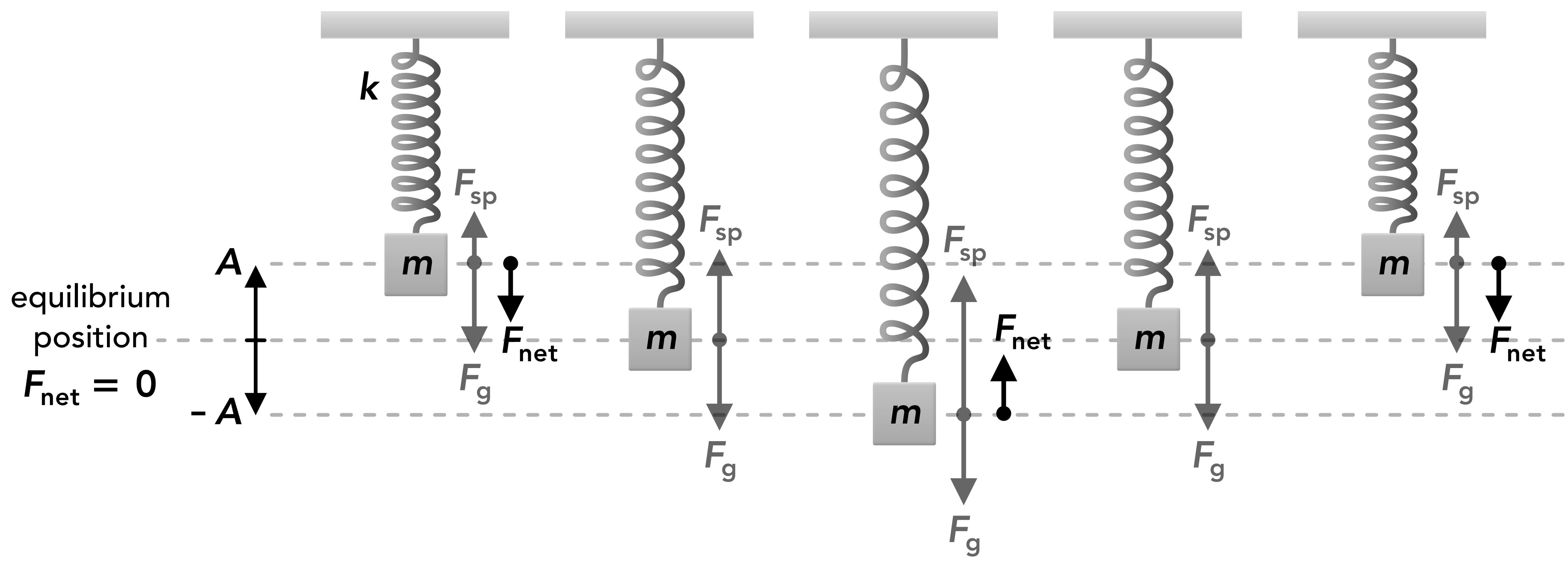


Spring force	Spring potential energy	Kinetic energy
$F_{sp} = k\Delta x$	$U_{sp} = \frac{1}{2}k\Delta x^2$	$K = \frac{1}{2}mv^2$

x	v	a	F_{sp}	U_{sp}	K
$x = -A$	$v = 0$	a_{max}	$F_{sp\ max}$	$U_{sp\ max}$	$K = 0$
$x = eq$	v_{max}	$a = 0$	$F_{sp} = 0$	$U_{sp} = 0$	K_{max}
$x = A$	$v = 0$	a_{max}	$F_{sp\ max}$	$U_{sp\ max}$	$K = 0$
$x = eq$	v_{max}	$a = 0$	$F_{sp} = 0$	$U_{sp} = 0$	K_{max}
$x = -A$	$v = 0$	a_{max}	$F_{sp\ max}$	$U_{sp\ max}$	$K = 0$

- The addition of a gravitational force and gravitational potential energy makes a vertical mass-spring system slightly more complex than a horizontal mass-spring system, but the motion behaves the same way.
- For a **vertical** mass-spring system the equilibrium position is where the upwards spring force and the downwards gravitational force are equal in magnitude so the net force on the mass is zero. The spring is already stretched some initial displacement due to the gravitational force on the mass. Therefore the actual spring force is not based on the object's displacement from the equilbrium position but from the displacement from the original unstretched length of the spring. We're going to assume the spring is always stretched some amount.
- The restoring force is a combination of the spring force (always upwards) and the gravitational force (always downwards). When the spring is at the equilbrium position the two forces are equal and the net force is zero. When the mass is above the equilibrium position the spring force is decreased and the net force is downwards. When the mass is below the equilibrium position the spring force is increased and the net force is upwards.
- The postion, velocity, acceleration, net force and kinetic energy are at their maximum and zero values at the same points in the motion as a horizontal mass-spring system.
- The spring potential energy is at its maximum at the lowest position and its minimum at the highest position.
- The gravitational potential energy is at its maximum at the highest position and its minimum at the lowest position.

Vertical mass-spring system



y	$y = A$	$y = eq$	$y = -A$	$y = eq$	$y = A$
v	$v = 0$	v_{max}	$v = 0$	v_{max}	$v = 0$
a	a_{max}	$a = 0$	a_{max}	$a = 0$	a_{max}
F_{sp}	$F_{sp\ min}$		$F_{sp\ max}$		$F_{sp\ min}$
F_g	F_g	F_g	F_g	F_g	F_g
U_{sp}	$U_{sp\ min}$		$U_{sp\ max}$		$U_{sp\ min}$
U_g	$U_{g\ max}$		$U_{g\ min}$		$U_{g\ max}$
K	$K = 0$	K_{max}	$K = 0$	K_{max}	$K = 0$

Pendulums

Variables		SI Unit
T	period	s
f	frequency	$\text{Hz} = \frac{\text{cycles}}{\text{s}}$
θ	angle	rad
L	length	m
g	grav. acceleration	$\frac{\text{m}}{\text{s}^2}$
U_g	grav. potential energy	J
K	kinetic energy	J

- Another example of simple harmonic motion is a simple pendulum which is a mass hanging from a rope (or other long object) which swings back and forth.
- The restoring force for a pendulum is the component of the gravitational force which acts tangentially to the circular path of the mass.
- We're going to look at a **simple pendulum** which includes a few assumptions: the rope is massless and **the maximum angle of the pendulum (the amplitude) is small** (<~10°). This means that the restoring force will be approximately proportional to the displacement from the equilibrium position.
- The period of a pendulum depends on the length of the pendulum and the acceleration due to gravity. Note that it does not depend on the mass or the amplitude.
- The equilibrium position for a pendulum is when the mass it at the lowest height.
- The velocity of the mass is at its maximum magnitude at the equilibrium position and zero at the maximum displacement when the mass is momentarily at rest while it reverses direction.
- The acceleration of the mass is at its maximum at the maximum displacement and zero at the equilibrium position.
- The gravitational potential energy is at its maximum when the mass it at the maximum displacement (which is the point of maximum height) and is at its minimum at the equilibrium position (lowest height).
- The kinetic energy depends on the velocity of the mass and is at its maximum at the equilibrium position and zero at the maximum displacement.

