PURE MATHEMATICS

FOR ADVANCED LEVEL

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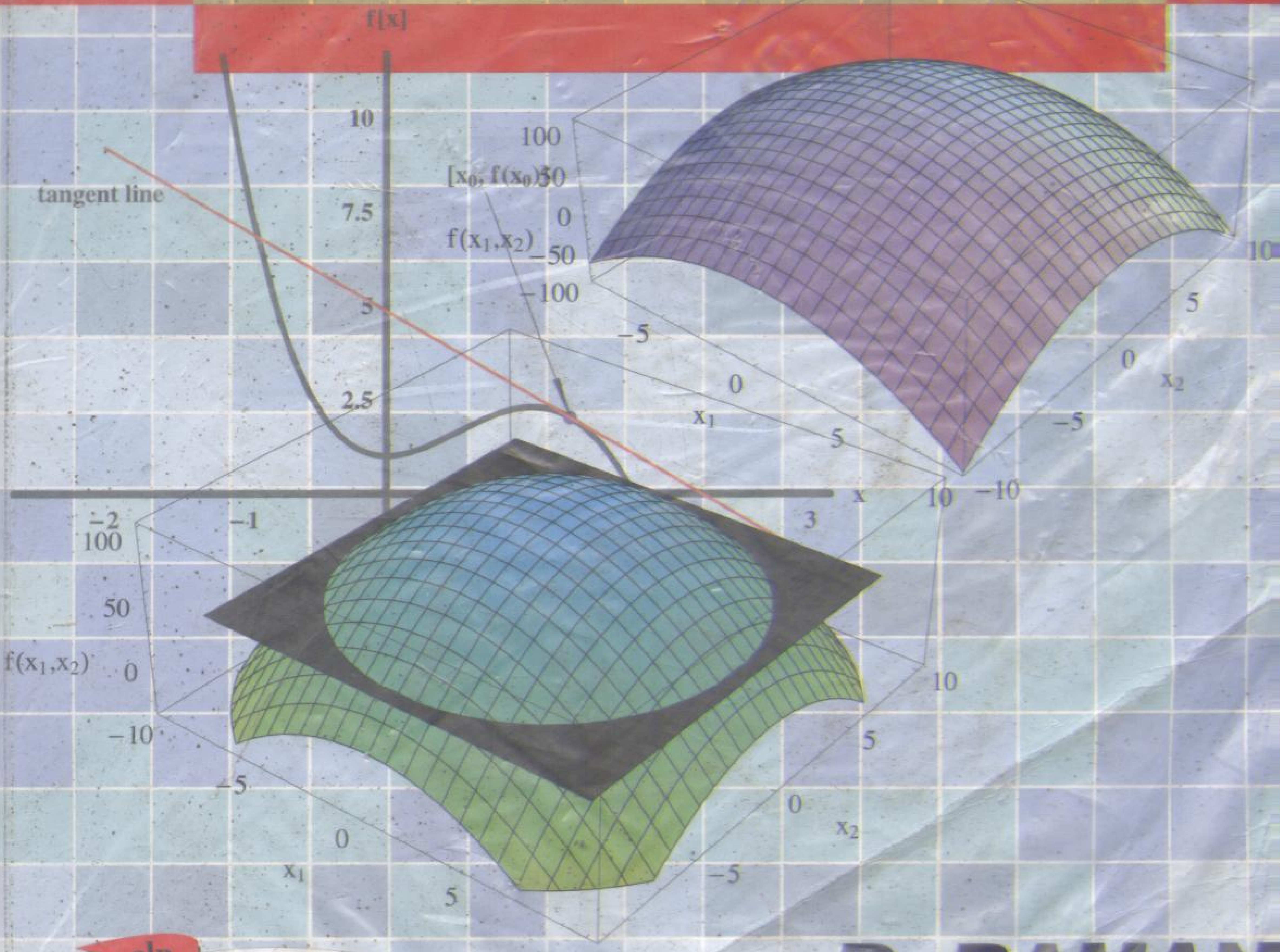

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R. RAMAN

PURE MATHEMATICS
FOR ADVANCED LEVEL

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This Second Edition of Pure Mathematics for Advanced Level is intended for students taking the University of Cambridge A-Level Mathematics at Principal Level as from 2008.

Rashid Raman, formerly Associate Professor at the Mauritius Institute of Education (MIE), has taught mathematics for the last 40 years. He taught at the Islamic Cultural College and the Queen Elizabeth College before joining the MIE in 1976 and retired in 1997. He is the author of several mathematics textbooks.

> "Pure Mathematics for Advanced Level *is an excellent text targeted for all SADC (Southern African Development Community) countries and is also suitable for the Singapore Mathematics Advanced Level Syllabus 9233 (Pure Mathematics)."*
>
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Rashid Raman

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PREFACE

This book has been written specifically for students taking principal level mathematics at the Higher School Certificate Examinations and who have taken Additional Mathematics at the Ordinary (School Certificate) Level. For the sake of simplicity, the chapters have been written in the same order as they appear in the syllabus but it is not intended that they should be taught or learned in the same order.

This is the second edition of this book. Further to comments made by teachers and students alike, alterations have been made to the bookwork in a few chapters and exercises at the end of these chapters.

Rashid RAMAN

CHAPTER 1

> QUADRATICS

1.1 Reduction of $ax^2 + bx + c$ to the form $a(x + h)^2 + k$

We know that a quadratic polynomial is of the form $ax^2 + bx + c$ where a, b, c are constants and $a \neq 0$.
Any quadratic polynomial $ax^2 + bx + c$ can be reduced to the form $a(x + h)^2 + k$ by completing the square.

1, 4, 9, 16, etc. are perfect squares as they are squares of whole numbers.
In algebra also, expressions like x^2, y^2, $(x + 2)^2$, $(x - 5)^2$ are perfect squares.

$$\begin{aligned} \text{Since } (x + y)^2 &= (x + y)(x + y) \\ &= x^2 + 2xy + y^2, \end{aligned}$$

$x^2 + 2xy + y^2$ is a perfect square, but in an expanded form.

In the expression $(x^2 + 2x + 1)$, the coefficient of $x^2 = 1$, the coefficient of $x = 2$, and 1 is the constant term.

In general, an expression in the form $(x^2 + bx + c)$ is a perfect square if $\left(\frac{b}{2}\right)^2 = c$. In the above expression, we can verify that $\left(\frac{2}{2}\right)^2 = 1$.

Thus, if we want to make the expression $(x^2 + 6x)$ become a perfect square, we find the constant term as follows:

$$c = \left(\frac{6}{2}\right)^2 = 9$$ *(Note: Coefficient of x divided by 2, then squared = 9)*

The new expression becomes $x^2 + 6x + 9$.

Note that $x^2 + 6x + 9 = (x + 3)^2$. So, $x^2 + 6x + 9$ is a complete square.

Example 1

Reduce each of the following to the form $a(x + h)^2 + k$.

(a) $x^2 + 8x$ (b) $x^2 + 10x$ (c) $x^2 + 7x$ (d) $x^2 + 3x + 4$

Solution

$$\begin{aligned} \text{(a) } x^2 + 8x &= x^2 + 8x + \left(\frac{8}{2}\right)^2 - \left(\frac{8}{2}\right)^2. \\ &= x^2 + 8x + 16 - 16 \\ &= (x + 4)^2 - 16. \end{aligned}$$

(b) $x^2 + 10x = x^2 + 10x + \left(\frac{10}{2}\right)^2 - \left(\frac{10}{2}\right)^2$

$= x^2 + 10x + 25 - 25$

$= (x + 5)^2 - 25.$

(c) $x^2 + 7x = x^2 + 7x + \left(\frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2$

$= x^2 + 7x + \frac{49}{4} - \frac{49}{4}$

$= \left(x + \frac{7}{2}\right)^2 - \frac{49}{4}.$

(d) $x^2 + 3x + 4 = x^2 + 3x + \left(\frac{3}{2}\right)^2 + 4 - \left(\frac{3}{2}\right)^2$

$= x^2 + 3x + \frac{9}{4} + 4 - \frac{9}{4}$

$= \left(x + \frac{3}{2}\right)^2 + \frac{7}{4}$

Example 2

Rewrite $2x^2 - 9x + 11$ in the form $a(x + h)^2 + k$.

Solution

$$2x^2 - 9x + 11 = 2\left(x^2 - \frac{9}{2}x + \frac{11}{2}\right)$$

$$= 2\left[x^2 - \frac{9}{2}x + \left(\frac{9}{4}\right)^2 + \frac{11}{2} - \left(\frac{9}{4}\right)^2\right]$$

$$= 2\left[\left(x - \frac{9}{4}\right)^2 + \frac{7}{16}\right]$$

$$= 2\left(x - \frac{9}{4}\right)^2 + \frac{7}{8}$$

Example 3

Rewrite $-3x^2 + 11x - 14$ in the form $a(x + h)^2 + k$.

Solution

$$-3x^2 + 11x - 14 = -3\left(x^2 - \frac{11}{3}x + \frac{14}{3}\right)$$

$$= -3\left[x^2 - \frac{11}{3}x + \left(\frac{11}{6}\right)^2 + \frac{14}{3} - \left(\frac{11}{6}\right)^2\right]$$

$$= -3\left[\left(x - \frac{11}{6}\right)^2 + \frac{47}{36}\right]$$

$$= -3\left(x - \frac{11}{6}\right)^2 - \frac{47}{12}$$

More generally

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right]$$

$$= a\left[\left(x + \frac{b^2}{2a^2}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

Exercise 1 A

Reduce each of the following quadratic polynomials to the form $a(x + h)^2 + k$.

1. $a^2 + 8a + 17$.
2. $x^2 - 18x$.
3. $2x^2 - 11x + 10$.
4. $3x^2 - 10x$.
5. $x^2 + 4x$.
6. $x^2 - 4x$.
7. $x^2 + \frac{1}{4}x$.
8. $2x^2 - 4x - 5$.
9. $3x^2 + 4x - 1$.
10. $8 - 9x - 10x^2$.
11. $x^2 + 5x - 6$.
12. $2x^2 + 3x - 2$.
13. $12 - 2x - 3x^2$.
14. $25 - 4x - 4x^2$.
15. $3x^2 + 11x + 20$.
16. $5x^2 - 12x + 7$.

1.2 Maximum or minimum value of $ax^2 + bx + c$

(a) We know also that if x is a real number, $x^2 \geqslant 0$.

So the minimum value of x^2 is 0 and it occurs when $x = 0$.

Also, $-x^2 \leqslant 0$.

So the maximum value of $-x^2$ is 0 and it occurs when $x = 0$.

It follows that $x^2 + k \geqslant k$ and $-x^2 + k \leqslant k$.

Hence, *the polynomial $x^2 + k$ has a minimum value of k and the polynomial $-x^2 + k$ has a maximum value of k. Both occur when $x = 0$.*

Example 4

Find the minimum value of $x^2 + 3$ and the value of x for which it occurs.

Solution

For any value of x, $x^2 \geqslant 0$.

Adding 3 to both sides, $x^2 + 3 \geqslant 3$

Hence the minimum value of $x^2 + 3$ is 3, and this occurs when $x = 0$.

(b) *If $a > 0$, the polynomial $ax^2 \geqslant 0$ and has therefore a minimum value of 0 when $x = 0$.*
If $a < 0$, the polynomial $ax^2 \leqslant 0$ and has therefore a maximum value of 0 when $x = 0$.
Thus $-2x^2$ has a maximum value of 0 when $x = 0$ and $5x^2$ has a minimum value of 0 when $x = 0$.
Hence, $ax^2 + k$ has a minimum value of k when $x = 0$ if $a > 0$, and a maximum value of k when $x = 0$, if $a < 0$.
Thus $2x^2 - 7$ has a minimum value of -7 when $x = 0$ and $-5x^2 + 2$ has a maximum value of 2 when $x = 0$.

(c) If $a > 0$, the polynomial $a(x + h)^2 \geqslant 0$ and has therefore a minimum value of 0 when $x + h = 0$, i.e. when $x = -h$.
If $a < 0$, the polynomial $a(x + h)^2 \leqslant 0$ and has therefore a maximum value of 0 when $x + h = 0$, i.e. when $x = -h$.
So, $a(x + h)^2 + k \geqslant k$ for $a > 0$ and $a(x + h)^2 + k \leqslant k$ for $a < 0$.
It follows that $a(x + h)^2 + k$ has minimum value k when $x = -h$, if $a > 0$.
Also $a(x + h)^2 + k$ has maximum value k when $x = -h$, if $a < 0$.

Let us consider the expression $3(x - 1)^2 + 5$. The minimum value of $(x - 1)^2$ is 0.
So the minimum value of $3(x - 1)^2 + 5$ is $(3 \times 0) + 5$ or $0 + 5$ or 5.
This occurs when $(x - 1)^2 = 0$, $x = 1$.

Similarly, the minimum value of $4(x + 2)^2 + 7$ is 7 when $x + 2 = 0$, i.e. when $x = -2$.

More generally, if $a > 0$, the minimum value of $a(x + h)^2 + k$ is k when $x = -h$ and if $a < 0$, the maximum value of $a(x + h)^2 + k$ is k when $x = -h$.

(d) To find the maximum or the minimum value of a quadratic polynomial, we reduce the polynomial to the form $a(x + h)^2 + k$ as illustrated in the following example:

Example 5

State whether the expression $2x^2 - 7x - 10$ has a maximum value or a minimum value. Find the maximum value or the minimum value and state the value of x for which it occurs.

Solution

$2x^2 - 7x - 10$ has a minimum value as the coefficient of $x^2 > 0$.

$$2x^2 - 7x - 10 = 2\left(x^2 - \frac{7}{2}x - 5\right)$$

$$= 2\left[x^2 - \frac{7}{2}x + \left(\frac{7}{4}\right)^2 - 5 - \left(\frac{7}{4}\right)^2\right]$$

$$= 2\left[\left(x - \frac{7}{4}\right)^2 - \frac{129}{16}\right]$$

$$= 2\left(x - \frac{7}{4}\right)^2 - \frac{129}{8}$$

The minimum value = $-\frac{129}{8}$ when $x = \frac{7}{4}$.

Example 6

Find the minimum or maximum value of $12 - 6x - 5x^2$ and state the value of x for which it occurs.

Solution

$12 - 6x - 5x^2$ has a maximum value as the coefficient of $x^2 < 0$.

$$12 - 6x - 5x^2 = -5\left(x^2 + \frac{6}{5}x - \frac{12}{5}\right)$$

$$= -5\left[x^2 + \frac{6}{5}x + \left(\frac{3}{5}\right)^2 - \frac{12}{5} - \left(\frac{3}{5}\right)^2\right]$$

$$= -5\left[\left(x + \frac{3}{5}\right)^2 - \frac{69}{25}\right]$$

$$= -5\left(x + \frac{3}{5}\right)^2 + \frac{69}{5}$$

The minimum value = $\frac{69}{5}$ when $x = -\frac{3}{5}$.

Exercise 1 B

Reduce each of the following quadratic polynomials to the form $a(x + h)^2 + k$.
In each case, state whether the polynomial has a miximum or a minimum value. Find this value and the value of x for which it occurs.

1. $x^2 + 8x + 19$.
2. $x^2 - 6x + 10$.
3. $-x^2 + 6x - 15$.
4. $-x^2 - 10x + 11$.
5. $2x^2 - 6x + 15$.
6. $3x^2 + 11x + 20$.
7. $12 - 2x - 3x^2$.
8. $25 - 4x - 4x^2$.
9. $3x^2 - 8x$.
10. $12x - 5x^2$.

1.3 Graph of $y = ax^2 + bx + c$

We know that the graphs of $y = x^2$ and $y = -x^2$ are as shown in Figures 1.1(a) and 1.1(b)

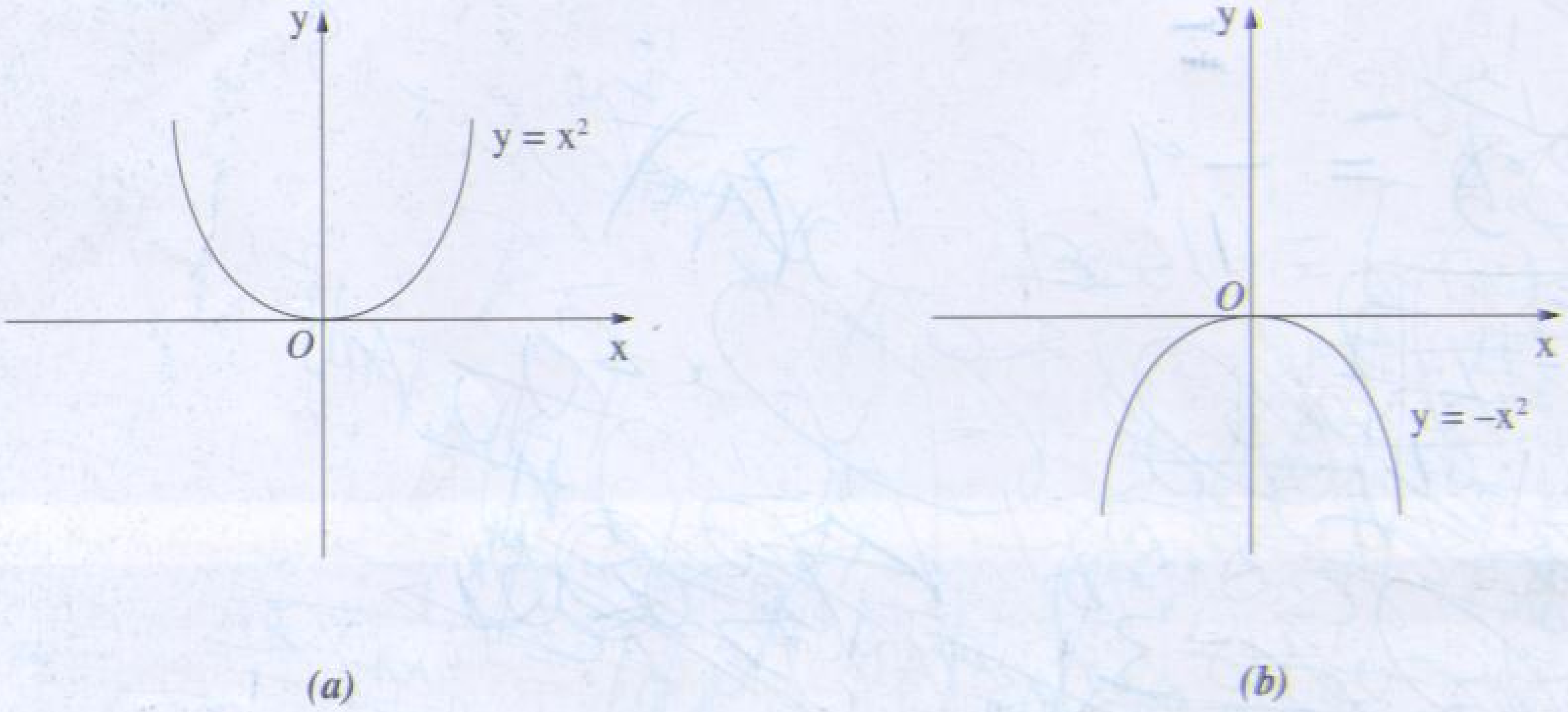


Figure 1.1

Generally, the graph of $y = ax^2 + bx + c$ can be obtained by determining its maximum or its minimum value according to whether $a < 0$ or $a > 0$.

To sketch the graph of $y = x^2 - 4x + 7$, we find its minimum value.

$$x^2 - 4x + 7 = x^2 - 4x + 2^2 + 7 - 2^2$$

$$= (x - 2)^2 + 3.$$

$x^2 - 4x + 7$ has therefore a minimum value of 3 when $x = 2$.

The graph of $y = x^2 - 4x + 7$ has therefore a minimum at (2, 3) as shown in Figure 1.2.

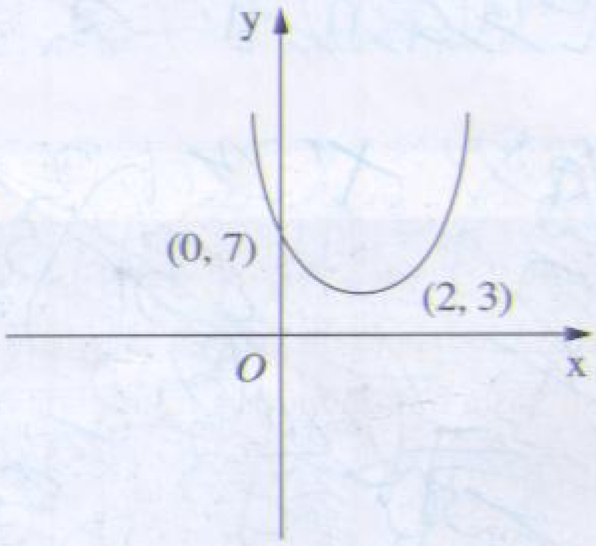


Figure 1.2

Note: The graph cuts the y-axis at (0, 7) and it does not cut the x-axis.

Example 7

Sketch the graph of $y = -2x^2 + 8x - 17$.

Solution

$-2x^2 + 8x - 17$ has a maximum value as the coefficient of $x^2 < 0$.

$$\begin{aligned}-2x^2 + 8x - 17 &= -2\left(x^2 - 4x + \frac{17}{2}\right)\\ &= -2\left[x^2 - 4x + 2^2 + \frac{17}{2} - 2^2\right]\\ &= -2\left[(x-2)^2 + \frac{9}{2}\right]\\ &= -2(x-2)^2 - 9\end{aligned}$$

Maximum value = -9 when $x = 2$.

The graph of $-2x^2 + 8x - 17$ has therefore a maximum value at $(2, -9)$.

The graph of $y = -2x^2 + 8x - 17$ is as shown in Figure 1.3.

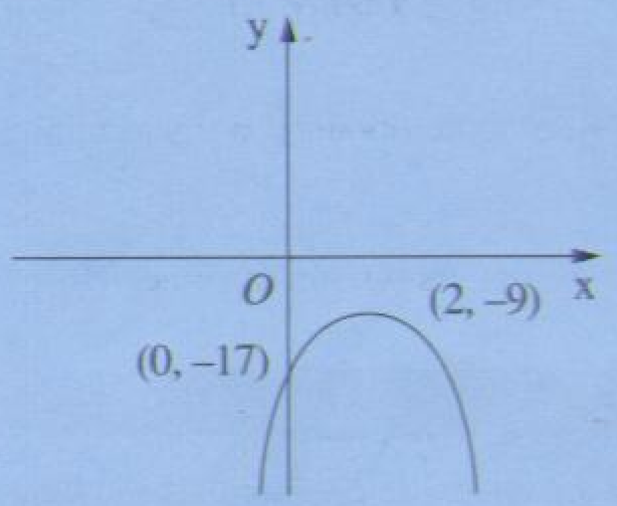


Figure 1.3

Note: The graph cuts the y-axis at (0, –17) and it does not cut the x–axis.

Example 8

Sketch the graph of $y = 3x^2 + 2x - 7$, *showing the coordinates of its points of intersection with the axes, if any.*

Solution

$3x^2 + 2x - 7$ has a maximum value as the coefficient of $x^2 > 0$.

$$\begin{aligned}3x^2 + 2x - 7 &= 3\left[x^2 + \frac{2}{3}x - \frac{7}{3}\right]\\ &= 3\left[x^2 + \frac{2}{3}x + \left(\frac{1}{3}\right)^2 - \frac{7}{3} - \left(\frac{1}{3}\right)^2\right]\\ &= 3\left[\left(x + \frac{1}{3}\right)^2 - \frac{7}{3} - \frac{1}{9}\right]\end{aligned}$$

$$= 3\left[\left(x + \frac{1}{3}\right)^2 - \frac{22}{9}\right]$$

$$= 3\left(x + \frac{1}{3}\right)^2 - \frac{22}{3}$$

The minimum value = $-\frac{22}{3}$ when $x = -\frac{1}{3}$.

When y = 0

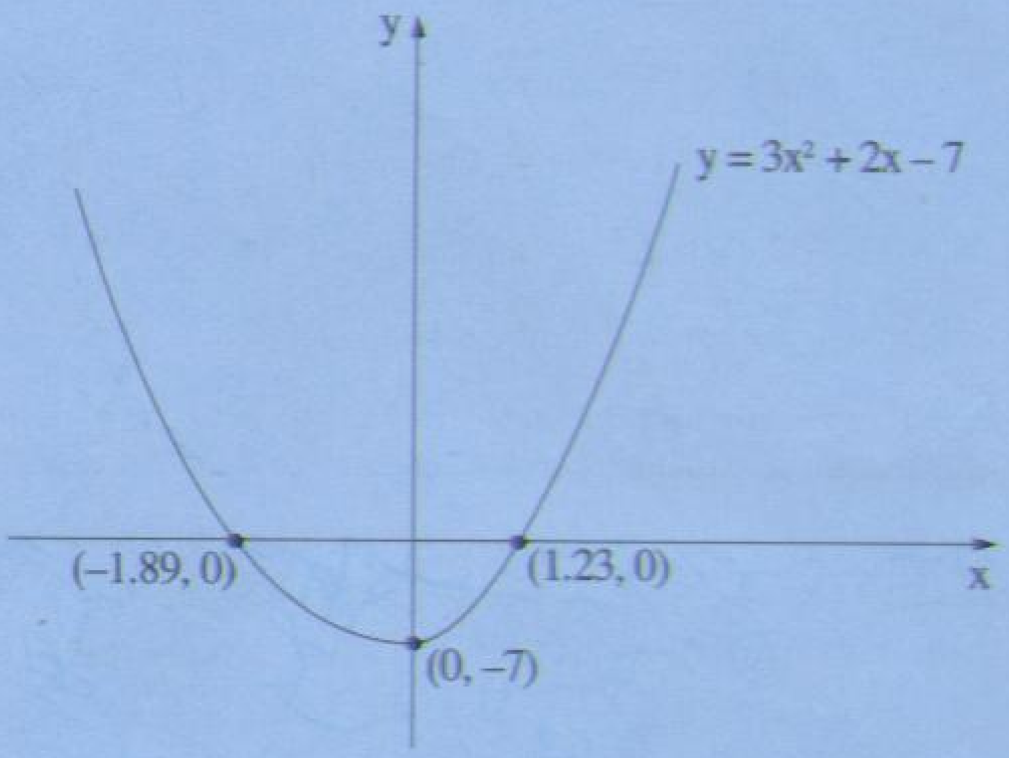
$$x = \frac{-2 \pm \sqrt{4 - 4 \times 3 \times -7}}{6}$$

$$= \frac{-2 \pm \sqrt{4 + 84}}{6}$$

$$= \frac{-2 \pm 9.38}{6}$$

$$= \frac{7.38}{6} \text{ or } \frac{-11.38}{6}$$

$= 1.23$ or -1.89



Exercise 1 C

Sketch the graph of each of the following quadratic polynomials showing the coordinates of the points of intersection with the axes, if any.

1. $y = x^2 - 4x + 7$.
2. $y = x^2 - 6x + 4$.
3. $y = x^2 + 8x + 17$.
4. $y = x^2 + 5x - 9$.
5. $y = 2x^2 - 6x + 9$.
6. $y = 3x^2 + 9x + 13$.
7. $y = 15 - 4x - x^2$.
8. $y = 8 + 6x - x^2$.
9. $y = 13 - 2x - 2x^2$.
10. $y = 12 - 5x - 3x^2$.

1.4 Nature of roots of a quadratic equation

We know from earlier work in the mathematics course that the roots of the equation $ax^2 + bx + c = 0$ are $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

The nature of the roots depends upon the value of $b^2 - 4ac$ which is known as the *discriminant.*

(i) If $b^2 - 4ac = 0$, then both roots are $-\dfrac{b}{2a}$ and the equation has two equal roots.

(ii) If $b^2 - 4ac > 0$, the roots are real and unequal.

(iii) If $b^2 - 4ac < 0$, then $\sqrt{b^2 - 4ac}$ is not real as there is no real number whose square is negative. We say that the roots are not real or that they are imaginary.

It follows that the equation $ax^2 + bx + c$ has real roots if $b^2 - 4ac \geqslant 0$.

Example 9

Determine the nature of the roots of each of the following quadratic equations:

(a) $3x^2 + 2x + 1 = 0$ *(b) $9x^2 - 12x + 4 = 0$* *(c) $2x^2 - 7x - 3 = 0$*

Solution

(a) $a = 3, b = 2, c = 1$

$b^2 - 4ac = 4 - (4 \times 3 \times 1) < 0$

So, the equation has imaginary roots.

(b) $a = 9, b = -12, c = 4$

$b^2 - 4ac = 144 - (4 \times 9 \times 4) = 0$

Equation has equal roots.

(c) $a = 2, b = -7, c = -3$

$b^2 - 4ac = 49 - (4 \times 2 \times -3) > 0$

Roots are real and unequal.

Example 10

Find the values of k for which the equation $3x^2 + kx + 12 = 0$ has equal roots.

Solution

$a = 3, \quad b = k, \quad c = 12$

$$\begin{aligned} b^2 - 4ac &= k^2 - 4 \times 3 \times 12 \\ &= k^2 - 144 \end{aligned}$$

As equation has equal roots

$$\begin{aligned} k^2 - 144 &= 0 \\ k^2 &= 144 \\ k &= \pm 12. \end{aligned}$$

Exercise 1 D

1. Find the nature of the roots of each of the following equations:
   (a) $3x^2 + x + 2 = 0$
   (b) $2x^2 + x - 3 = 0$
   (c) $3x^2 - x - 4 = 0$
   (d) $x^2 + 6x + 9 = 0$
   (e) $x^2 - 10x + 25 = 0$
   (f) $5 + 3x - x^2 = 0$
   (g) $7 + 4x + 2x^2 = 0$
   (h) $-64 + 16x - x^2 = 0$
   (i) $9 + 2x - 3x^2 = 0$
   (j) $5 - 2x - 4x^2 = 0$.

2. Find the values of m for which the equation $mx^2 + 10x - 20 = 0$ has equal roots.

3. Find the values of m for which the equation $3x^2 + mx - m = 0$ has equal roots.

4. Show that the equation $3x^2 + kx - 4 = 0$ has real roots for all real values of k.

5. Find the value of k for which $(x + k)^2 = x$ has equal roots.

1.5 Quadratic Inequalities

1.5.1 Method 1 – Graphical

To solve a quadratic inequality such as $x^2 - 4x + 3 \geqslant 0$, we sketch the graph of $y = x^2 - 4x + 3$ paying particular attention to its points of intersection with the x-axis.

The graph intersects the x-axis where $x^2 - 4x + 3 = 0$.

i.e. $(x - 1)(x - 3) = 0$

$x = 1$ or 3.

