So far, we have looked at the general theory theory and tools for working with curvilinear coordinates, as well as doing integrals in curvilinear coordinates.

This is already enough for a lot of interesting applications in vector calculus. However, there is still one thing we are missing - how do the standard vector operators like gradients, divergence, curl and Laplacian work in curvilinear coordinates?

That is the topic we are diving in with this lesson. The key idea we'll discover is that the interpretation and geometric meaning of all the vector calculus operations remain exactly the same (which we've discussed previously), but their *form* looks different when expressed in different coordinates.

We will first begin by discussing the general theory behind how these operators work in different coordinate systems. Then, we will look at various examples of how to actually apply these in physics - the main applications for us being for electromagnetics.

Lesson Contents:

1. Nabla Operators In Curvilinear Coordinates

In many of the previous lessons, we've discussed (and used) the various operators in vector calculus. The most important ones include the gradient (∇f), divergence ($\nabla \cdot \vec{F}$), curl ($\nabla \times \vec{F}$) and Laplacian ($\nabla^2 f$). Collectively, we will refer to these here as "nabla operators", originating from the commonly used name *Nabla* for the symbol ∇.

However, we've mainly only seen them when expressed in Cartesian coordinates. In Cartesian coordinates - just as a little reminder - here's how each of these operators look like in coordinate form (i.e. expressed in terms of derivatives with respect to coordinates):

$$\begin{cases} \nabla f = \dfrac{\partial f}{\partial x}\hat{x} + \dfrac{\partial f}{\partial y}\hat{y} + \dfrac{\partial f}{\partial z}\hat{z} \\ \nabla \cdot \vec{F} = \dfrac{\partial F_x}{\partial x} + \dfrac{\partial F_y}{\partial y} + \dfrac{\partial F_z}{\partial z} \\ \nabla \times \vec{F} = \left(\dfrac{\partial F_z}{\partial y} - \dfrac{\partial F_y}{\partial z}\right)\hat{x} + \left(\dfrac{\partial F_x}{\partial z} - \dfrac{\partial F_z}{\partial x}\right)\hat{y} + \left(\dfrac{\partial F_y}{\partial x} - \dfrac{\partial F_x}{\partial y}\right)\hat{z} \\ \nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} \end{cases}$$

The key thing to understand from this lesson is that the **coordinate form** of these nabla operators will look very different in different coordinates. For example, compare the Laplacians of a scalar function in Cartesian and spherical coordinates:

$$\begin{cases} \nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} \\ \nabla^2 f = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial f}{\partial r}\right) + \dfrac{1}{r^2 \sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta\dfrac{\partial f}{\partial \theta}\right) + \dfrac{1}{r^2 \sin^2\theta}\dfrac{\partial^2 f}{\partial \varphi^2} \end{cases}$$

Both of these describe exactly the same thing geometrically - the *average difference* in the values of f at a particular point, compared to the neighboring points.

However, what is clearly different is how the two expressions look like when expressed in different coordinates. In the case of spherical coordinates, there are these additional factors like r^2 and $\sin\theta$.

These arise exactly from the fact that the spherical coordinate system is a curvilinear one. This is indeed where the heart of the matter lies - the nabla operators have very different expressions in different curvilinear coordinates.

In this lesson, we'll find out how to derive these expressions as well as use them to do physics.

In the following sections, we'll look at each operator, specifically for orthogonal coordinate systems here. We'll first look at the general cases and dive into more specific examples like spherical and cylindrical coordinates thereafter.

1.1. Gradient

The gradient is an operator that acts on a scalar field or multivariable function (it can also act on vector fields, though) and produces a vector. This vector describes the "direction of largest rate of change" of the scalar field. In Cartesian coordinates, it has the simple form:

$$\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

But what does the gradient look like in a different, more general u, v, w -coordinate system? Does it involve just derivatives like $\partial f / \partial u$? Well, not quite. Here is the formula for the gradient in any **orthogonal curvilinear coordinate system**:

$$\boxed{\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{e}_v + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{e}_w}$$

Notice that it involves these first partial derivatives with respect to each coordinate like we would probably expect. The formula also involves the unit basis vectors like $\hat{e}_u$ and so on.

However, you can also see the scale factors h_u, h_v and h_w appearing once again. The main reason for this is the fact that the lengths of basis vectors in curvilinear coordinates are not constant - as you'll see in the derivation down below.

Derivation of The General Gradient Formula

With what we know so far about scale factors, basis vectors and all that, deriving the above formula is really just a simple exercise in multivariable calculus. What we will do is basically "transform" the Cartesian gradient to the more general u, v, w - coordinates:

$$\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

The way we can do this, specifically, is express these partial derivatives like $\partial f / \partial x$ in the new coordinates. It's actually enough to focus on just one of the terms for now, as you'll see soon.

We imagine that $f = f(u, v, w)$ is a function of the new coordinates, and u, v and w are thus functions of the old, Cartesian coordinates x, y and z. The partial derivative $\partial f / \partial x$ then becomes by the **multivariable chain rule**:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} \qquad (1)$$

I've marked this as equation (1) *since we'll come back to it shortly.*

Let's think about these partial derivatives like $\partial u / \partial x$. Partial derivatives of the *opposite order*, such as $\partial x / \partial u$, would correspond to terms we have in the definition of the basis vectors like $\vec{e}_u = \partial \vec{r} / \partial u = \partial x / \partial u \hat{x} + \ldots$ In fact, if you consider the opposite, partial derivatives of $\vec{r}$ with respect to the Cartesian coordinates instead, we have again by the chain rule:

$$\hat{x} = \frac{\partial \vec{r}}{\partial x} = \frac{\partial \vec{r}}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \vec{r}}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial \vec{r}}{\partial w}\frac{\partial w}{\partial x}$$

Here, the partial derivatives of $\vec{r}$ with respect to u, v and w are the basis vectors in the u, v, w -coordinate system, which means that:

$$\hat{x} = \frac{\partial u}{\partial x}\vec{e}_u + \frac{\partial v}{\partial x}\vec{e}_v + \frac{\partial w}{\partial x}\vec{e}_w$$

Now, for **orthogonal coordinates**, we generally have $\vec{e}_u \cdot \vec{e}_v = \vec{e}_u \cdot \vec{e}_w = 0$. Therefore, we could take the dot product with $\vec{e}_u$ on both sides to find:

$$\hat{x} \cdot \vec{e}_u = \frac{\partial u}{\partial x}\underbrace{\vec{e}_u \cdot \vec{e}_u}_{=h_u^2} + \frac{\partial v}{\partial x}\underbrace{\vec{e}_v \cdot \vec{e}_u}_{=0} + \frac{\partial w}{\partial x}\underbrace{\vec{e}_w \cdot \vec{e}_u}_{=0} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{1}{h_u^2}\hat{x} \cdot \vec{e}_u$$

Here we have an expression for these partial derivatives we were looking for!

We typically want to express these in terms of the unit basis vectors, $\hat{e}_u$, which we can do by writing $\vec{e}_u = h_u \hat{e}_u$, so:

$$\frac{\partial u}{\partial x} = \frac{1}{h_u}\hat{x} \cdot \hat{e}_u$$

In the exact same way, we would find for the other partial derivatives:

$$\frac{\partial v}{\partial x} = \frac{1}{h_v}\hat{x} \cdot \hat{e}_v$$
$$\frac{\partial w}{\partial x} = \frac{1}{h_w}\hat{x} \cdot \hat{e}_w$$

Now, let's get back to our equation (1) from earlier.

Inserting the above expressions into it, we find:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{1}{h_u}\hat{x}\cdot\hat{e}_u + \frac{\partial f}{\partial v}\frac{1}{h_v}\hat{x}\cdot\hat{e}_v + \frac{\partial f}{\partial w}\frac{1}{h_w}\hat{x}\cdot\hat{e}_w$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = \left(\frac{1}{h_u}\frac{\partial f}{\partial u}\hat{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{e}_v + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{e}_w\right)\cdot\hat{x}$$

What we have here inside the parentheses is some vector, which when taken the dot product of with $\hat{x}$, results in the x-component of the gradient in Cartesian coordinates:

$$\nabla f\cdot\hat{x} = \frac{\partial f}{\partial x}$$

So, what we have inside the parentheses is indeed the full gradient vector, ∇f - however, now expressed in terms of the coordinates u, v and w! Therefore, we find the expected result as:

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{e}_v + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{e}_w$$

If you trace back our calculation to the point these scale factors appeared at, they came from the fact that $\vec{e}_u\cdot\vec{e}_u = h_u^2$ and $\vec{e}_u = h_u\hat{e}_u$. In other words, these scale factors are a result of the fact that these curvilinear basis vectors have length h_u instead of 1, like they do in Cartesian coordinates.