Period of a pendulum oscillation

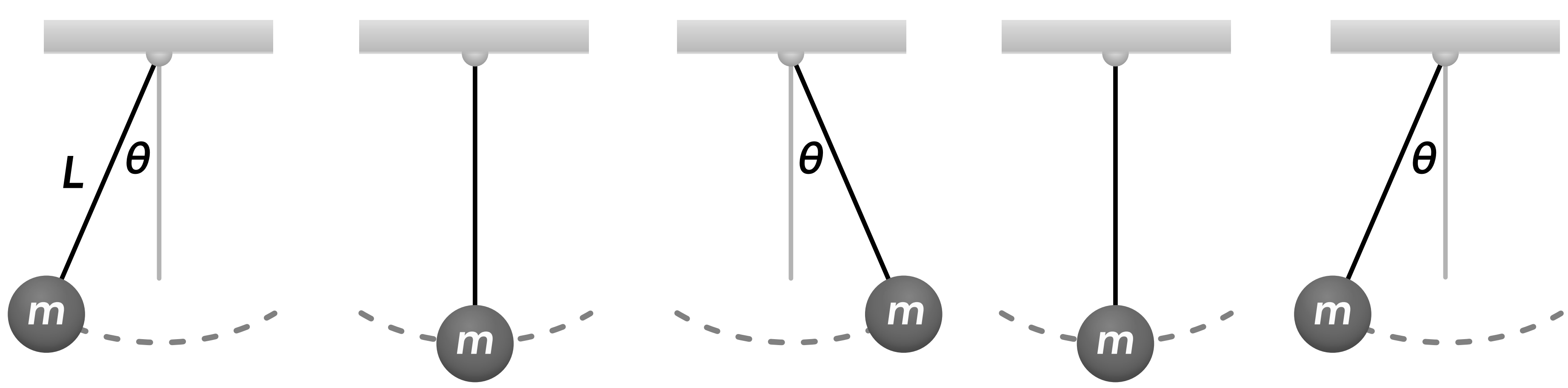
$$T_p = 2\pi\sqrt{\frac{L}{g}}$$

Frequency of a pendulum oscillation

$$f_p = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

Maximum velocity of a pendulum oscillation

$$v_{max} = \theta_{max}\sqrt{gL}$$



θ	$-\theta_{max}$	$\theta = 0$	θ_{max}	$\theta = 0$	$-\theta_{max}$
y	y_{max}	y_{min}	y_{max}	y_{min}	y_{max}
v	$v = 0$	v_{max}	$v = 0$	v_{max}	$v = 0$
a	a_{max}	$a = 0$	a_{max}	$a = 0$	a_{max}
U_g	$U_{g\ max}$	$U_{g\ min}$	$U_{g\ max}$	$U_{g\ min}$	$U_{g\ max}$
K	$K = 0$	K_{max}	$K = 0$	K_{max}	$K = 0$

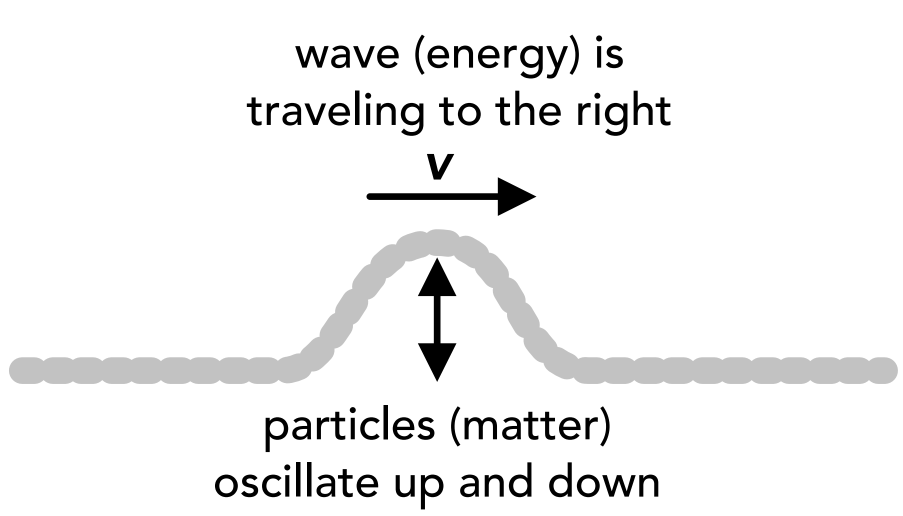
WAVES

Waves

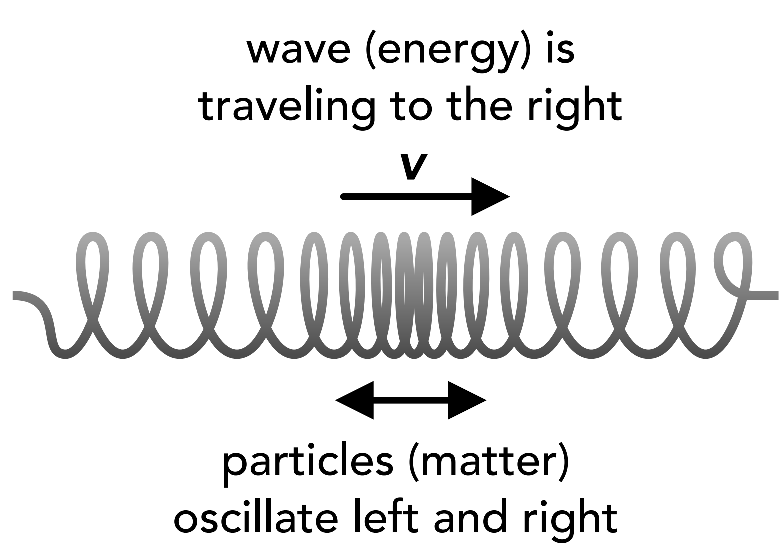
- There are many different types of waves which behave in different ways, but they all share similar characteristics.
- All waves carry or transport energy, and some waves also carry matter.
- **Transverse waves** are waves where the physical material moves **perpendicular** to the direction of the wave. If the wave is traveling to the right, the particles in the medium move up and down. Examples include water waves and waves traveling in a string.
- **Longitudinal waves** are waves where the physical material moves **parallel** to the direction of the wave. The particles in the medium do not travel with the wave, they just oscillate back forth within a small distance. Examples include sound waves and longitudinal waves traveling in a spring.

Variables		SI Unit
λ	wavelength	m
T	period	s
f	frequency	$\text{Hz} = \frac{\text{cycles}}{\text{s}}$
A	amplitude	m, ...
v	velocity	$\frac{\text{m}}{\text{s}}$