As the graph has a minimum, it is as shown in figure 1.4

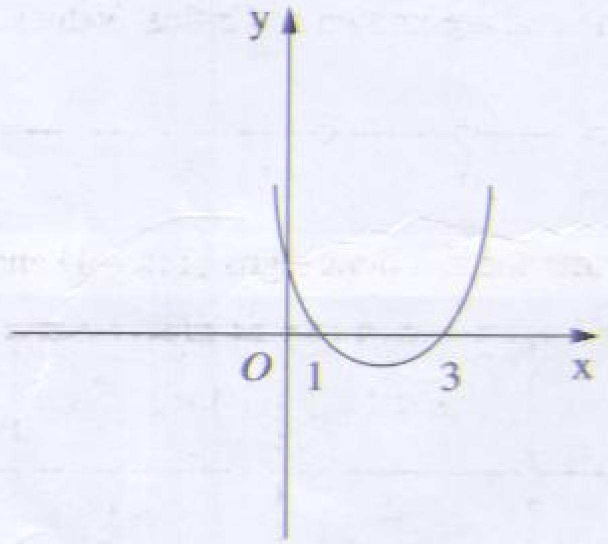


Figure 1.4

$x^2 - 4x + 3 \geqslant 0$, i.e. $y \geqslant 0$ for the parts of the graph where $x \leqslant 1$ or $x \geqslant 3$.

So, $x^2 - 4x + 3 \geqslant 0$

$x \leqslant 1$ or $x \geqslant 3$.

Example 11

Find the range of values of x for which $20 - 3x - 2x^2 > 0$.

Solution

The graph of $y = 20 - 3x - 2x^2$ cuts the x-axis at

$$\begin{aligned} 20 - 3x - 2x^2 &= 0 \\ 2x^2 + 3x - 20 &= 0 \\ (2x - 5)(x + 4) &= 0 \\ x &= \frac{5}{2} \quad \text{or} \quad -4. \end{aligned}$$

As the graph of $20 - 3x - 2x^2$ has a maximum, it is as shown in Figure 1.5.

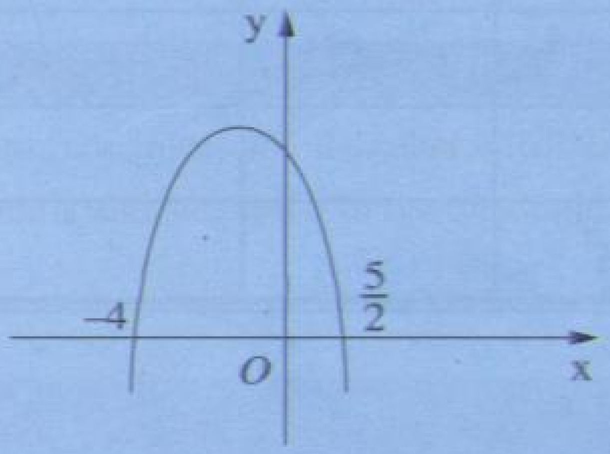


Figure 1.5

From the graph $20 - 3x - 2x^2 > 0$, i.e. $y > 0$ for $-4 < x < \frac{5}{2}$.

1.5.2 Numerical Method

To find the values of x for which $x^2 - 4x + 3 \geq 0$, we find the values of x for which $x^2 - 4x + 3 = 0$ which are 1 and 3.

These values are shown on a number line and are known as Critical Values.

As $x^2 - 4x + 3 = (x - 1)(x - 3)$, we consider the algebraic signs of $(x - 1)$ and of $(x - 3)$ for $x < 1$, $1 < x < 3$, $x > 3$.

	$x < 1$	$1 < x < 3$	$x > 3$
$x - 1$	–	+	+
$x - 3$	–	–	+
$x^2 - 4x + 3$ i.e. $(x - 1)(x - 3)$	+	–	+

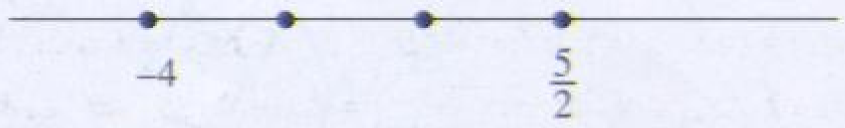
From the table $x^2 - 4x + 3 > 0$ for $x < 1$ or $x > 3$.

So, $x^2 - 4x + 3 \geqslant 0$ for $x \leqslant 1$ or $x \geqslant 3$.

Similarly, to find the values of x for which $20 - 3x - 2x^2 > 0$, we find the values of x for which $20 - 3x - 2x^2 = 0$.

i.e. $\quad (5 - 2x)(4 + x) = 0$

$$x = \frac{5}{2} \text{ or } -4.$$



	$x < -4$	$-4 < x < \frac{5}{2}$	$x > \frac{5}{2}$
$5 - 2x$	+	+	–
$4 + x$	–	+	+
$20 - 3x - 2x^2$	–	+	–

From the table, $20 - 3x - 2x^2 > 0$ for $-4 < x < \frac{5}{2}$.

Example 12

Find the range of values of x for which $10 - 5x - x^2 \leqslant 0$.

Solution

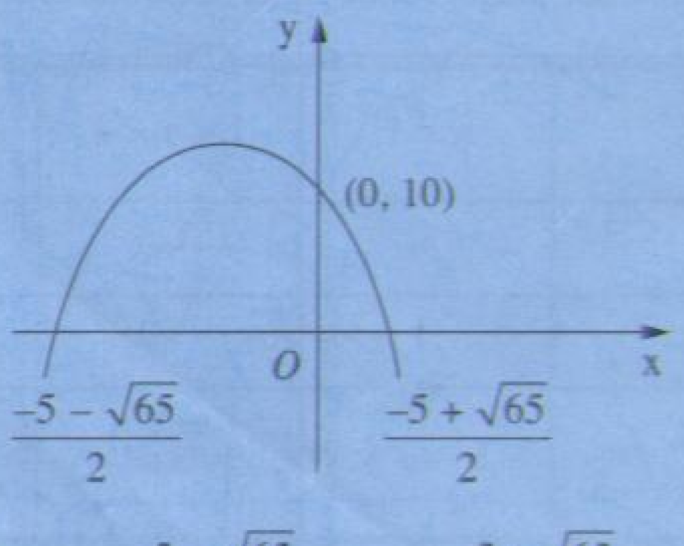
$10 - 5x - x^2 = 0$

$x^2 + 5x - 10 = 0$

$$x = \frac{-5 \pm \sqrt{25 + 40}}{2}$$

$$= \frac{-5 + \sqrt{65}}{2} \text{ or } \frac{-5 - \sqrt{65}}{2}$$

In this case, it is more suitable to use the graphical method.



From the graph $10 - 5x - x^2 \leqslant 0$ for $x \leqslant \frac{-5 - \sqrt{65}}{2}$ or $x \geqslant \frac{-5 + \sqrt{65}}{2}$.

Exercise 1 E

Use either a graphical method or a numerical method to solve the following quadratic inequalities.

1. $x^2 - 8x + 12 \leqslant 0$
2. $2x^2 + x - 10 \geqslant 0$
3. $3x^2 - 10x + 3 < 0$
4. $4x^2 + x - 3 > 0$
5. $10 - 3x - x^2 \leqslant 0$
6. $2 + x - 3x^2 \geqslant 0$
7. $25 - x^2 < 0$
8. $144 - x^2 \leqslant 0$
9. $10 - 2x - 3x^2 > 0$
10. $5x^2 - 3x - 5 > 0$
11. Find the values of k for which the equation $5x^2 + kx + 5 = 0$ has real roots in x.
12. Find the values of k for which the equation $x(x + 2k) = 4x - k^2$ has real roots in x.
13. Find the values of k for which the equation $x(k - 3x) = 3$ has no real roots in x.
14. Find the values of k for which the equation $x(3x - 2) + kx + 12 = 0$ has no real roots in x.

1.6 Simultaneous equations, one linear and one quadratic

To solve a pair of simultaneous equations, one linear and the other quadratic, we make one variable in the linear equation become the subject of the formula and substitute in the quadratic. The method is illustrated in the following example:

Example 13

Solve the equations

$$x + 2y = 1 \qquad (i)$$

$$x^2 - xy + 3y^2 = 15 \qquad (ii)$$

Solution

From the linear equation $x = 1 - 2y$ (A)

Replacing in (ii)

$$\begin{aligned}(1 - 2y)^2 - (1 - 2y)y + 3y^2 &= 15\\ 1 - 4y + 4y^2 - y + 2y^2 + 3y^2 &= 15\\ 9y^2 - 5y - 14 &= 0\\ (9y - 14)(y + 1) &= 0\\ y &= \frac{14}{9} \text{ or } -1\end{aligned}$$

From (A) above, $y = \frac{14}{9}, x = 1 - \frac{28}{9} = -\frac{19}{9}$

$y = -1, x = 1 - (-2) = 3.$

Solutions are $x = -\frac{19}{9}, y = \frac{14}{9}$ and $x = 3, y = -1.$

Example 14

Solve the equations

$x - y = 3$ (i)

$(x - 2y)^2 = 81$ (ii)

Solution

From (i) $x - y = 3$

$x = y + 3$

Replacing in (ii)

$(x + 2y)^2 = 81$

$(y + 3 + 2y)^2 = 81$

$(3y + 3)^2 = 81$

$3y + 3 = \pm 9$

$3y + 3 = 9$ or $3y + 3 = -9$

$3y = 6$ or $3y = -12$

$y = 2$ or $y = -4$

$x = y + 3$

$x = 2 + 3$ or $x = -4 + 3$

$x = 5$ or $x = -1$

Exercise 1 F

1. Solve the simultaneous equations

(i) $2x + y = 4$
$x^2 - 2xy + y^2 = 25$

(ii) $3x^2 + 4xy - y^2 = 3$
$3x + 7y = 1$

(iii) $2x - 3y = 11$
$x^2 + y^2 - 2x + 4y = -4$

(iv) $3x + 2y = -5$
$3x^2 + y^2 + 7x - 3y = 0$

(v) $x^2 - 2xy + 4y^2 = 7$
$2x - 3y = 5$

(vi) $2x + 3y = 13$
$x^2 + 5xy + 6y^2 = 88$

(vii) $3x + 2y = 23$
$3x^2 + 2xy + 1 = 162$

(viii) $2x - y = 4$
$3x^2 - xy - y^2 = 17$

1.7 Disguised quadratics

The equation $x - 2\sqrt{x} - 3 = 0$ as such is not a quadratic equation. However, if we put $\sqrt{x} = y$, $x = y^2$.

So, $x - 2\sqrt{x} - 3 = 0$ becomes $y^2 - 2y - 3 = 0$

$$(y - 3)(y + 1) = 0$$

$$y = 3 \text{ or } - 1.$$

The value – 1 is rejected as $\sqrt{x} \geqslant 0$ for all real values of x.

so, $\sqrt{x} = 3$

$x = 9$

The solution can be checked as for $x = 9$, $x - 2\sqrt{x} - 3 = 9 - (2 \times 3) - 3 = 0$.

Note that if $y = -1$ is accepted, $x = 1$ and this does not satisfy the equation $x - 2\sqrt{x} -3 = 0$.

Equations which can be reduced to the form $ax^2 + bx + c = 0$ $(a \neq 0)$ are known as disguised quadratics.

Examples of disguised quadratics are $2^{2x+1} - 5 \times 2^x - 25 = 0$, $x^4 - 2x^2 - 6 = 0$, $x^6 + 2x^3 - 8 = 0$

Use in each case the appropriate substitution to reduce it to a quadratic polynomial and solve the equations. The solutions are given below.

Example 15

Solve the following equations

(a) $2^{2x+1} - 5 \times 2^x - 25 = 0$ (b) $x^4 - 2x^2 - 6 = 0$ (c) $x^6 + 2x^3 - 8 = 0$

Solution

(a)

$$\begin{aligned} \text{Put } 2^x &= y \\ 2^{2x+1} - 5 \times 2^x - 25 &= 0 \\ 2^{2x} \times 2^1 - 5 \times 2^x - 25 &= 0 \\ 2y^2 - 5y - 25 &= 0 \\ (2y + 5)(y - 5) &= 0 \end{aligned}$$

$y = -\frac{5}{2}$ (not possible as $2^x > 0$) or $y = 5$

$2^x = 5$

$x \lg 2 = \lg 5$

$$x = \frac{\lg 5}{\lg 2}$$

$= 2.32$ (to 3 significant figures).

(b) $x^4 - 2x^2 - 6 = 0$

Put $x^2 = y$

$y^2 - 2y - 6 = 0$

$$y = \frac{2 \pm \sqrt{4 - 4 \times 1 \times -6}}{2}$$

$$= \frac{2 \pm \sqrt{28}}{2}$$

$= 3.645...$ (or $-1.645...$ which is not possible as $x^2 \geq 0$).

So, $x^2 = 3.645...$

$x = \pm 1.91$ (to 3 significant figures).

(c) $x^6 + 2x^3 - 8 = 0$

Put $x^3 = y$

$y^2 + 2y - 8 = 0$

$(y + 4)(y - 2) = 0$

$y = -4$ or 2

$x^3 = -4$ or 2

$x = -1.59$ or 1.26 (to 3 significant figures).

Exercise 1 G

Solve the following equations:

1. $3 \times 3^{2x} - 4 \times 3^x + 1 = 0$
2. $2x - 5\sqrt{x} + 2 = 0$
3. $x^8 - 6x^4 + 8 = 0$
4. $e^{2x} - 8e^x + 15 = 0$
5. $(2x + 1)^4 - 6(2x + 1)^2 + 5 = 0$
6. $\frac{5}{(x+1)^4} - \frac{1}{(x+1)^2} - 4 = 0$
7. $3 \times 4^{-x} - 17 \times 2^{-x} + 10 = 0$
8. $2x^{-4} - x^{-2} - 3 = 0$
9. $(2^{2x} + 1)^4 - 5(2^{2x} + 1)^2 + 6 = 0$
10. $(x + \sqrt{x})^4 - 3(x + \sqrt{x})^2 + 2 = 0$

Miscellaneous Exercise 1

1. Solve the simultaneous equations

   $4x - 3y = 15$

   $8x^2 - 27y^2 = 45$ [C]

2. Find the range of values of x for which $2x^2 - 3x \geqslant 2$. [C]

3. Calculate the coordinates of the points of intersection of the straight line $2x + 3y = 10$ and the curve $\frac{2}{x} + \frac{3}{y} = 5$. [C]

4. Find the range of values of x for which $4x(4 - x) \geqslant 15$. [C]

5. Write $2x^2 - 6x + 25$ in the form $A(x + h)^2 + k$. Hence obtain the minimum value of $2x^2 - 6x + 25$ and the value of x for which it occurs. Sketch the graph of $y = 2x^2 - 6x + 25$.

6. Solve the simultaneous equations

   $x + y = xy$

   $2y = x + 2$ [C]

7. Find the range of values of x for which $x(x + 5) \geqslant -6$. [C]

8. Find the maximum value of $25 - 4x - 2x^2$ and the value of x for which it occurs.
Hence, sketch the graph of $y = 25 - 4x - 2x^2$.

9. Find the values of p for which the equation $(p + 1)x^2 + 4px + 9 = 0$ has equal roots.

10. Solve the equation $2x^4 - 3x^2 + 1 = 0$.

11. Find the coordinates of the minimum point of the curve $y = (3x - 5)^2 + 2$ and sketch the curve.

12. Solve the equation $3 \times 2^{4x} - 5 \times 2^{2x} - 2 = 0$.

13. Show that the equation $(p + 1)x^2 + (2p + 3)x + (p + 2) = 0$ has real roots for all real values of p.

14. The quadratic equation $x^2 + px + q = 0$ has roots –2 and 6. Find:
   (a) the value of p and of q,
   (b the range of values of r for which the equation $x^2 + px + q = r$ has no real roots.

15. (a) Find the range of values of x for which $8x + 3 < 3x^2$.
   (b) Show that the equation $x^2 + (2 - k)x + k = 3$ has real roots for all real values of k.

16. (a) Solve the simultaneous equations $y = x^2 - 3x + 2$, $y = 3x - 7$.
   (b) Interpret your solution to part (a) geometrically.

17. Solve the inequality $x^2 < (2x+1)^2$.

18. Solve the equations $x^3 + y^3 = 7$ and $x^6 - x^3y^3 - y^6 = 71$.

19. Solve the inequalities:
   (a) $x^4 - 2x^2 - 3 > 0$ (b) $x^4 + 2x^2 - 3 < 0$

20. Write $x^2 - 8x + 19$ in the form $(x + a)^2 + b$. Hence, obtain the minimum value of $x^2 - 8x + 19$ and the value of x for which it occurs.
Sketch the graphs of $y = x^2 - 8x + 19$ and $y = \dfrac{1}{x^2 - 8x + 19}$. [C]

21. Write down the roots of the equation $ax^2 + bx + c = 0$. Hence, show that the sum of the roots is $-\dfrac{b}{a}$ and the product of the roots is $\dfrac{c}{a}$.
*The equation $2x^2 + x + k = 0$ has distinct real roots. Given that $k < 0$, show that the roots have opposite algebraic signs. Given also that the roots are 2 and p, find the value of k. [C]

CHAPTER 2

> FUNCTIONS

2.1 Functions

2.1.1 Concept of a function

We are already familiar with the concept of a function. Starting with a set and a rule defined on the set, if this rule associates one and only one element 'b' with each element 'a' of the set we have a function.

Thus, for the set A = {1, 2, 3, 4}, if the rule is 'square each element', then with the element 1 we associate 1, with the element 2 we associate 4, etc.

With each element 'a' of A we associate only one element 'b', which may or not belong to A. Thus, we have in this case a function.

If on the other hand, the rule is 'find a number whose square is', with the element 1, we associate 1 and – 1. We do not have a function.

Note: a and b are not necessarily distinct. In the given example, if a = 1, b = 1.

2.1.2 Notation for a function

We know already that a function is written in the form $f{:}x \mapsto \ldots$ $g{:}x \mapsto \ldots$ etc.

Sometimes it can be expressed in terms of a mathematical formula, e.g. $f{:}x \mapsto x^2$ for the example above.

A function f has the important property that for every element $x \in$ the domain, $f(x)$ is uniquely defined.

For $f{:}x \mapsto 3x - 4$, $f(0) = -4$, $f(1) = -1$, etc.

Note: $g{:}x \mapsto \pm\sqrt{x}$ is not a function as $g(9) = \pm 3$ and $g(x)$ is not unique.

2.1.3 Range of a function

The value of $f(x)$ corresponding to an element x of the domain is called the f-image of x.

If $f{:}x \mapsto 3x - 4$ $(x \in \mathbb{R})$, $f(1) = -1$ and -1 is the f-image of 1.

Similarly, $f(2) = 2$ and 2 is the f-image of 2.

The set of f-images of all elements x of the domain is called the range of f.

If $g{:}x \mapsto x^3 + 1$ and the domain of g is $\{-3, -2, -1, 0, 1\}$, the range of g is $\{-26, -7, 0, 1, 2\}$ as $g(-3) = -26$, $g(-2) = -7$, etc.

If the domain of g is the set of all real numbers greater than or equal to zero, the range of g is the set of all real numbers greater than or equal to 1.

The range of a function can be obtained by sketching its graph as shown in the following examples:

Example 1

Sketch the graph of $f{:}x \mapsto x^2 - 6x + 7$ *for* $x \in \mathbb{R}$. *Hence find the range of f.*

Solution

$$\begin{aligned} x^2 - 6x + 7 &= x^2 - 6x + 3^2 + 7 - 3^2 \\ &= (x - 3)^2 - 2 \end{aligned}$$

The graph of f has therefore a minimum at (3, –2).

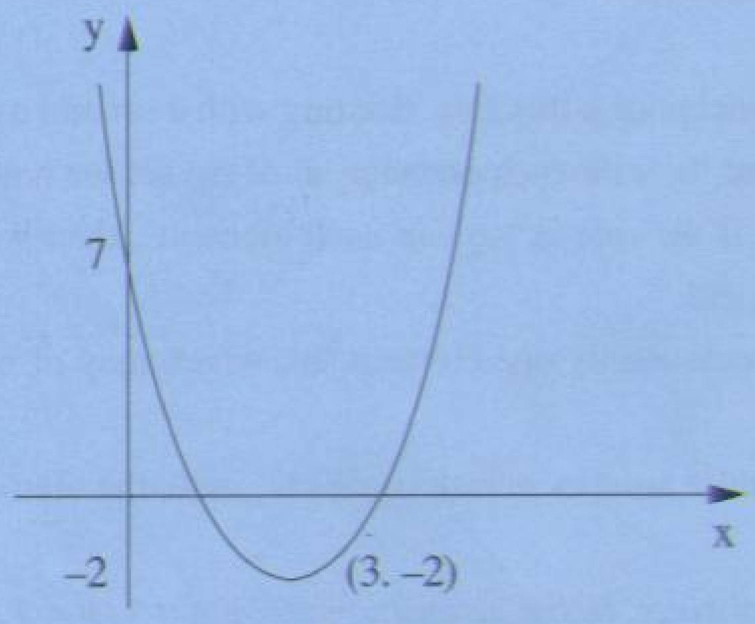


Figure 2.1

From the graph, the range = $\{f(x) \in \mathbb{R} : f(x) \geqslant -2\}$ or $[-2, \infty)$

Note: The range is a set and must be written as a set.

Example 2

Sketch the graph of $g{:}x \mapsto 2^x$ *for* $\{x \in \mathbb{R} : -1 \leqslant x \leqslant 3\}$ *and find its range.*

Solution

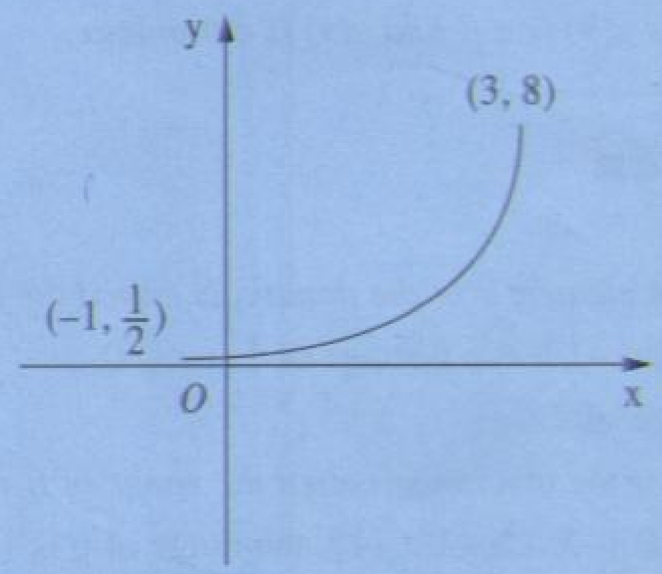


Figure 2.2

The graph of g is shown in Figure 2.2.

The range = $\left\{g(x) \in \mathbb{R} : \frac{1}{2} \leqslant g(x) \leqslant 8\right\}$ or $\left[\frac{1}{2}, 8\right]$

2.1.4 One-one functions

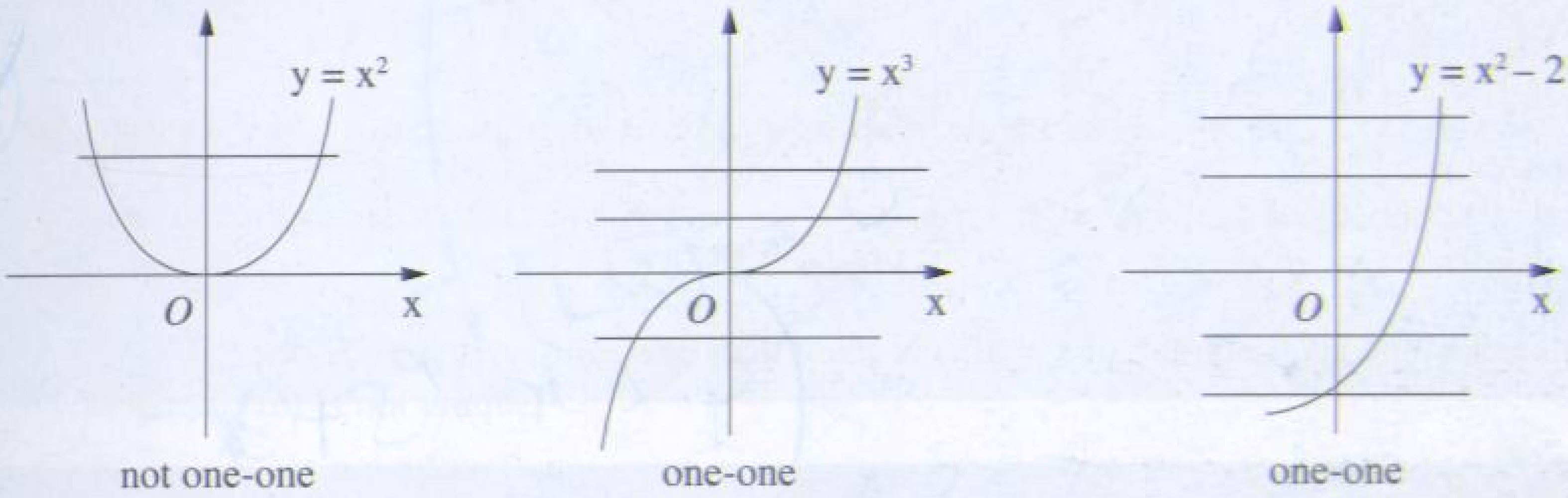
Consider the function $f:x \mapsto x^2$, $x \in \mathbb{R}$ where $\mathbb{R}$ is the set of real numbers. There exist more than one element $x \in \mathbb{R}$ which has the same f-image, e.g. 3 and –3 have the same f-image 9.

For the function $g:x \mapsto \cos x$ for $x \in \mathbb{R}$, there exist more than one element $x \in \mathbb{R}$ which has the same g-image, e.g. $\frac{\pi}{3}, \frac{5\pi}{3}, -\frac{\pi}{3}, -\frac{5\pi}{3}, \ldots$ have the same image $\frac{1}{2}$.

For the function $h:x \mapsto x^2$ for $x \in \mathbb{R}$, $x \geqslant 0$, no two elements of x have the same h-image, e.g. 4 is the image of 2 only, 16 is the image of 4 only, etc.

We say h is a one-one function, read as a one to one function.

A function f is said to be one-one if no two elements of the domain have the same f-image, i.e. if $f(x_1) = f(x_2)$, $x_1 = x_2$. Graphically, this implies that any y-line cuts the graph of $y = f(x)$ in not more than one point.



not one-one one-one one-one

Figure 2.3

Example 3

Determine whether $f:x \mapsto x^2 - 4x + 7$, $\{x \in \mathbb{R}, x \geqslant 2\}$ is one-one.

Solution

We sketch the graph of $y = f(x)$ for $x \in \mathbb{R}$, $x \geqslant 2$.

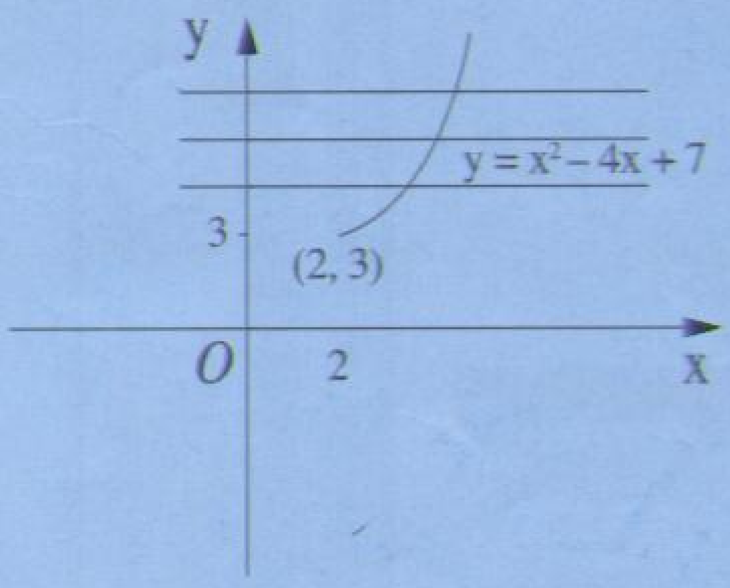


Figure 2.4

From the graph, we see that y-lines cut the graph at **most once**. So f is one-one.

Example 4

Show that $g:x \mapsto \sin x \{x \in \mathbb{R}, 0 \leqslant x \leqslant 2\pi\}$ is not one-one.

Solution

To show g is not one-one, it is sufficient to find two values of x in the interval $0 \leqslant x \leqslant 2\pi$ having the same g-image.

As $g(0) = 0$ and $g(\pi) = 0$, g is not one-one.

We may also sketch the graph of $y = g(x)$.

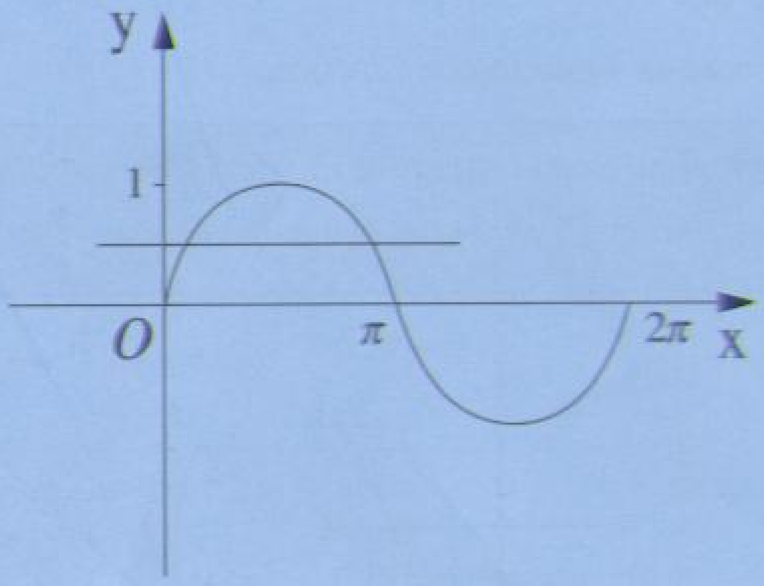


Figure 2.5

As there are y-lines which cut the graph of $y = g(x)$ in more than one point, g is not one-one.

Exercise 2 A

For each of the following functions sketch its graph and find its range. Determine whether the function is one-one. If it is not one-one, write down two elements of x which have the same f-image.

1. $f:x \mapsto x^2 + 4x + 10, x \in \mathbb{R}$
2. $f:x \mapsto \frac{1}{x}, x \in \mathbb{R}, x \neq 0$
3. $f:x \mapsto \frac{1}{x^2}, x \in \mathbb{R}, x \neq 0$
4. $f:x \mapsto \cos x, x \in \mathbb{R}, -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$
5. $f:x \mapsto |x|, x \in$ {real numbers $\geqslant 0$}
6. $f:x \mapsto 8x - x^2, x \in \mathbb{R}, x \leqslant 4$
7. $f:x \mapsto x^3, x \in \mathbb{R}$
8. $f:x \mapsto \frac{1}{x}, x \in \mathbb{R}^+$
9. $f:x \mapsto \tan x, 0 \leqslant x < \frac{\pi}{2}$
10. $f:x \mapsto x^2 + 6x - 9, x \in \mathbb{R}, x \leqslant -3.$

2.2 Inverse of a function

Consider the function $f:x \mapsto 2x + 1, x \in \{2, 3, 5, 7\}$.