1.2. Divergence

Let's discuss the divergence next. The divergence is an operator that acts on a vector field to produce a scalar. If we have a vector field $\vec{F}$, its divergence in any **orthogonal** coordinate system is calculated by:

$$\nabla \cdot \vec{F} = \frac{1}{h_u h_v h_w} \left(\frac{\partial (h_v h_w F_u)}{\partial u} + \frac{\partial (h_u h_w F_v)}{\partial v} + \frac{\partial (h_u h_v F_w)}{\partial w} \right)$$

The scale factors h_u, h_v and h_w appear here once again. The interesting thing now is that some of them appear *inside* the derivatives - this means that if, say h_v depends on the coordinate u, it will produce an extra contribution to the divergence.

The quantities F_u, F_v and F_w in the above formula are the components of the vector field $\vec{F}$ in the u, v, w -coordinate system. Therefore, it is important to first express this vector field in these coordinates to be able to calculate its divergence.

Derivation of The General Divergence Formula

The derivation of the above divergence formula is a bit more involved than the one for the gradient.

However, the reason I'm including it here is because it also functions as an exercise of using the divergence theorem.

Even more importantly, this derivation also illustrates the use of the kinds of "approximation techniques" that are also quite common in physics.

We can do the derivation by using the divergence theorem, since it contains the divergence $\nabla \cdot \vec{F}$ on the left-hand side.

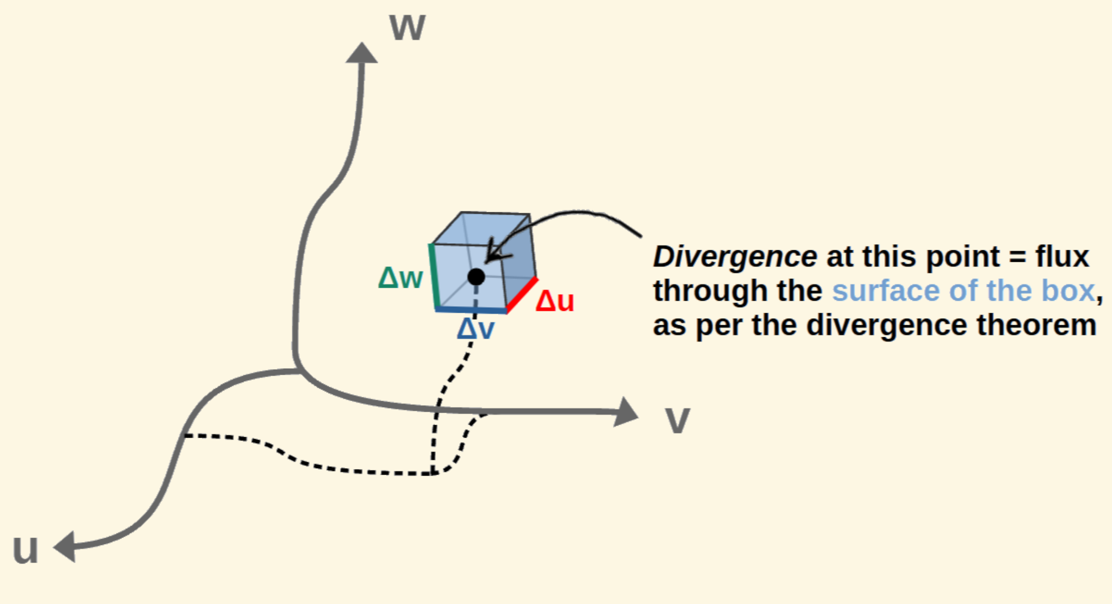
Therefore, finding a suitable expression for the right-hand side might give us a direct expression for the divergence:

$$\iiint_V \nabla \cdot \vec{F} dV = \oiint_{\partial V} \vec{F} \cdot \hat{n} dS$$

The way we're going to do this is by considering little approximations first and then taking limits at the end to make our results exact. Specifically, we want to find some approximation for the volume integral on the left.

Because the divergence is always calculated at a single point, what we can do is choose any point and approximate it by placing it inside a little (infinitesimal) box. If this box is small enough, it's effectively the same as the point itself but allows us to "approximate away" the volume integral.

The sides of this box are Δu, Δv and Δw along each of the u, v, w-coordinate axes. The box has volume V and surface ∂V, which consists of six sides:

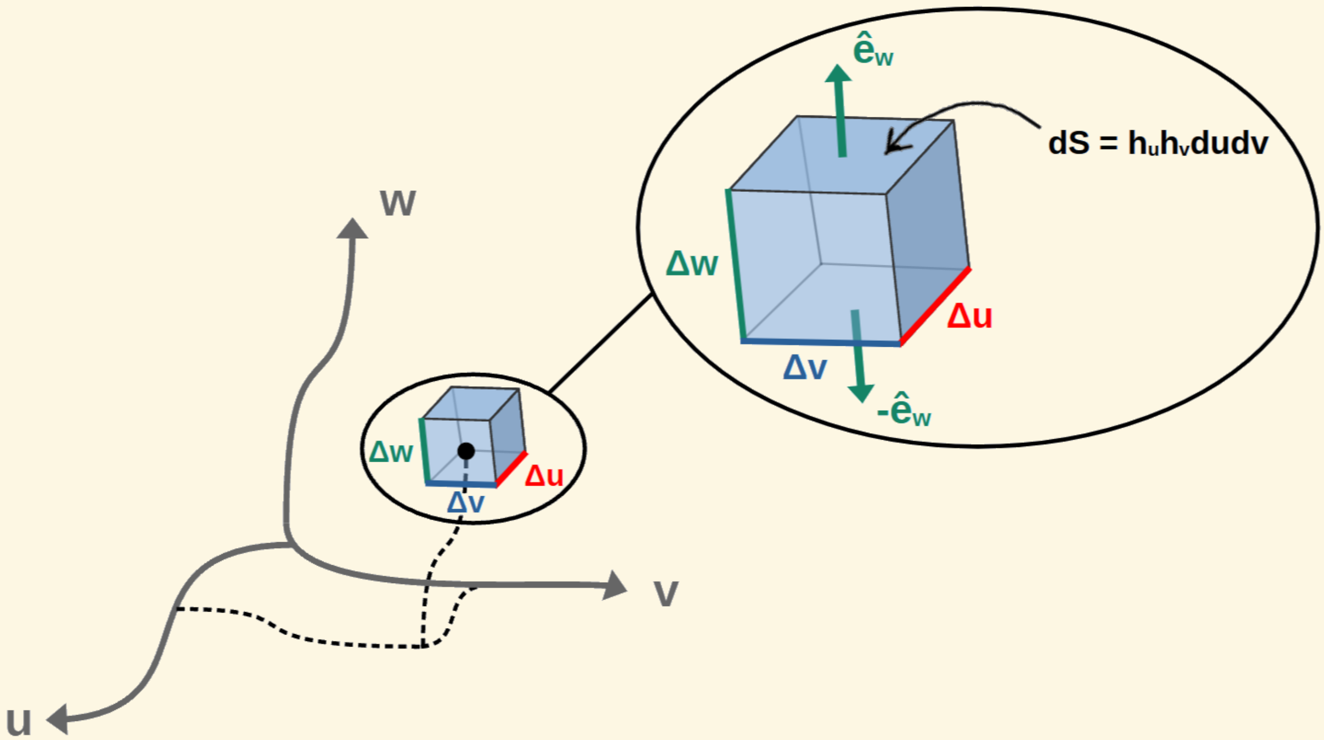


We can now approximate our volume integral on the left-hand side of the divergence theorem as (using the general form of the volume element $dV = h_u h_v h_w du dv dw$):

$$\iiint_V \nabla \cdot \vec{F} dV = \iiint_V \nabla \cdot \vec{F} h_u h_v h_w du dv dw \approx \nabla \cdot \vec{F} h_u h_v h_w \Delta u \Delta v \Delta w$$

For the right-hand side of the divergence theorem, we need to calculate the flux through the closed surface of the entire box, ∂V. Well, since it's a box, its surface consists of six separate pieces - each will be perpendicular to one of the coordinate planes.