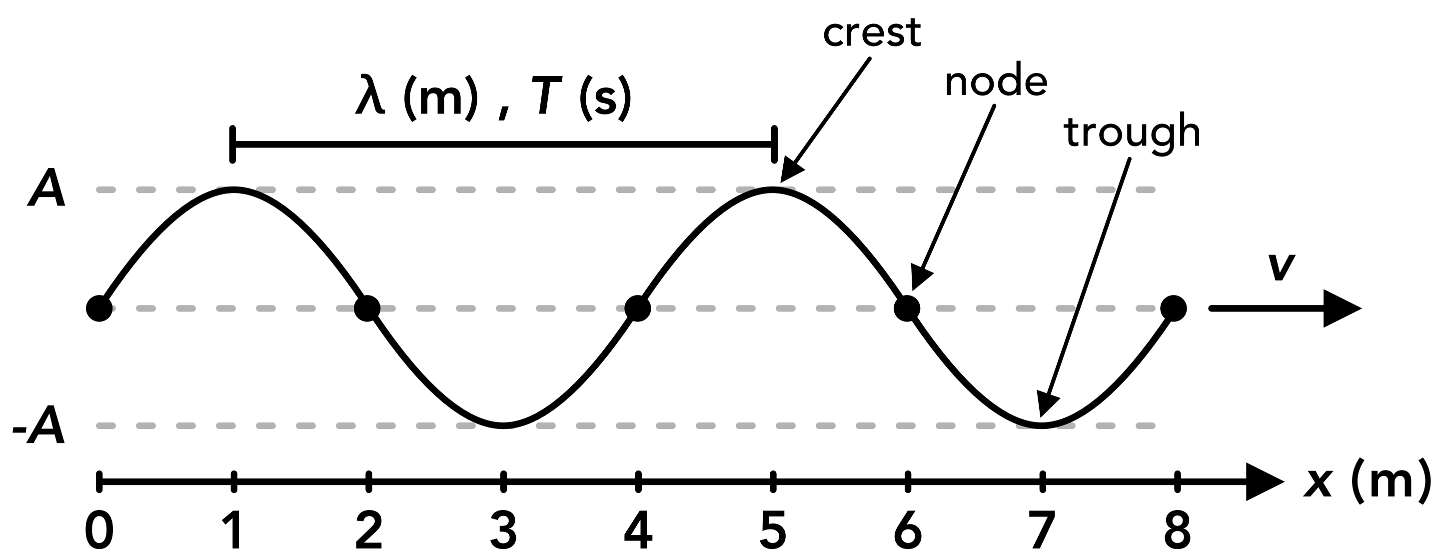
Transverse wave



Longitudinal wave

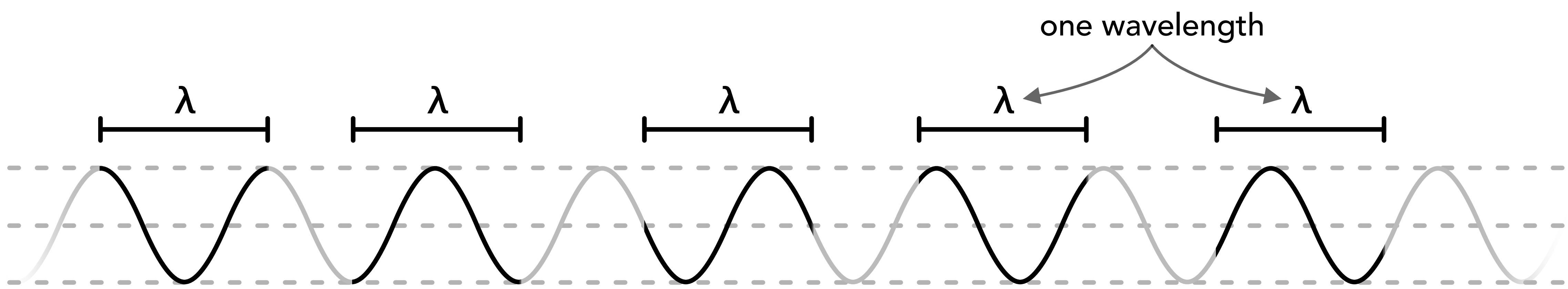


- A **crest** is the upper amplitude of a visual wave or graph of a wave. This is also an antinode.
- A **trough** is the lower amplitude of a visual wave or graph of a wave. This is also an antinode.
- A **node** is a point where the wave is at the center or equilibrium position.
- The **wavelength** is the length of a section that repeats and is easiest to measure as the distance between crests, the distance between troughs, or 3 nodes across.
- The **period** is the amount of time it takes the wave to travel one wavelength.
- The **wave speed** is the speed that the wave (energy) travels and is equal to the wavelength divided by the period.

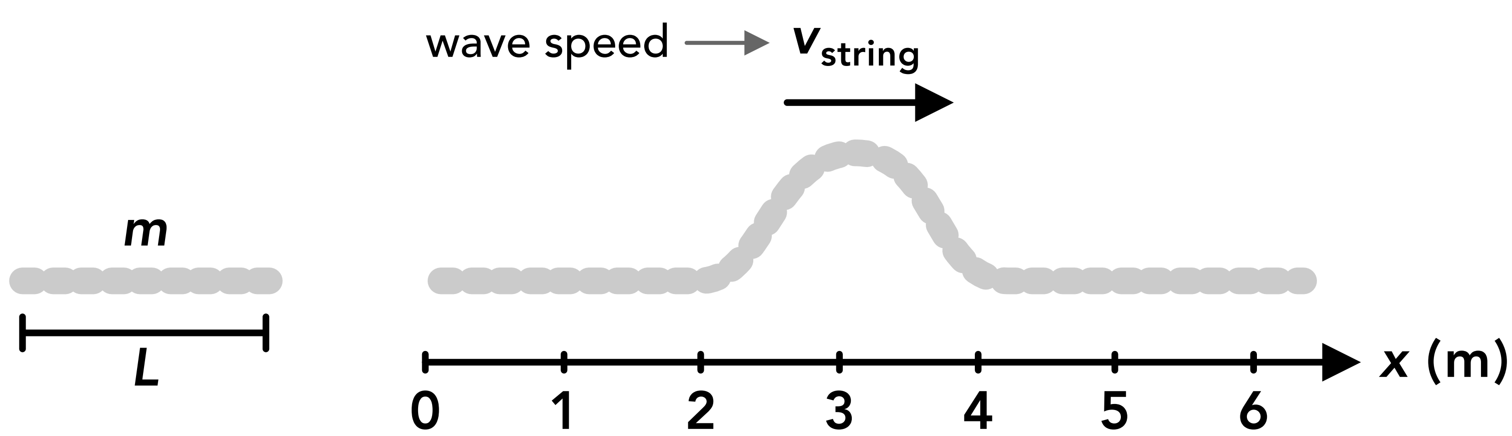


Wave speed

$$v = \lambda f = \frac{\lambda}{T}$$



- If a wave is traveling on a string, the wave speed depends on the tension in the string and the linear mass density of the string (the mass per unit length).



Variables		SI Unit
μ	linear density	$\frac{\text{kg}}{\text{m}}$
m	mass	kg
L	length	m
T_s	string tension	N
v	velocity	$\frac{\text{m}}{\text{s}}$