$f(2) = 5$

The element 2 of the domain which has image 5 is written $f^{-1}(5)$.

So, $f^{-1}(5) = 2$

Similarly $f(3) = 7, f^{-1}(7) = 3$.

$f(5) = 11, f^{-1}(11) = 5$

$f(7) = 15, f^{-1}(15) = 7$

f^{-1} is called the inverse of f.

The range of $f = \{5, 7, 11, 15\}$

The range of $f^{-1} = \{2, 3, 5, 7\}$

In this case, f^{-1} is a function as $f^{-1}(x)$ is unique for $x \in \{5, 7, 11, 15\}$.

The inverse of a function is not necessarily a function.

Consider $g:x \mapsto x^2, x \in \{-2, -1, 0, 1, 2\}$

$g(-2) = 4, g(-1) = 1, g(0) = 0, g(1) = 1$, and $g(2) = 4$.

The range of $g = \{0, 1, 4\}$

$g^{-1}(0) = 0, g^{-1}(1) = \pm 1, g^{-1}(4) = \pm 2$.

So, g^{-1} is not a function.

For the inverse f^{-1} of a function f to be itself a function, f must be one-one.

If f is not one-one, there are at least two elements a and b of the domain which have the same image c, i.e. $f(a) = c$ and $f(b) = c$.

So, $f^{-1}(c) = a$ or b.

Hence, $f^{-1}(c)$ is not unique.

So, f^{-1} is not a function.

If the domain of a one-one function f is A and its range is B, the inverse function f^{-1} has domain B and range A.

Generally if $f(x) = y$, $x = f^{-1}(y)$

To find the inverse of a function f, we find x in terms of y where $y = f(x)$.

Example 5

Find the inverse of the function $f:x \mapsto x^3 - 3, x \in \mathbb{R}$

Solution

$$f(x) = x^3 - 3$$

$$x^3 - 3 = y$$

$$x^3 = y + 3$$

$$x = \sqrt[3]{y + 3}$$

So, $f^{-1}(y) = \sqrt[3]{y + 3}$

$f^{-1}(x) = \sqrt[3]{x + 3}$

The inverse of $f:x \mapsto x^3 - 3$ is therefore $f^{-1}:x \mapsto \sqrt[3]{x + 3}$

Example 6

Show that the function $f{:}x \mapsto x^2 - 2x + 5, (x \in \mathbb{R}, x \geqslant 1)$ *is one-one. Obtain the domain and range of the inverse function* f^{-1} *and find* f^{-1}.

Solution

$$f(x) = x^2 - 2x + 5$$
$$= (x - 1)^2 + 4$$

The graph of $y = f(x)$ has a minimum at $(1, 4)$ and is shown in Figure 2.6.

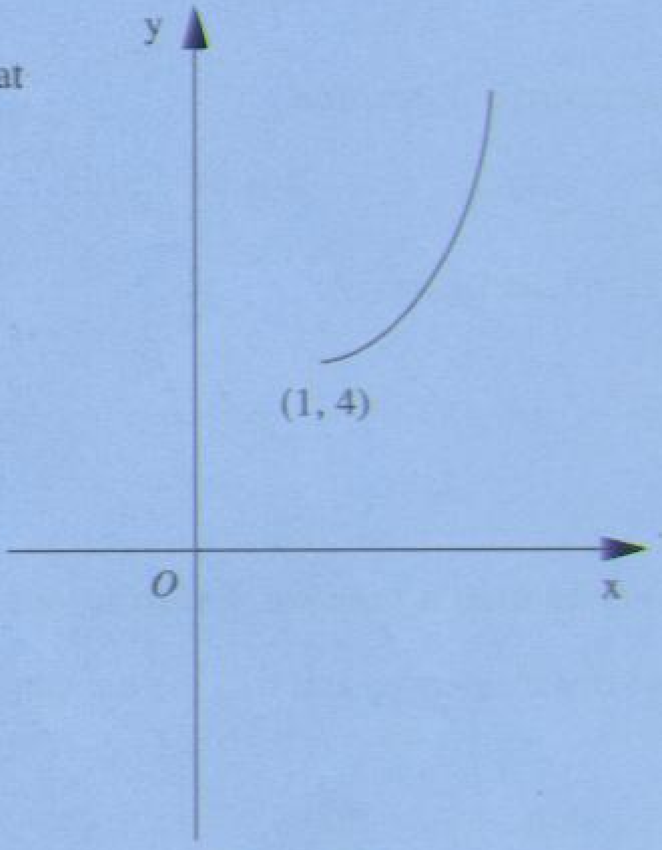


Figure 2.6

The range of $f = \{f(x) \in \mathbb{R} : f(x) \geqslant 4\}$

The domain of $f^{-1} = \{x \in \mathbb{R} : x \geqslant 4\}$

The range of $f^{-1} = \{f^{-1}(x) \in \mathbb{R} : f^{-1}(x) \geqslant 1\}$

To find $f^{-1}(x)$, we write $(x - 1)^2 + 4 = y$

$$(x - 1)^2 = y - 4$$
$$x - 1 = \pm\sqrt{y - 4}$$
$$x = 1 \pm \sqrt{y - 4}$$
$$\text{As } x \geqslant 1, x = 1 + \sqrt{y - 4}$$
$$f^{-1}(y) = 1 + \sqrt{y - 4}$$
$$f^{-1}(x) = 1 + \sqrt{x - 4}$$

So, $f^{-1}(x) \mapsto 1 + \sqrt{x - 4}, x \geqslant 4$

Exercise 2 B

1. Find the inverse of each of the following functions where $x \in \mathbb{R}$:

(a) $f{:}x \mapsto \dfrac{2x + 3}{x - 2}, x \neq 2$

(b) $f{:}x \mapsto \dfrac{3}{x + 3}, x \neq -3$

(c) $f{:}x \mapsto 4 - 3\sqrt{x-2}\ , x \geq 2$

(d) $f{:}x \mapsto \dfrac{1+\dfrac{2}{x}}{3-\dfrac{5}{2x}}\ x \neq 0 \text{ or } \dfrac{5}{6}$

(e) $f{:}x \mapsto 2x^3 + 3$

(f) $f{:}x \mapsto 3 - 2x$

(g) $f{:}x \mapsto 3 + \dfrac{2}{x}, (x \neq 0)$

(h) $f{:}x \mapsto x^2 + 6x - 4\ (x \geq -3)$

(i) $f{:}x \mapsto (2x-1)^2 + 3, (x \leq \dfrac{1}{2})$

(j) $f{:}x \mapsto 9 - (x-3)^2, (x \geq 3)$

2. Show that each of the following functions is one-one. In each case, find the domain, range and the rule of the inverse function:

(a) $f{:}x \mapsto \dfrac{1}{x}, x > 0$

(b) $f{:}x \mapsto \dfrac{1}{x}, x \neq 0$

(c) $f{:}x \mapsto \dfrac{1}{x^2}, x > 0$

(d) $f{:}x \mapsto \dfrac{1}{x^2}, x < 0$

(e) $f{:}x \mapsto x^2, x \geq 0$

(f) $f{:}x \mapsto -x^2, x \geq 0$

(g) $f{:}x \mapsto 3x^3 + 1, x \geq 0$

(h) $f{:}x \mapsto 4 - (x-2)^2, (x \leq 2)$

(i) $f{:}x \mapsto x^2 + 2x - 3, x \leq -1$

(j) $f{:}x \mapsto x^2 - 8x + 9, (x \geq 4)$

2 . 3 Graph of the inverse of a function

If (a, b) is any point on the graph of a function f, then the corresponding point on the graph of its inverse f^{-1} is (b, a).
Hence, a point (a, b) becomes (b, a).
But (a, b) is transformed onto (b, a) by a reflection in the line y = x.

Hence, the graph of $y = f^{-1}(x)$ is obtained from the graph of $y = f(x)$ by a reflection in the line $y = x$ as illustrated in the following examples.

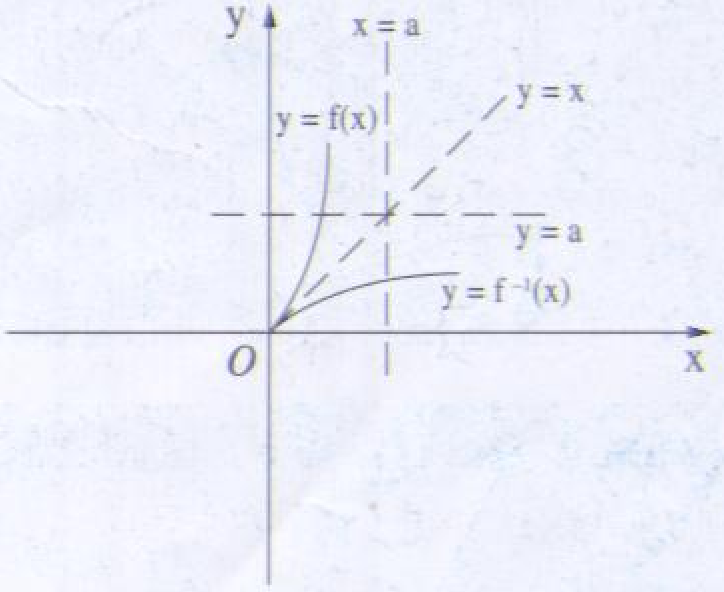


Figure 2.7

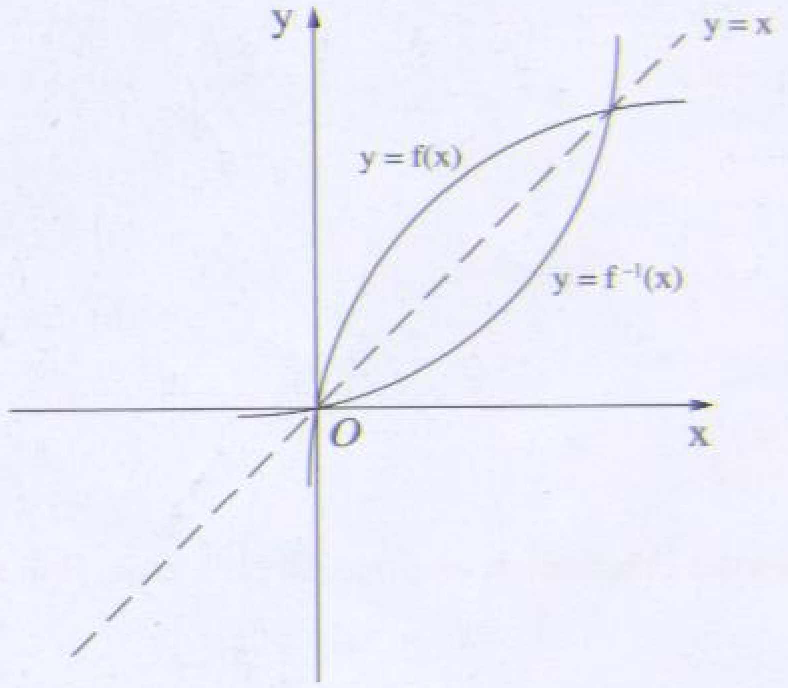
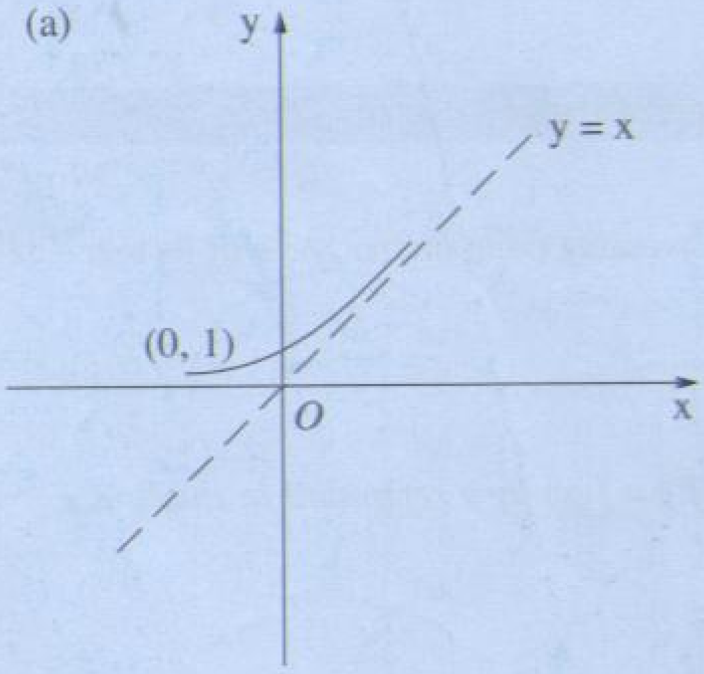


Figure 2.8

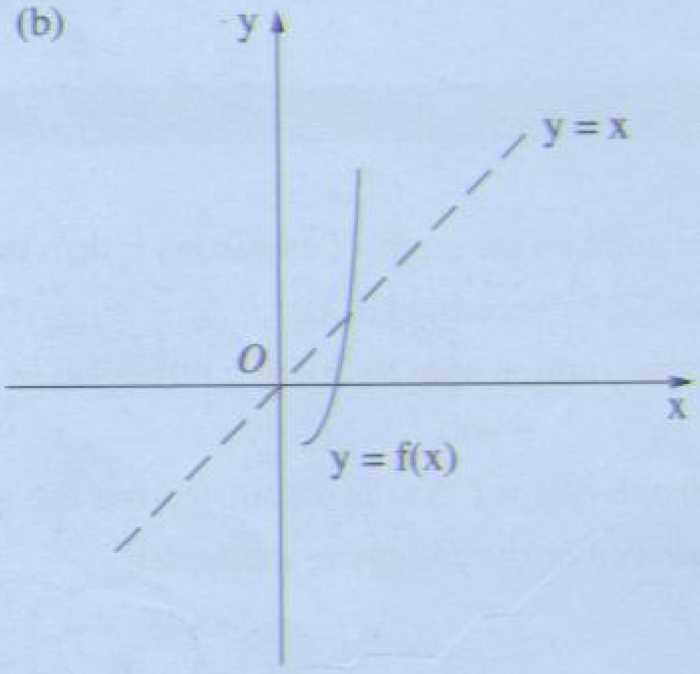
Exercise 2 C

1. In each of the following diagrams, the graph of $y = f(x)$ and the line $y = x$ are shown. Sketch in each case the graph of $y = f^{-1}(x)$

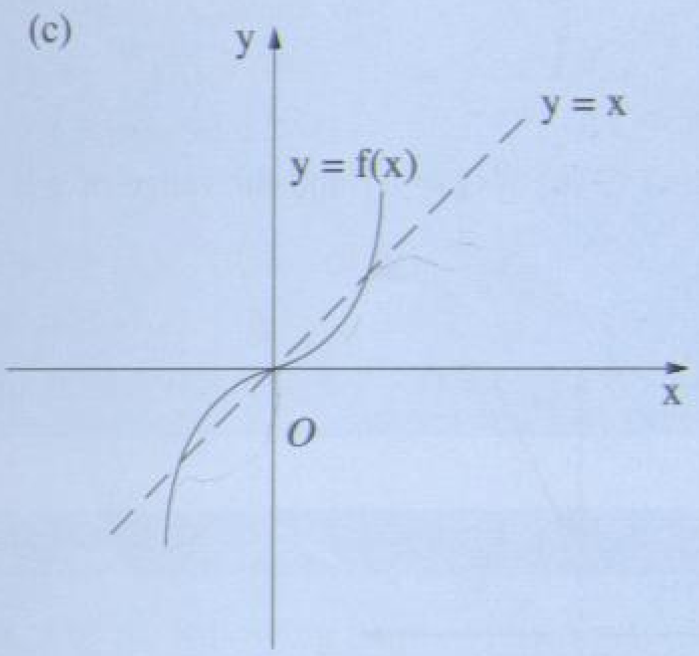
(a)



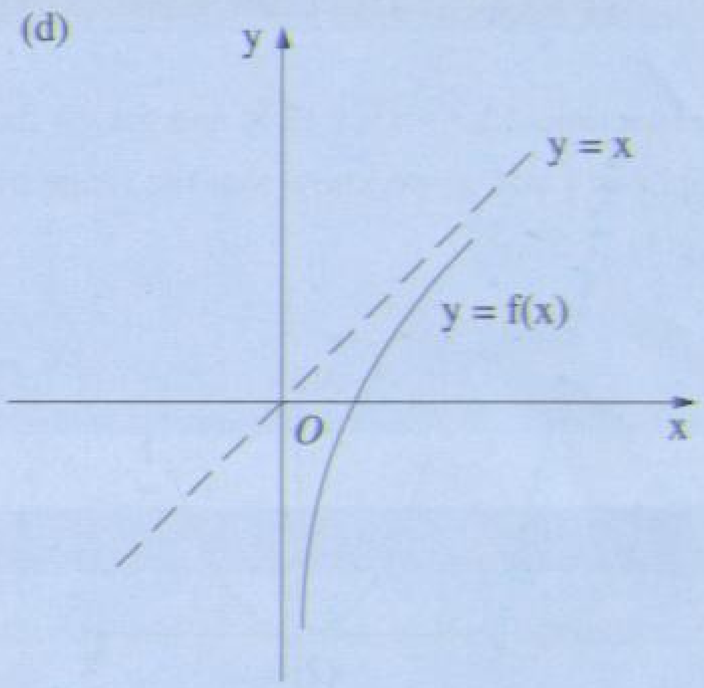
(b)



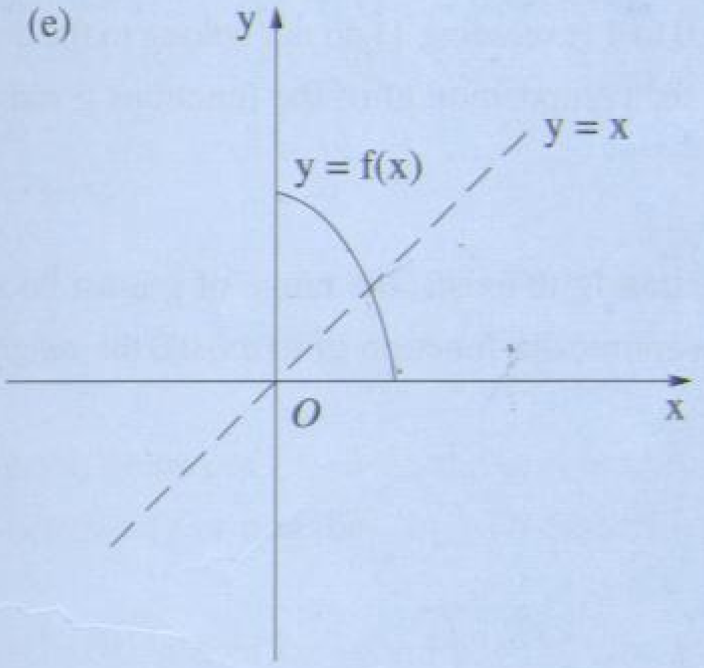
(c)



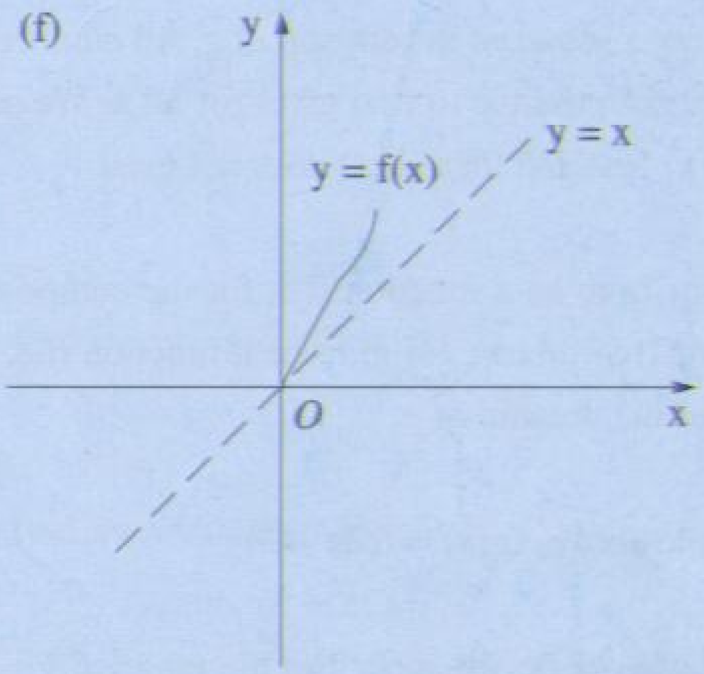
(d)



(e)



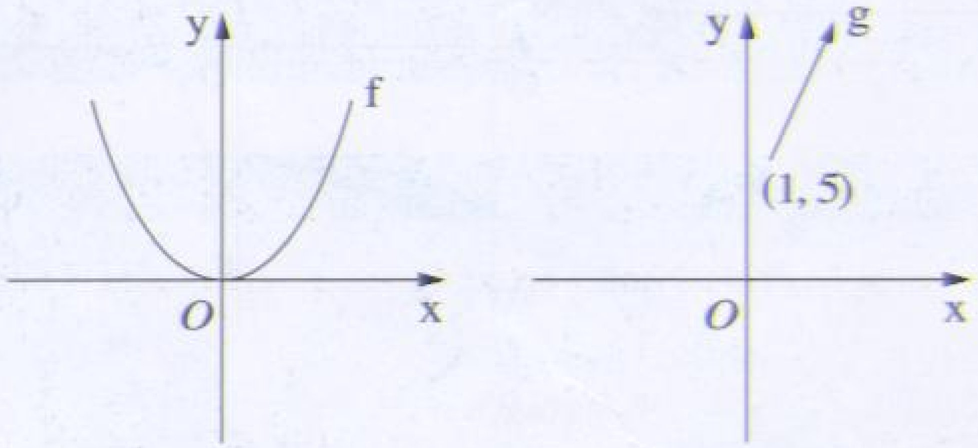
(f)



2. Given that $f(x) = \sin(x)$, $(0 \leqslant x \leqslant \frac{\pi}{2})$, sketch the graph of $y = f^{-1}(x)$.
3. If $g(x) = \cos x$, $(0 \leqslant x \leqslant \pi)$, sketch the graph of $g^{-1}(x)$.
4. Sketch the graph of $h^{-1}(x)$, given that $h(x) = x^2$; $x \geqslant 0$
5. Sketch the graph of $f^{-1}(x)$, given that $f(x) = x^3$.

2.4 Composition of functions

Consider the functions $f:x \mapsto x^2, x \in \mathbb{R}$ and $g:x \mapsto 2x + 3, (x \geqslant 1)$.
From the graphs of f and g, we know that the range of f is $\{f(x) \in \mathbb{R} : f(x) \geqslant 0\}$ and the range of g is $\{g(x) \in \mathbb{R} : g(x) \geqslant 5\}$.



We note that the range of g is a subset of the domain of f. It is therefore possible to find fg(x) for all $x \geqslant 1$. We say that the composition fg of the functions f and g obtained by finding g(x) first and then fg(x) exists. However the range of f is not a subset of the domain of g. All numbers from 0 to 1 (excluding 1) do not belong to the domain of g. It is therefore not possible to find gf(x) for all x. We say that the composition gf of the functions g and f obtained by finding f(x) first and then gf(x) does not exist.

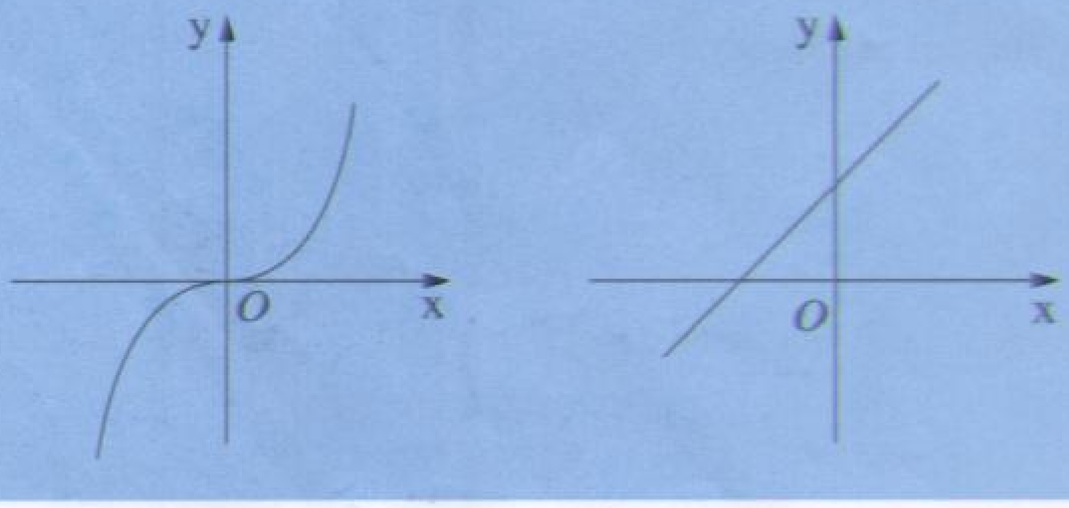
In general, for fg to be a function (i.e. for the composite function fg to exist), the range of g must be a subset of the domain of f. Similarly, for gf to be a function (i.e. for the composite function gf to exist), the range of f must be a subset of the domain of g.

In the example above, $fg(x) = f(2x + 3)$
$= (2x + 3)^2$
To find the range of fg, we apply f to the range of g.
As $f(5) = 25$, the range of fg $= \{fg(x) \in \mathbb{R} : fg(x) \geqslant 25\}$

Example 7

Given that $x \in \mathbb{R}$, $f:x \mapsto x^3$, $g:x \mapsto 3x + 2$, find (a) fg and (b) gf. Obtain the range of each.

Solution



(a) Range of g = $\mathbb{R}$ = domain of f.

So, fg exists and $fg(x) = f(3x + 2) = (3x + 2)^3$.

Range of fg = Range of f = $\mathbb{R}$.

(b) Range of f = $\mathbb{R}$ = domain of g.

So gf exists and $gf(x) = 3x^3 + 2$.

Range of gf = Range of g = $\mathbb{R}$.

Exercise 2 D

1. For each of the following pairs of functions, find whether fg and gf exist for x. If either exists, find its rule in the form fg : x ↦ ... or gf : x ↦ ... .

   (a) $f:x \mapsto 3x - 2$, $g:x \mapsto 5 - 3x$

   (b) $f:x \mapsto 2x + 1$, $g:x \mapsto \frac{1}{x}$ $(x \neq 0)$

   (c) $f:x \mapsto \frac{1}{x-3}$ $(x \neq 3)$, $g:x \mapsto 4 - 5x$

   (d) $f:x \mapsto 3x - 4$, $g:x \mapsto |x|$

   (e) $f:x \mapsto 1 - 2x$, $g:x \mapsto \ln x, x > 0$

   (f) $f:x \mapsto \sqrt{x}$, $(x \geqslant 0)$, $g:x \mapsto 2x + 3$

   (g) $f:x \mapsto \sqrt[3]{x}$, $g:x \mapsto 3x - 1$

   (h) $f:x \mapsto 2^x$, $g:x \mapsto x^2$

2. Sketch the graphs of $f:x \mapsto 3^x$ and $g:x \mapsto 3 - 2x$ for $x \in \mathbb{R}$.
   Find whether fg or gf exists. In each case, if it exists, find its rule and its range.

3. Sketch the graphs of $f:x \mapsto 5^x$ and $g:x \mapsto x^2$ for $x \in \mathbb{R}$.
   Find whether fg or gf exists. In each case, if it exists, find its rule and its range.

4. Sketch the graphs of $f:x \mapsto \sin x$, $(0 \leqslant x \leqslant \pi)$ and $g:x \mapsto \cos x$, $(0 \leqslant x \leqslant \pi)$. Show that gf exists but fg does not exist and find the range of gf.

Miscellaneous Exercise 2

1. Express $x^2 + 4x$ in the form $(x + a)^2 + b$, stating the numerical values of a and b.
   The functions f and g are defined as follows:

   $f:x \mapsto x^2 + 4x, x \geqslant -2$

   $g:x \mapsto x + 6, x \in \mathbb{R}$

   (a) Show that the equation $gf(x) = 0$ has no real roots.

   (b) State the domain of f^{-1}, and find an expression in terms of x for $f^{-1}(x)$.

   (c) Sketch, in a single diagram, the graphs of $y = f(x)$ and $y = f^{-1}(x)$, making clear the relationship between these graphs. [C]

2. The functions f and g are defined as follows:

$$f:x \mapsto \frac{1}{x}, x \in \mathbb{R}, x \neq 0$$

$$g:x \mapsto 1 - x, x \in \mathbb{R}$$

Write down expressions for fg(x) and gf(x) where $x \neq 0, x \neq 1$ and hence show that gfg(x) = fgf(x) for all such x. [C]

3. The function f is defined by

$$f:x \mapsto (x-1)^2 + 2, x \in \mathbb{R}, x \geqslant 1.$$

On a single clearly labelled diagram, sketch the graphs of $y = f(x)$, $y = f^{-1}(x)$ and $y = ff^{-1}(x)$.

The function g is defined by

$$g:x \mapsto 3x + 2, x \in \mathbb{R}.$$

Find gf(x), for $x \geqslant 1$.

The function h is defined by $h:x \mapsto ax + b, x \in \mathbb{R}$; where a and b are constants. Find the values of a and b such that $fh:x \mapsto 4x^2 + 16x + 18, x \in \mathbb{R}, x \geqslant -2$. [C]

4. The function f is defined by $f:x \mapsto 4x^3 + 3, (x \in \mathbb{R})$.
Give the corresponding definition of f^{-1}.
State a relationship between the graphs of f and f^{-1}. [C]

5. The functions f, g and h are defined for x by

$$f(x) = x^2,$$

$$g(x) = \sqrt{x}\ (x \geqslant 0)$$

$$h(x) = x + 2.$$

Sketch on separate, clearly labelled diagrams, the graphs of
(a) $y = f(x)$
(b) $y = g(x)$
(c) $y = gf(x)$
(d) $y = fh(x)$
(e) $y = hf(x)$
Find $(gh)^{-1}(y)$, given $y \geqslant 0$. [C]

6. The subset of $\mathbb{R}$ given by $\{x: -1 \leqslant x \leqslant 3\}$ is denoted by S. The function $f:S \mapsto R$ is defined by $f:x \mapsto x^2 - x - 2$.
State whether f is one–one, giving a reason for your answer, and determine the range of f. [C]

7. Functions f and g, each with domain, are defined as follows:
$f{:}x \mapsto 3x + 2$, $g{:}x \mapsto x^2 + 1$
For each of f and g, state the range of the function and give a reason to show whether or not the function is one–one. Give explicit definitions, in the above form, of each of the composite functions fg and gf and find the values of x for which $fg(x) = gf(x)$.
State the domain of the inverse relation $(fg)^{-1}$ and give an explicit definition of this relation.
Explain briefly why $(fg)^{-1}$ is not a function. [C]

8. The function f has domain $\{x \in \mathbb{R}: x \geqslant -2\}$ and is defined by $f(x) = x^2 + 4x + 7$.
Define the inverse function f^{-1}. [C]

9 The function f has domain $\{x \in \mathbb{R}: x \leqslant 3\}$ and is defined by $f(x) = x^2 - 6x + 10$.
Define the inverse function f^{-1}. [C]

10. The function f is given by $f{:}x \mapsto x^2 - 3x - 4$, where $x \in \mathbb{R}$.
Find the range of f and the values of x for which $f(x) = 0$.
Sketch the graphs of $y = x^2 - 3x - 4$ and $y = \dfrac{1}{x^2 - 3x - 4}$. [C]

11. Show that $f{:}x \mapsto x^2 - 4x + 3$, where $x \in \mathbb{R}$, is not a one–one function.
The one–one function g has domain D where $D \subset \mathbb{R}^+$ and is defined by $g{:}x \mapsto x^2 - 4x + 3$. Given that the range of g is the same as the range of f, find D. Find also g^{-1} in the form $g^{-1}{:}x \mapsto$ .... giving its domain and range. [C]

12. The function f is defined by $f{:}x \mapsto \dfrac{x}{x+1}$, $x \neq -1$. Obtain simplified expressions for:

(a) $f\left(\dfrac{1}{x}\right)$ (b) $f^{-1}(x)$ (c) $f^{-1}\left(\dfrac{1}{x}\right)$.