Let's focus on the surfaces perpendicular to the uv-plane for now. For the box, there are two of these surfaces, one with unit normal vector $\hat{e}_w$ (= basis vector in the w-direction) and one with $-\hat{e}_w$, with the surface element for this plane being $dS = h_u h_v du dv$:



The contributions to the total flux integral through these two sides then consists of, well, two parts. First, the flux through the surface $S \perp uv$ (this is to indicate the surface is perpendicular to the uv-plane) - the "bottom" of the box.

Second, there is also the flux through the same surface, but offset by the amount Δw in the w-direction, which we denote as $S \perp uv + \Delta w$ (this is the "top" of the box). These surfaces have the unit normals $\hat{n} = \hat{e}_w$ and $\hat{n} = -\hat{e}_w$ like stated above.

The flux integral through these two sides can then be written as:

$$\iint_{S\perp uv} \vec{F}\cdot\hat{n}dS = \iint_{S\perp uv} \underbrace{\vec{F}\cdot\hat{e}_w}_{F_w} h_u h_v du dv + \iint_{S\perp uv+\Delta w} \underbrace{\vec{F}\cdot(-\hat{e}_w)}_{-F_w} h_u h_v du dv$$

$$= \iint_{S\perp uv} F_w h_u h_v du dv - \iint_{S\perp uv+\Delta w} F_w h_u h_v du dv$$

Like was done earlier, we can now approximate these integrals as follows:

$$\iint_{S\perp uv} \vec{F}\cdot\hat{n}dS \approx F_w h_u h_v \Delta u \Delta v\Big|_{w} - F_w h_u h_v \Delta u \Delta v\Big|_{w+\Delta w}$$

$$= \left(F_w h_u h_v\Big|_{w} - F_w h_u h_v\Big|_{w+\Delta w}\right)\Delta u \Delta v$$

Here, $\Delta u \Delta v$ do not depend on the value of w, so they can be pulled outside the limits.

Inside the parentheses, we now have the *difference* in a quantity at two nearby points, so essentially $\Delta(F_w h_u h_v)$.

Now, if these two nearby points are "nearby enough", which they are because we assume this box is infinitesimal (this will become exact later when we take limits) then this quantity can be approximated with a *derivative* as:

$$\Delta(F_w h_u h_v) \approx \frac{\partial(F_w h_u h_v)}{\partial w}\Delta w$$

Thus, we have that the contribution to the flux integral from the sides with $w = \text{constant}$ is:

$$\iint_{S\perp uv} \vec{F}\cdot\hat{n}dS \approx \frac{\partial(F_w h_u h_v)}{\partial w}\Delta w \Delta u \Delta v$$

We can follow a similar train of thought to find the contributions through the other coordinate surfaces as:

$$\iint_{S\perp uw} \vec{F} \cdot \hat{n} dS \approx \frac{\partial(F_v h_u h_w)}{\partial v} \Delta v \Delta u \Delta w$$

$$\iint_{S\perp vw} \vec{F} \cdot \hat{n} dS \approx \frac{\partial(F_u h_v h_w)}{\partial u} \Delta u \Delta v \Delta w$$

What this means is that the total flux through the closed surface of the entire box is (i.e. what we have on the right-hand side of the divergence theorem):

$$\oiint_{\partial V} \vec{F} \cdot \hat{n} dS = \iint_{S\perp uv} \vec{F} \cdot \hat{n} dS + \iint_{S\perp uw} \vec{F} \cdot \hat{n} dS + \iint_{S\perp vw} \vec{F} \cdot \hat{n} dS$$
$$\approx \frac{\partial(F_w h_u h_v)}{\partial w} \Delta w \Delta u \Delta v + \frac{\partial(F_v h_u h_w)}{\partial v} \Delta v \Delta u \Delta w + \frac{\partial(F_u h_v h_w)}{\partial u} \Delta u \Delta v \Delta w$$
$$= \left(\frac{\partial(h_v h_w F_u)}{\partial u} + \frac{\partial(h_u h_w F_v)}{\partial v} + \frac{\partial(h_u h_v F_w)}{\partial w} \right) \Delta u \Delta v \Delta w$$

We now have the following approximation as a result of the divergence theorem:

$$\iiint_V \nabla \cdot \vec{F} dV = \oiint_{\partial V} \vec{F} \cdot \hat{n} dS$$
$$\Rightarrow \quad \nabla \cdot \vec{F} h_u h_v h_w \Delta u \Delta v \Delta w \approx \left(\frac{\partial(h_v h_w F_u)}{\partial u} + \frac{\partial(h_u h_w F_v)}{\partial v} + \frac{\partial(h_u h_v F_w)}{\partial w} \right) \Delta u \Delta v \Delta w$$

Dividing now by $h_u h_v h_w \Delta u \Delta v \Delta w$ on both sides and taking the limit $\Delta u \Delta v \Delta w \to 0$, this approximation indeed becomes exact - describing the divergence at exactly a single point - and we have:

$$\nabla \cdot \vec{F} = \frac{1}{h_u h_v h_w} \left(\frac{\partial(h_v h_w F_u)}{\partial u} + \frac{\partial(h_u h_w F_v)}{\partial v} + \frac{\partial(h_u h_v F_w)}{\partial w} \right)$$

1.3. Curl

The curl is an operation that takes a vector field to another vector field. It represents, in some sense, the "circulation" of the vector field at each point.

Here, we won't derive the general formula for the curl. However, it has a similar structure to the gradient and divergence in the sense that it also contains the scale factors (both inside and outside the derivatives).

Anyway, here is the general formula for the curl of a vector field, again in any *orthogonal* curvilinear coordinate system (written component-wise):

$$\begin{aligned}(\nabla\times\vec{F})_u &= \frac{1}{h_v h_w}\left(\frac{\partial(h_w F_w)}{\partial v}-\frac{\partial(h_v F_v)}{\partial w}\right)\\ (\nabla\times\vec{F})_v &= \frac{1}{h_u h_w}\left(\frac{\partial(h_u F_u)}{\partial w}-\frac{\partial(h_w F_w)}{\partial u}\right)\\ (\nabla\times\vec{F})_w &= \frac{1}{h_u h_v}\left(\frac{\partial(h_v F_v)}{\partial u}-\frac{\partial(h_u F_u)}{\partial v}\right)\end{aligned}$$

The full curl vector field is then just $\nabla\times\vec{F}=(\nabla\times\vec{F})_u\hat{e}_u+(\nabla\times\vec{F})_v\hat{e}_v+(\nabla\times\vec{F})_w\hat{e}_w$ *, however, it's much more convenient to write it component-wise like above due to the length of the resulting expression.*

1.4. Laplacian

Lastly, we have the Laplacian. This is an operator that acts on a scalar field to give another scalar field (although it could also act on a vector field).

The Laplacian essentially describes the *average difference* in values of the scalar field at a particular point, compared to the neighboring points. This is why it describes things like heat diffusion, which happens if there is a temperature difference between nearby points.

In any case, here is the general formula for the Laplacian in any *orthogonal* coordinate system:

$$\nabla^2 f = \frac{1}{h_u h_v h_w}\left(\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial f}{\partial w}\right)\right)$$

Again, it contains lots of these scale factors, both inside and outside the partial derivatives. Notably, like it should, the Laplacian has two derivatives with respect to each coordinate.

Derivation of The General Formula For The Laplacian

Given that we already know the general formulas for the gradient and divergence, calculating the Laplacian is very simple. This is because the Laplacian of a scalar field is *defined* as the divergence of the gradient of the scalar field, $\nabla^2 f = \nabla \cdot \nabla f$.