Linear density

$$\mu = \frac{m}{L}$$

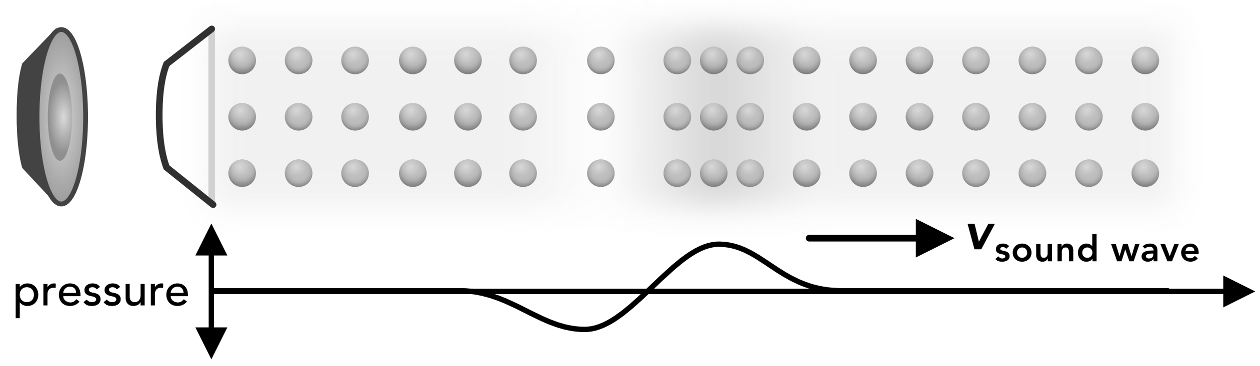
Speed of a wave in a string

$$v_{\text{string}} = \sqrt{\frac{T_s}{\mu}}$$

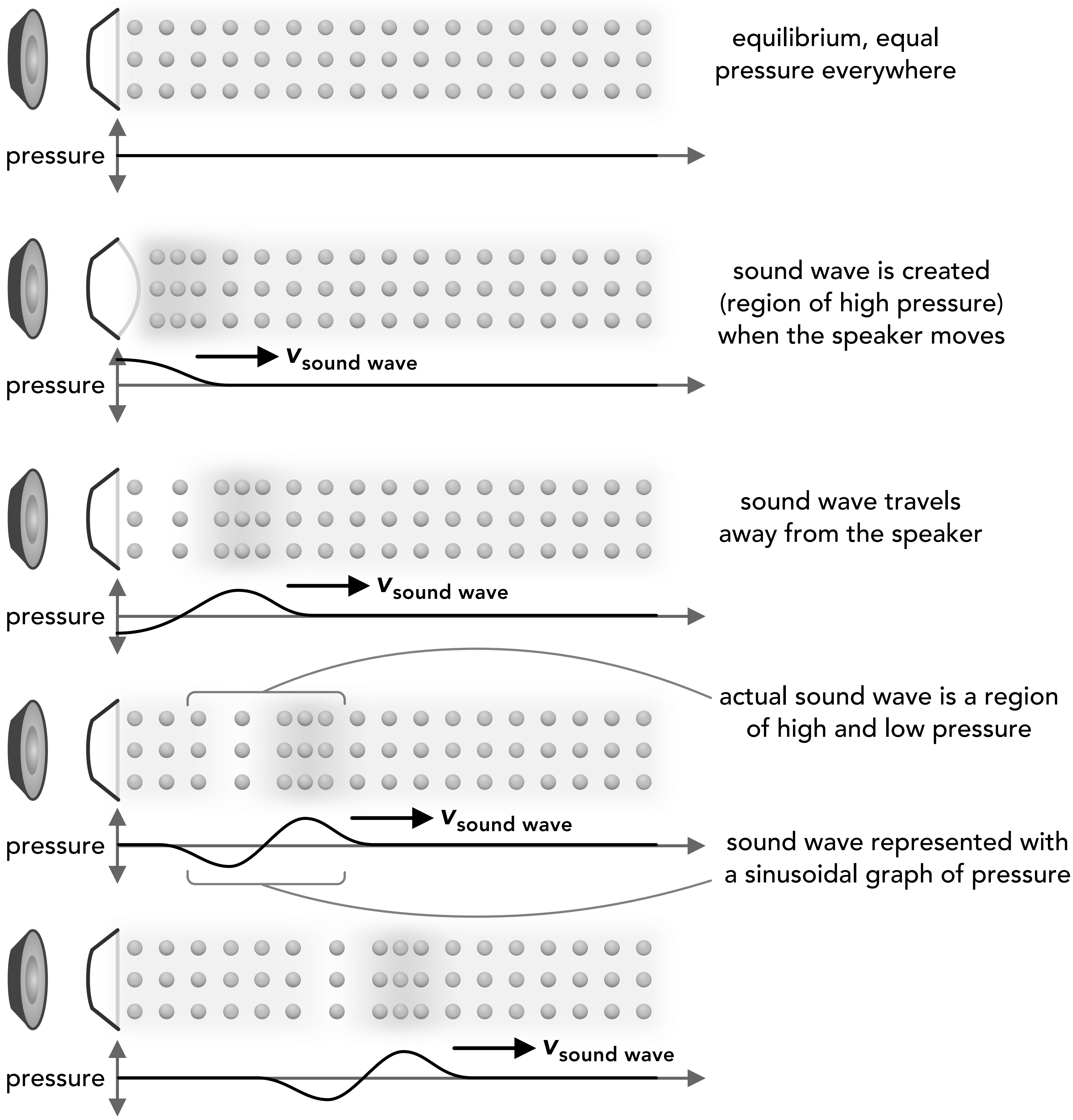
SOUND

Sound Waves

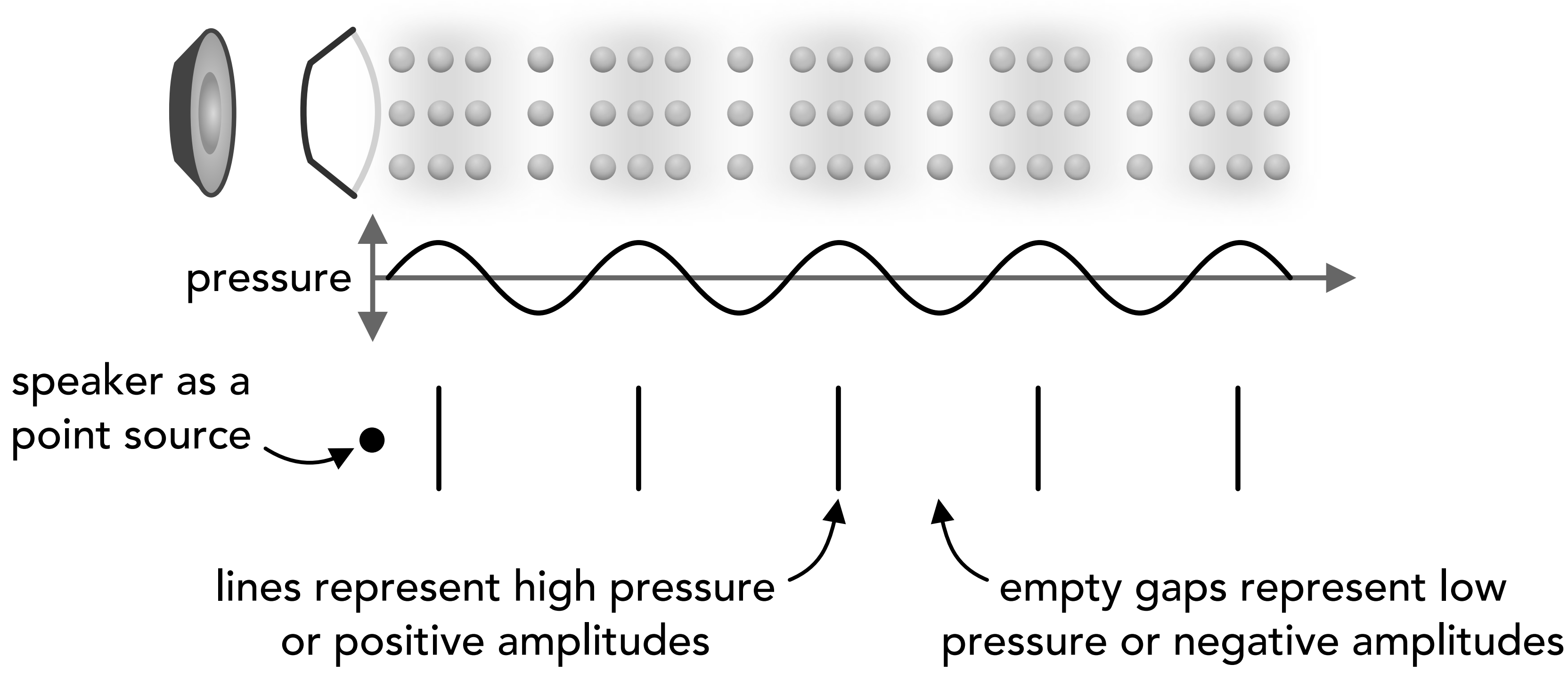
- **Sound waves** are longitudinal pressure waves where regions of high and low air pressure move as a wave.
- A volume of air consists of empty space and gas molecules (oxygen, nitrogren and more) which are constantly moving around. When the air is in a normal equilibrium state (no sound waves or other disturbances are present) the gas molecules are moving around but they are evenly spaced and the air pressure is the same everywhere.
- If the gas molecules get closer to each other there are more gas molecules per volume of space and the pressure in that region is higher. If the gas molecules get farther from each other there are less gas molecules per volume of space and the pressure in that region is lower.
- If something causes a disturbance such as a moving object or a speaker playing music, the gas molecules directly next to the moving surface will also move, either towards or away from the neighboring gas molecules. This creates a region of high or low pressure. Soon after, the gas molecules (and their neighbors) will move from a high pressure region towards a low pressure region to reach equilibrium pressure again. However, this causes a "chain reaction" of moving gas molecules, and the result is that the region of high and low pressure moves in one direction away from the source of the disturbance. This is a sound wave.
- A physical sound wave is a moving region (or many regions in a row) of high and low pressure, but sound waves (and other longitudinal waves) are often represented as a visual sinusoidal wave where the vertical axis represents the air pressure. This may look like a transverse wave but it's just a representation of air pressure at each position.



A sound wave is a moving region of high pressure and low pressure

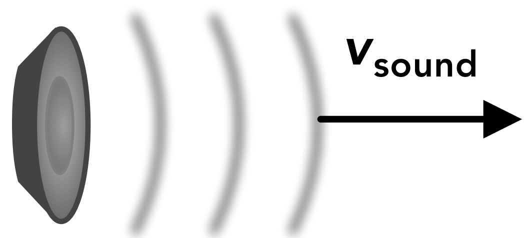


Repeating sound waves can be represented as lines called "wave fronts", where the lines represent high pressure regions (positive amplitudes) and the empty gaps represent low pressure regions (negative amplitudes)



- The **speed** that a sound wave moves through a medium depends on several things. For a sound wave moving in a gas, the speed of sound depends on the temperature, the molar mass of the gas and the adiabatic index (also known as the heat capacity ratio) of the gas which is a number between 1 and 2 for most gases.