Hence, solve the equation $f\left(\dfrac{1}{x}\right) + f^{-1}\left(\dfrac{1}{x}\right) = f(x)$. [C]

CHAPTER 3

> COORDINATE GEOMETRY

3.1 Straight lines

3.1.1 Distance between two points

We know from earlier work that the distance d between two points with coordinates (x_1, y_1) and (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 1

Find the distance between the points with coordinates (3, –1) and (–1, 2).

Solution

$$\begin{aligned} \text{Distance} &= \sqrt{(-1-3)^2 + (2 - -1)^2} \\ &= \sqrt{4^2 + 3^2} \\ &= 5 \end{aligned}$$

Example 2

Show that the points with coordinates (2, 3) and (–2, –1) are equidistant from the point with coordinates (–4, 5).

Solution

$$\begin{aligned} \text{Distance between point } (2, 3) \text{ and point } (-4, 5) &= \sqrt{(-4-2)^2 + (5-3)^2} \\ &= \sqrt{36+4} \\ &= \sqrt{40} \end{aligned}$$

$$\begin{aligned} \text{Distance between point } (-2, -1) \text{ and point } (-4, 5) &= \sqrt{(-4+2)^2 + (5+1)^2} \\ &= \sqrt{4+36} \\ &= \sqrt{40} \end{aligned}$$

Hence points (2, 3) and (2, –1) are equidistant from point (–4, 5).

3.1.2 Mid-point of a line segment

If A and B have coordinates (x_1, y_1) and (x_2, y_2), the coordinates of the mid-point of line segment AB are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Example 3

Find the coordinates of the mid-point of the line segment AB where $A = (2, -3)$, $B = (-3, 4)$.

Solution

$$\text{Coordinates of mid-point} = \left(\frac{2 + -3}{2}, \frac{-3 + 4}{2}\right)$$

$$= \left(-\frac{1}{2}, \frac{1}{2}\right)$$

Example 4

A, B, C are the vertices of a triangle ABC with $A\left(-\frac{1}{2}, 1\right)$, $B(-2, 3)$ *and* $C(3, -2)$. *Find the length of the median AM of the triangle ABC. (A median of a triangle is a line joining one vertex to the mid-point of the opposite side).*

Solution

$$M = \left(\frac{-2 + 3}{2}, \frac{3 + -2}{2}\right)$$

$$= \left(\frac{1}{2}, \frac{1}{2}\right)$$

$$AM = \sqrt{\left(\frac{1}{2} - -\frac{1}{2}\right)^2 + \left(\frac{1}{2} - 1\right)^2}$$

$$= \sqrt{1^2 + \left(-\frac{1}{2}\right)^2}$$

$$= \sqrt{1 + \left(\frac{1}{4}\right)}$$

$$= \sqrt{\left(\frac{5}{4}\right)}$$

$$= \frac{\sqrt{5}}{2}$$

3.1.3 Gradient of a line

(i) We know already that the gradient of a line is the ratio $\frac{y_1 - y_2}{x_1 - x_2}$ where (x_1, y_1) and (x_2, y_2) are the coordinates of any two given points on the line.

The gradient of the straight line through $(-2, 3)$ and $(2, -3)$ is $\frac{-3 - 3}{2 - -2} = -\frac{-6}{4} = -\frac{3}{2}$.

(ii) All lines parallel to the x-axis, i.e. y lines, have gradient 0. Thus, $y = 4$, $y = -5$, $y = \frac{7}{2}$ have zero gradient.

(iii) All lines parallel to the y-axis, i.e. x lines, have infinite gradient. Thus, $x = 1, x = -5, x = -\frac{3}{4}$ have infinite gradient.

(iv) If a straight line has positive gradient, it makes an acute angle with the positive direction of the x-axis and if it has negative gradient, it makes an obtuse angle with the positive direction of the x-axis.

3.1.4 Equations of straight lines

The Cartesian equation of a straight line is the equation satisfied by the x-coordinate and the y-coordinate of any point on the line.
To find the equation of a straight line, we need to know the coordinates of one point on the line and its gradient or the coordinates of two points on the line.

3.1.5 Equation of the straight line through a point (x_1, y_1) and with gradient m

We take the general point (x, y) on the line

$$\frac{y - y_1}{x - x_1} = m \text{ as gradient is m.}$$

$$y - y_1 = m(x - x_1)$$

Note: If the point is the origin, $x_1 = y_1 = 0$, the equation is $y = mx$.

Example 5

Find the equation of the straight line with gradient $-\frac{4}{5}$ and passing through the point with coordinate (–2,.1).

Solution

$$\text{Equation is } \frac{y - 1}{x + 2} = -\frac{4}{5}$$

$$5(y - 1) = -4(x + 2)$$

$$5y - 5 = -4x - 8$$

$$5y + 4x + 3 = 0$$

3.1.6 Equation of the straight line through two given points

To find the equation of the straight line through (2, –3) and (5, 4), we find the gradient

$$\text{Gradient} = \frac{4 - -3}{5 - 2}$$

$$= \frac{7}{3}$$

Taking the general point (x, y) on the line and one of the two given points say (2, –3), the equation is then

$$\frac{y - -3}{x - 2} = \frac{7}{3}$$

$$\frac{y + 3}{x - 2} = \frac{7}{3}$$

$$3y + 9 = 7x - 14$$

$$3y = 7x - 23$$

Note: Instead of using (2, –3), we could have used (5, 4) and obtained the same equation.

Example 6

Find the equation of the line through (1, –2) and (–3, 1).

Solution

$$\text{Gradient of line} = \frac{1 - -2}{-3 - 1}$$

$$= -\frac{3}{4}$$

$$\text{Equation of straight line is} \quad \frac{y + 2}{x - 1} = -\frac{3}{4}$$

$$4y + 8 = -3x + 3$$

$$4y = -3x - 5$$

$$4y + 3x + 5 = 0$$

Example 7

Find the equation of the line through (–2, 3) and (4, 3).

Solution

$$\text{Gradient of line} = \frac{3 - 3}{4 - -2}$$

$$= \frac{0}{6}$$

The line has gradient 0.

So, it is a *y-line*.

$y = 3$

Its equation is $y = 3$ (as both points have y-coordinate 3)

Example 8

Find the equation of the line through (–2, 0) and (–2, 5).

Solution

Gradient of line = $\frac{5 - 0}{-2 - -2}$

$= \frac{5}{0}$ i.e. is infinite

Equation of line is $x = -2$.

Exercise 3 A

1. Find the length of the line segment joining each of the following pairs of points with given coordinates:
   (a) (2, –1), (–1, –5)
   (b) (0, –5), (3, 0)
   (c) (–1, 4), (3, 1)
   (d) (–1, –5), (–2, 3)
   (e) (3, 2), (–2, 1)
   (f) (4, –1), (–1, 4)
   (g) (–2, 0), (4, 0)
   (h) (0, –2), (0, 7)
   (i) (a, b), (b, a)
   (j) (–2, 1), (a, –2)

2. Find the coordinates of the mid-point of each of the line segments in question 1.

3. The point P (a, 4) lies on the circle with AB as diameter. Given that A and B have coordinates (–2, 3) and (4, 11) respectively, find the coordinates of the centre of the circle and the values of a.

4. Show that the points (3, 5), (–5, 1), (–1, 5) lie on a circle with centre (1, –1).

5. ABC is a triangle with coordinates A(2, –1), B(0, 3) and C(5, 9). Find the lengths of the medians of the triangle.

6. The vertices A, B and C of a parallelogram ABCD have coordinates (4, 8), (2, 3) and (–2, –6) respectively. Find the coordinates of the point of intersection of the diagonals. Hence, find the coordinates of the vertex D.

7. A parallelogram ABCD has vertices A(x_1, y_1), B(x_2, y_2), C(x_3, y_3) and D(x_4, y_4). Show that $x_1 + x_3 = x_2 + x_4$ and $y_1 + y_3 = y_2 + y_4$.

8. A and B have coordinates (–1, 2) and (3, 4) respectively. Find by calculation whether the following points lie on the mediator of AB: (a) (1, 3) (b) (–2, 9) (c) (2, –1).

9. ABCD is a quadrilateral with vertices A(1, 2), B(6, 4), C(4, 9) and D(–1, 7).
   Show that all sides are of equal length.
   Find the lengths of the diagonals BD and AC. Hence, show that all the angles of the quadrilateral are right angles.
   What is the special name given to this quadrilateral?

10. ABCD is a quadrilateral with A(–1, 3), B(4, 5), C(6, 10) and D(1, 8).
Show that all the sides are of equal length.
Show that the diagonals AC and BD have the same mid-point M.
By considering the lengths of MC, MB and BC, show that the diagonals AC and BD intersect at right angles.

11. A, B and C have coordinates (7, 3), (2, 3) and (4, –1) respectively.
Show that the triangle ABC is isosceles.
Find the coordinates of the mid-point M of BC.
By considering the lengths of AM, MB and AB, show that angle AMB is a right angle.

12. Find the equations of the straight lines with given gradients, passing through the given points:
(a) through (1, –4), gradient 4
(b) through (2, –1), gradient –4
(c) through (–4, 3), gradient –6
(d) through (–5, –2), gradient $\frac{3}{4}$
(e) through $\left(\frac{1}{3}, -\frac{1}{4}\right)$, gradient –5
(f) through $\left(-\frac{3}{5}, -\frac{1}{2}\right)$, gradient $-\frac{3}{4}$
(g) through $\left(\frac{5}{3}, -\frac{3}{5}\right)$, gradient $-\frac{1}{4}$
(h) through (a, b), gradient m
(i) through $(e, e^2 + 1)$, gradient $\frac{1}{e}$
(j) through $(at^2, 2at)$, gradient –t
(k) through $\left(ct, \frac{c}{t}\right)$, gradient $-\frac{1}{t^2}$
(l) through $\left(\sqrt{2}, 1 - \sqrt{2}\right)$, gradient $\frac{1}{\sqrt{2}}$

13. Find the equations of the straight lines through the following pairs of points:
(a) (1, –3), (4, 3)
(b) (–1, 5), (3, 7)
(c) (–1, 2), (3, 5)
(d) $\left(2, -\frac{1}{2}\right), \left(\frac{1}{2}, 2\right)$
(e) $\left(\frac{3}{5}, -\frac{5}{4}\right), \left(\frac{1}{2}, -\frac{1}{4}\right)$
(f) (a, b), (b, a)
(g) $(ap^2, 2ap), (aq^2, 2aq)$
(h) (3, –1), (5, –1)
(i) (–4, 3), (–4, 5)
(j) $\left(\frac{1}{p}, -\frac{1}{p^2}\right), \left(\frac{1}{q}, -\frac{1}{q^2}\right)$

14. ABC is a triangle with vertices A(2, 4), B(4, –2) and C(3, 6). Find the equations of the medians of the triangle.

15. A(–1, 2), B(3, 5) and C(1, 8) are the vertices of the triangle ABC. D, E, F are the mid-points of AB, BC and AC respectively. Find the equations of DE, DF and EF.

3.2 Equation of straight lines

3.2.1 The equation of y = mx + c

Consider the equation of the straight line through (0, c) and gradient m which is:

$$\frac{y-c}{x-0} = m$$
$$y - c = mx$$
$$y = mx + c$$

Conversely, the equation $y = mx + c$ represents for a given value of m and of c a straight line through (0, c) and gradient m.

Thus, the line $y = 3x + 4$ represents a straight line of gradient 3 passing through (0, 4) and the line $y = -\frac{1}{2}x - 5$ represents a straight line of gradient $-\frac{1}{2}$ passing through (0, –5).

The equation $ax + by = p$ can be reduced to the form $y = mx + c$ as

$$ax + by = p$$
$$by = -ax + p$$
$$y = -\frac{a}{b}x + \frac{p}{b}$$

This represents a straight line of gradient $-\frac{a}{b}$ passing through the point $\left(0, \frac{p}{b}\right)$.

Note: 'c' is called the y-intercept of the line.

Example 9

Find the gradient of the line with equation 3y + 2x = –9 and its y-intercept.

Solution

$$3y + 2x = -9$$
$$3y = -2x - 9$$
$$y = -\frac{2}{3}x - 3$$

Gradient is $-\frac{2}{3}$ and y-intercept is –3.

3.2.2 The equation $y - y_1 = m(x - x_1)$

We have seen that the equation of the straight line with gradient m passing through the point (x_1, y_1) is $\frac{y - y_1}{x - x_1} = m$ or $y - y_1 = m(x - x_1)$.

Conversely, the equation $y - y_1 = m(x - x_1)$ represents a straight line of gradient m passing through the point (x_1, y_1).

Thus, $y - 3 = 2(x - 4)$ is a straight line of gradient 2 through the point (4, 3) and $y + 1 = -4(x - 3)$ is a straight line of gradient –4 and passing through the point (3, –1).

3.2.3 Parallel lines

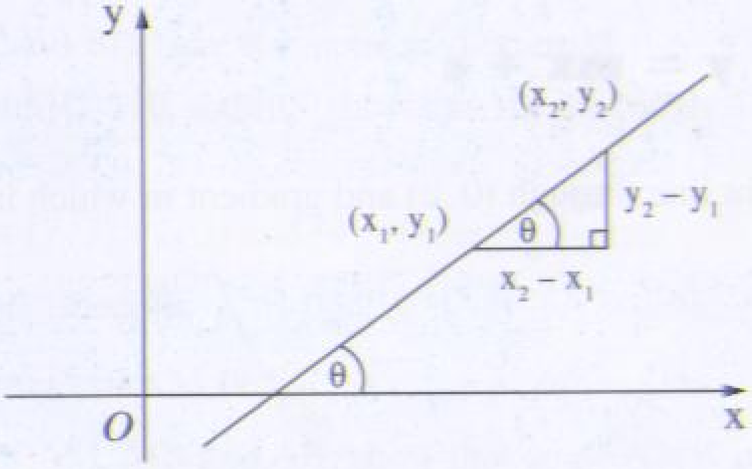


Figure 3.1

From figure 3.1, the gradient of the line $\frac{y_2 - y_1}{x_2 - x_1}$ = tan θ, where θ is the angle between the line and the positive direction of the x-axis.

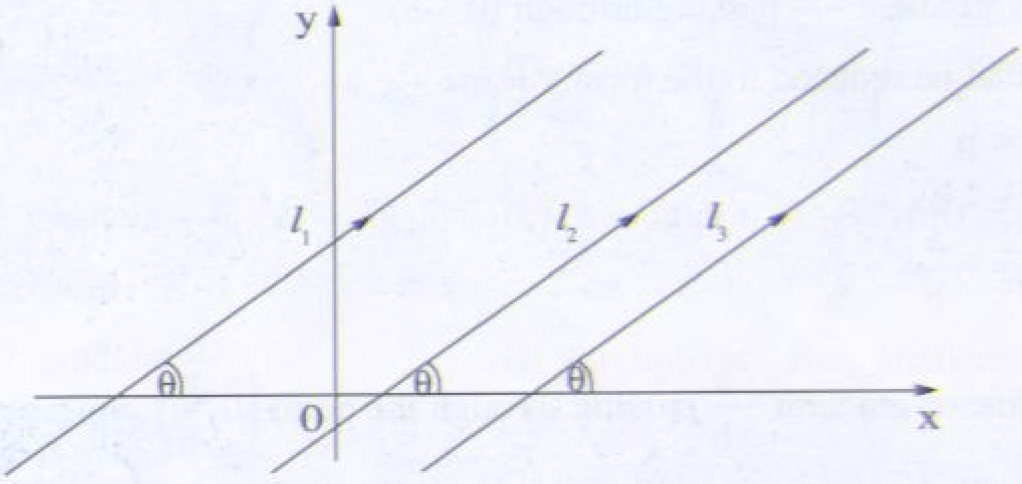


Figure 3.2

It follows that if lines are parallel, they make equal angles with the positive direction of the x-axis, and so, their gradients are equal.

Conversely, if two or more straight lines have equal gradients, they are parallel.

Example 10

Show that lines 3y + 2x = 7 and 6y + 4x = 11 are parallel.

Solution

$$3y + 2x = 7$$
$$3y = -2x + 7$$
$$y = -\frac{2}{3}x + \frac{7}{3}$$

$$\text{Gradient} = -\frac{2}{3}$$

$$6y + 4x = 11$$
$$6y = -4x + 11$$
$$y = -\frac{2}{3}x + \frac{11}{6}$$

As gradients are the same, the lines are parallel.

Example 11

Find the equation of the line through (1, –3) parallel to the line through (–1, –2) and (2, 5).

Solution

Gradient of line through (–1, –2) and (2, 5) = $\frac{5+2}{2+1} = \frac{7}{3}$

Gradient of parallel line = $\frac{7}{3}$

Equation of line is $\frac{y+3}{x-1} = \frac{7}{3}$

$$3y + 9 = 7x - 7$$
$$3y = 7x - 16$$

Example 12

Find the equation of the line through (5, –7) parallel to the line 2y + 3x = 5.

Solution

$$2y + 3x = 5$$
$$2y = -3x + 5$$
$$y = -\frac{3}{2}x + \frac{5}{2}$$

Gradient of line = $-\frac{3}{2}$

Equation of parallel line is $\frac{y+7}{x-5} = -\frac{3}{2}$

$$2y + 14 = -3x + 15$$

$$2y + 3x = 1$$

3.2.4 Perpendicular lines

Consider two lines parallel to the axes (x-lines and y-lines), e.g. $x = 3$ and $y = -4$. These 2 lines are perpendicular. Next, consider two perpendicular lines which are not parallel to the axes (Figure 3.3).

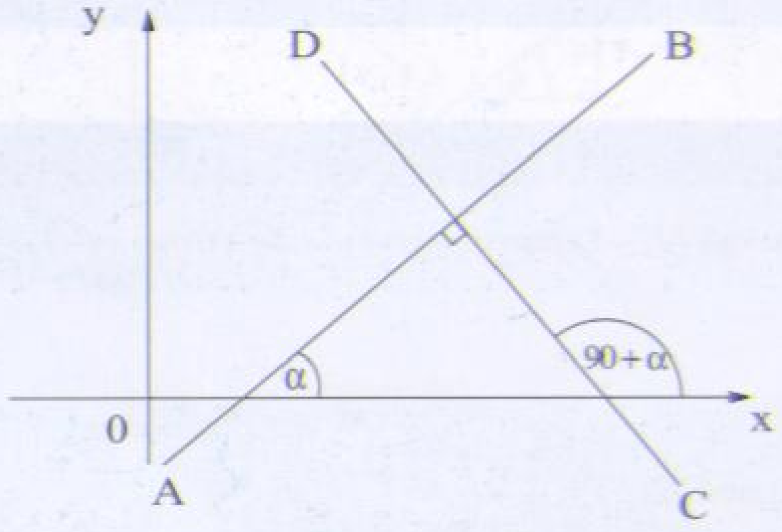


Figure 3.3

If AB makes an angle α with the positive direction of the x-axis, its gradient = $\tan \alpha$.
CD then makes an angle of $(90 + \alpha)$ with the positive direction of the x-axis and its gradient is $\tan (90 + \alpha) = -\cot \alpha$.
Product of gradients is then $\tan \alpha \times -\cot \alpha = -1$.
So, if two lines are perpendicular and they are not parallel to the axes, the product of their gradients is -1.
Conversely, if the product of the gradients of two lines is -1, the lines are perpendicular.

Example 13

Find the gradient of a straight line which is perpendicular to the line $2y + 3x = 7$.

Solution

$$\begin{aligned} 2y + 3x &= 7 \\ 2y &= -3x + 7 \\ y &= -\frac{3}{2}x + \frac{7}{2} \end{aligned}$$

Gradient of line is $-\frac{3}{2}$

Gradient of perpendicular line $= -1 \div -\frac{3}{2} = \frac{2}{3}$.

Example 14

Find the equation of the straight line through the point $(1, 4)$ perpendicular to the line joining $A(2, -1)$ and $B(-1, 3)$.

Solution

Gradient of AB $= \frac{3 - -1}{-1 - 2}$

$$= -\frac{4}{3}$$

Gradient of perpendicular line $= \frac{3}{4}$

Equation of perpendicular line is $\frac{y-4}{x-1} = \frac{3}{4}$

$$4y - 16 = 3x - 3$$
$$4y = 3x + 13$$

Example 15

Find the equation of the mediator of the line segment joining (1, 2) and (4, 6).

Solution

Gradient of line segment $= \frac{6-2}{4-1}$

$$= \frac{4}{3}$$

Gradient of mediator $= -1 \div \frac{4}{3} = -\frac{3}{4}$

Midpoint of line segment $= \left(\frac{1+4}{2}, \frac{2+6}{2}\right)$

$$= \left(\frac{5}{2}, 4\right)$$

Equation of mediator is $\frac{y-4}{x-\frac{5}{2}} = -\frac{3}{4}$

$$4y - 16 = -3x + \frac{15}{2}$$
$$8y - 32 = -6x + 15$$
$$8y + 6x = 47$$

Example 16

Find the equation of the line through (1, 2) perpendicular to the line segment joining (–1, 3) and (5, 3).

Solution

Gradient of line $= \frac{3-3}{5+1}$

$$= 0$$

The line is therefore a y-line.

The perpendicular line is therefore an x-line.

Since it passes through (1, 2), its equation is $x = 1$.

Note: When the gradient of a line is 0, we cannot use the fact that the product of the gradients of perpendicular lines is –1.

Exercise 3 B

1. Find the gradient of each of the following lines and the y-intercept:

(a) $2y - 5x = 7$

(b) $3y + 2x = 9$

(c) $\frac{2x}{3} + \frac{3y}{2} = 5$

(d) $\frac{x}{a} + \frac{y}{b} = 1$

(e) $ax + by = c$

(f) $x \cos \theta + y \sin \theta = p$

(g) $y - 3 = 2(x + 1)$

(h) $y + 3 = -\frac{5}{2}(x - 5)$

(i) $2(y - 3) = -5(x + 7)$

(j) $\frac{y-5}{x+4} = -\frac{5}{2}$

2. Find the gradients of the lines which are (i) parallel and (ii) perpendicular to each of the following lines:

(a) $3y - 4x = 9$

(b) $4x + 3y = 8$

(c) $y - 2 = 3(x + 4)$

(d) $y - 3 = p(x + 2)$

(e) $\frac{3x}{2} + \frac{5y}{3} = 1$

(f) $x \cos \theta + y \sin \theta = p$

(g) $3(y + 4) = 5(x - 7)$

(h) $x = 8$

(i) $y = -5$

(j) $\frac{y-3}{x+2} = \frac{3}{7}$

3. Find the value of a in each of the following:

(a) A(a, 3), B(1, 4), C(–1, 2), D(3, 5) and AB is parallel to CD

(b) A(1, 2), B(3, 5), C(0, 1), D(a, –2) and AB is perpendicular to CD.

(c) A(–1, 1), B(3, 4), C(–4, 2), D(1, a) and AD is parallel to BC.

(d) A(2, 0), B(–1, 1), C(a, 3), D(5, 7) and AD is perpendicular to BC.

(e) A(1, 1), B(2, 3), C(–1, a), D(3, 5) and AD is perpendicular to BC.

4. A parallelogram is defined as a quadrilateral which has its 2 pairs of opposite sides parallel. Use this definition to show that the quadrilateral ABCD with vertices A(2, 3), B(5, 8), C(7, 9) and D(4, 4) is a parallelogram.

5. A rectangle is defined as a parallelogram which has one of its angles 90°.
Use this definition to show that the quadrilateral ABCD with vertices A(–1, 2), B(3, 5), C(0, 9) and D(–4, 6) is a rectangle.

6. $A(x_1, y_1)$, $B(x_2, x_2)$ and $C(x_3, x_3)$ are the vertices of a triangle ABC. L and M are the mid-points of AB and AC respectively. Show that LM is parallel to BC and its length is $\frac{1}{2}$ length of BC.
Given N, the mid-point of BC has coordinates (2, –1) and L and M have coordinates (3, 4) and (5, 7) respectively, find the equation of BC.

7. Find the equation of the line through:
   (a) (–1, 2), parallel to the line $2y + x = 3$
   (b) (2, –1), perpendicular to the line $x - 3y = 4$
   (c) (0, 2), parallel to the line $\frac{x}{3} + \frac{y}{4} = 5$
   (d) (1, –3), perpendicular to the line $y - 3 = -4(x + 1)$
   (e) (1, 3), parallel to the line $y = -4$
   (f) (1, 3), perpendicular to the line $y = -4$
   (g) (–1, 5), parallel to the line $x = 3$
   (h) (–1, 5), perpendicular to the line $x = 3$
   (i) (2, 3), perpendicular to the line $x \cos 30° + y \sin 30° = 1$
   (j) $(at^2, 2at)$, parallel to the line $y - 3 = t(x + 2)$
   (k) $\left(ct, \frac{c}{t}\right)$, perpendicular to the line $\frac{y + 3}{x - 1} = t^2$
   (l) (cp^2, cp), parallel to the line $y + 3 = \frac{1}{p}(x - 4)$

8. ABCD is a rhombus. A has coordinates (2, –4) and C (–2, 4). Find the equation of the diagonal BD.

9. Find the equations of the three mediators of the triangle ABC where A, B and C have coordinates (1, 2), (5, 2) and (3, 6) respectively.

10. The line AB makes an angle of 120° with the positive direction of the x-axis. Find the equations of the straight lines through $(\sqrt{3}, 1 - \sqrt{3})$ respectively parallel and perpendicular to AB.

3.3 Intersection of two straight lines

(i) To find the coordinates of the point of intersection of a straight line with the y-axis, we substitute $x = 0$ in the equation of the line.
To find the coordinates of the point of intersection of the line with the x-axis, we substitute $y = 0$ in the equation.

Thus, $2x + 3y + 6 = 0$ intersects the y-axis when $x = 0$.

i.e. $3y + 6 = 0$

$y = -2$

Point of intersection with the y-axis is (0, –2).

It intersects the x-axis when $y = 0$

i.e. $2x + 6 = 0$

$x = -3$

It therefore intersects the x-axis at (–3, 0)

(ii) To find the point of intersection of any two given lines, we solve their equations.

To find the point of intersection of the lines with equations $2x + 3y = 5$ and $3x + 2y = 10$, we solve the two equations.

$$\begin{aligned} 2x + 3y &= 5 && (1) \times 2 \\ 3x + 2y &= 10 && (2) \times 3 \\ 4x + 6y &= 10 && (A) \\ 9x + 6y &= 30 && (B) \\ -5x &= -20 && (A) - (B) \\ x &= 4 \end{aligned}$$

Substitute in (1) $8 + 3y = 5$

$y = -1$

The two lines intersect at (4, –1).

Example 17

The straight line $2x + 3y = 12$ intersects the y-axis at A. Through A, a line AC is drawn at right angles to this line and through B(1, –2) a line BC of gradient 3 is drawn. Find the coordinates of C.

Solution

At A $x = 0$, $3y = 12$

$y = 4$

A has coordinates (0, 4)

Gradient of line $2x + 3y = 12$ is $-\frac{2}{3}$

Gradient of AC $= \frac{3}{2}$

Equation of AC is $\frac{y-4}{x-0} = \frac{3}{2}$

$2y - 8 = 3x$

$2y - 3x = 8$

Equation of BC is $\frac{y+2}{x-1} = 3$

$y + 2 = 3x - 3$

$y - 3x = -5$

To obtain coordinates of C, we solve the equations

$$2y - 3x = 8 \qquad (1)$$

$$y - 3x = -5 \qquad (2)$$

$$y = 13 \qquad (1) - (2)$$

From (1) $$26 - 3x = 8$$

$$3x = 18$$

$$x = 6$$

Coordinates of C = (6, 13)

Exercise 3 C

1. From the point A(1, –2), a perpendicular is drawn to the line $2x + 3y = 9$ to meet it at M. Find the coordinates of M and the length of AM.

2. The straight line through the point (1, –2) parallel to the line $2y + 3x = 6$ meets the x-axis at A and the y-axis at B. Find the length of AB.

3. The mediator of AB where A and B have coordinates (1, 5) and (2, 6) meets the line through (2, –3) perpendicular to the line $x + y = 5$ at C. Find the coordinates of C.

4. Find the length of the perpendicular from the point (1, –3) to the straight line $5x + 12y + 5 = 0$.

5. Show that if A, B, C and D have coordinates (2, 1), (5, 4), (4, 5) and (1, 2) respectively, then ABCD is a rectangle. Find the coordinates of the foot of the perpendicular from C to BD and find the length of this perpendicular.

   A line of gradient $\frac{1}{2}$ is drawn through C to cut AB at E. Find the length of AE.

6. A(3, 1), B(–5, 7) and C(1, 11) are the coordinates of a triangle ABC. M and N are the mid-points of BC and AC respectively. Find the equations of the medians AM and BN of the triangle. Hence obtain the coordinates of the point of intersection G of the medians.

   Find the equation of CL where L is the mid-point of AB and show that G lies on CL.

7. Find the equations of the mediators of the triangle ABC with A(–1, 2), B(5, 2) and C(–1, 8). Show that the point of intersection of the mediators of AB and AC lies on BC.

8. A triangle is enclosed by the lines $y = 3x$, $y = \frac{1}{3}x$ and $x + y = 8$. Find the coordinates of its vertices. Find also the coordinates of the point of intersection of its mediators.

3.4 Intersection of a line and a curve

To find the coordinates of the points of intersection (if any) of a curve and a straight line, we solve their equations. A curve may not intersect a line at all. It may intersect it in one point only, in which case, the line is a tangent to the curve or it may intersect it in more than one point. The 3 different cases are illustrated in Figure 3.4.