The divergence of *any* vector field, which we derived earlier, is:

$$\nabla \cdot \vec{F} = \frac{1}{h_u h_v h_w}\left(\frac{\partial(h_v h_w F_u)}{\partial u} + \frac{\partial(h_u h_w F_v)}{\partial v} + \frac{\partial(h_u h_v F_w)}{\partial w}\right)$$

Thus, this formula will also apply for the vector field $\vec{F} = \nabla f$. The components of this vector field are the components of the gradient in this curvilinear coordinate system, which we found earlier as:

$$F_u = (\nabla f)_u = \frac{1}{h_u}\frac{\partial f}{\partial u},\ F_v = (\nabla f)_v = \frac{1}{h_v}\frac{\partial f}{\partial v},\ F_w = (\nabla f)_w = \frac{1}{h_w}\frac{\partial f}{\partial w}$$

Thus, the Laplacian is:

$$\nabla \cdot \vec{F} = \nabla \cdot \nabla f = \nabla^2 f = \frac{1}{h_u h_v h_w}\left(\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial f}{\partial w}\right)\right)$$

2. Examples of Nablas In Different Coordinates

We've now looked at the general formulas for each of the nabla operators in *orthogonal* curvilinear coordinates. For faster reference later, here are the results:

Gradient	$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{e}_v + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{e}_w$
Divergence	$\nabla \cdot \vec{F} = \frac{1}{h_u h_v h_w}\left(\frac{\partial(h_v h_w F_u)}{\partial u} + \frac{\partial(h_u h_w F_v)}{\partial v} + \frac{\partial(h_u h_v F_w)}{\partial w}\right)$
Curl (component-wise)	$(\nabla \times \vec{F})_u = \frac{1}{h_v h_w}\left(\frac{\partial(h_w F_w)}{\partial v} - \frac{\partial(h_v F_v)}{\partial w}\right)$ $(\nabla \times \vec{F})_v = \frac{1}{h_u h_w}\left(\frac{\partial(h_u F_u)}{\partial w} - \frac{\partial(h_w F_w)}{\partial u}\right)$ $(\nabla \times \vec{F})_w = \frac{1}{h_u h_v}\left(\frac{\partial(h_v F_v)}{\partial u} - \frac{\partial(h_u F_u)}{\partial v}\right)$
Laplacian	$\nabla^2 f = \frac{1}{h_u h_v h_w}\left(\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial f}{\partial w}\right)\right)$

Next, we will apply these formulas to obtain expressions for the gradient, divergence, curl and Laplacian in the two most common coordinate systems used in physics - spherical and cylindrical coordinates.

2.1. Spherical Coordinates

Let's begin with *spherical coordinates*. All we really need to know are the following:

1. **The coordinates being used**, which in this case, we are going to set as $u = r$, $v = \theta$ and $w = \varphi$ for spherical coordinates.

2. **The scale factors for spherical coordinates**, which are $h_u = h_r = 1$, $h_v = h_\theta = r$ and $h_w = h_\varphi = r \sin\theta$.

Also, the unit basis vectors are simply $\hat{e}_u = \hat{r}$, $\hat{e}_v = \hat{\theta}$ and $\hat{e}_w = \hat{\varphi}$. With these, we can write out the expression for the **gradient** as:

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{e}_v + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{e}_w \quad \Rightarrow \quad \boxed{\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\varphi}}$$

Quite a simple calculation at this point, isn't it? This looks a bit similar in form to the Cartesian gradient, except the factors of $1/r$ and $1/r\sin\theta$ - these are just additional factors that need to be included in the case of curvilinear coordinates.

Anyway, let's do the **divergence** next. Using out general formula from above, this becomes:

$$\nabla \cdot \vec{F} = \frac{1}{h_u h_v h_w}\left(\frac{\partial(h_v h_w F_u)}{\partial u} + \frac{\partial(h_u h_w F_v)}{\partial v} + \frac{\partial(h_u h_v F_w)}{\partial w}\right)$$

$$\Rightarrow \quad \nabla \cdot \vec{F} = \frac{1}{r^2 \sin\theta}\left(\frac{\partial\left(r^2 \sin\theta F_r\right)}{\partial r} + \frac{\partial(r\sin\theta F_\theta)}{\partial \theta} + \frac{\partial(rF_\varphi)}{\partial \varphi}\right)$$

We can simplify this a bit by pulling out the factor of $\sin\theta$ from the r-derivative (since $\sin\theta$ is a constant with respect to r) and similarly for the two factors of r from the θ- and φ-derivatives.

The result we get is:

$$\nabla \cdot \vec{F} = \frac{1}{r^2}\frac{\partial\left(r^2 F_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta F_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial\varphi}$$

For the components of the **curl**, we get by similar simplifications:

$$\begin{cases} (\nabla\times\vec{F})_u = \dfrac{1}{h_v h_w}\left(\dfrac{\partial(h_w F_w)}{\partial v} - \dfrac{\partial(h_v F_v)}{\partial w}\right) \\ (\nabla\times\vec{F})_v = \dfrac{1}{h_u h_w}\left(\dfrac{\partial(h_u F_u)}{\partial w} - \dfrac{\partial(h_w F_w)}{\partial u}\right) \\ (\nabla\times\vec{F})_w = \dfrac{1}{h_u h_v}\left(\dfrac{\partial(h_v F_v)}{\partial u} - \dfrac{\partial(h_u F_u)}{\partial v}\right) \end{cases} \Rightarrow \begin{cases} (\nabla\times\vec{F})_r = \dfrac{1}{r\sin\theta}\left(\dfrac{\partial(\sin\theta F_\varphi)}{\partial\theta} - \dfrac{\partial F_\theta}{\partial\varphi}\right) \\ (\nabla\times\vec{F})_\theta = \dfrac{1}{r\sin\theta}\left(\dfrac{\partial F_r}{\partial\varphi} - \dfrac{\partial(r\sin\theta F_\varphi)}{\partial r}\right) \\ (\nabla\times\vec{F})_\varphi = \dfrac{1}{r}\left(\dfrac{\partial(rF_\theta)}{\partial r} - \dfrac{\partial F_r}{\partial\theta}\right) \end{cases}$$

Lastly, for the **Laplacian**, we have:

$$\nabla^2 f = \frac{1}{h_u h_v h_w}\left(\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial f}{\partial w}\right)\right)$$

$$\Rightarrow \quad \nabla^2 f = \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{r\sin\theta}{r}\frac{\partial f}{\partial\theta}\right) + \frac{\partial}{\partial\varphi}\left(\frac{r}{r\sin\theta}\frac{\partial f}{\partial\varphi}\right)\right)$$

Once again, we can pull some of the terms outside the derivatives to simplify this. The final result is:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

Examples

Let's have a look at some simple examples of using these formulas (we'll discuss some more detailed examples and applications later). First, consider the following scalar field, which only depends on the radial coordinate r:

$\phi = \phi(r) = \frac{k}{r}$, where k is a constant.

This type of scalar field can describe, for example, the electric potential of a point particle. Now, since this is a scalar field, we can calculate its gradient and Laplacian using the formulas above:

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat{\varphi} = \frac{\partial}{\partial r}\left(\frac{k}{r}\right)\hat{r} = -\frac{k}{r^2}\hat{r}$$

$$\begin{aligned}\nabla^2\phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{k}{r}\right)\right)\\ &= 0\end{aligned}$$

Notice how the Laplacian here gives zero because of the factor of r^2 *inside* the derivative cancelling with that resulting from the expression for ϕ itself.

The relevance of this result is that the potential satisfies (assuming $r \neq 0$) Laplace's equation, $\nabla^2\phi = 0$ - an important equation in electrostatics - which we will discuss more later.

Next, consider the following vector field with only a radial component (that also depends on the r-coordinate only):

$\vec{E} = E_r(r)\hat{r} = \frac{k}{r^2}\hat{r}$, where k is a constant again.

This describes, for example, the electric field of the same point particle as the potential ϕ we considered above.

For this vector field, we can calculate its divergence and curl as:

$$\nabla \cdot \vec{E} = \frac{1}{r^2}\frac{\partial\left(r^2 E_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta E_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial E_\varphi}{\partial\varphi} = \frac{1}{r^2}\frac{\partial\left(r^2\frac{k}{r^2}\right)}{\partial r} = 0$$

$$\nabla \times \vec{E} = \frac{1}{r\sin\theta}\left(\frac{\partial E_r}{\partial\varphi} - \frac{\partial(r\sin\theta E_\varphi)}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial(rE_\theta)}{\partial r} - \frac{\partial E_r}{\partial\theta}\right)\hat{\varphi} = 0$$

The curl here gives zero because there are no derivatives of the form $\partial E_r / \partial r$.

These results are important, because Maxwell's equations for electrostatics state that in a charge-free region, the electric field must satisfy $\nabla \cdot \vec{E} = \nabla \times \vec{E} = 0$ - which our electric field of the form $\vec{E} \sim \hat{r}/r^2$ indeed does (again, outside $r = 0$). Thus, this would be a valid solution to Maxwell's equations!

Exercise 3.1

The electric potential of a dipole (consisting of two charges, q and $-q$, separated by a distance d) can be expressed in spherical coordinates as:

$$\phi = \phi(r,\theta) = \frac{qd\cos\theta}{4\pi\varepsilon_0 r^2}$$

Here, q, d, π, ε_0 are just constants.

Calculate the electric field of the dipole in spherical coordinates as $\vec{E} = -\nabla\phi$. Also show that $\nabla \cdot \vec{E} = 0$.