Constants		Unit	Name
R	8.3145	$\frac{\text{J}}{\text{mol}\cdot\text{K}}$	ideal gas constant



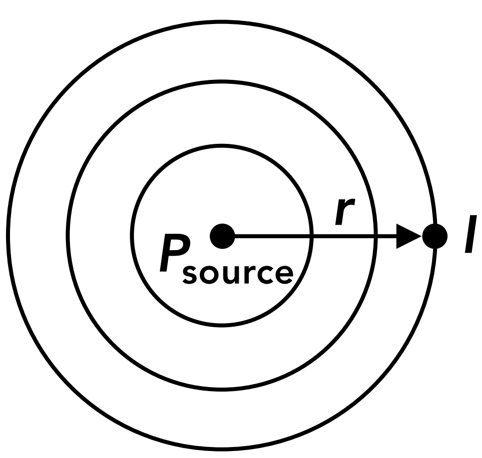
Speed of sound in a gas

$$v_{sound} = \sqrt{\frac{\gamma R T}{M}}$$

Variables		SI Unit
v	velocity	$\frac{\text{m}}{\text{s}}$
γ	adiabatic index	
T	temperature	K
M	molar mass	$\frac{\text{kg}}{\text{mol}}$

- As a circular sound wave moves away from a point source, the **intensity** or volume of the sound decreases.
- The **sound intensity level** is relationship between the sound intensity and the human threshold of hearing, and describes sound intensity in a way that's more relevant to the way humans perceive sound. Sound intensity level is measured in decibels (dB).

Constants		Unit	Name
I_0	1×10^{-12}	$\frac{\text{W}}{\text{m}^2}$	threshold of hearing



Sound intensity

$$I = \frac{P_{source}}{4\pi r^2}$$

Sound intensity level

$$\beta = (10\text{ dB})\log_{10}\left(\frac{I}{I_0}\right)$$

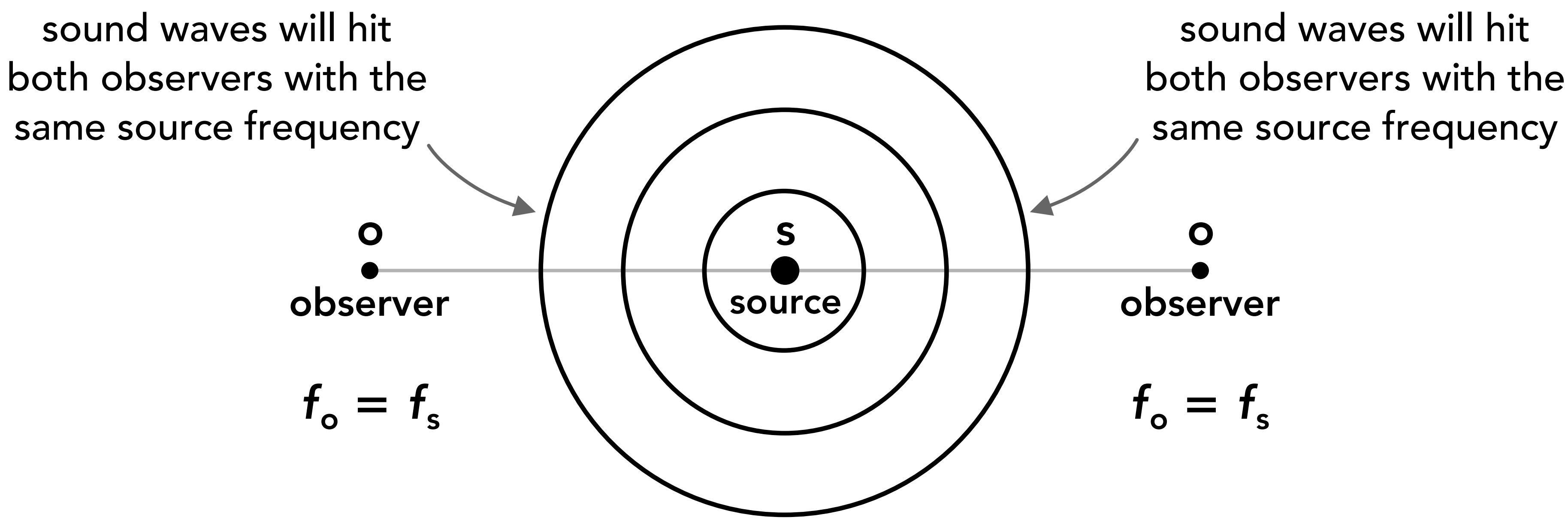
Variables		SI Unit
I	sound intensity	$\frac{\text{W}}{\text{m}^2}$
P	power	$\frac{\text{J}}{\text{s}}$
r	distance from source	m
β	sound intensity level	dB

Doppler Effect

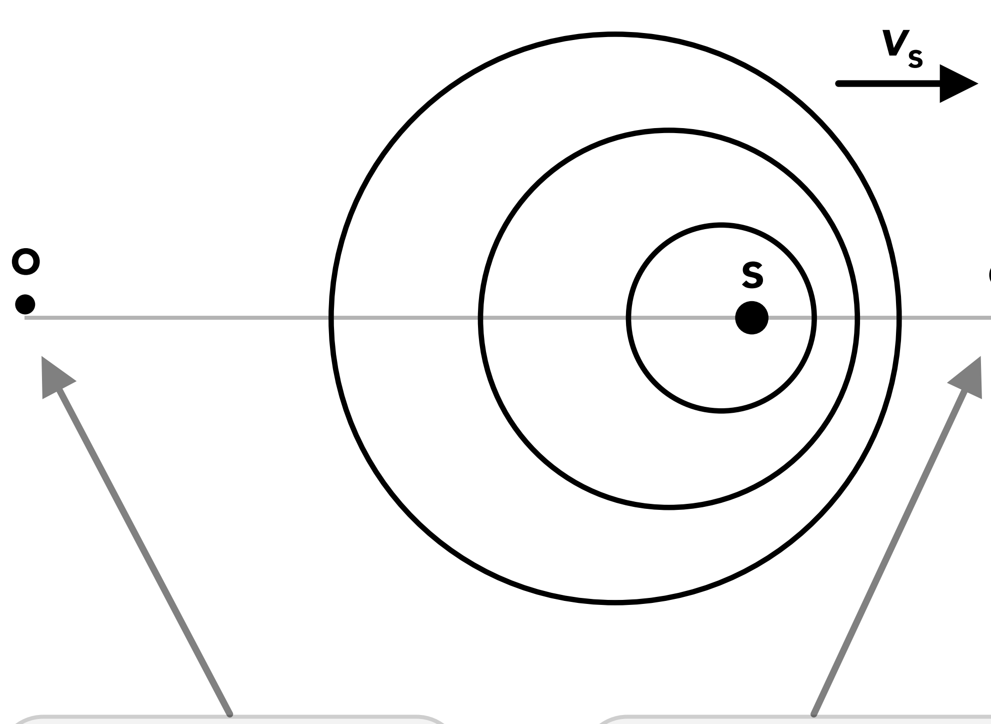
Variables		SI Unit
f_s	source frequency	Hz
f_o	observed frequency	Hz
v_s	source speed	$\frac{m}{s}$
v_o	observer speed	$\frac{m}{s}$
v	speed of sound	$\frac{m}{s}$

- When a sound source and an observer are moving relative to each other, the frequency that the observer hears is different than the true frequency of the sound source. The sound waves become compressed (closer together) or decompressed (farther apart) from the perspective of the observer which changes the wavelength and frequency.
- If the source and the observer are moving towards each other the observed frequency is higher. If the source and the observer are moving away from each other the observed frequency is lower.