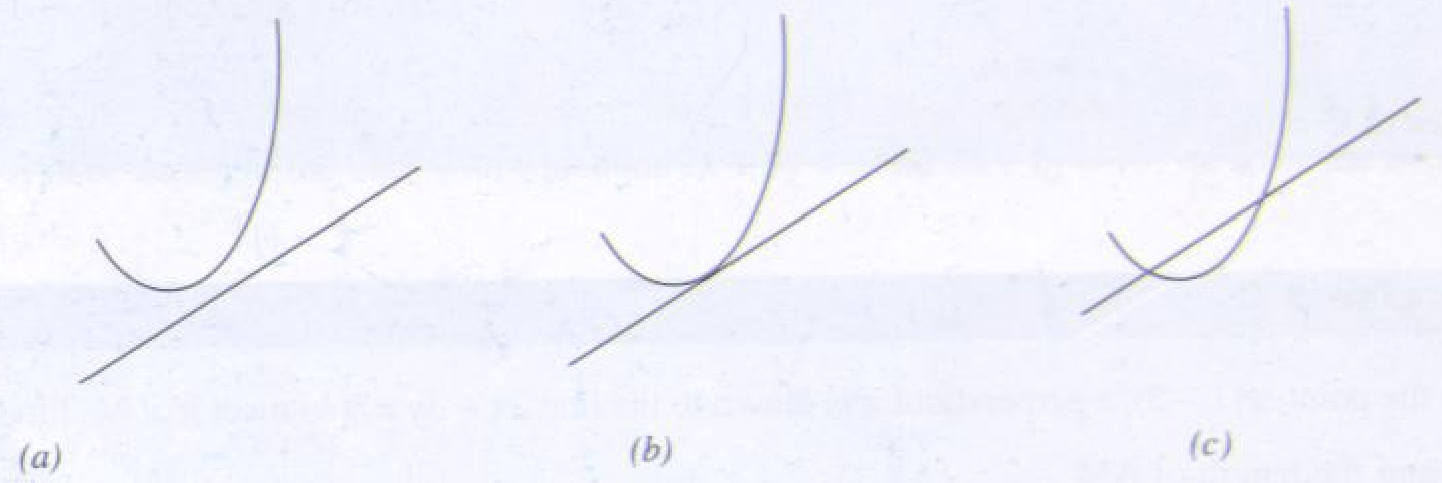


Figure 3.4

For a quadratic curve $y = ax^2 + bx + c$ and a line $y = px + q$, we obtain:

$$ax^2 + bx + c = px + q \text{ at points of intersection}$$

$$ax^2 + x(b - p) + c - q = 0$$

This is a quadratic equation which has 2 unequal roots (thus giving 2 points of intersection) if the discriminant > 0, i.e.:

$$(b - p)^2 - 4a(c - q) > 0$$

In this case, the line cuts the curve in two distinct points.

It has 2 equal roots giving only one point of intersection if discriminant = 0, in which case the line is a tangent to the curve.

It has no real roots giving no point of intersection if discriminant < 0.

These cases are illustrated in Figure 3.5.

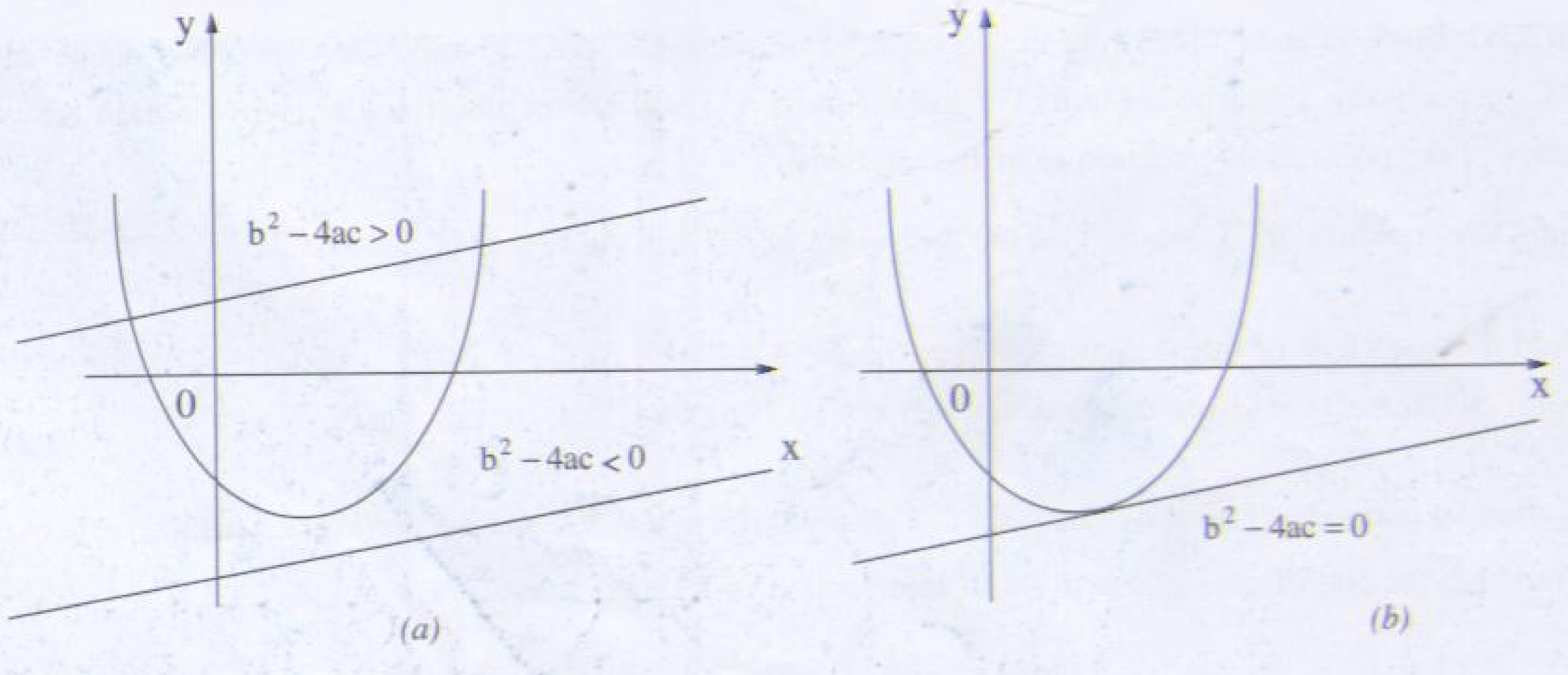


Figure 3.5

Example 18

Find the coordinates of the points of intersection of the straight line $x + y = 4$ *and the curve* $\frac{(x+1)^2}{8} + \frac{(y+4)^2}{5} = 7$

Solution

We solve $x + y = 4$ (1) and $\frac{(x+1)^2}{8} + \frac{(y+4)^2}{5} = 7$ (2)

From (1) $y = 4 - x$ (A)

Substituting in (2)

$$\frac{(x+1)^2}{8} + \frac{(4-x+4)^2}{5} = 7$$

$$\frac{x^2+2x+1}{8} + \frac{64-16x+x^2}{5} = 7$$

$$5x^2 + 10x + 5 + 512 - 128x + 8x^2 = 280$$

$$13x^2 - 118x + 237 = 0$$

$$(x-3)(13x-79) = 0$$

$$x = 3 \text{ or } \frac{79}{13}$$

$x = 3$, $y = 4 - 3$ from (A)

$= 1$

$x = \frac{79}{13}, y = 4 - \frac{79}{13}$

$= -\frac{27}{13}$

The line intersects the curve at (3, 1), $\left(\frac{79}{13}, -\frac{27}{13}\right)$.

Example 19

Find the value of m, given that the line $y = mx + 1$ *is a tangent to the curve* $y^2 = 8x$.

Solution

Solving the two equations, we have $(mx + 1)^2 = 8x$

$m^2x^2 + 2mx + 1 = 8x$

$m^2x^2 + (2m - 8)x + 1 = 0$

This equation is a quadratic equation which has 2 real roots if the line cuts the curve in 2 points, 2 equal roots if the line is a tangent to the curve and non-real or 'imaginary' roots if the line does not intersect the curve.

In this case, roots are equal.

$$b^2 - 4ac = 0$$
$$(2m - 8)^2 - 4m^2 = 0$$
$$4m^2 - 32m + 64 - 4m^2 = 0$$
$$m = 2$$

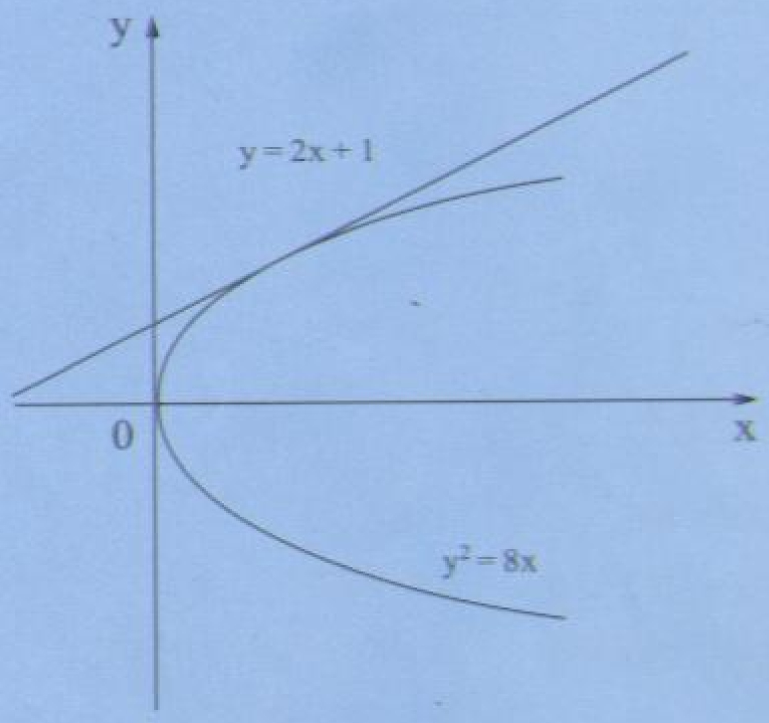


Figure 3.6

Example 20

The line y = 2x + 3 intersects the curve y = x² + 7x + m in two distinct points. Find the range of values of m.

Solution

Solving $y = 2x + 3$ and $y = x^2 + 7x + m$

$$x^2 + 7x + m = 2x + 3$$
$$x^2 + 5x + (m - 3) = 0$$

This quadratic has 2 real distinct roots if

$$b^2 - 4ac > 0$$
$$b^2 > 4ac$$
$$25 > 4(m - 3)$$
$$25 + 12 > 4m$$
$$m < \frac{37}{4}$$
$$m < 9\frac{1}{4}$$

Exercise 3 D

1. Find the coordinates of the point(s) of intersection of the following curves with the x-axis:
   (a) $y = x^2 + x - 6$
   (b) $y = 2x^2 - x - 3$
   (c) $y = x^2 - 9x + 18$
   (d) $y = (x + 1)(x - 2)^2$
   (e) $y = 2x^3 - 16x^2$

2. The line $y = x - 2$ cuts the curve $y = x^2 + 2x - 14$ at A and B. Find the length of AB.

3. The line $x + y = 5$ intersects the curve $\frac{(x+1)^2}{4} - \frac{(y+3)^2}{5} = -1$ at A and B. Find the length of AB.

4. Find the range of values of m for which the line $2y + x = m$ intersects the curve $x^2 + xy + 8 = 0$ in two distinct points.

5. Find the values of m for which the line $y = 8$ is a tangent to the curve $y = x^2 + (m + 3)x + (5m - 1)$.

6. Find the coordinates of the point(s) of intersection of:
   (a) $y = 6 - x - x^2$ and $y = x^2 + x + 2$
   (b) $y = x^2$ and $y^2 = 4x$
   (c) $y = x^2 + x - 3$ and $y = x^2 - x + 1$
   (d) $y = 8 + x - 2x^2$ and $y = x^2 - 9x + 11$
   (e) $y^2 = 9x$ and $x^2 = 9y$.

Miscellaneous Exercise 3

1. The point A has coordinates (7, 4). The straight lines with equations $x + 3y + 1 = 0$ and $2x + 5y = 0$ intersect at the point B. Find the coordinates of B and hence show that one of these two lines is perpendicular to AB. [C]

2. The line $y = 2x + 3$ intersects the y-axis at A. The points B and C on this line are such that AB = BC. The line through B perpendicular to AC passes through the point D(–1, 6). Calculate:
   (a) the equation of BD
   (b) the coordinates of B
   (c) the coordinates of C. [C]

3. The line $y = ax + 7$ is parallel to the line $y = 2x - 3$.
   The line $y = bx + 7$ is perpendicular to the line $y = 2x - 3$.
   (a) State the value of a and of b
   (b) Calculate the perpendicular distance between the pair of parallel lines. [C]

4. A has coordinates (1, 1) and B(–1, 4). C is a point in the plane and the gradients of AB, AC and BC are –3m, 3m and m respectively.
   (a) Find the value of m
   (b) Find the coordinates of C
   (c) Show that AC = 2AB. [C]

5. (a) Find the value of p for which the line $y = 6$ is a tangent to the curve
$y = x^2 + (1 - p)x + 2p$
(b) Find the range of values of q for which the line $x + 2y = q$ meets the curve
$x(x + y) + 8 = 0$ [C]

6. $A(2t, t + 1)$, $B(4t, t + 5)$, $C(10 + t, 0)$, $O(0, 0)$ are 4 points. Given that t only takes positive values, calculate the value of t for which the mid-point of AB is equidistant from O and C. [C]

7. In a rectangle ABCD, A and B are the points (4, 2) and (2, 8) respectively. Given that the equation of AC is $y = x - 2$, find:
(a) the equation of BC
(b) the coordinates of C
(c) the coordinates of D
(d) the area of the rectangle ABCD. [C]

8. The line $2x + 3y = 1$ intersects the curve $x(x + y) = 10$ at A and B. Calculate the coordinates of the mid-point of AB. [C]

9. ABCD is a parallelogram, lettered anticlockwise, such that A and C are the points (–1, 5) and (5, 1) respectively. Find the coordinates of the mid-point of AC.
Given that BD is parallel to the line whose equation is $y + 5x = 2$, find the equation of BD.
Given that BC is perpendicular to AC, find the equation of BC.
Calculate:
(a) the coordinates of B
(b) the coordinates of D [C]

10. (a) Given that the straight line $y = c - 3x$ does not intersect the curve $xy = 3$, find the range of values of c.
(b) Find the value of k for which the line $y + 3x = k$ is a tangent to the curve $y = x^2 + 5$. [C]

CHAPTER 4

> CIRCULAR MEASURE

4.1 Circular Measure

4.1.1 The radian

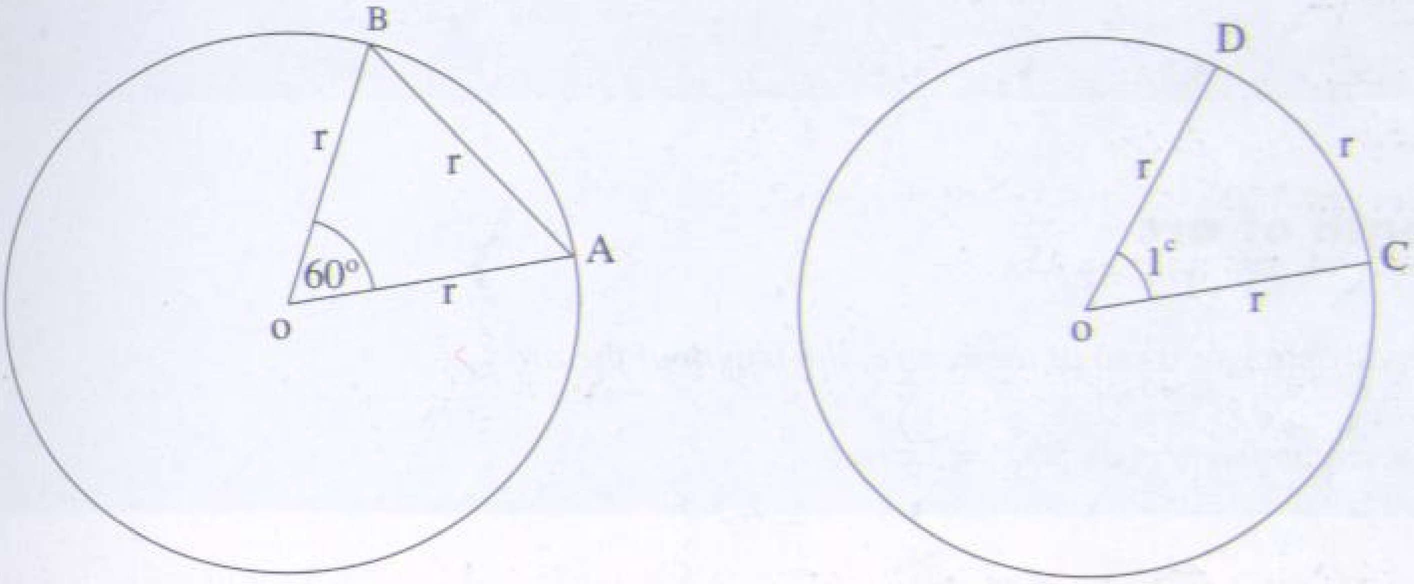


Figure 4.1

Consider a circle O with circumference 20 cm. If A and B are points on the circle and arc AB is of length 4 cm, the angle AOB = $\frac{4}{20} \times 360° = 72°$.

Generally, if the circumference of a circle is of length C units and the length of an arc is L units, the angle subtended by the arc at the centre of the circle = $\frac{L}{C} \times 360°$.

Next, consider a circle of radius r. Its circumference $C = 2\pi r$. If arc AB is of length equal to r, the angle which arc AB subtends at the centre of the circle is $\frac{r}{2\pi r} \times 360° = \frac{180°}{\pi}$.

This angle is a constant, it is called a *radian*.

The radian is the angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.

The symbol for radian is rad.

We write $1 \text{ rad} = \frac{180°}{\pi}$

or $\pi \text{ rad} = 180°$

Note: Care must be taken not to write $\pi = 180°$.

As the value of π is not exact, a radian cannot be obtained exactly in degrees.

It is approximately 57.3°.

Example 1

Convert $\frac{\pi}{8}$ radian to degrees and convert 60° to radians.

Solution

$$\pi \text{ rad} = 180°$$
$$\frac{\pi}{8} \text{ rad} = \frac{180°}{8}$$
$$= 22.5°$$
$$180° = \pi \text{ rad}$$
$$60° = \frac{\pi}{180} \times 60 \text{ rad} = \frac{\pi}{3} \text{ rad}$$

4.1.2 Length of arc

If an arc subtends an angle θ rad *at the centre, the length of the arc*

$$= \frac{\theta}{2\pi} \times \textit{circumference, as } 360° = 2\pi \textit{ rad}$$

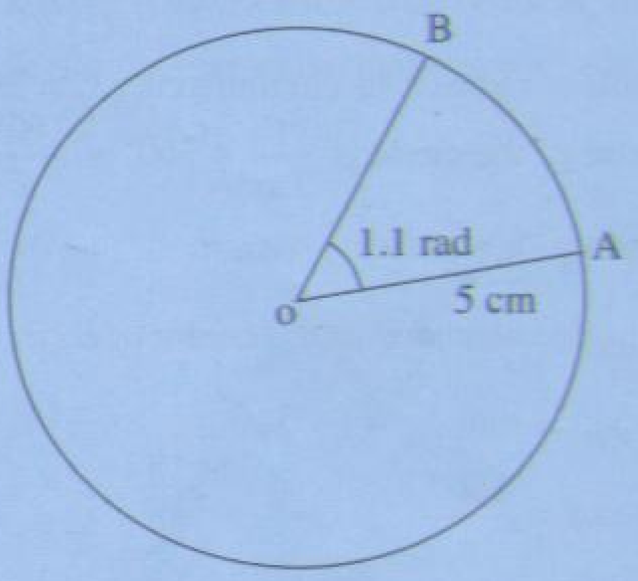
$$= \frac{\theta}{2\pi} \times 2\pi r \textit{ where } r \textit{ is the radius}$$

$$= r\theta .$$

This formula is remembered as $s = r\theta$ where s is the length of arc, r is the radius and θ is the angle in radians subtended by the arc at the centre of the circle.

Example 2

In the given diagram, ∠AOB = 1.1 radian and the radius is 5 cm. Find the perimeter of the sector AOB.



Solution

Using $s = r\theta$

Arc AB = 5 × 1.1 cm

= 5.5 cm

Perimeter of AOB = OA + arc AB + BO

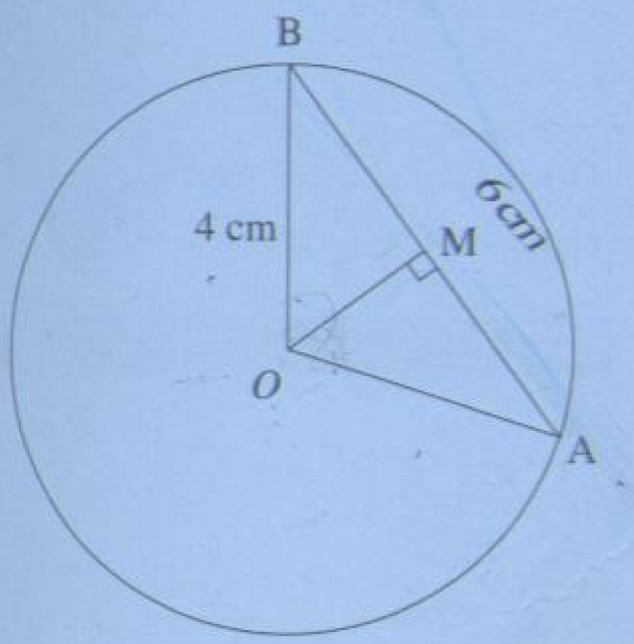
= 5 cm + 5.5 cm + 5 cm

= 15.5 cm

Example 3

AB is a chord of a circle centre O and radius 4 cm. Given AB = 6 cm, find the length of arc AB.

Solution



We find ∠AOB

As triangle OAB is isosceles, we draw the line of symmetry OM.

$$\sin \angle AOM = \frac{AM}{OA} = \frac{3}{4}$$

$$\angle AOM = \sin^{-1} \frac{3}{4} = 0.848 \text{ ... rad (calculator)}$$

$$\angle AOB = 2\sin^{-1} \frac{3}{4} = 1.696 \text{ rad}$$

$$\text{Arc } AB = r\theta$$

$$= 4 \times 1.696$$

$$= 6.78 \text{ cm.}$$

Exercise 4 A

1. Convert to degrees:

(a) $\frac{\pi}{6}$ rad (b) $\frac{2\pi}{3}$ rad (c) $\frac{5\pi}{4}$ rad (d) $\frac{3\pi}{8}$ rad (e) $\frac{11\pi}{16}$ rad

(f) 0.4 rad (g) 2.5 rad (h) 3.9 rad (i) 1.3 rad (j) 0.82 rad

2. Convert to radians leaving your answers in terms of :

(a) 90° (b) 150° (c) 210° (d) 75° (e) 67.5°

(f) 140° (g) 28° (h) 37.1° (i) 138.6° (j) 290.6°

3. Find the length of an arc of a circle which subtends an angle of:

(a) $\frac{2\pi}{5}$ radians at the centre of a circle of radius 5 cm.

(b) $\frac{5\pi}{8}$ radians at the centre of a circle of radius 12 cm.

(c) 1.9 radians at the centre of a circle of radius 6.4 cm.

(d) 0.5 radian at the centre of a circle of radius 12.5 cm.

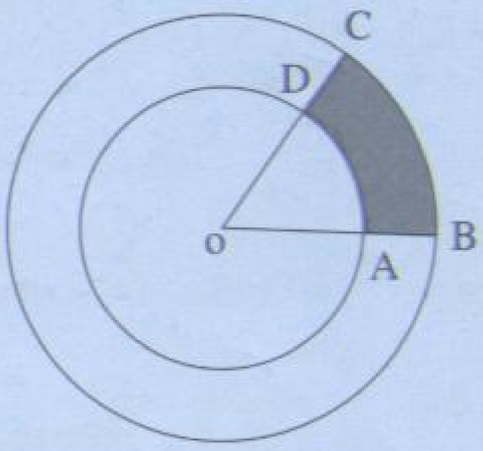
(e) 1.56 radians at the centre of a circle of radius 4.3 cm.

4. An arc of a circle of radius 8 cm subtends an angle of 0.8 radian at a point on the remaining part of the circle. Find the length of the arc.

5. A chord AB of a circle subtends an angle of $\frac{3\pi}{4}$ radians at the centre. Find the ratio of the length of the minor arc AB to the length of the chord AB.

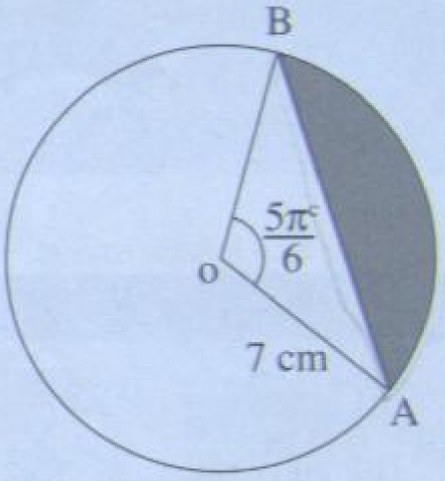
6. Find the perimeter of each of the shaded shapes:

(a)



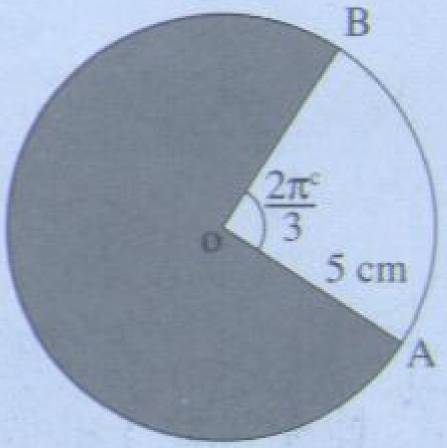
OA = 4 cm, AB = 3 cm and $\angle BOC = \frac{\pi}{4}$ rad

(b)



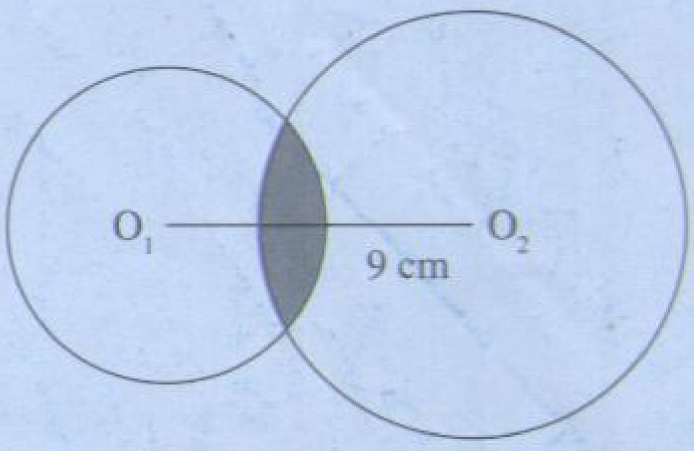
OA = 7 cm, $\angle AOB = \frac{5\pi}{6}$ rad

(c)



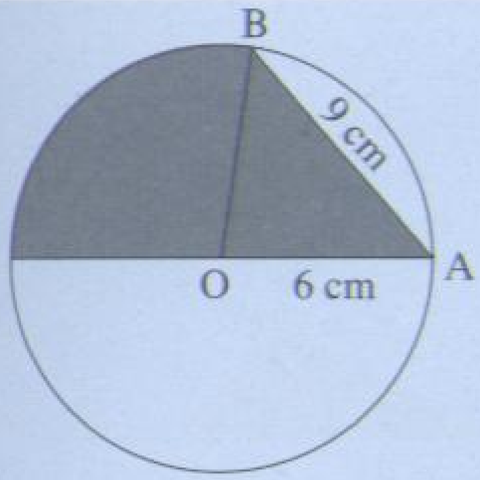
OA = 5 cm, obtuse $\angle AOB = \frac{2\pi}{3}$ rad

(d)



O_1, O_2 are centres of circles of radius 5 cm and 7 cm respectively and $O_1O_2 = 9$ cm.

(e)



O is the centre of circle of radius 6 cm and chord AB is of length 9 cm.

4.2 The area of a sector

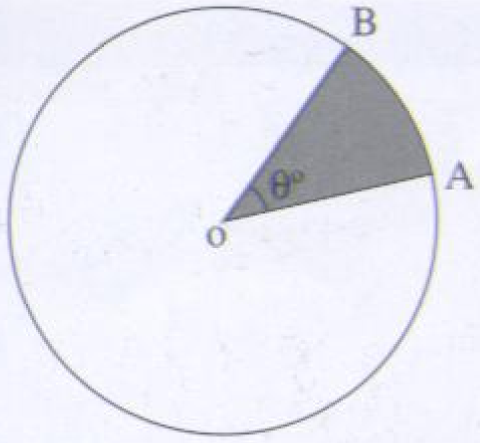


Figure 4.2

If ∠AOB = θ°, the area of sector AOB = $\dfrac{\theta}{360} \times \pi r^2$ where r is the radius of the circle.

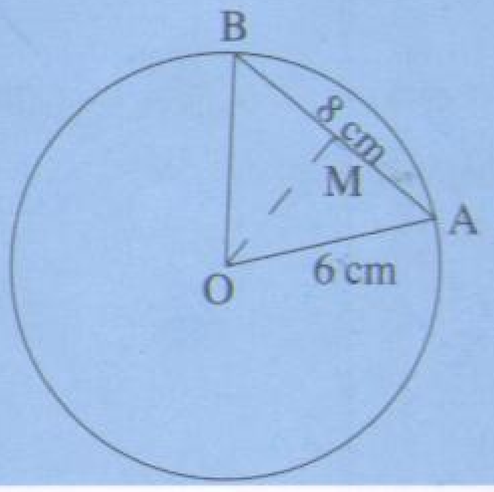
If ∠AOB = θ rad, the area of sector AOB = $\dfrac{\theta}{2\pi} \times \pi r^2 = \dfrac{1}{2} r^2 \theta$.

This formula is written as $A = \dfrac{1}{2} r^2 \theta$ where A is the area of a sector containing an angle θ radians at the centre.

Example 4

AB is a chord of a circle of radius 6 cm. Given that the chord AB is of length 8 cm, find the area of the minor segment AB.

Solution



OM is the perpendicular from O to AB

$$\sin \angle AOM = \frac{4}{6} = \frac{2}{3}$$

$$\angle AOM = \sin^{-1} \frac{2}{3}$$

$$\angle AOB = 2 \times \sin^{-1} \frac{2}{3}$$

$$= 1.459 \ldots \text{ rad (calculator)}$$

$$\text{Area of sector AOB} = \frac{1}{2} \times 6^2 \times 1.459 \text{ cm}^2$$
$$= 26.26 \text{ cm}^2 \text{ (calculator)}$$

$$\text{Area of triangle AOB} = \frac{1}{2} \times 6^2 \sin 1.459$$
$$= 17.89 \text{ cm}^2 \text{ (calculator)}$$

$$\text{Area of minor segment} = 26.26 - 17.89 \text{ cm}^2$$
$$= 8.37 \text{ cm}^2 \text{ (3 s.f)}$$

Exercise 4 B

1. Find the area of the sector containing an angle of:
   - (a) 0.8 rad at the centre of a circle of radius 6 cm.
   - (b) 1.2 rad at the centre of a circle of radius 9 cm.
   - (c) 2.5 rad at the centre of a circle of radius 12.6 cm.
   - (d) 1.8 rad at the centre of a circle of radius 9.4 cm.