2.2. Cylindrical Coordinates

Let's look at the cylindrical coordinate system next. In this case, we have the coordinates $u = \rho$, $v = \varphi$ and $w = z$ as well as the scale factors $h_\rho = 1$, $h_\varphi = \rho$ and $h_z = 1$. I'll just tell you what the results we get from the general formulas are:

- **Gradient:** $\nabla f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\hat{\varphi} + \frac{\partial f}{\partial z}\hat{z}$
- **Divergence:** $\nabla \cdot \vec{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\varphi}{\partial \varphi} + \frac{\partial F_z}{\partial z}$
- **Curl:** $\nabla \times \vec{F} = \left(\frac{1}{\rho}\frac{\partial F_z}{\partial \varphi} - \frac{\partial F_\varphi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right)\hat{\varphi} + \frac{1}{\rho}\left(\frac{\partial(\rho F_\varphi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \varphi}\right)\hat{z}$
- **Laplacian:** $\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$

Example: Consider the following vector field in cylindrical coordinates:

$\vec{B} = \frac{\mu_0 I}{2\pi\rho}\hat{\varphi}$, where μ_0 and I (and π, of course) are constants.

This describes the magnetic field around a straight wire located at $\rho = 0$, carrying a current I. We can calculate the divergence and curl of this magnetic field using the formulas from above (the magnetic field only has a φ-component, $B_\varphi = \mu_0 I / 2\pi\rho$):

$$\nabla \cdot \vec{B} = \frac{1}{\rho}\frac{\partial(\rho B_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial B_\varphi}{\partial \varphi} + \frac{\partial B_z}{\partial z} = 0$$

$$\nabla \times \vec{B} = \left(\frac{1}{\rho}\frac{\partial B_z}{\partial \varphi} - \frac{\partial B_\varphi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho}\right)\hat{\varphi} + \frac{1}{\rho}\left(\frac{\partial(\rho B_\varphi)}{\partial \rho} - \frac{\partial B_\rho}{\partial \varphi}\right)\hat{z} = 0$$

Now, according to Maxwell's equations, in a system with static fields and no currents, all magnetic fields must satisfy $\nabla \cdot \vec{B} = \nabla \times \vec{B} = 0$. We see that this isthe case (outside $\rho = 0$), so this would be a valid *free-space solution* to Maxwell's equations!

Exercise 3.2

Consider the following vector potential in cylindrical coordinates:

$\vec{A} = -\frac{\mu_0 I}{2\pi} \ln \rho \hat{z}$, where μ_0 and I are constants again.

Calculate the magnetic field in cylindrical coordinates, defined as $\vec{B} = \nabla \times \vec{A}$. Based on your result and the example above, what would this vector potential describe?

We've now discussed all the key ideas of this lesson! Next, it's time to move on to some real physics applications and examples.

However, before we do that, on the next page, you'll find a summary of all the results we've found for **spherical coordinates** and **cylindrical coordinates** in this lesson. Feel free to bookmark these for later reference!

Spherical Coordinates	
Gradient	$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\varphi}$
Divergence	$\nabla \cdot \vec{F} = \frac{1}{r^2}\frac{\partial\left(r^2 F_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta F_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial\varphi}$
Curl	$\nabla \times \vec{F} = \frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta F_\varphi)}{\partial\theta} - \frac{\partial F_\theta}{\partial\varphi}\right)\hat{r} + \frac{1}{r\sin\theta}\left(\frac{\partial F_r}{\partial\varphi} - \frac{\partial(r\sin\theta F_\varphi)}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right)\hat{\varphi}$
Laplacian	$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$

Cylindrical Coordinates	
Gradient	$\nabla f = \frac{\partial f}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial\varphi}\hat{\varphi} + \frac{\partial f}{\partial z}\hat{z}$
Divergence	$\nabla \cdot \vec{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial F_\varphi}{\partial\varphi} + \frac{\partial F_z}{\partial z}$
Curl	$\nabla \times \vec{F} = \left(\frac{1}{\rho}\frac{\partial F_z}{\partial\varphi} - \frac{\partial F_\varphi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial\rho}\right)\hat{\varphi} + \frac{1}{\rho}\left(\frac{\partial(\rho F_\varphi)}{\partial\rho} - \frac{\partial F_\rho}{\partial\varphi}\right)\hat{z}$
Laplacian	$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial^2 f}{\partial z^2}$

3. Application: Laplace's Equation In Electrostatics

In this section, we'll dive into some applications in electrostatics that use the Laplacian operator in different coordinates. In particular, we will solve the so-called Laplace's equation for a couple simple example systems in electrostatics.

Electrostatics is the "sub-branch" of electromagnetism in which all electric fields we study are time-independent - that is, they are only functions of position. Usually, we also take magnetic fields to be zero in the context of electrostatics.

In this case, with these assumptions, the two relevant Maxwell's equations for the electric field have the form:

$$\begin{cases} \nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0} \\ \nabla \times \vec{E} = 0 \end{cases}$$

Now, we could just start solving these for the electric field right away like this (for a particular charge configuration). However, in a lot of cases, solving for the electric field is much simpler if done through the **electric potential**.

The electric potential is a scalar function - denoted here as ϕ - that is used to define the electric field in electrostatics as:

$$\boxed{\vec{E} = -\nabla \phi}$$

The "in-many-cases-simpler" approach would then be to first solve for the potential and then calculate the electric field from this definition. But how do we solve for the potential? We can derive an equation for it! First, substitute the definition $\vec{E} = -\nabla \phi$ into Gauss's law:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \quad \nabla \cdot (-\nabla \phi) = \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \quad \nabla^2 \phi = -\frac{\rho}{\varepsilon_0}$$

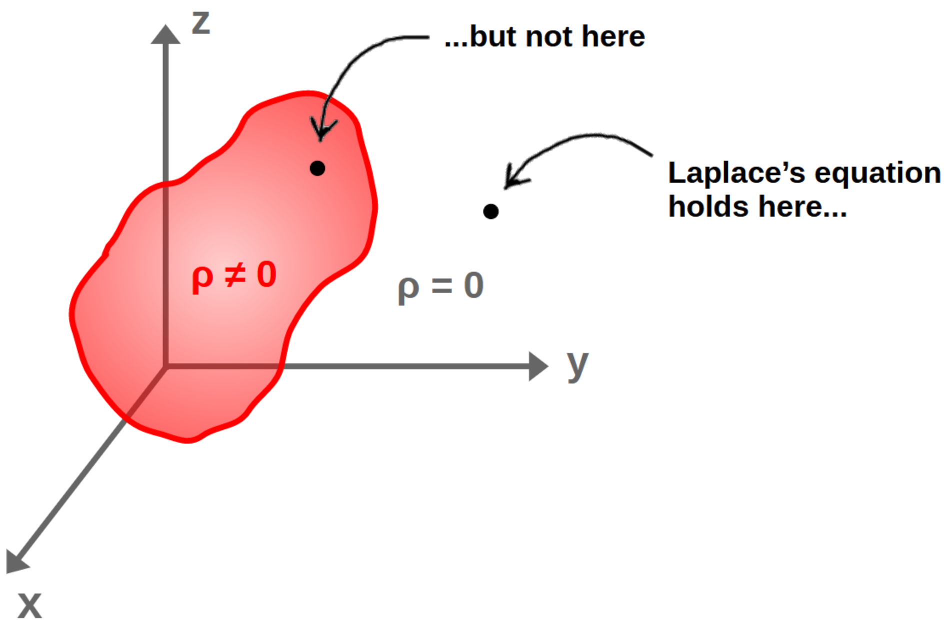
Here, $\nabla \cdot \nabla = \nabla^2$ *is the Laplacian operator.*

This is the so-called *Poisson's equation*, which the electric potential of any electrostatic system must satisfy. However, in a lot of cases, we can get away by solving a much simpler equation if we just set $\rho = 0$. In this case, we get **Laplace's equation**:

$$\boxed{\nabla^2 \phi = 0}$$

Okay, this is a simpler equation, but does setting $\rho = 0$ actually make any sense? Well, it does. In many electrostatic problems, we are dealing with regions in which there are no electric charges directly. In these regions, $\rho = 0$ is true.

In fact, take any arbitrary charge distribution - if we only want to know the fields *outside* of that charge distribution (and don't care about what is going on inside it), then we have $\rho = 0$ and Laplace's equation holds.



Therefore, Laplace's equation is incredibly useful if we want to find the fields at points where there are no charge at exactly that point. In fact, I would argue that there are more problems where the fields outside some charge region are more useful to know than inside it.