Stationary source, stationary observers
(no doppler effect)



Moving source, stationary observers



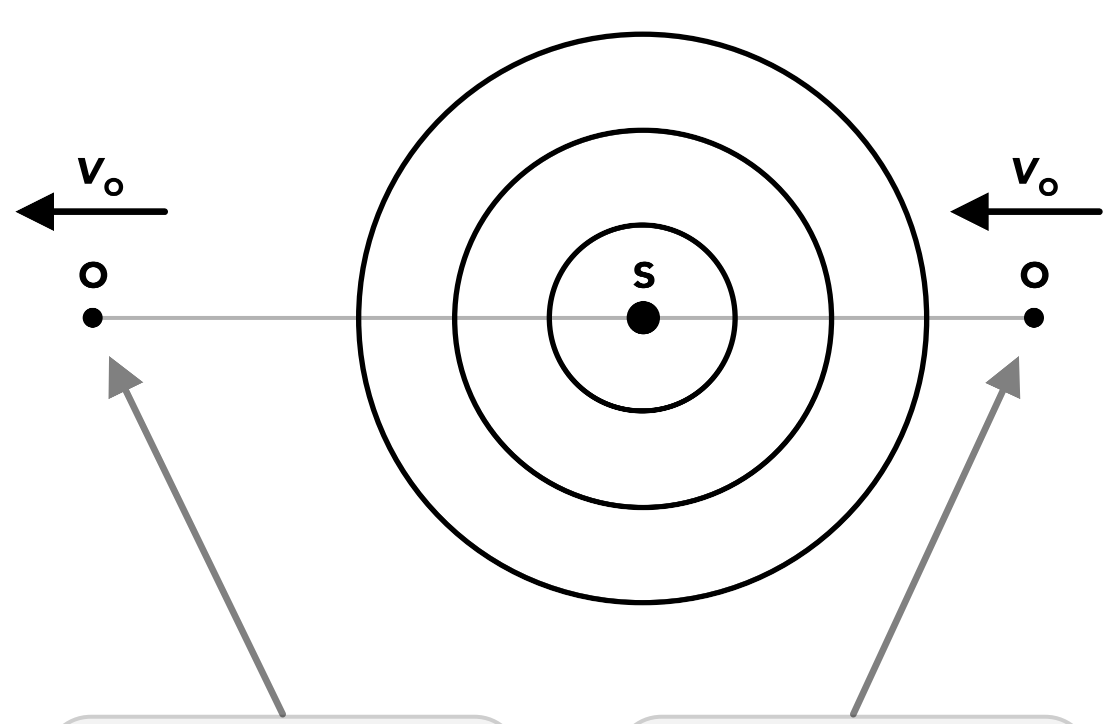
$$f_o = \frac{f_s}{1 + (v_s/v)}$$

Observed frequency, receding sound source

$$f_o = \frac{f_s}{1 - (v_s/v)}$$

Observed frequency, approaching sound source

Stationary source, moving observers



$$f_o = \left(1 - \frac{v_o}{v}\right) f_s$$

Observed frequency, receding observer

$$f_o = \left(1 + \frac{v_o}{v}\right) f_s$$

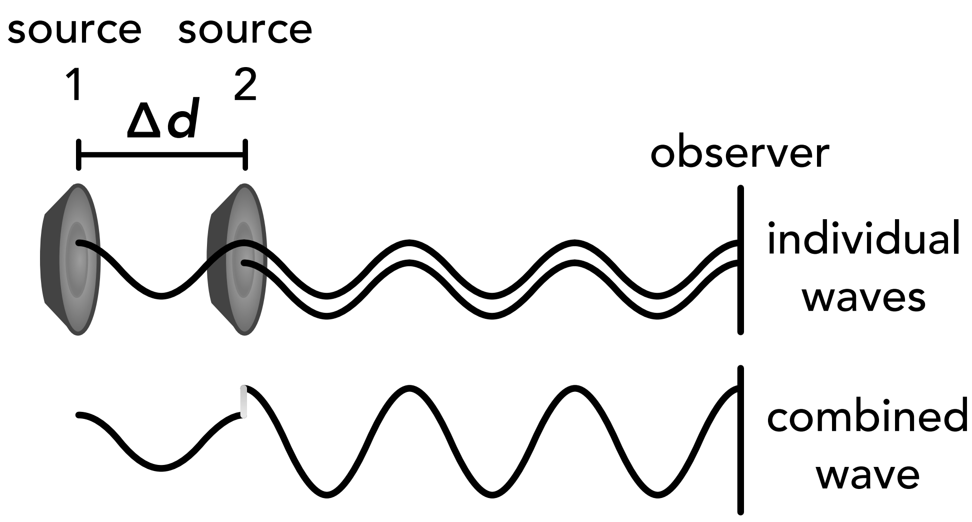
Observed frequency, approaching observer

Sound Wave Interference

- When multiple sound waves overlap (interfere), their values at every position are added together, resulting in a new wave. At every point:
  - If the values of each wave have the same sign (positive or negative) the result is **constructive interference** and the waves "build" on each other, creating a larger wave.
  - If the values of each wave have opposite signs, the result is **destructive interference** and the waves "subtract" from each other, creating a smaller wave (or no wave if they completely cancel out).

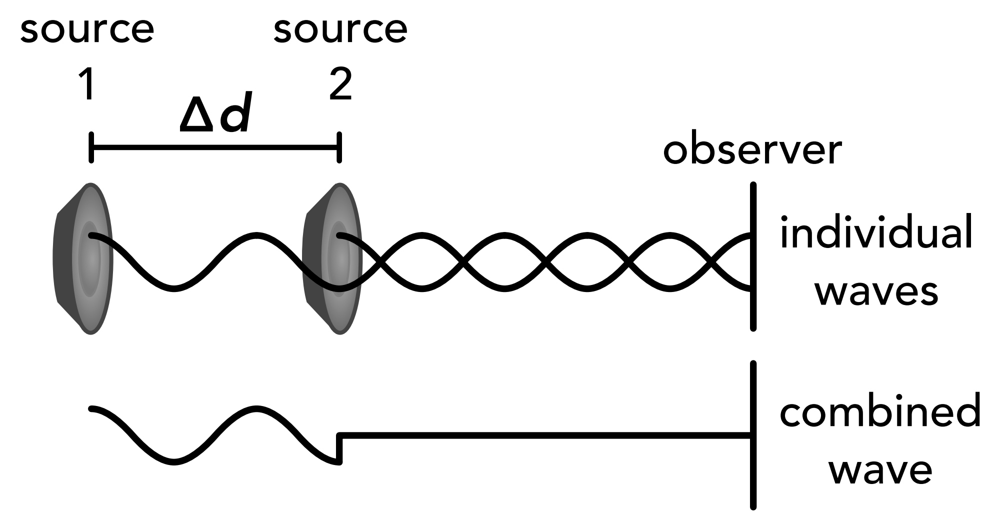
Variables		SI Unit
d	in-line path length	m
r	radial path length	m
λ	wavelength	m
m	number of wavelengths	