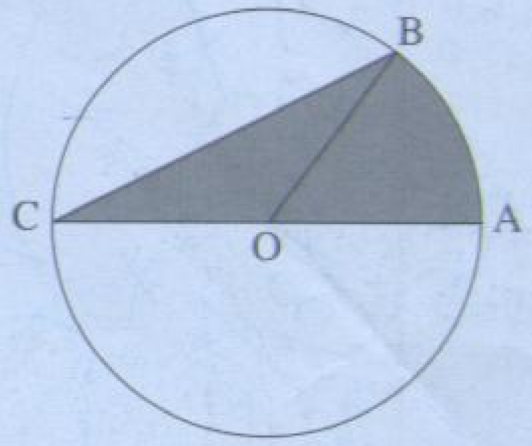
2. An arc of circle of radius 6 cm is 9 cm long. Find the area of the sector.

3. O is the centre of a circle and A, B are points on the circle. OA = 9 cm and the area of sector OAB = 8.82 cm^2. Find the area of the minor segment AB.

4. An arc AB of length 8 cm subtends an angle of 2 radians at the centre of the circle. Find the area of:
   - (a) the sector AOB
   - (b) the major segment AB.

5. Find the area of each of the shaded regions:

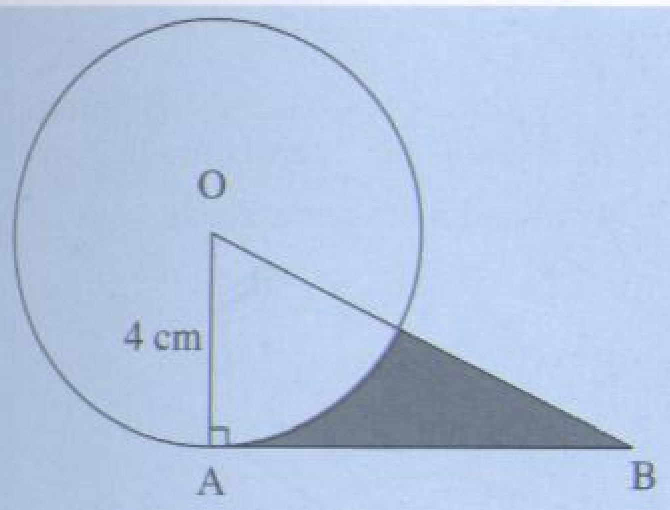
(a)



O is the centre of the circle,

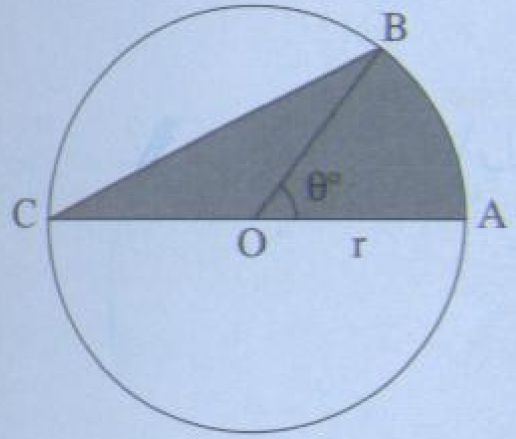
OA = 6 cm, ∠AOC = 1.6 radians.

(b)



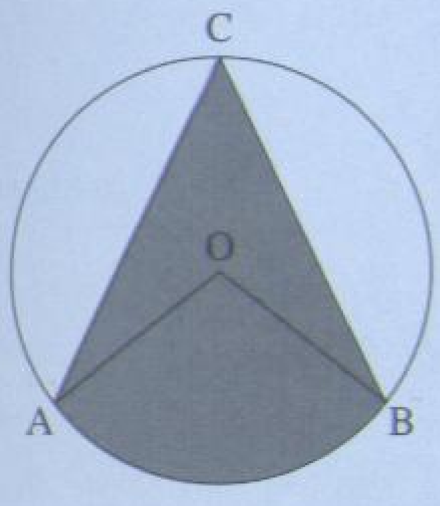
O is the centre of the circle.
OA = 4 cm and
$\angle AOC = 0.6$ radians.

(c)



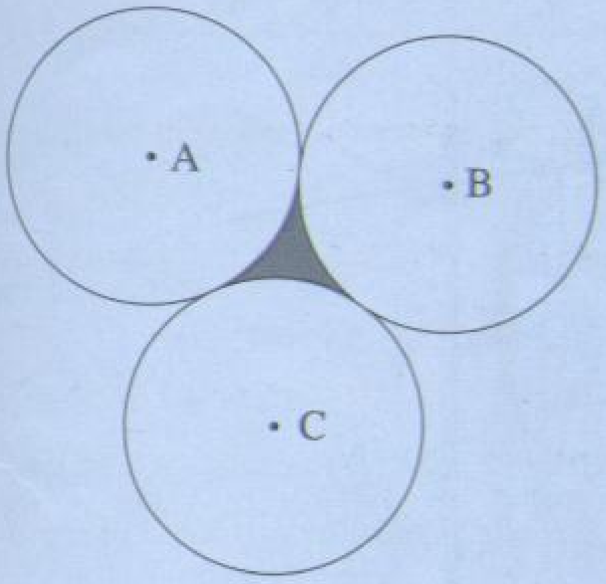
O is the centre of a circle of radius r units and $\angle AOB = \theta$ radians.

(d)



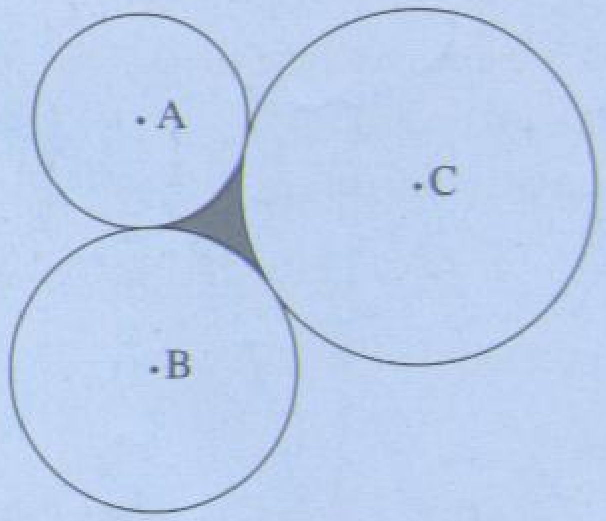
O is the centre of the circle of radius r.
$\angle AOB = \theta$ radians and CA = CB

(e)



A, B and C are centres of equal circles, each of radius 4 cm.

(f)

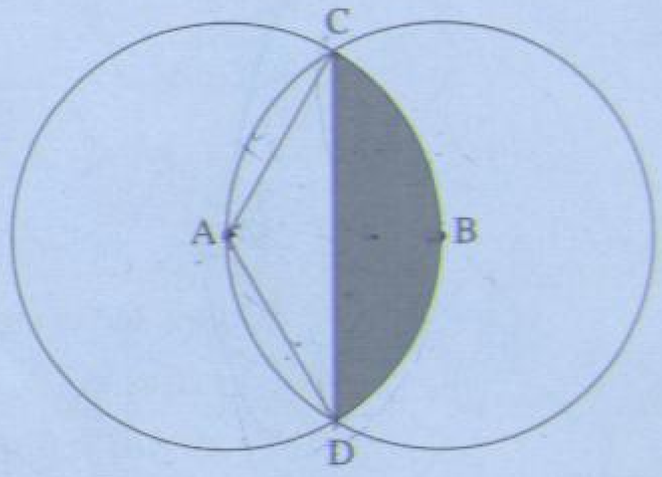


A, B and C are circles of radii 3 cm, 4 cm and 5 cm respectively.

Miscellaneous Exercise 4

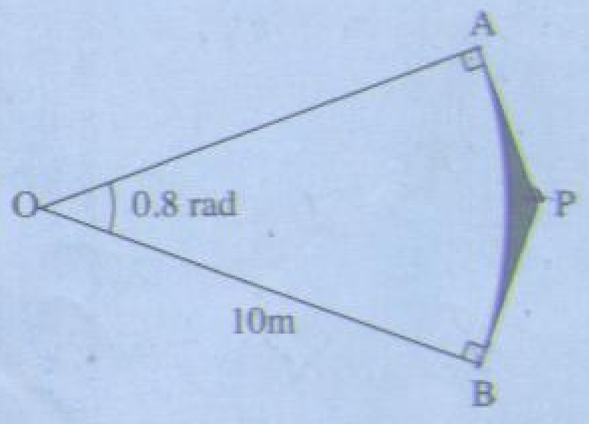
1. A sector OAB of a circle, of radius a and centre O, has $\angle AOB = \theta$ radians. Given that the area of the sector OAB is twice the square of the length of the arc AB, find θ. [C]

2.



The diagram shows two circles with centres A and B, intersecting at C and D in such a way that the centre of each lies on the circumference of the other. The radius of each circle is 1 unit. Write down the size of angle CAD and calculate the area of the shaded region bounded by the arc CBD and the straight line CD. Hence show that the area of the region common to the interiors of the two circles is approximately 39% of the area of one circle. [C]

3.

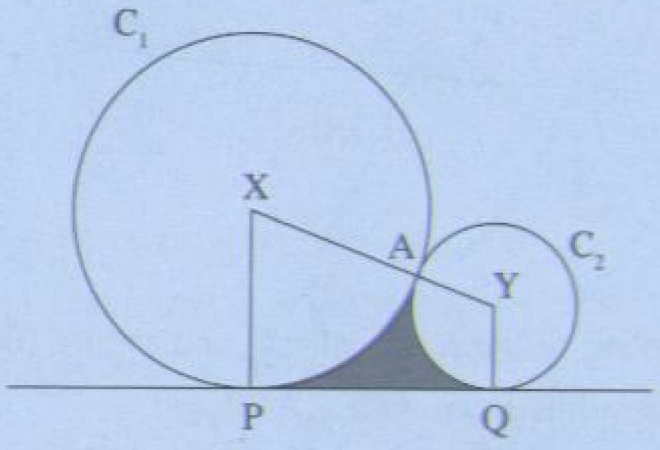


The diagram shows part of a circle, centre O, of radius 10 m. The tangents at the points A and B on the circumference of the circle meet at the point P and the angle AOB is 0.8 radian. Calculate:

(a) the perimeter of the shaded region

(b) the area of the shaded region. [C]

4.



The diagram shows two circles, C_1 and C_2, touching at A. Circle C_1 has radius 9 cm and centre X; circle C_2 has radius 4 cm and centre Y. A tangent touches the circles C_1 and C_2 at the points P and Q respectively. Calculate the length of PQ and show that angle PXY, to 3 decimal places, is 1.176 radians.

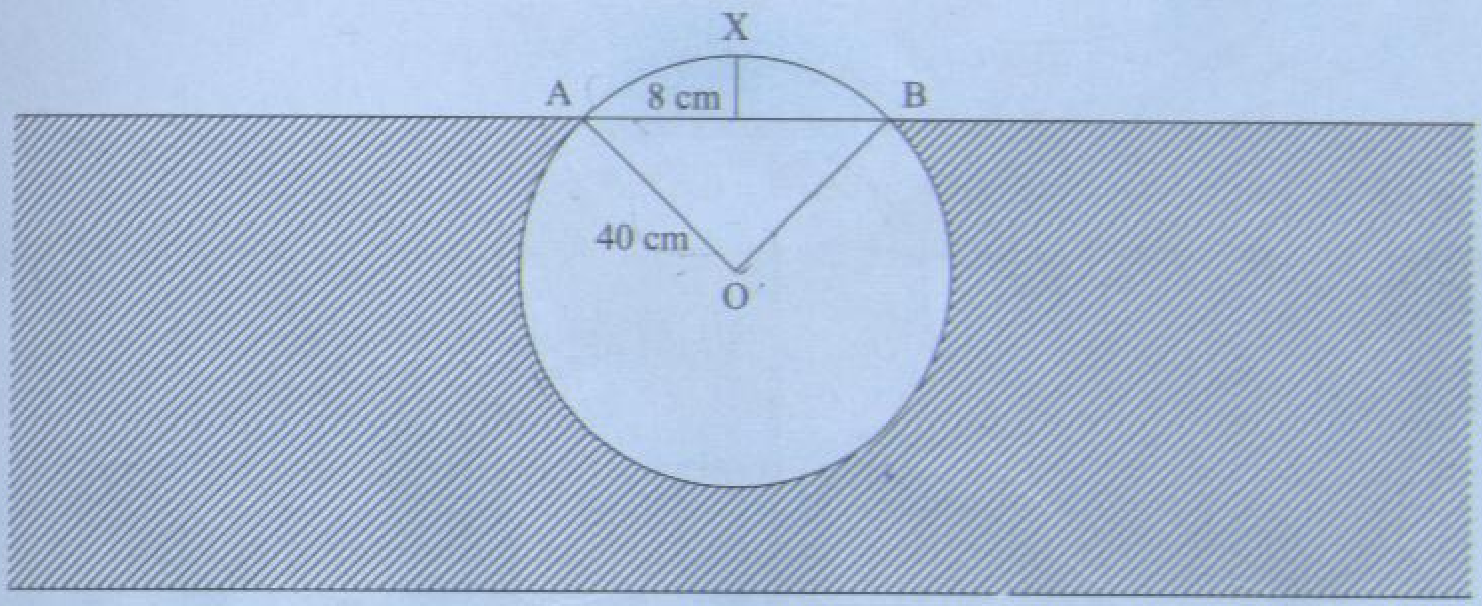
Find:

(a) the length of the minor arc AP of circle C_1,

(b) the length of the minor arc AQ of circle C_2,

(c) the area of the shaded region. [C]

5.



The figure shows the circular cross-section of a uniform log of radius 40 cm floating in water. The points A and B are on the surface and the highest point X is 8 cm above the surface.

Show that ∠AOB is approximately 1.29 radians.

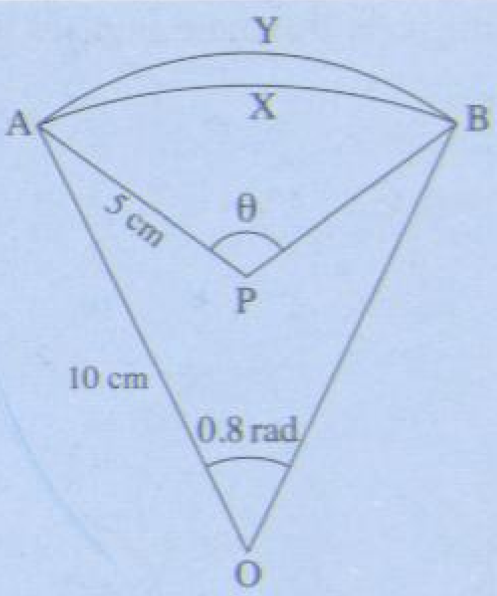
Calculate:

(a) the length of the arc AXB,

(b) the area of the cross-section below the surface

(c) the percentage of the volume of the log below the water. [C]

6.



In the given diagram AXB is an arc of a circle centre O and radius 10 cm with $\angle AOB = 0.8$ radian. AYB is an arc of a circle centre P and radius 5 cm with $\angle APB = \theta$.

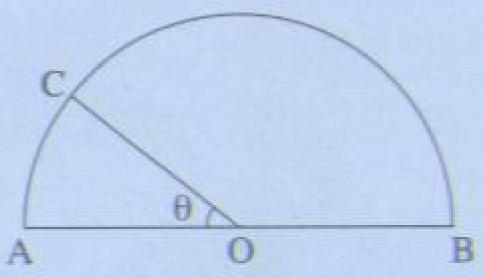
Calculate:

(a) the length of the chord AB

(b) the value of θ in radians

(c) the difference in length between the arcs AYB and AXB. [C]

7.

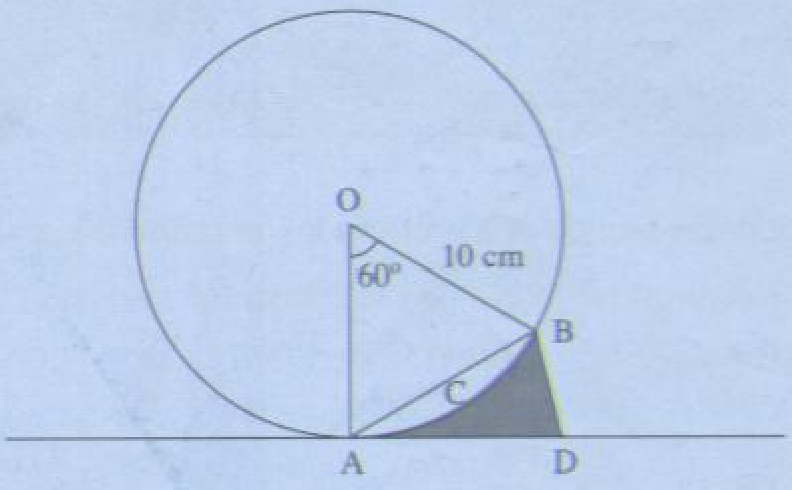


The diagram shows part of a circle centre O of radius 6 cm.

(a) Calculate the area of the sector BOC when $\theta = 0.8$ radian.

(b) Find the value of θ in radians for which the arc length BC is equal to the sum of the arc length CA and the diameter AB. [C]

8.



The diagram shows three points A, B and C on a circle, centre O and radius 10 cm. The line AD is a tangent to the circle. Given that angle $AOB = 60°$, find, to one decimal place,

(a) the length of the arc ACB

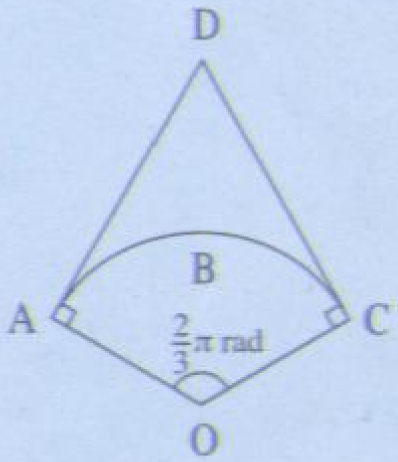
(b) the area of the segment ACB.

Given also that the length of AD equals the length of the arc ACB, find:

(c) the area of the shaded region ABCD

(d) the length of BD. [C]

9.



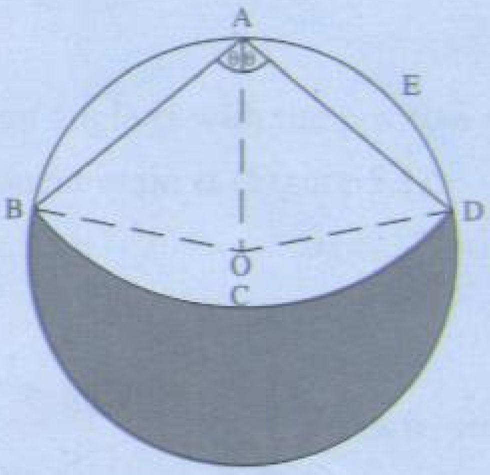
In the diagram, ABC is an arc of a circle with centre O and radius 5 cm. The lines AD and CD are tangents to the circle at A and C respectively. Angle AOC = $\frac{2}{3}\pi$ radians

(a) Show that the exact length of AD is $(5\sqrt{3})$ cm.

(b) Find the area of the sector AOC, giving your answer in terms of π.

(c) Calculate the area of the region enclosed by AD, DC and the arc ABC, giving your answer correct to 2 significant figures. [C]

10.



The diagram shows a circle with centre O and radius r. The three points A, B, D on the circle are such that $B\hat{A}O = D\hat{A}O = \theta$ radians, and BCD is an arc of the circle with centre A and radius AB. The point E lies on the arc AD. Find, in terms of r and , expressions for:

(a) the length of AD

(b) the area of the sector of the circle (centre A) bounded by AB and AD and the arc BCD.

(c) the area of triangle OAD

(d) the area of the segment ADE.

Hence, show that the area of the shaded region may be expressed as $r^2 (\sin 2\theta - 2\theta \cos 2\theta)$.

Deduce that, when $\theta = \frac{1}{4}\pi$, this region has an area approximately one third that of the circle with centre O.

[C]

CHAPTER 5

> TRIGONOMETRY 1

5.1 Trigonometrical ratios of angles of any magnitude

5.1.1 Acute angles

We are familiar with the three trigonometrical ratios sine, cosine and tangent.

We recall that in a right-angled triangle

$$\sin \alpha = \frac{\text{OPPOSITE}}{\text{HYPOTENUSE}}$$

$$\cos \alpha = \frac{\text{ADJACENT}}{\text{HYPOTENUSE}}$$

$$\tan \alpha = \frac{\text{OPPOSITE}}{\text{ADJACENT}}$$

If **OP** is a vector of length 1 unit making an angle α with the positive direction of the x-axis, the x-component = $\cos \alpha$ and the y-component = $\sin \alpha$ (Figure 5.1).

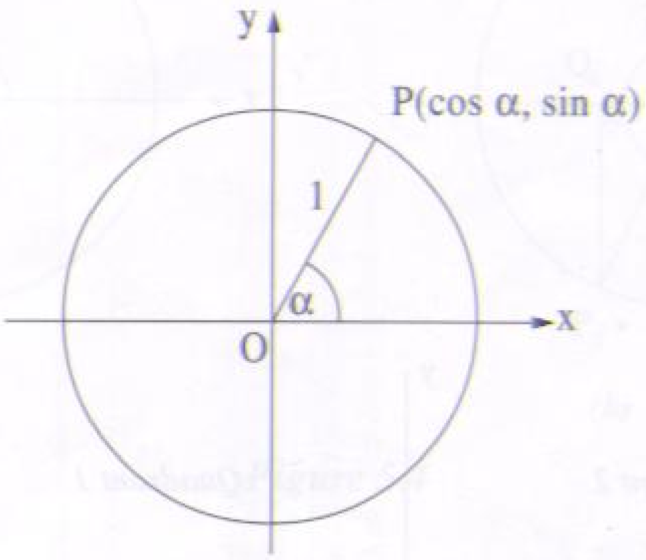
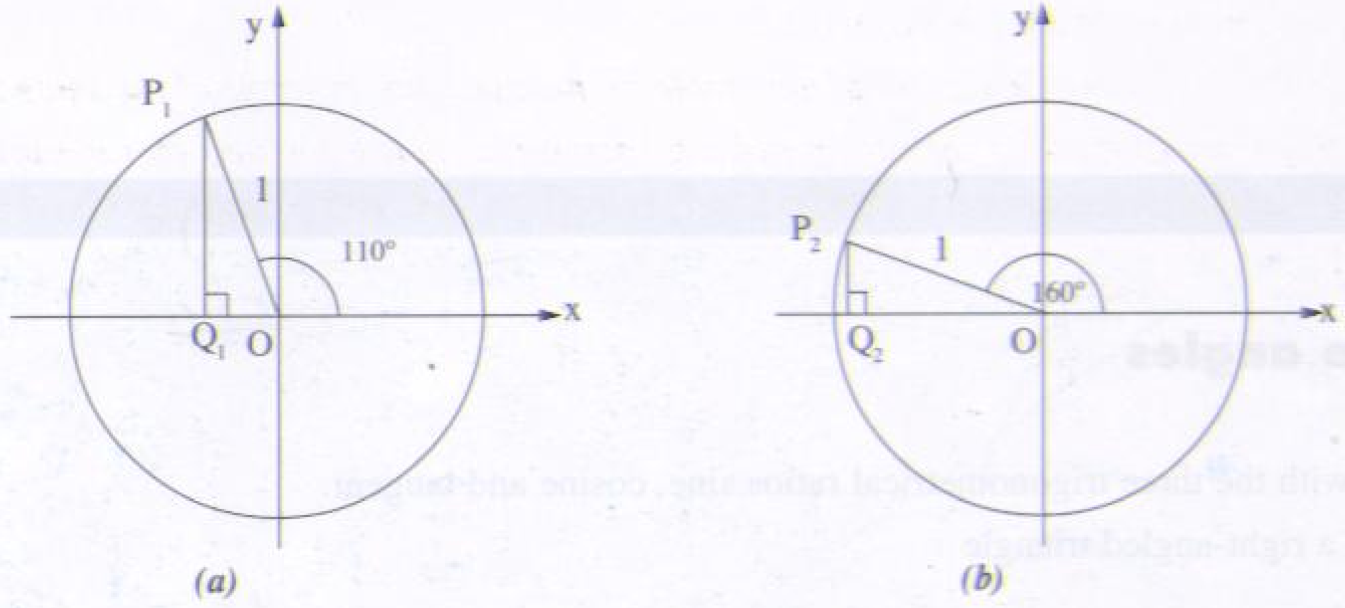


Figure 5.1

It follows that the coordinates of **P** are $(\cos \alpha, \sin \alpha)$.

5.1.2 Angles between 90° and 180°



(a) *(b)*

Figure 5.2

If $\mathbf{OP_1}$ and $\mathbf{OP_2}$ are vectors of unit length making angles of 110° and 160° respectively with the positive direction of the x-axis, the coordinates of P_1 and P_2 are (cos 110°, sin 110°) and (cos 160°, sin 160°) respectively. The coordinates of P_1 and P_2 can also be written as (–cos 70°, sin 70°) and (–cos 20°, sin 20°) respectively as OQ_1 = cos 70°, Q_1P_1 = sin 70°, OQ_2 = cos 20°, Q_2P_2 = sin 20°.

It follows that $\cos 110° = -\cos 70°$

$\cos 160° = -\cos 20°$

$\sin 110° = \sin 70°$

$\sin 160° = \sin 20°$

It can be checked that cos 132° = –cos 48°, sin 132° = sin 48°, cos 143° = –cos 37° and sin 143° = sin 37°. In general, $\cos \alpha° = -\cos (180 - \alpha)°$ and $\sin \alpha° = \sin (180 - \alpha)°$

$$\text{Also, } \tan (180 - \alpha)° = \frac{\sin(180 - \alpha)°}{\cos(180 - \alpha)°}$$

$$= \frac{\sin\alpha°}{-\cos\alpha°}$$

$$= -\tan \alpha°$$

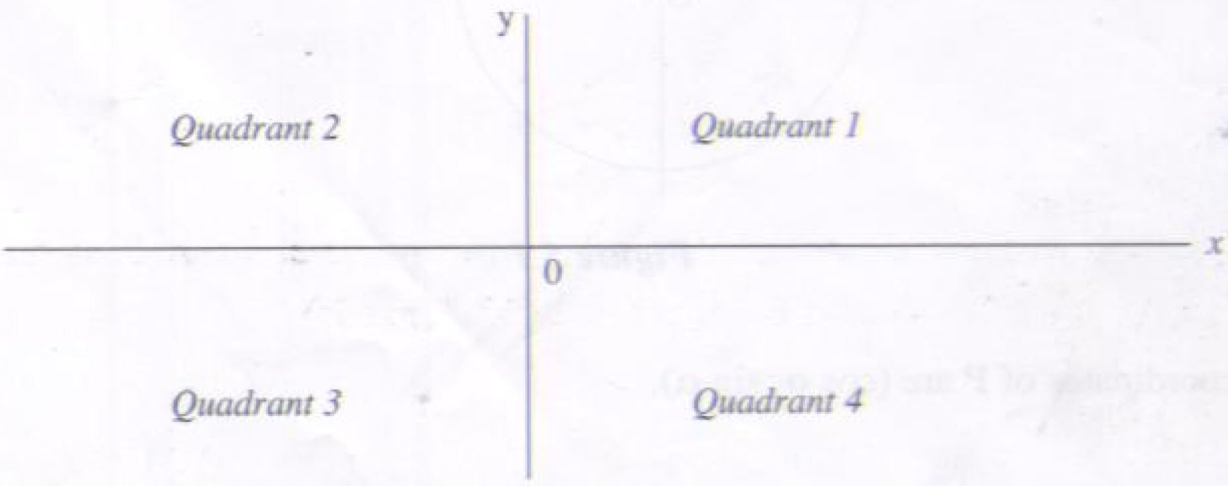


Figure 5.3

The axes of coordinates divide the cartesian plane into four quadrants as shown in Figure 5.3.

An angle between 90° and 180° is found in quadrant 2.

The sine of any angle in this quadrant is positive.

Note: We can find the trigonometrical ratio of any angle in this quadrant in terms of the same ratio of an acute angle by subtracting from 180° and using the appropriate algebraic sign, + for sin, – for cos and tan.

Example 1

Express as a ratio of an acute angle:

(a) sin 116° *(b) cos 166°* *(c) tan 138°*

Solution

(a) $\sin 116° = +\sin (180 - 116)°$
$= +\sin 64°$

(b) $\cos 166° = -\cos (180 - 166)°$
$= -\cos 14°$

(c) $\tan 138° = -\tan (180 - 138)°$
$= -\tan 42°$

5.1.3 Angles between 180° and 270°

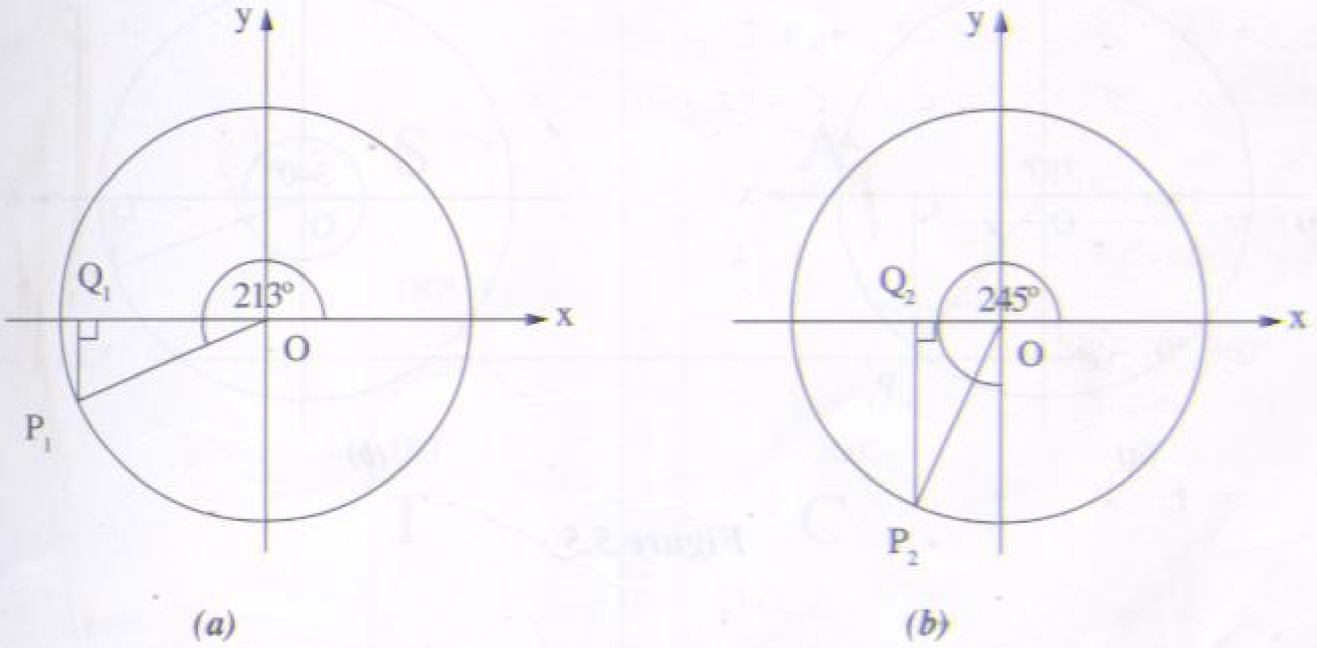


Figure 5.4

In Figure 5.4, we can write the coordinates of P_1 and P_2 in two different ways, P_1 as (cos 213°, sin 213°) or (–cos 33°, –sin 33°) and P_2 as (cos 245°, sin 245°) or (–cos 65°, –sin 65°).