3.1. Laplace's Equation In Spherical Coordinates

In its most general form, Laplace's equation states that $\nabla^2 \phi = 0$. But, as we know based on this lesson, the **Laplacian operator** looks different in coordinate systems.

Therefore, if we want to write Laplace's equation in terms of a specific set of coordinates, we need to express the Laplacian in that coordinate system.

Let's begin by looking at Laplace's equation in **spherical coordinates**. We know from earlier that the Laplacian for any function f has the following form in spherical coordinates:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}$$

Thus, Laplace's equation for the electric potential $\phi = \phi(r, \theta, \varphi)$ takes the following form:

$$\nabla^2 \phi = 0 \quad \Rightarrow \quad \boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \varphi^2} = 0}$$

In its most general form, this equation looks pretty complicated - and solving it is even more complicated. Moreover, solving partial differential is not the aim of this lesson anyway.

To get a basic idea for what its solutions may describe, we'll resort to some simplifications. In particular, let's take a *spherically symmetric* potential - so, one that only depends on the radial distance, $\phi = \phi(r)$.

In this case, the derivatives with respect to θ and φ go to zero and we get:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = 0 \quad \Rightarrow \quad \frac{d}{dr}\left(r^2\frac{d\phi(r)}{dr}\right) = 0$$

Here, the $\partial/\partial r$-derivatives turn into ordinary derivatives, since $\phi(r)$ is just a single-variable function now.

This can now be solved quite easily. First, we just integrate once with respect to r to get:

$r^2\frac{d\phi(r)}{dr} = A$, where A is an arbitrary integration constant.

We can then divide by r^2 and integrate once more to get another integration constant B:

$$\phi(r) = \int \frac{A}{r^2}dr = \boxed{-\frac{A}{r} + B}$$

This is the most general expression for a spherically symmetric electric potential. Notice how it is of the same form as the potential of a single charged particle, $\phi \sim r^{-1}$.

The interesting thing, however, is that we didn't even mention anything about a single charged particle - this $\sim 1/r$ dependence is purely a result of Laplace's equation in spherical coordinates. It is true for any radial potential that only depends on the coordinate r (outside regions of charges).

3.2. Solving Laplace's Equation For Different Capacitors

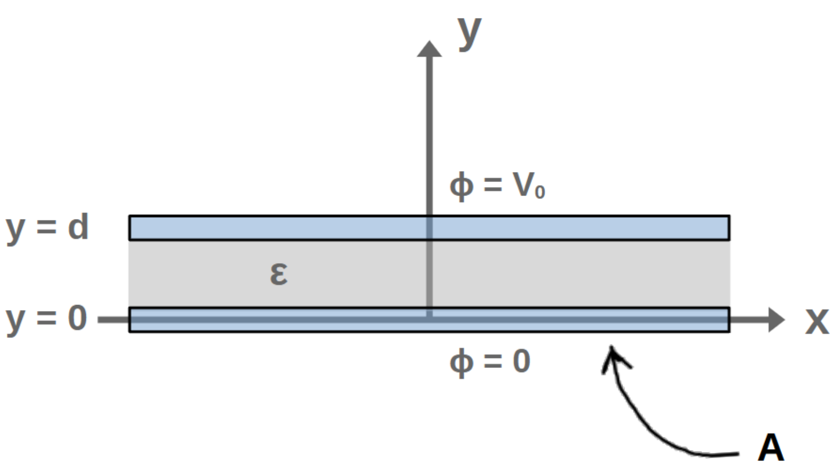
One of the many useful applications of Laplace's equation (in electrostatics) is for analyzing various types of **capacitors**. Capacitors are electronic devices that are able to "store" energy (or electric charge) and release it when needed, kind of like a battery.

Virtually all modern electronic circuits use capacitors in some form. In DC (direct current) circuits, the main use for capacitors is to "stabilize" voltage fluctuations.

We'll begin by analyzing one of the simplest types of capacitors - the **parallel plate capacitor**. It consists of two plates of some conducting material, with an insulating dielectric layer between them.

The purpose of the dielectric is that electricity cannot flow between the plates, meaning that if the plates become electrically charged, the charge will remain on the plates and there is a voltage between them. This is how the capacitor "stores charge".

We'll assume here that the plates have a surface A and separation d between them. One of the plates is charged to a constant potential or voltage V and the other is grounded (so, its potential is zero). The permittivity of the dielectric layer is ε - a constant material parameter.



Our goal here is to find the electric potential and electric field between the plates. Now, something important to realize is that inside the dielectric, there are not free charges - that's what it means for something to be a dielectric.

Therefore, in this region between the plates, Laplace's equation will apply. We can write it down for the potential ϕ between the plates in (2D) Cartesian coordinates as:

$$\nabla^2\phi = 0 \quad \Rightarrow \quad \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

When we are well inside the plates, we would expect the potential to not depend on x, only on y.

At the edges of the plates, this assumption is not exactly true - but what we can do is assume the length of the plates in the x-direction is much much larger than the separation d between them (which is a reasonable assumption for actual real-world capacitors).

Thus, well inside the plates, we have $\phi = \phi(y)$ and Laplace's equation becomes:

$$\frac{d^2\phi(y)}{dy^2} = 0$$

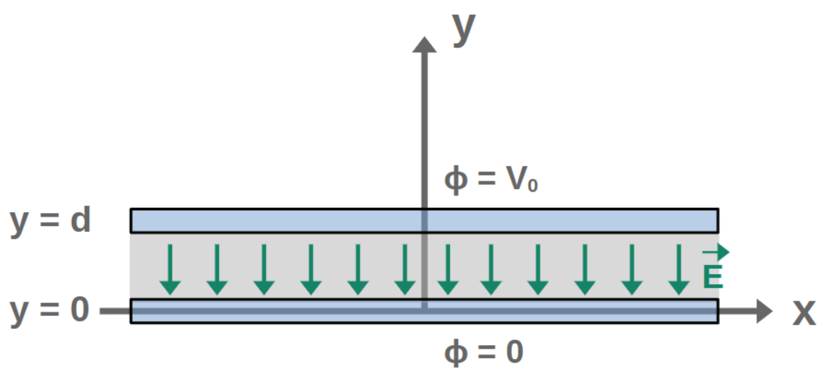
The solution to this is a linear function of y, so $\phi(y) = Ay + B$. In order to match this with our boundary conditions - $\phi(0) = 0$ and $\phi(d) = V_0$ - we need $A = V_0 / d$ and $B = 0$, so our solution for the potential is:

$$\phi(y) = \frac{V_0}{d} y$$

The electric field can then be calculated as:

$$\vec{E} = -\nabla\phi = -\frac{\partial\phi}{\partial y}\hat{y} = -\frac{V_0}{d}\hat{y}$$

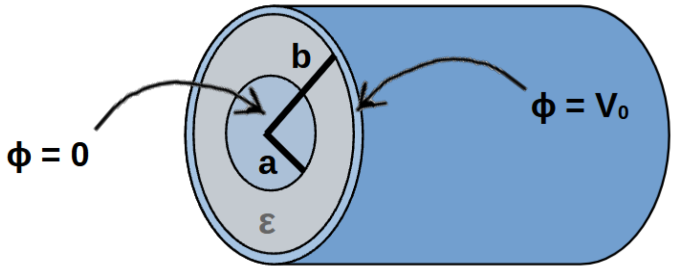
What this means is that the electric field, first of all, is a constant. As for its direction, it points from the plate with higher potential to the grounded plate (in the negative y-direction):



Okay, that's the parallel plate capacitor. This wasn't a very complicated example of using Laplace's equation, since we were able to do everything in Cartesian coordinates. Therefore, let's look at something a bit more complicated.

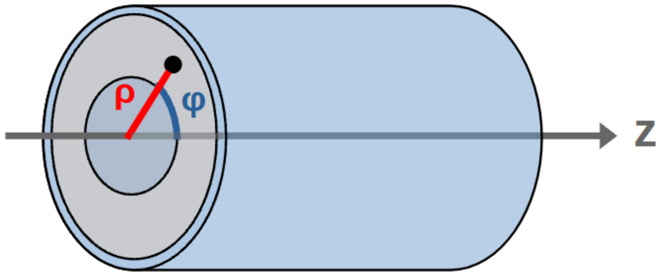
In particular, we'll analyze the so-called **cylindrical capacitor**. As the name suggests, this is a capacitor that consists of two conducting cylinders (of different radii), again with a dielectric layer between them.

We'll label the two radii as a and b. The outer cylinder is charged to voltage $\phi = V_0$ and the inner cylinder is grounded, so $\phi = 0$ there. The permittivity of the dielectric between the conductors is ε.