In-line sound wave interference



$$\Delta d = m\lambda \quad m = 0, 1, 2, \ldots$$

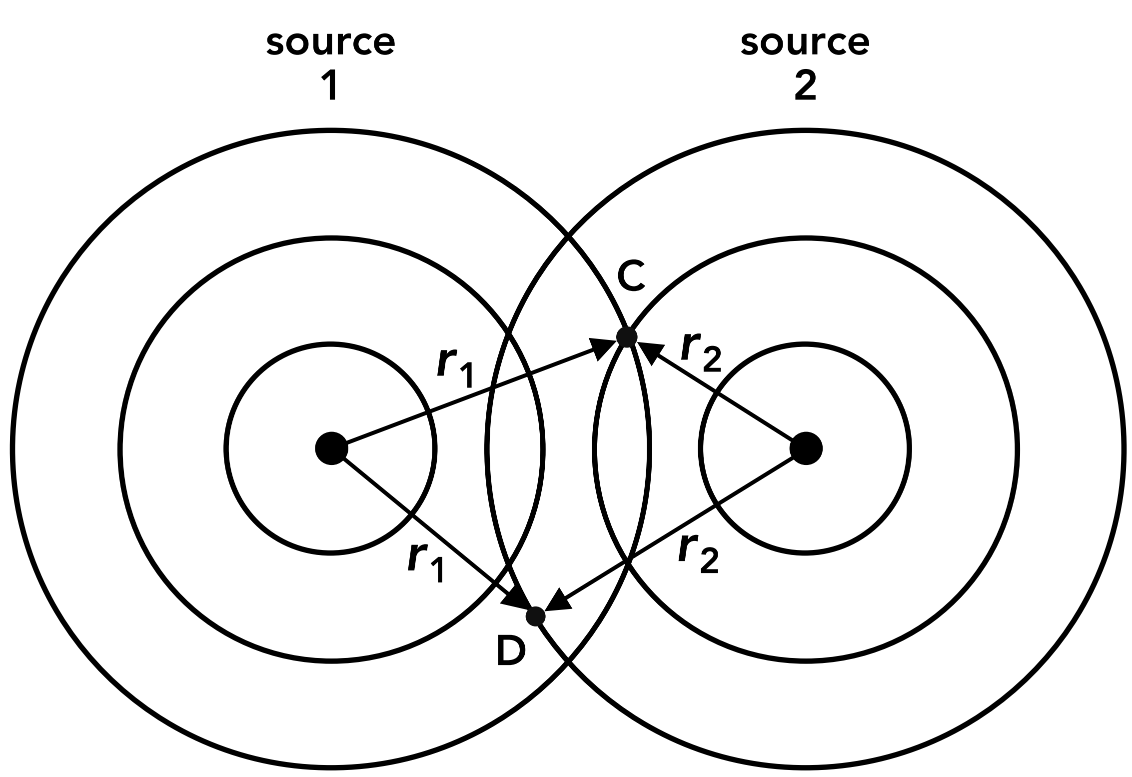
Constructive interference



$$\Delta d = \left(m + \frac{1}{2}\right)\lambda \quad m = 0, 1, 2, \ldots$$

Destructive interference

Radial (spherical) sound wave interference



$$\Delta r = |r_1 - r_2|$$

$$\Delta r = m\lambda \quad m = 0, 1, 2, \ldots$$

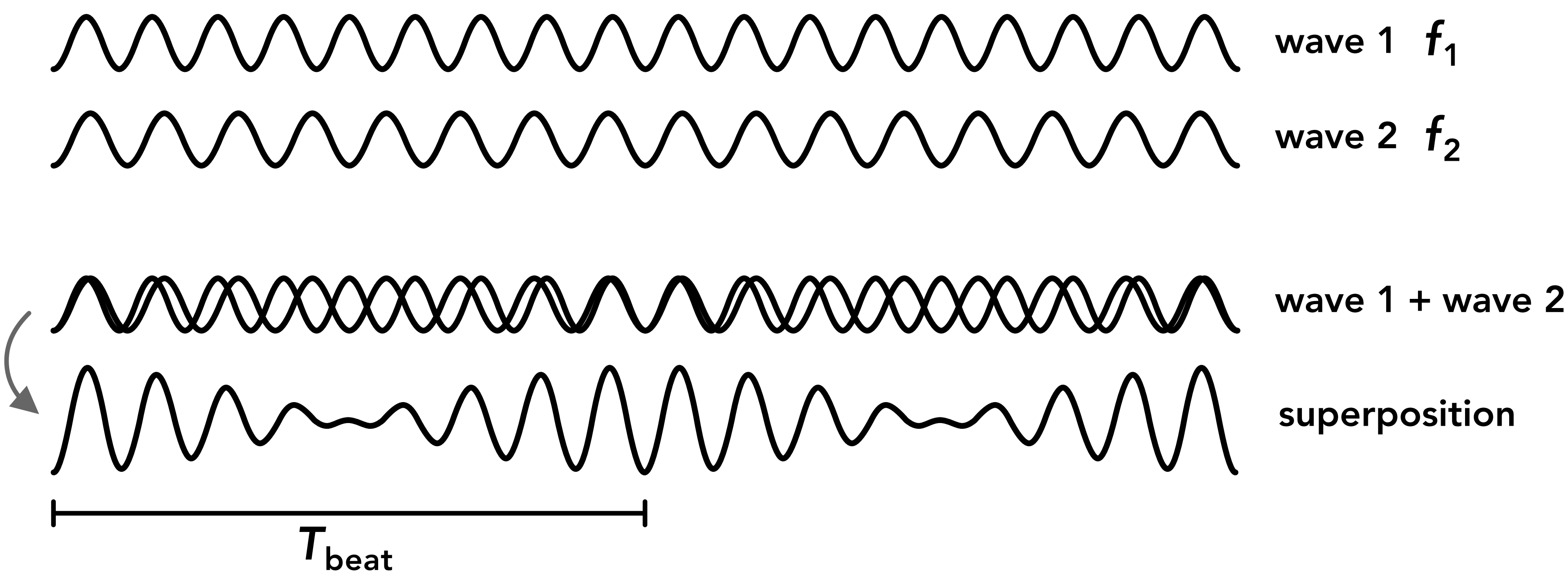
Constructive interference (point C)

$$\Delta r = \left(m + \frac{1}{2}\right)\lambda \quad m = 0, 1, 2, \ldots$$

Destructive interference (point D)

- When two sound waves with different frequencies interfere, the combined sound wave (the superposition of the two waves) will alternate between constructive and destructive interference.
- The frequency of this oscillation between high amplitude and zero amplitude is called the **beat frequency**.
- A listener will hear the sounds of wave 1 and wave 2 at the same time but the amplitude (volume or intensity) of the sound will oscillate at the beat frequency.

Variables		SI Unit
f	frequency	Hz



Beat frequency

$$f_{beat} = |f_1 - f_2|$$

STUDY GUIDE

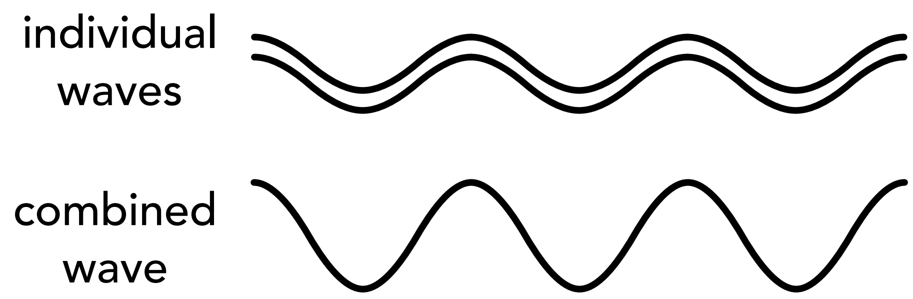
STANDING WAVES

Standing Waves

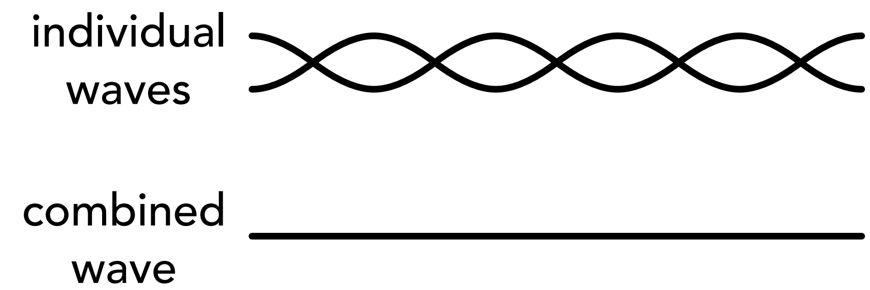
Variables		SI Unit
λ	wavelength	m
f	frequency	Hz
L	length	m
v	velocity	$\frac{m}{s}$
m	mode	

- When two waves overlap or interfere (they occupy the same space in the medium), their values at every position are added together, resulting in a new wave. At every point:
  - If the values of each wave have the same sign (positive or negative) the result is **constructive interference** and the waves "build" on each other, creating a larger wave.
  - If the values of each wave have opposite signs, the result is **destructive interference** and the waves "subtract" from each other, creating a smaller wave (or no wave if they completely cancel out).