Hence, $\cos 213° = -\cos 33°$
$\cos 245° = -\cos 65°$
$\sin 213° = -\sin 33°$
$\sin 245° = -\sin 65°$

As both sine and cosine are negative in quadrant 3, tangent is positive, i.e. tan 213° = + tan 33° and tan 245° = + tan 65°.

In quadrant 3, the trigonometrical ratio of any angle can be obtained in terms of the same ratio of an acute angle by subtracting 180° and using the appropriate algebraic sign, + for tan, – for sin and cos.

Example 2

Express as a ratio of an acute angle:

(a) sin 227° *(b) cos 218°* *(c) tan 193°*

Solution

(a) $\sin 227° = -\sin(227 - 180)°$
$= -\sin 47°$

(b) $\cos 218° = -\cos(218 - 180)°$
$= -\cos 38°$

(c) $\tan 193° = +\tan(193 - 180)°$
$= +\tan 13°$

5.1.4 Angles between 270° and 360°

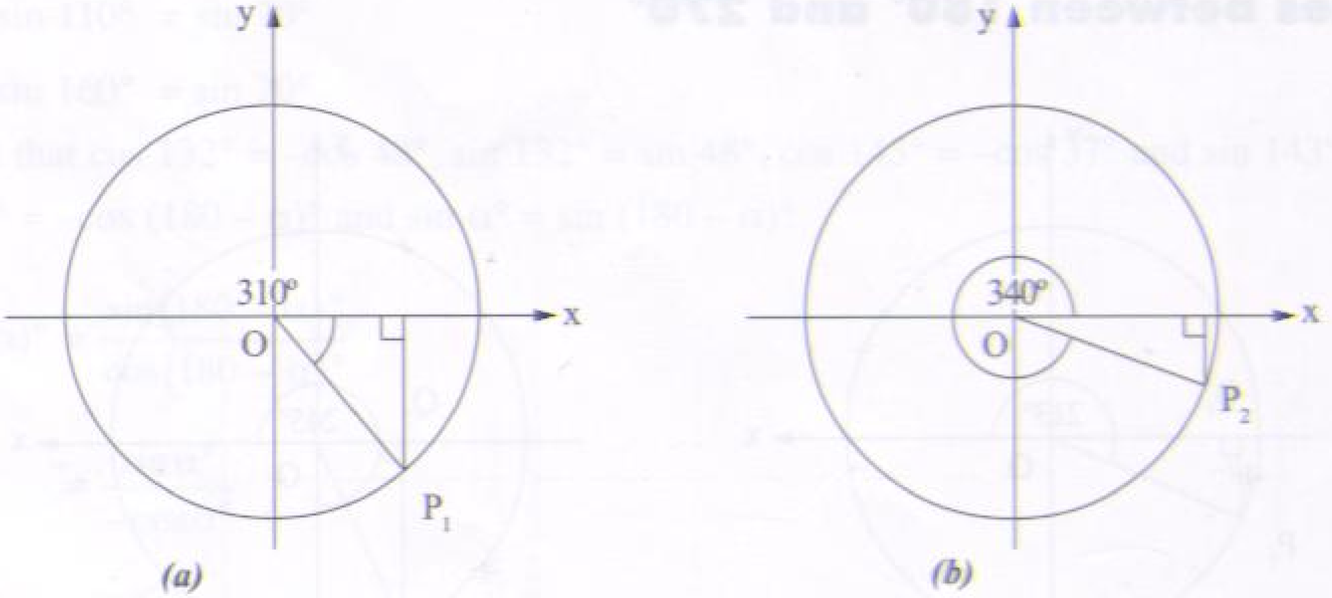


Figure 5.5

In Figure 5.5, the coordinates of P_1 can be written either as (cos 310°, sin 310°) or (cos 50°, –sin 50°) and the coordinates of P_2 as (cos 340°, sin 340°) or (cos 20°, – sin 20°).

Hence, $\cos 310° = +\cos 50°$
$\cos 340° = +\cos 20°$
$\sin 310° = -\sin 50°$
$\sin 340° = -\sin 20°$

As sine is negative and cosine is positive, tan is negative,
i.e. tan 310° = –tan 50° and tan 340° = –tan 20°.

In quadrant 4, the trigonometrical ratio of any angle can be obtained in terms of the same ratio of an acute angle by subtracting from 360° and using the appropriate algebraic sign, + for cos and – for sin and tan.

Example 3

Express each of the following ratios as a trigonometrical ratio of an acute angle:

(a) sin 331° *(b) cos 286°* *(c) tan 312°*

Solution

(a) $\sin 331^\circ = -\sin(360 - 331)^\circ$
$= -\sin 29^\circ$

(b) $\cos 286^\circ = +\cos(360 - 286)^\circ$
$= +\cos 74^\circ$

(c) $\tan 312^\circ = -\tan(360 - 312)^\circ$
$= -\tan 48^\circ$

5.1.5 The CAST diagram

To summarise, the trigonometric ratio of an angle of any magnitude can be obtained in terms of the same ratio of an acute angle from Figure 5.6.

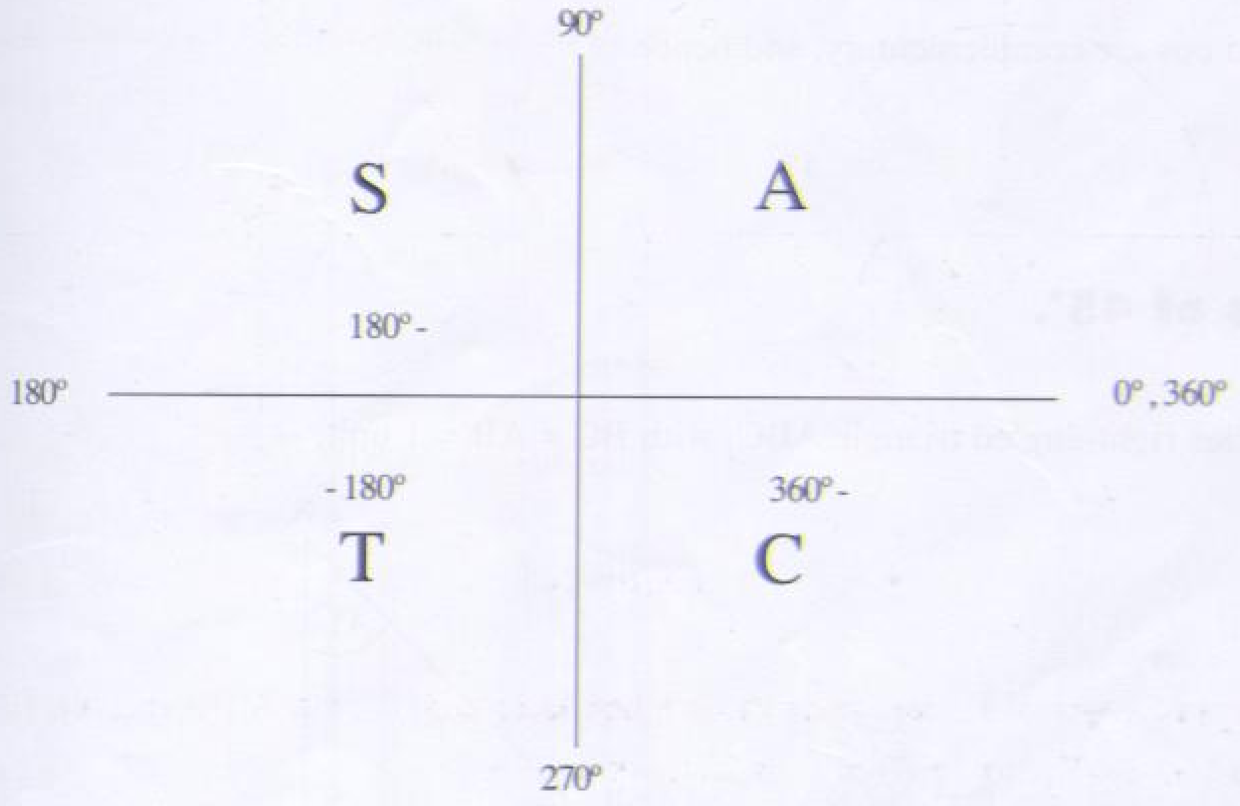
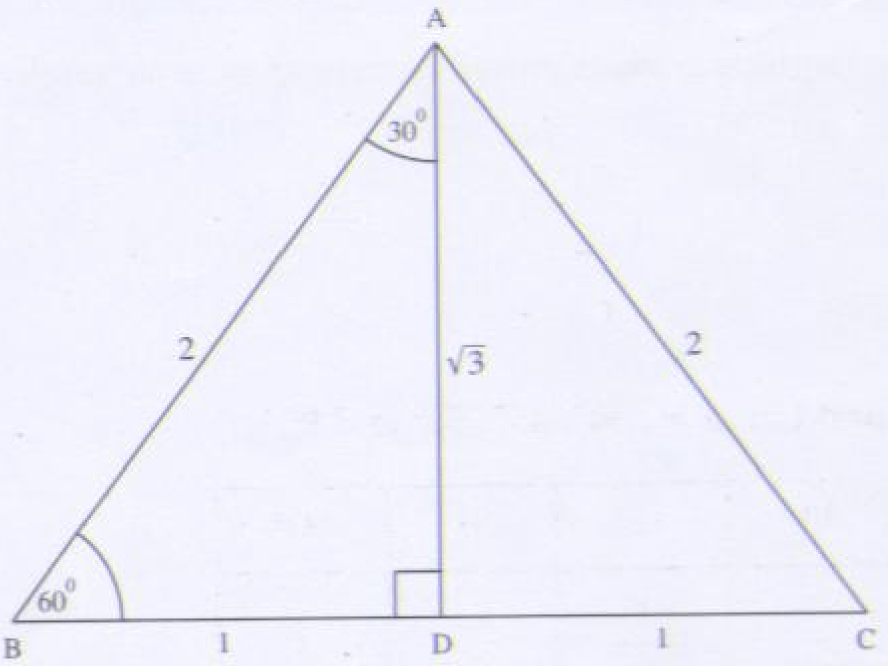


Figure 5.6

In the 1st quadrant, **A** stands for all, meaning all the ratios are positive.
In the 2nd quadrant, **S** stands for sine which is positive, in the 3rd quadrant, **T** for tan which is positive, in the 4th quadrant, **C** for cos which is positive.
180° – means subtract from 180°, **–180°** subtract 180° and **360° –** for subtract from 360°.

5.1.6 Angles of 60° and 30°.

Consider the equilateral triangle ABC having all its sides of length 2 units.



$$\sin 60° = \frac{AD}{AB} = \frac{\sqrt{3}}{2} \qquad \sin 30° = \frac{1}{2}$$

$$\cos 60° = \frac{1}{2} = \frac{1}{2} \qquad \cos 30° = \frac{\sqrt{3}}{2}$$

$$\tan 60° = \sqrt{3} \qquad \tan 30° = \frac{1}{\sqrt{3}}$$

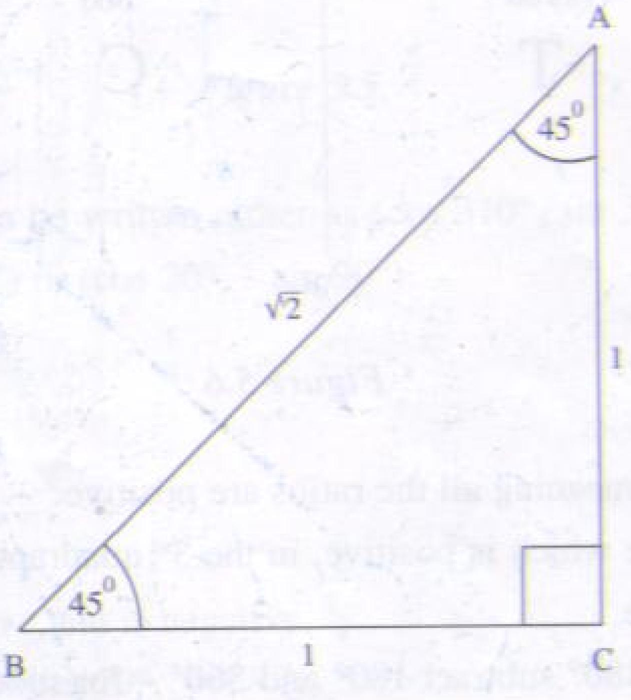
we note that sin and cos are complementary, and hence

$\sin 30° = \cos 60°$

$\sin 60° = \cos 30°$

5.1.7 Angles of 45°.

Consider the isosceles right-angled triangle ABC, with BC = AB = 1 unit.



$$AB^2 = AC^2 + BC^2$$
$$= 1^2 + 1^2$$
$$= 2$$

$$AB = \sqrt{2}$$
$$\sin 45° = \frac{1}{\sqrt{2}}$$
$$\cos 45° = \frac{1}{\sqrt{2}}$$
$$\tan 45° = 1$$

Table for sin, cos and tan of 0°, 30°, 45°, 60° and 90°

	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Example 4

Express each of the following as the same ratio of an acute angle:

(a) sin 210° *(b) cos 165°* *(c) tan 312°* *(d) sin 160°*
(e) cos 219° *(f) tan 312°* *(g) tan 226°* *(h) sin 315°*
(i) cos 346°

Solution

(a) Using figure 5.6, sin 210° < 0 as 210° is in quadrant 3.

So, $\sin 210° = -\sin(210° - 180°)$
$= -\sin 30°$

(b) $\cos 165° = -\cos(180° - 165°)$ *(cos is negative in quadrant 2)*
$= -\cos 15°$

(c) $\tan 312° = -\tan (360° - 312°)$ *(tan is negative in quadrant 4)*
$= -\tan 48°$

(d) $\sin 160° = +\sin (180° - 160°)$ *(sin is positive in quadrant 2)*
$= +\sin 20°$

(e) $\cos 219° = -\cos(219° - 180°)$
$= -\cos 39°$

(f) $\tan 226° = +\tan(226° - 180°)$
$= +\tan 46°$

(g) $\tan 146° = -\tan(180° - 146°)$
$= -\tan 34°$

(h) $\sin 315° = -\sin(360° - 315°)$
$= -\sin 45°$

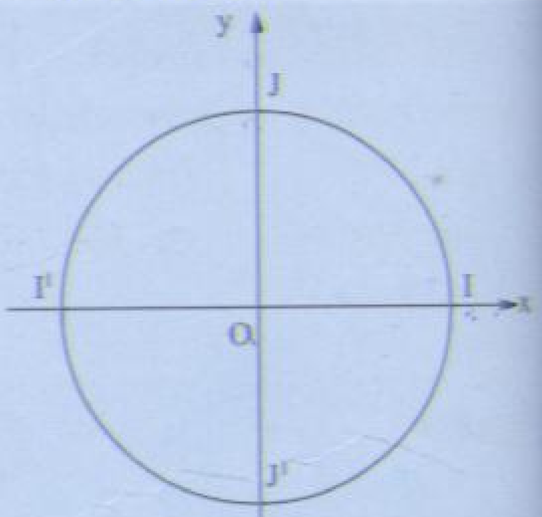
(i) $\cos 346° = +\cos(360° - 346°)$
$= +\cos 14°$

Exercise 5 A

1. Find each of the following ratios in terms of the same ratio of an acute angle:

(a) sin 168°	(b) cos 342°	(c) tan 224°	(d) sin 219°
(e) cos 209°	(f) tan 316°	(g) sin 301°	(h) cos 113°
(i) cos 108°	(j) tan 119°	(k) sin 352°	(l) cos 272°
(m) tan 207°	(n) sin 136°	(o) cos 303°	(p) tan 141°

2. In the given diagram, O is the centre of a circle of radius 1 unit. **QI**, **OJ**, **OI^1** and **OJ^1** make angles of 0°, 90°, 180° and 270° respectively with the positive direction of the x-axis.



Write the coordinates of each of I, J, I^1 and J^1 in two different ways and deduce the values of cos 0°, sin 0°, tan 0°, cos 90°, sin 90°, tan 90°, cos 180°, sin 180°, tan 180°, cos 270°, sin 270° and tan 270°.

3. Use the table on page 71 to obtain the exact values of:

(a) sin 135°	(b) cos 225°	(c) tan 315°
(d) sin 210°	(e) cos 330°	(f) tan 300°
(g) sin 240°	(h) cos 210°	(i) tan 330°
(j) sin 150°	(k) cos 120°	(l) tan 240°

5.2 Trigonometric ratios of angles greater than 360° or less than 0°

A rotation through any given angle is equivalent to a rotation obtained by a further rotation of any multiple of 360°, clockwise or anticlockwise.

Thus, a rotation through an angle of 30° is equivalent to a rotation of $30° + 360°$, $30° + 720°$, $30° - 360°$, $30° - 720°$, etc.

To obtain the ratio of any angle greater than 360°, we subtract 360° continuously until we obtain an angle between 0° and 360°. For an angle less than 0°, we add 360° continuously until we obtain an angle between 0° and 360°.

Example 5

Express each of the following as the same ratio of an acute angle:

(a) $\sin 1310°$ *(b)* $\cos 2050°$ *(c)* $\tan (-800°)$

(d) $\sin (-610°)$ *(e)* $\cos (-840°)$ *(f)* $\tan 910°$

Solution

(a)
$$\begin{aligned}\sin 1310° &= \sin (1310° - 360°)\\ &= \sin 950°\\ &= \sin (950° - 360°)\\ &= \sin 590°\\ &= \sin (590° - 360°)\\ &= \sin 230°\\ &= -\sin (230° - 180°)\\ &= -\sin 50°\end{aligned}$$

Note: It is preferable to use a calculator to obtain the angle between 0° and 360°.

(b)
$$\begin{aligned}\cos 2050° &= \cos 250° && \text{(subtracting 360° continuously)}\\ &= -\cos (250° - 180°)\\ &= -\cos 70°\end{aligned}$$

(c)
$$\begin{aligned}\tan (-800°) &= \tan 280° && \text{(adding 360° continuously)}\\ &= -\tan (360° - 280°)\\ &= -\tan 80°\end{aligned}$$

(d)
$$\begin{aligned}\cos (-840°) &= \cos 240°\\ &= -\cos (240° - 180°)\\ &= -\cos 60°\end{aligned}$$

(e)
$$\begin{aligned}\tan 910° &= \tan 190°\\ &= +\tan (190° - 180°)\\ &= +\tan 10°\end{aligned}$$

Note: It is advisable to refer to Figure 5.6 when finding the ratio of an angle between 0° and 360°.

Exercise 5 B

Find each of the following ratios in terms of the same ratio of an acute angle. If its value is exact, write down this value.

(a) $\sin 750$ (b) $\cos 855^\circ$ (c) $\tan 1815^\circ$ (d) $\sin 1860^\circ$

(e) $\cos 405^\circ$ (f) $\tan 1200^\circ$ (g) $\sin(-225^\circ)$ (h) $\cos(-600^\circ)$

(i) $\tan(-1325^\circ)$ (j) $\sin(-390^\circ)$ (k) $\cos(-1775^\circ)$ (l) $\tan(-855^\circ)$

5.3 Solutions of equations involving trigonometrical ratios

(a)

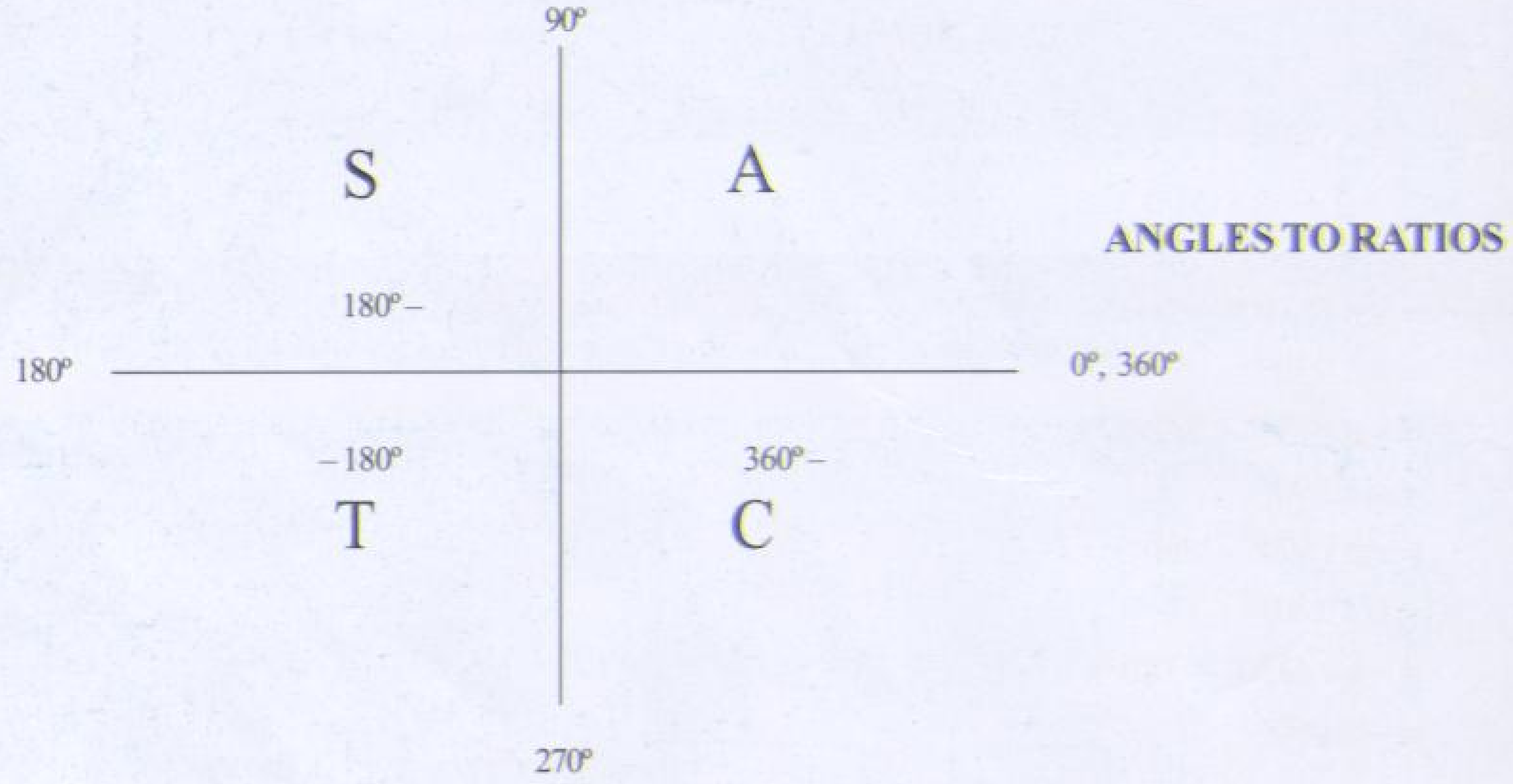


Figure 5.7(a)

(b)

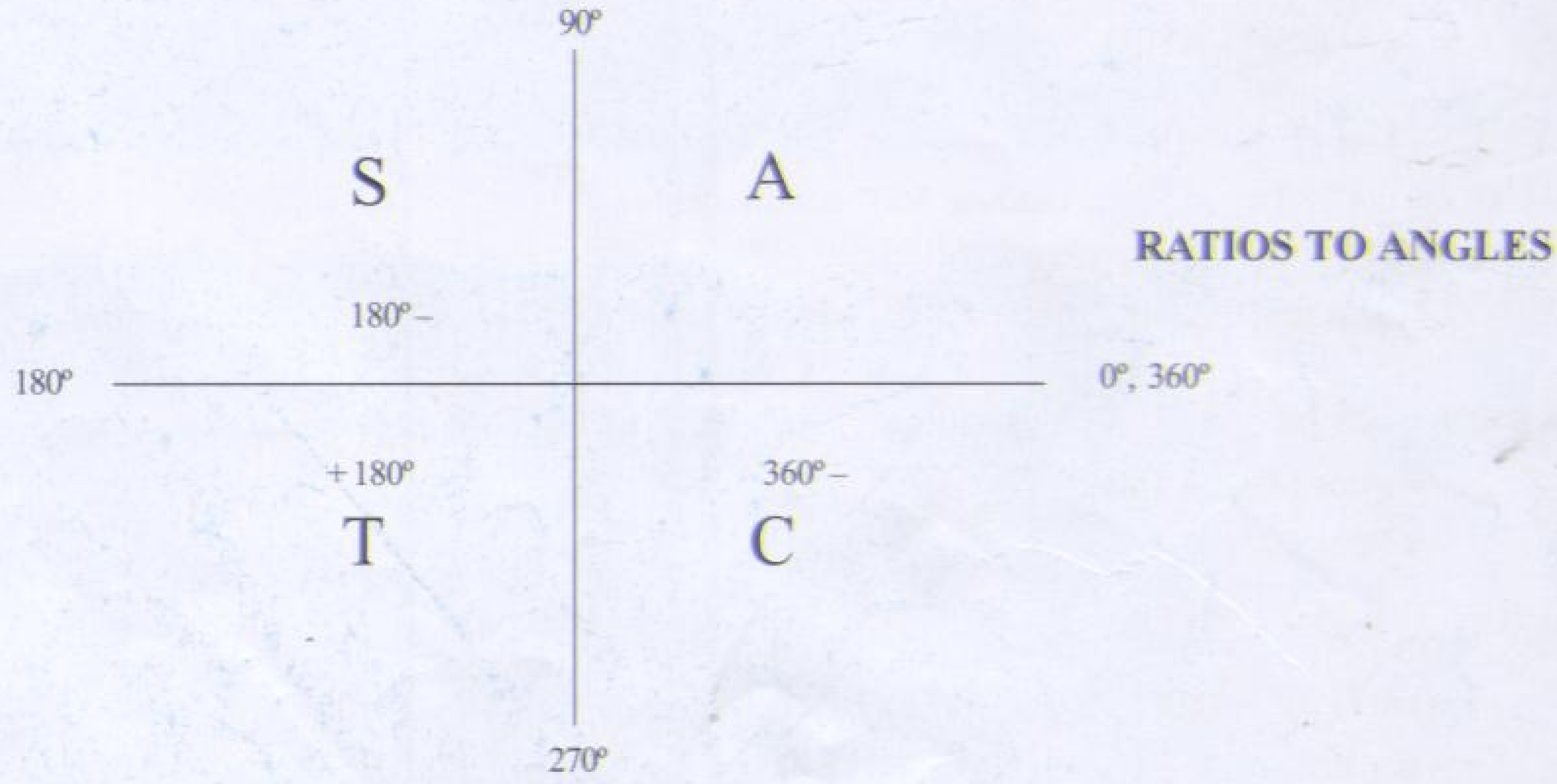


Figure 5.7(b)

5.3.1 Simple equations

To find the ratio of any angle, we use Figure 5.6 shown in Figure 5.7(a). To find the angle if we are given the value of the ratio, we use Figure 5.7(b) as illustrated in the following examples.

Example 6

Solve the equation $\sin\theta = -0.8660$ *for* $0° \leq \theta \leq 360°$

Solution

The – sign indicates that the angle is found in quadrant 3 or quadrant 4.
We consider the acute angle having its sine to be 0.8660. We refer to this angle as the *key angle* which is 60° in this case.

θ Q3 $180° + 60° = 240°$
Q4 $360° - 60° = 300°$

So, $\theta = 240°, 300°$

Example 7

Solve the equation $\cos\theta = -\frac{1}{\sqrt{2}}$ *for* $0° \leq \theta \leq 360°$

Solution

Key angle = 45°
θ is in quadrant 2 or 3
θ Q2 $180° - 45° = 135°$
Q3 $180° + 45° = 225°$

So, $\theta = 135°, 225°$

Example 8

Solve $\tan\theta = \frac{2}{9}$ *for* $-180° \leq \theta \leq 180°$

Solution

Key angle = 12.5° (calculator)
θ Q1 12.5°
Q3 $192.5° \equiv -167.5°$

So, $\theta = -167.5°, 12.5°$

Example 9

Solve $\sin\theta = -0.6$ *for* $-180° \leq \theta \leq 180°$

Solution

Key angle = 36.9° (calculator)

θ Q3 $180° + 36.9° = 216.9° \equiv -143.1°$

Q4 $360° - 36.9° = 323.1° \equiv -36.9°$

So, $\theta = -143.1°, -36.9°$

5.3.2 Harder equations

We consider equations involving multiples and fractions of θ.

Example 10

Solve $\cos 2\theta = -\frac{1}{\sqrt{2}}$ *for* $0° \leq \theta \leq 360°$

Solution

We find first the range of values of 2θ

As $0° \leq \theta \leq 360°$

$0° \leq 2\theta \leq 720°$

We take all the values of 2θ in this range.

Key angle = 45°

2θ Q2 $135°, 495°$

Q3 $225°, 1585°$

$\theta = 67.5°, 112.5°, 247.5°, 292.5°$

Example 11

Solve the equation $\tan\frac{1}{2}\theta = 1$ *for* $0° \leq \theta \leq 360°$

Solution

As $0° \leq \theta \leq 360°$

$0° \leq \frac{1}{2}\theta \leq 180°$

Key angle = 45°

$\frac{1}{2}\theta$ Q1 45°

Note: We ignore Q3 as $\frac{1}{2}\theta \leq 180°$

$\theta = 90°$

Example 12

Solve the equation $\sin(2\theta - 15°) = -\frac{1}{2}$ *for* $0° \leq \theta \leq 360°$

Solution

As $0° \leq \theta \leq 360°$

$-15° \leq 2\theta - 15° \leq 705°$

Key angle = 30°

$2\theta - 15$ Q3 $210°, 570°$

Q4 $330°, 690°$

$2\theta = 225°, 345°, 585°, 705°$

$\theta = 112.5°, 172.5°, 292.5°, 352.5°$

Example 13

Solve the equation $2\sin\theta = 3\tan\theta$ *for* $0° \leq \theta \leq 360°$

Solution

$2\sin\theta = 3\frac{\sin\theta}{\cos\theta}$

$2\sin\theta\cos\theta = 3\sin\theta$

$2\sin\theta\cos\theta - 3\sin\theta = 0$

$\sin\theta(2\cos\theta - 3) = 0$

either $\sin\theta = 0$ or $\cos\theta = \frac{3}{2}$ (inadmissible)

$\theta = 0°, 180°, 360°$

Example 14

Solve the equation $8\sin^2\theta + 2\cos\theta = 7$ *for* $-180° \leq \theta \leq 180°$

Solution

We use the identity $\cos^2\theta + \sin^2\theta = 1$ to obtain an equation containing $\cos\theta$ only.