What is important to realize again is that since the layer between the two conducting cylinders is a dielectric (a *perfect* dielectric here), Laplace's equation applies in this region. So, inbetween the cylinders, we have $\nabla^2\phi = 0$

In this case, it definitely makes sense to use **cylindrical coordinates**. We'll place the origin at the center of the capacitor, in which case the region of interest to us is $a \leq \rho \leq b$:



We can now write down Laplace's equation for the potential in cylindrical coordinates:

$$\nabla^2 \phi = 0 \quad \Rightarrow \quad \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \phi}{\partial \varphi^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Due to the cylindrical symmetry of this problem, we would expect the potential to NOT depend on the angle φ.

Moreover, if the length (or height) of the cylinder in the z-direction is large enough and we are well inside the cylinder, the potential does not depend on z either. Thus, we have $\phi = \phi(\rho)$ and Laplace's equation becomes:

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{d\phi(\rho)}{d\rho}\right) = 0 \quad \Rightarrow \quad \frac{d}{d\rho}\left(\rho \frac{d\phi(\rho)}{d\rho}\right) = 0$$

Integrating this once, we have $\rho \dfrac{d\phi(\rho)}{d\phi} = A$, where A is an integration constant. Dividing by ρ and integrating once more to get another rintegration constant B, we have:

$$\phi(\rho) = \int \frac{A}{\rho} d\rho = A \ln \rho + B$$

This definitely has a very different position-dependence than the parallel plate capacitor from earlier. Now, in order for this to also match our boundary conditions - $\phi(a) = 0$ and $\phi(b) = V_0$ - we need to define $A = V_0 / \ln(b/a)$ and $B = -A \ln a$, such that:

$$\phi(\rho) = V_0 \frac{\ln(\rho/a)}{\ln(b/a)}$$

Lastly, we can calculate the electric field using the formula for the gradient in cylindrical coordinates:

$$\vec{E}(\rho) = -\nabla \phi = -\frac{\partial \phi}{\partial \rho}\hat{\rho} = -\frac{V_0}{\ln(b/a)}\frac{\hat{\rho}}{\rho}$$

Again, we see that the electric field points from the conductor with higher potential to the grounded one, in the negative ρ-direction. In this case, however, the electric field is not constant - it depends on the radial distance as $\sim 1/\rho$.

4. Application: Helmholtz Equation & Electromagnetic Waves

Above, we looked at some examples from electrostatics - that is, in cases where the fields do not change with time. At this point, however, we are in a good place to also study more complicated situations, such as time-dependent fields.

The plan for this section is to first derive the wave equation for **electromagnetic waves** from Maxwell's equations. Then, we will apply it further for the special case of *time-harmonic fields*. This gives us the so-called **Helmholtz equation**, which we will solve for a few example cases.

4.1. Derivation of The Helmholtz Equation

If we want to study full time-dependent electric and magnetic fields, we need the full four Maxwell's equations:

$$\begin{cases} \nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0} \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{B} = \mu_0 \vec{J} + \dfrac{1}{c^2}\dfrac{\partial \vec{E}}{\partial t} \end{cases}$$

What we would ideally like is an equation for only one of the fields, say the electric field $\vec{E}$. The way we can get such an equation is by manipulating the above four Maxwell's equations a little bit.

First, let's take the curl of the third Maxwell's equation:

$$\nabla \times \nabla \times \vec{E} = -\nabla \times \frac{\partial \vec{B}}{\partial t}$$

On the left-hand side, we can apply the vector calculus identity for the "double-curl", which states that $\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$. On the right-hand side, since partial derivatives are assumed interchangeable here, we can change the order of the $\partial / \partial t$ and $\nabla \times$ operations. Thus, we have:

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

We can then insert the first Maxwell equation - $\nabla \cdot \vec{E} = \rho / \varepsilon_0$ - on the left-hand side and the fourth Maxwell equation - $\nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$ - on the right:

$$\nabla\left(\frac{\rho}{\varepsilon_0}\right) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}\right) \quad \Rightarrow \quad \nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{1}{\varepsilon_0} \nabla \rho - \mu_0 \frac{\partial \vec{J}}{\partial t}$$

This is the general form of the wave equation for electromagnetic waves, with the effects of sources also included. In our case, however, we will study the vacuum solutions for this wave equation, which means that $\rho = 0$ and $\vec{J} = 0$.

With this, we then get the **free space wave equation** in the form:

$$\boxed{\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0}$$

This differential equation describes all electromagnetic waves in free space. But why does it describe *waves*, exactly? Well, it has the mathematical form of a standard wave equation. Also, what we will find out soon is that the solutions to it have the form of sine waves.

The equation above is the general wave equation in free space. However, many of the most useful solutions to it are a result of something called the *Helmholtz equation*.

Specifically, the wave equation can be reduced to the Helmholtz equation in the special case of fields that can be "separated" as $\vec{E}(\vec{r},t) = f(\vec{r})T(t)\hat{u}$, where $\hat{u}$ is some constant unit vector.

This is commonly known as the technique of **separation of variables** - perhaps the most common strategy for solving partial differential equations.

In any case, the idea is to write the field as a product of two functions, with one containing all of the time-dependence and the other all the spatial dependence. Plugging this ansatz into the wave equation, we find:

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$$
$$\Rightarrow \quad \nabla^2(f(\vec{r})T(t)\hat{u}) - \frac{1}{c^2}\frac{d^2(f(\vec{r})T(t)\hat{u})}{dt^2} = 0$$
$$\Rightarrow \quad \left(T(t)\nabla^2 f(\vec{r}) - \frac{1}{c^2}f(\vec{r})\frac{d^2T(t)}{dt^2}\right)\hat{u} = 0$$
$$\Rightarrow \quad \frac{1}{f(\vec{r})}\nabla^2 f(\vec{r}) = \frac{1}{c^2}\frac{1}{T(t)}\frac{d^2T(t)}{dt^2}$$

Here, we went through a couple of steps quite quickly. First, we can pull $T(t)$ outside the ∇^2-operation since the Laplacian only contains spatial derivatives (and similarly for the d^2/dt^2-derivative). We also "divided" out the $\hat{u}$ from both sides and rearranged our equation to have f and T on different sides.

We now have this equation in an interesting form - all the spatial dependence is contained on the left-hand side, while all the time-dependence is on the right.

The only way for an equation like this to be true is if both sides are actually *constant*.

If we call this constant $-k^2$, the above equation splits into two equations as follows:

$$\begin{cases} \dfrac{1}{f(\vec{r})}\nabla^2 f(\vec{r}) = -k^2 & \Rightarrow \quad \nabla^2 f(\vec{r}) + k^2 f(\vec{r}) = 0 \\ \dfrac{1}{c^2}\dfrac{1}{T(t)}\dfrac{d^2T(t)}{dt^2} = -k^2 & \Rightarrow \quad \dfrac{d^2T(t)}{dt^2} + k^2c^2T(t) = 0 \end{cases}$$

The second equation here is simple to solve, since it has the form of a standard differential equation describing **harmonic motion**. The solutions to it are waves with frequency $\omega = ck$, so for example:

$$\boxed{T(t) = \cos \omega t}$$

More generally, complex exponentials $e^{-i\omega t}$ *would also work as solutions here. In fact, complex exponentials like* $e^{-i\omega t}$ *lead to travelling wave solutions for the full electric field, while a solution like* $\cos \omega t$ *leads to a standing wave - but either one is a valid solution.*

We've now solved for the time-dependence of our electric field! As promised, it indeed describes a wave. Generally, fields that have a sinusoidal time-dependence of this form are called **time-harmonic**. Time-harmonic fields are by far the most useful electromagnetic wave solutions to Maxwell's equations.

Perhaps the more interesting part of our discussion is the first equation we derived above. This is called the **Helmholtz equation** and its solutions give us the spatial dependence of our electromagnetic waves:

$$\boxed{\nabla^2 f(\vec{r}) + k^2 f(\vec{r}) = 0}$$

What makes this equation interesting is the fact that the Laplacian has a different form in every coordinate system. Therefore, solutions to the Helmholtz equation are also going to be very different, depending on the coordinate system we choose.

So, unlike the time-dependence (which always has the same form of $\sim \cos(\omega t)$), solutions to the Helmholtz equation can be much more unique and interesting.