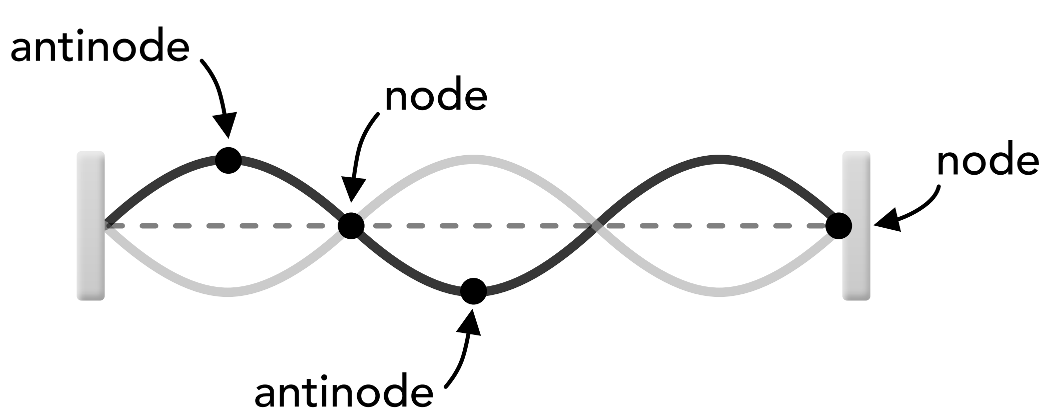
Constructive wave interference

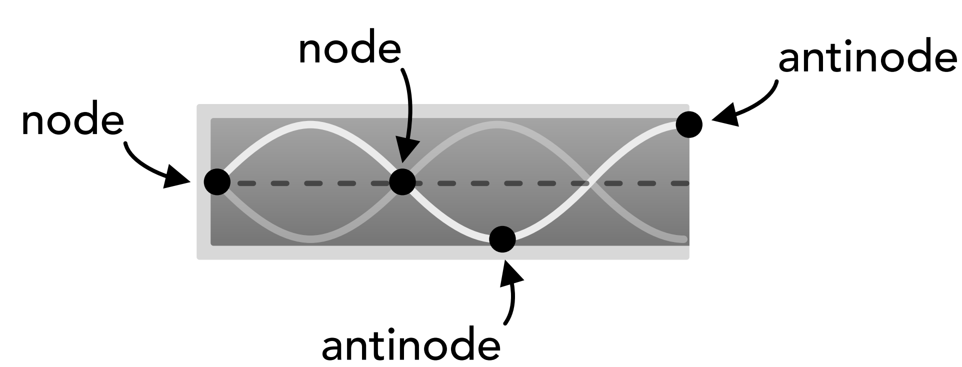


Destructive wave interference



- When a wave is traveling in a medium (like air or a string) and it's **reflected** at one end, it travels back in the opposite direction. A wave may be reflected at both ends (of a tube or a string), moving back and forth.
- If multiple reflecting waves overlap we get standing waves.
- A **standing wave** is just the superposition of two waves reflecting back and forth, which results in the amplitudes of the waves appearing to switch between positive and negative but the wave doesn't travel anywhere.
- A **node** is a point on a standing wave that does not move (zero amplitude).
- An **antinode** is a point on a standing wave that moves the maximum amount (maximum amplitude).
- The wavelength of a standing wave, like any other wave, is the length of a section that repeats: the distance between two crests, the distance between two troughs, or the distance of 3 nodes across.
- A **mode** is the wave shape or the fractions of a wavelength that fit into the length of the medium. As the wavelength changes and more wavelengths fit into the length of the medium, new wave shapes are formed.





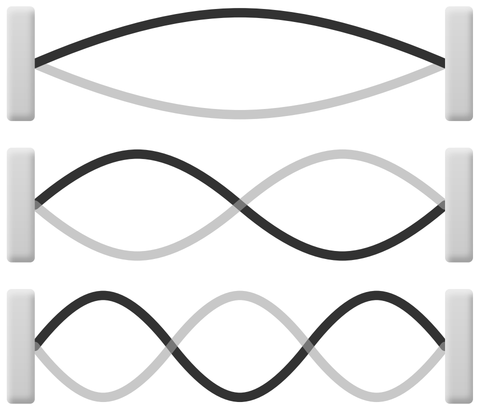
Both ends are either nodes or antinodes

String
(fixed at both ends)

$m = 1$

$m = 2$

$m = 3$



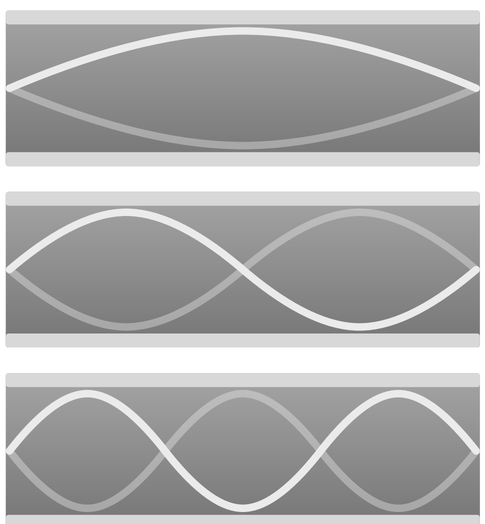
Tube
(open at both ends)

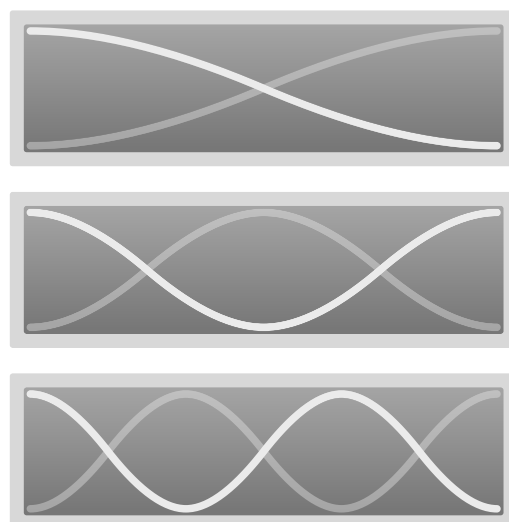
Tube
(closed at both ends)

$m = 1$

$m = 2$

$m = 3$





Wavelengths

$$\lambda_m = \frac{2L}{m} \quad m = 1, 2, 3, \ldots$$

Frequencies

$$f_m = \frac{v}{\lambda_m} = m\left(\frac{v}{2L}\right) \quad m = 1, 2, 3, \ldots$$

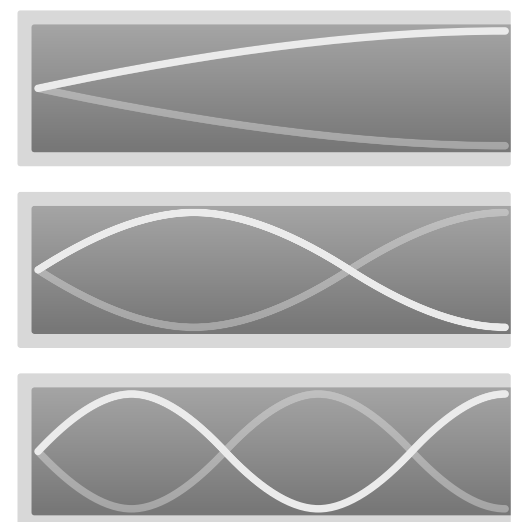
One end is a node, one end is an antinode

Tube
(open/closed ends)

$m = 1$

$m = 2$

$m = 3$



Wavelengths

$$\lambda_m = \frac{4L}{m} \quad m = 1, 3, 5, \ldots$$

Frequencies

$$f_m = \frac{v}{\lambda_m} = m\left(\frac{v}{4L}\right) \quad m = 1, 3, 5, \ldots$$