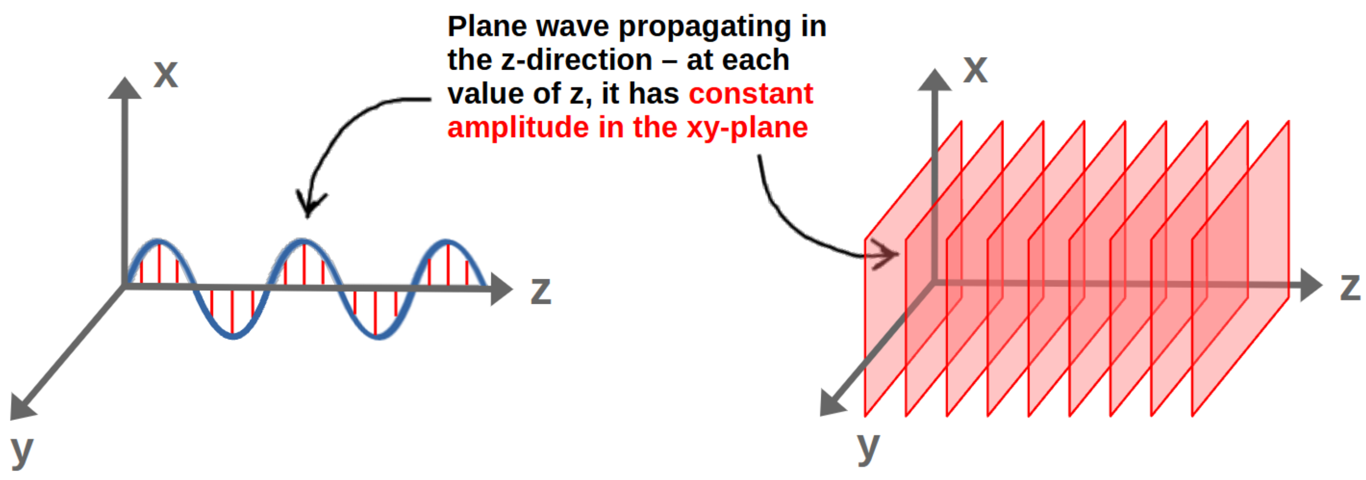
Now, these solutions will also generally be waves - but more interesting kinds of waves, like *spherical waves*.

In the following section, we will solve the Helmholtz equation for some example cases and see what kind of electromagnetic wave solutions we find.

4.2. Plane Waves & Spherical Waves

The first, and also the simplest, type of wave the Helmholtz equation describes is called a **plane wave**. This is a wave that propagates in a single spatial direction, and at any point, has a constant magnitude in a plane perpendicular to the propagation direction.

The term 'plane wave' comes from the fact that you can think of the wave as a "plane" that moves forward in its propagation direction, with the amplitude being constant throughout that plane:



Following the picture above, let's assume we have a plane wave propagating in the z-direction. As is probably clear already, plane waves are best described in Cartesian coordinates (which consists of, well, planes).

By definition, the amplitude of the plane wave can then only depend on the z-coordinate, so it will be of the form $f(\vec{r}) = f(z)$. The Laplacian also has the simple form $\nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2$.

In Cartesian coordinates and with the function $f(z)$, Helmholtz's equation gives us:

$$\nabla^2 f(\vec{r}) + k^2 f(\vec{r}) = 0$$
$$\Rightarrow \quad \underbrace{\frac{\partial^2 f(z)}{\partial x^2}}_{=0} + \underbrace{\frac{\partial^2 f(z)}{\partial y^2}}_{=0} + \frac{\partial^2 f(z)}{\partial z^2} + k^2 f(z) = 0$$
$$\Rightarrow \quad \frac{d^2 f(z)}{dz^2} + k^2 f(z) = 0$$

Notice how this is now of the exact same form as the time-dependence equation we had earlier. Thus, this will have solutions of the form:

$$\boxed{f(z) = A\cos kz}$$

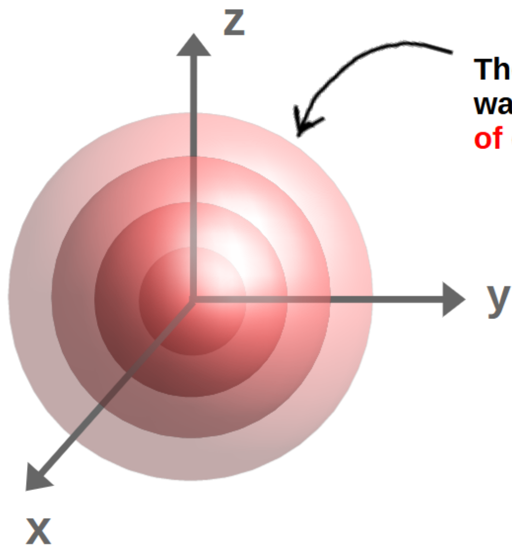
If you were to write the full electric field solution using this, it would be of the form $\vec{E}(z,t) \propto \cos kz \cos \omega t$. This actually describes a standing wave, which consists of plane waves propagating in both the $+z$ and $-z$ directions.

We can obtain another, perhaps more interesting solution to the Helmholtz equation by solving it in spherical coordinates instead of Cartesian. For this specific solution, we will assume our function f is of the form $f(r)$ - it only depends on the radial coordinate.

But what would such a solution describe? Well, at a given fixed value of r, $f(r)$ is constant - in other words, the amplitude of the wave is constant on the surface of a sphere of fixed radius.

Such a thing is called a **spherical wave** (it'll become clear why a *wave* once we actually solve the Helmholtz equation). This is kind of like a plane wave but instead of planes, the "planes" of constant magnitude are spheres of a given radius.

Because these spheres become larger with increasing r, we would also expect the wave to somehow "spread out", the further we go in the radial direction. Due to this "spreading out" -phenomenon, spherical waves actually decrease in magnitude with the r-coordinate (as we will see very soon).



With $f(\vec{r}) = f(r)$, the Helmholtz equation will only contain derivatives with respect to r. Using the spherical coordinate form of the Laplacian from earlier, we get:

$$\nabla^2 f(\vec{r}) + k^2 f(\vec{r}) = 0$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f(r)}{\partial r}\right) + \underbrace{\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f(r)}{\partial\theta}\right)}_{=0} + \underbrace{\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f(r)}{\partial\varphi^2}}_{=0} + k^2 f(r) = 0$$

$$\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{df(r)}{dr}\right) + k^2 f(r) = 0$$

To solve this, we will write out the derivative of the parentheses (using the product rule) and then multiply by r:

$$\frac{1}{r^2}\left(2r\frac{df(r)}{dr} + r^2\frac{d^2 f(r)}{dr^2}\right) + k^2 f(r) = 0 \quad \Rightarrow \quad 2\frac{df(r)}{dr} + r\frac{d^2 f(r)}{dr^2} + k^2 r f(r) = 0$$

If you stare at this for a bit, you might notice that the first two terms can be written as $d^2(rf(r))\,/\,dr^2$:

$$\frac{d^2}{dr^2}(rf(r)) + k^2 r f(r) = 0$$

Now, you might notice that both terms contain the same expression $rf(r)$. Therefore, doing a substitution of the form $u(r) = rf(r)$ actually turns this into an ordinary harmonic motion differential equation, which has a possible solution $u(r) = A\cos kr$:

$$\frac{d^2u(r)}{dr^2} + k^2u(r) = 0 \quad \Rightarrow \quad u(r) = A\cos kr$$

But we are not quite done yet! We still need to substitute back in $u(r) = rf(r)$ and solve for $f(r)$, which is what we care about. Doing so gives us:

$$rf(r) = A\cos kr \quad \Rightarrow \quad \boxed{f(r) = \frac{A}{r}\cos kr}$$

This solution is interesting in a couple of ways:

- It exhibits the same **wave-like behaviour** with $f(r) \propto \cos kr$ as the standard plane waves we discussed earlier - only now, the wave spreads out in all directions, radially, instead of in the direction of a single axis.

- Notably, the **amplitude of the wave also decreases** as $f(r) \propto r^{-1}$. The reason for this is that the spherical wave spreads out more and more with increasing r, leading to its amplitude having to decrease in a particular direction the further out the wave is.

The key difference here is exactly the $\sim r^{-1}$ decrease in the amplitude of the wave, compared to standard plane waves. In fact, most electromagnetic waves in the real world behave more like spherical waves instead of plane waves.

Now, perfect spherical waves do not really exist, as they would require a perfectly point-like radiating source. However, radiation from things like omnidirectional antennas - one of the more common types antennas used, for example, for WiFi - can be approximately modeled by spherical waves when far away from the antenna.