$8(1 - \cos^2\theta) + 2\cos\theta = 7$

$-8\cos^2\theta + 2\cos\theta + 1 = 0$

$8\cos^2\theta - 2\cos\theta - 1 = 0$

$(2\cos\theta - 1)(4\cos\theta + 1) = 0$

$\cos\theta = \frac{1}{2}$ or $-\frac{1}{4}$

For $\cos\theta = \frac{1}{2}$

Key angle = 60°

θ Q1 60°

Q2 300°, –60°

$\theta = 60°, -60°$

For $\cos\theta = -\frac{1}{4}$

Key angle = 75.5°

θ Q2 104.5°

Q3 255.5° or –104.5°

$\theta = 104.5°, -104.5°$

Complete solution $\theta = \pm 60°, \pm 104.5°$

Exercise 5 C

1. Solve the following equations for $0° \leqslant \theta \leqslant 360°$:

(a) $\sin\theta = \frac{\sqrt{3}}{2}$
(b) $\cos\theta = \frac{1}{\sqrt{2}}$
(c) $\tan\theta = -\frac{1}{\sqrt{3}}$
(d) $\sin\theta = \frac{1}{2}$
(e) $\cos\theta = 0$
(f) $\tan\theta = 0$
(g) $\sin\theta = -\frac{1}{\sqrt{2}}$
(h) $\cos\theta = 1$
(i) $\tan\theta = -1$
(j) $\sin\theta = -1$
(k) $\cos\theta = -1$
(l) $\tan\theta = -\sqrt{3}$

2. Solve the following equations for $0° \leqslant \theta \leqslant 360°$:

(a) $\sin\theta = 0.6000$
(b) $\cos\theta = -0.6428$
(c) $\tan\theta = -6.314$
(d) $\sin\theta = -0.6157$
(e) $\cos\theta = 0.6820$
(f) $\tan\theta = 0.5543$
(g) $\sin\theta = 0.4327$
(h) $\cos\theta = -0.2653$
(i) $\tan\theta = -2.4125$
(j) $\sin\theta = -\frac{8}{11}$
(k) $\cos\theta = \frac{13}{17}$
(l) $\tan\theta = \frac{19}{12}$
(m) $\sin\theta = \frac{\sqrt{3}}{6}$
(n) $\cos\theta = -\frac{\sqrt{2}}{9}$
(o) $\tan\theta = -\frac{5\sqrt{3}}{4}$

3. Solve the following equations for $-180° \leq \theta \leq 180°$:

(a) $\sin\theta = -0.6428$
(b) $\cos\theta = 0.8123$
(c) $\tan\theta = -3.526$
(d) $\sin\theta = -\frac{9}{11}$
(e) $\cos\theta = -0.6428$
(f) $\tan\theta = 1.742$
(g) $\sin\theta = -\frac{11}{13}$
(h) $\cos\theta = 0.4629$
(i) $\tan\theta = 0.8265$
(j) $\sin\theta = -\frac{\sqrt{3}}{2}$
(k) $\cos\theta = -\frac{\sqrt{3}}{2}$
(l) $\tan\theta = -1$

4. Solve the following equations for $0° \leq \theta \leq 360°$:

(a) $\sin 3\theta = \frac{\sqrt{3}}{2}$
(b) $\cos 2\theta = -\frac{1}{\sqrt{2}}$
(c) $\tan 3\theta = -1$
(d) $\sin(2\theta + 15°) = -\frac{1}{2}$
(e) $\cos\left(\frac{3}{2}\theta - 10°\right) = \frac{\sqrt{3}}{2}$
(f) $\tan(2\theta - 30°) = \frac{1}{\sqrt{3}}$
(g) $\sin\left(\frac{1}{2}\theta - 10°\right) = \frac{1}{\sqrt{2}}$
(h) $\cos(3\theta - 20°) = \frac{1}{\sqrt{2}}$
(i) $\tan\left(\frac{1}{2}\theta + 15°\right) = \frac{1}{\sqrt{3}}$
(j) $\sin\left(\frac{1}{2}\theta - 15°\right) = -0.2$
(k) $\cos\left(\frac{1}{2}\theta - 10°\right) = 0.8$
(l) $\tan(2\theta - 15°) = -3$

5. Solve the following equations for $-180° \leq \theta \leq 180°$:

(a) $\sin\frac{1}{2}\theta = -\frac{\sqrt{3}}{2}$
(b) $\cos\frac{2}{3}\theta = -\frac{1}{\sqrt{2}}$
(c) $\tan(2\theta + 15°) = -\sqrt{3}$
(d) $\sin(2\theta - 10°) = -\frac{1}{2}$
(e) $\cos(3\theta + 10°) = \frac{\sqrt{3}}{2}$
(f) $\tan\left(\frac{1}{2}\theta - 20°\right) = 1$
(g) $\sin(3\theta + 18°) = -0.925$
(h) $\cos\left(\frac{1}{2}\theta - 24°\right) = 0.6428$
(i) $\tan(3\theta - 16°) = -3.52$
(j) $\sin\left(\frac{1}{3}\theta + 15°\right) = -0.6321$
(k) $\cos(2\theta + 40°) = -0.5662$
(l) $\tan(2\theta - 35°) = 0.623$

6. Solve the following equations for $0° \leq \theta \leq 360°$:

(a) $2\sin^2\theta + \sin\theta - 1 = 0$
(b) $2\cos^2\theta + \sin\theta - 2 = 0$
(c) $4\cos^2\theta + 3\cos\theta - 1 = 0$
(d) $2\tan^2\theta + \tan\theta - 1 = 0$
(e) $6\cos^2\theta + 2\sqrt{2}\sin\theta - 5 = 0$
(f) $4\sin^2\theta - 2\cos\theta - 1 = 0$
(g) $2\cos\theta - 3\sin\theta = 0$
(h) $3\cos^2\theta - \sin\theta = 1$
(i) $2\tan y = 3\sin y$
(j) $4\sin\theta\tan\theta = 3$.

5.4 Graphs of trigonometrical functions

If we plot the points with coordinates (x, sin x) for values of x from 0 ° to 360 °, we obtain the graph of y = sin x which is as shown in Figure 5.8(a).

(A) Graphing *sin x*

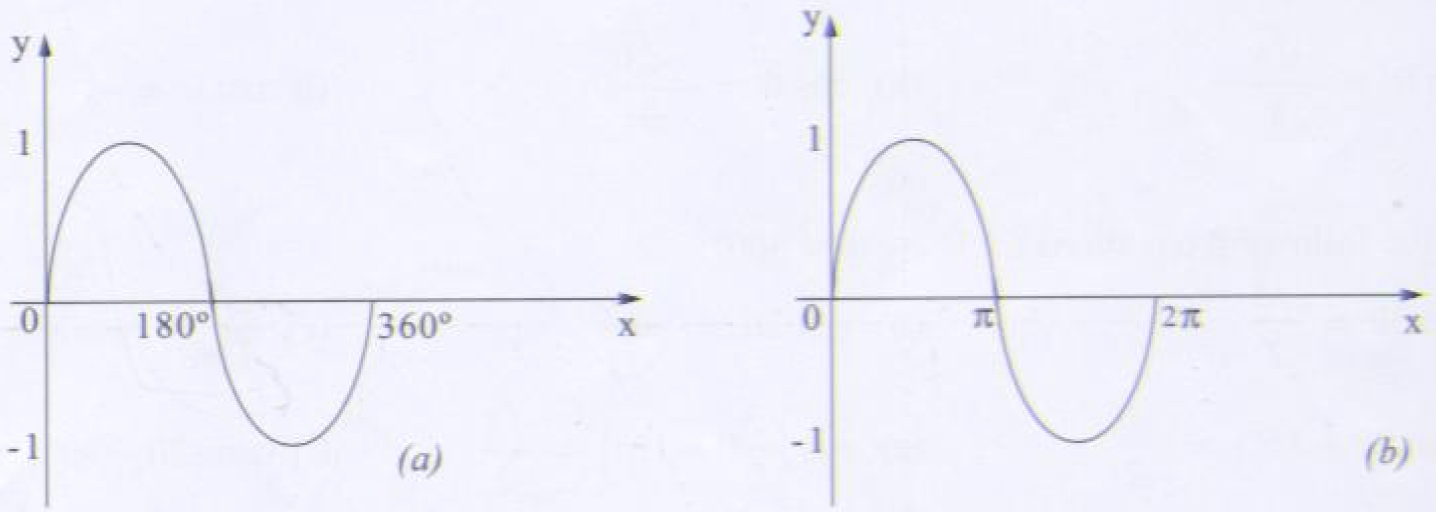


Figure 5.8

Instead of taking x in degrees, we may also take x in radians and we obtain Figure 5.8(b). If the graph of y = sin x is drawn for x ∈ ℝ, the graph is as shown in Figure 5.9.

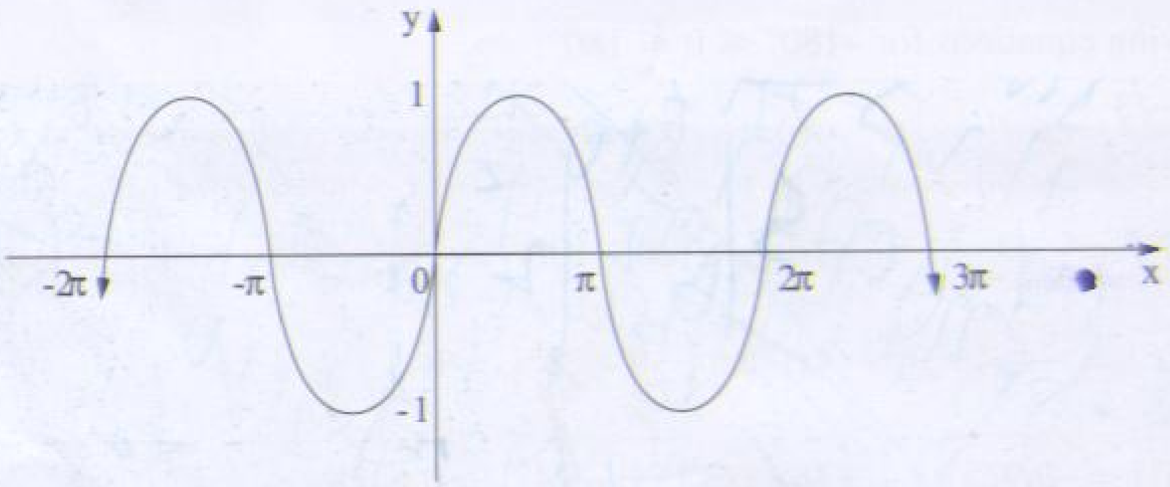
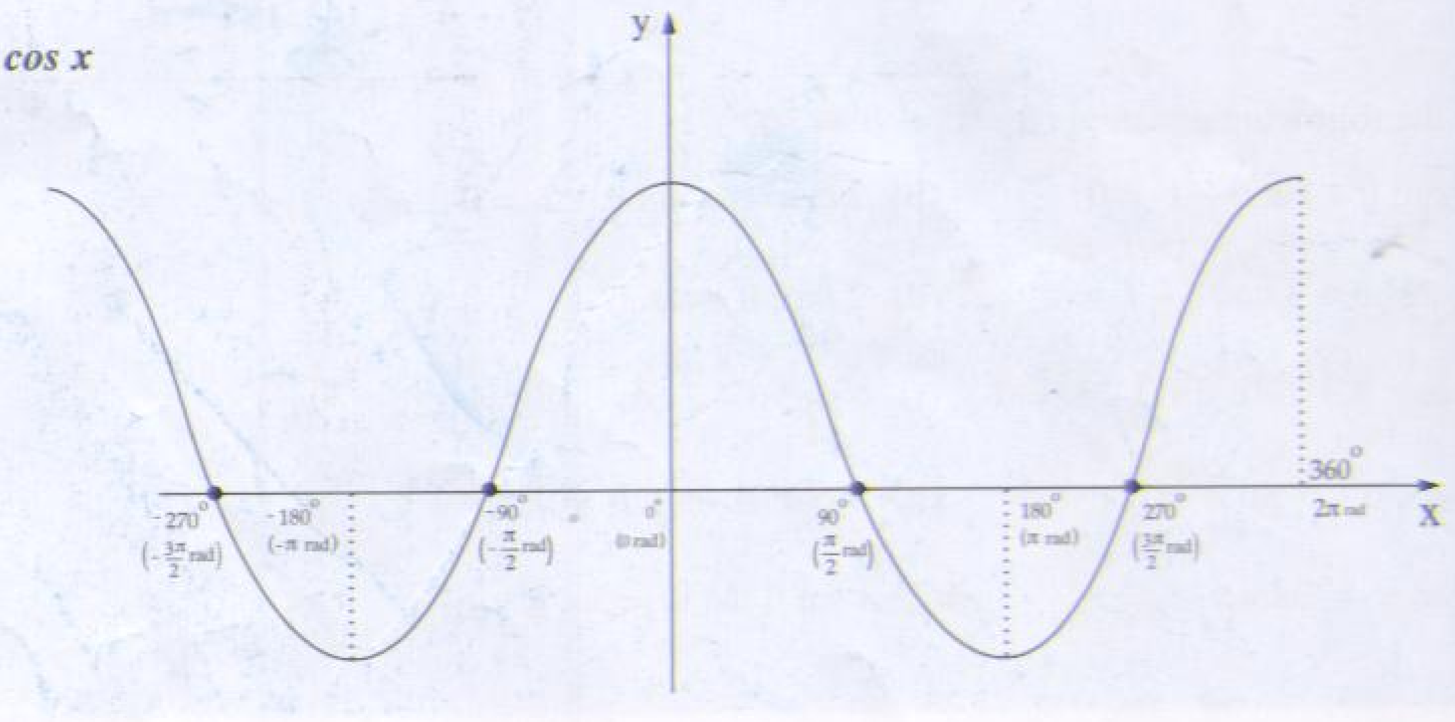


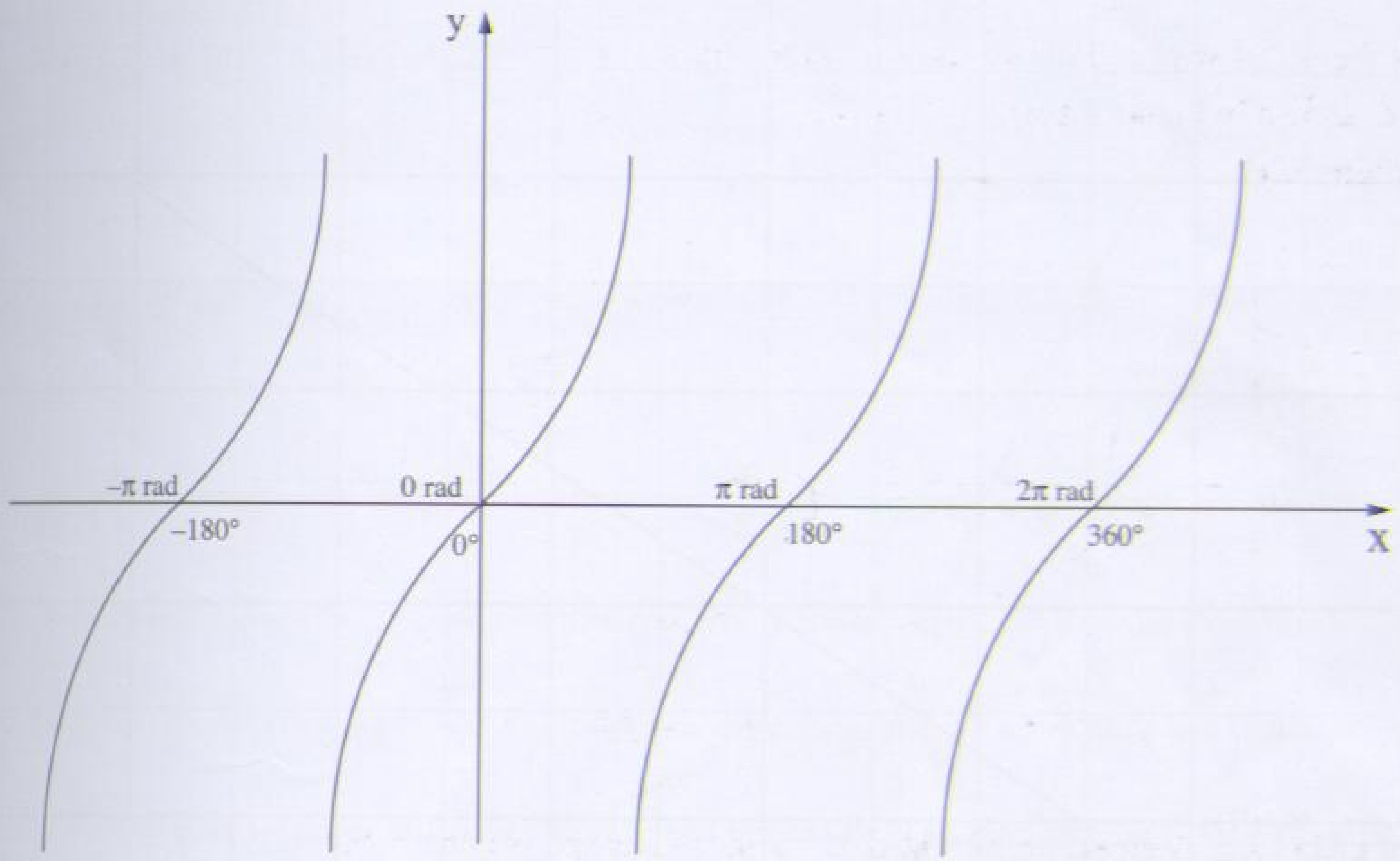
Figure 5.9

We note that the graph is repeated after an interval of 2π radians since sin (2 + x) = sin x.

(B) Graphing *cos x*



(C) Graphing *tan x*



Example 15

Draw the graph of $y = \cos 2x$ *for* $0 \leqslant x \leqslant \frac{\pi}{2}$ *and hence solve the equation* $2\cos 2x = x$ *for* $0 \leqslant x \leqslant \frac{\pi}{2}$

Solution

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
2x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$y = \cos 2x$	1	0.87	0.50	0	−0.50	−0.87	−1

The graph is as shown in Figure 5.12

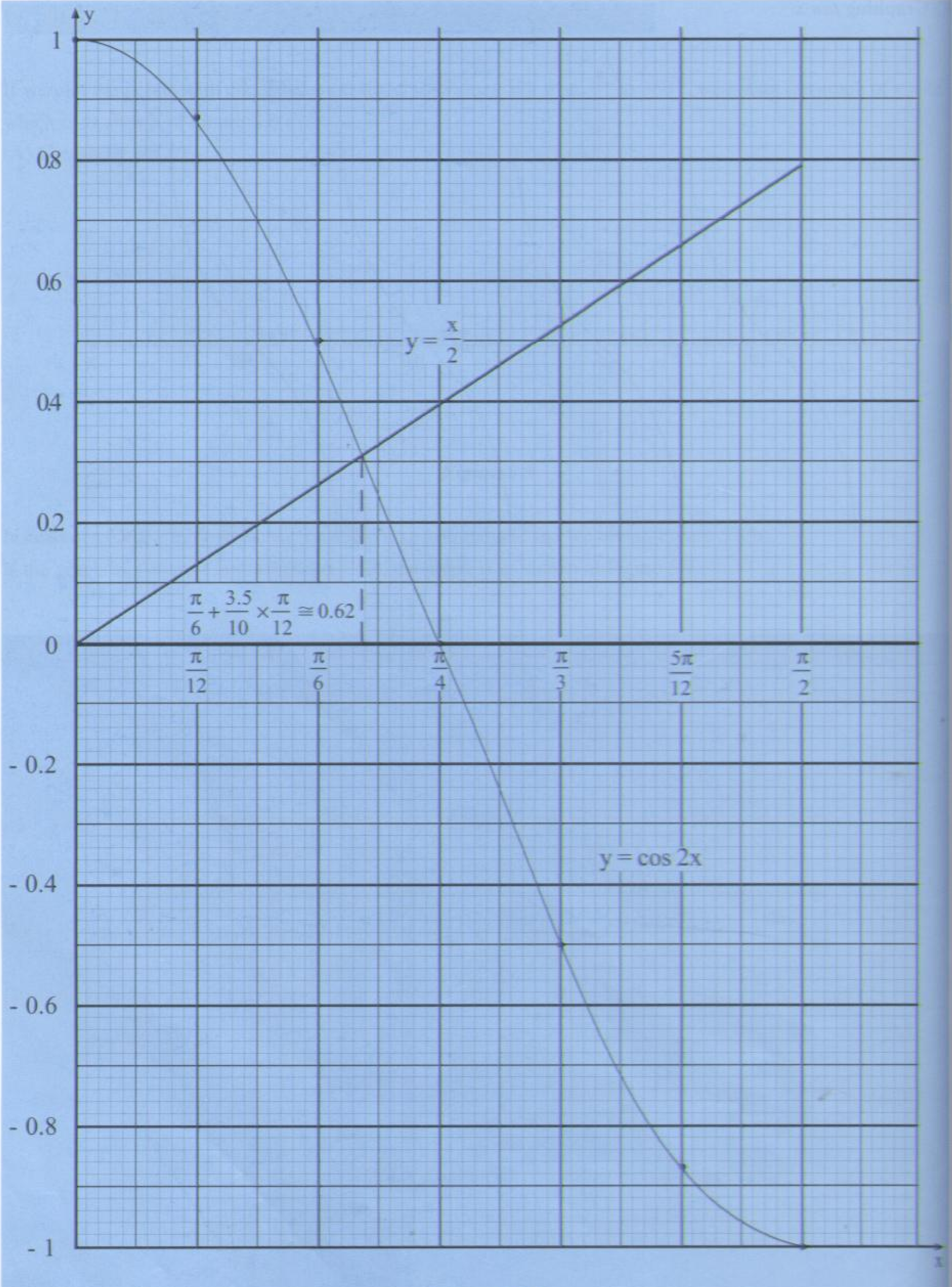

y
1
0.8
0.6
0.4
0.2
0
- 0.2
- 0.4
- 0.6
- 0.8
- 1
$y=\frac{x}{2}$
$\frac{\pi}{6}+\frac{3.5}{10}\times\frac{\pi}{12}\cong 0.62$
$\frac{\pi}{12}$
$\frac{\pi}{6}$
$\frac{\pi}{4}$
$\frac{\pi}{3}$
$\frac{5\pi}{12}$
$\frac{\pi}{2}$
y = cos 2x
x

Figure 5.12

To solve the equation $2\cos 2x = x$, we write $\cos 2x = \frac{x}{2}$. We therefore draw the line $y = \frac{x}{2}$ as shown. The line intersects the curve $y = \cos 2x$ at A.

The solution of the equation is the x-coordinate of A which is approximately 0.62.

Exercise 5 D

1. Draw the graph of $y = \tan x$ for $0 \leq x < \frac{\pi}{2}$. Hence solve the equation $\tan x = x + 1$ for $0 \leq x < \frac{\pi}{2}$.
2. Draw the graph of $y = \sin 2x$ for $0 \leq x \leq \frac{\pi}{2}$. Hence solve the equation $4 \sin 2x = 1$ for $0 \leq x \leq \frac{\pi}{2}$.
3. Draw the graph of $y = \cos^2 x$ for $0 \leq x \leq \pi$. Hence solve the equation $x = 2 \cos^2 x$ for $0 \leq x \leq \pi$.
4. Draw the graph of $y = \sin x$ for $0 \leq x \leq 2\pi$. Hence solve the equation $x = 4 \sin x$ for $0 \leq x \leq 2\pi$.
5. An arc AB of a circle subtends an angle 2x radians at the centre O and the tangents to the circle at A and B meet at C. If the length of arc AB is the same as the length of AC, show that $\tan x = 2x$.
   Draw the graph of $y = \tan x$ and hence find angle AOB.
6. A chord AB subtends an angle θ radians at the centre of a circle. The chord divides the area of the circle in the ratio 2 : 3. Show that $\sin \theta = \theta - 0.8\pi$.
   Use a graphical method to find the value of θ.
7. Draw the graph of $y = \cos x$ for $0 \leq x \leq 2\pi$.
8. Draw the graph of $y = \tan x$ for $0 \leq x \leq \pi$, $x \neq \frac{\pi}{2}$.

5.5 Principal values of inverse trigonometrical functions

5.5.1 $\sin^{-1} x$

We saw earlier that if $\sin y = \frac{1}{2}$, $y = 30° \left(\frac{\pi}{6}\right)$ rad, $150° \left(\frac{5\pi}{6}\right)$ rad, $-330° \left(\frac{-11\pi}{6}\right)$ rad, $-210° \left(\frac{-7\pi}{6}\right)$ rad.

In fact, there is an infinite number of values of y satisfying $\sin y = \frac{1}{2}$.

In general, there is an infinite number of values of y satisfying $\sin y = x$ $(-1 \leq x \leq 1)$.

The value of $y \left(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\right)$, *satisfying* $\sin y = x$ *is called the principal value and is denoted by* $\sin^{-1} x$.

Thus, $\sin^{-1} \frac{1}{2} = \frac{\pi}{6}$ and $\sin^{-1} \left(-\frac{1}{2}\right) = \left(-\frac{\pi}{6}\right)$ rad.

Note: $sin^{-1}\left(-\frac{1}{2}\right) \neq \frac{7\pi}{6}$ *as* $\frac{7\pi}{6}$ *does not lie in the specified interval.*

5.5.2 cos⁻¹ x

If $\cos y = x$*, then the value of* y*,* $(0 \leqslant y \leqslant \pi)$ *is called the principal value and is denoted by* $\cos^{-1} x$*.*

Thus, $\cos^{-1}\frac{1}{2} = \frac{\pi}{3}$ and $\cos^{-1}\left(-\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

5.5.3 tan⁻¹ x

If $\tan y = x$*, the value of* $y\ \left(-\frac{\pi}{2} < y < \frac{\pi}{2}\right)$ *is called the principal value and is denoted by* $\tan^{-1} x$*.*

Thus, $\tan^{-1}(1) = \frac{\pi}{4}$ and $\tan^{-1}(-1) = \left(-\frac{\pi}{4}\right)$

Hence, $-\frac{\pi}{2} \leqslant \sin^{-1}x \leqslant \frac{\pi}{2}, -\frac{\pi}{2} < \tan^{-1}x < \frac{\pi}{2}$ **and** $0 \leqslant \cos^{-1}x \leqslant \pi$

Example 16

Find: (a) $\sin^{-1}\frac{1}{\sqrt{2}}$ (b) $\cos^{-1}(-1)$ (c) $\tan^{-1}-\frac{1}{\sqrt{3}}$

(d) $\sin^{-1}(0.64)$ (e) $\cos^{-1}(-0.34)$ (f) $\tan^{-1}(1.35)$

Solution

(a) $\sin^{-1}\frac{1}{\sqrt{2}} = \left(\frac{\pi}{4}\right)$ rad

(b) $\cos^{-1}(-1) = \pi$ rad

(c) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \left(-\frac{\pi}{6}\right)$ rad

(d) $\sin^{-1}(0.64) = 0.694$ rad (using calculator and radian mode)

(e) $\cos^{-1}(-0.34) = 1.92$ rad

(f) $\tan^{-1}(1.35) = 0.933$ rad

Exercise 5 E

1. Find the principal value of each of the following, giving your result in terms of π:

(a) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (b) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (c) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(d) $\sin^{-1}(-1)$ (e) $\cos^{-1}(0)$ (f) $\tan^{-1}(-1)$

(g) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (h) $\cos^{-1}\left(-\frac{1}{2}\right)$ (i) $\tan^{-1}\left(-\sqrt{3}\right)$

(j) $\sin^{-1}(0)$
(k) $\cos^{-1}(1)$
(l) $\tan^{-1}(0)$

(m) $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$
(n) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$
(o) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

2. Find the principal value of each of the following:

(a) $\sin^{-1}(0.6428)$
(b) $\cos^{-1}(-0.9362)$
(c) $\tan^{-1}(1.824)$

(d) $\sin^{-1}(-0.9162)$
(e) $\cos^{-1}(0.1463)$
(f) $\tan^{-1}(-7.182)$

(g) $\sin^{-1}\left(\frac{\sqrt{2}}{3}\right)$
(h) $\cos^{-1}\left(-\frac{3}{7}\right)$
(i) $\tan^{-1}\left(\frac{13}{7}\right)$

(j) $\sin^{-1}\left(-\frac{1}{\sqrt{5}}\right)$
(k) $\cos^{-1}\left(\frac{2}{\sqrt{11}}\right)$
(l) $\tan^{-1}\left(-\frac{11}{\sqrt{23}}\right)$

(m) $\sin^{-1}\left(\frac{2\sqrt{2}}{5}\right)$
(n) $\cos^{-1}\left(-\frac{3\sqrt{2}}{7}\right)$
(o) $\tan^{-1}\left(\frac{11}{\sqrt{5}}\right)$

3. Draw the graph of $y = \sin^{-1} x$.

4. Draw the graph of $y = \cos^{-1} x$.

5. Draw the graph of $y = \tan^{-1} x$.

Miscellaneous Exercise 5

1. Given $\sin \alpha = p$ where $90° < \alpha < 180°$, find $\cos \alpha$ and $\tan \alpha$ in terms of p.

2. Given $\tan \alpha = p$ where $180° < \alpha < 270°$, find $\sin \alpha$ and $\cos \alpha$ in terms of p.

3. Given $\cos \alpha = p$ where $270° < \alpha < 360°$, find $\sin \alpha$ and $\tan \alpha$ in terms of p.

4. Given $\sin \alpha = p \left(0 < \alpha < \frac{\pi}{2}\right)$, find in terms of α, two values of θ $(0 < \theta < 2\pi)$ for which $\sin \theta = -p$.

5. Find all the angles between 0° and 360° which satisfy the equations
(a) $\sin x + 3\cos x = 0$
(b) $\sin(2y + 60°) = -0.5$ [C]

6. Find all angles between 0° and 360° which satisfy the equations
(a) $2\cos 2x = 1$
(b) $3\cos^2 z - \sin z = 1$ [C]

7. Find all the angles between 0° and 360° inclusive which satisfy the equations
(a) $2\cos^2 z + \sin z = 1$
(b) $2\tan y = 3\sin y$ [C]

8. Find all the angles from 0° to 360° inclusive which satisfy the equations
   (a) $\tan(x - 30°) - \tan 50° = 0$
   (b) $3\sin y + \tan y = 0$
   (c) $2\sin^4 z + 7\cos^2 z = 4$ [C]

9. A particle moves along a straight line and t seconds after starting its distance s m measured from O is given by $s = 5\sin\frac{1}{2}t$. Find:
   (a) its distance from O after $\frac{\pi}{2}$ seconds
   (b) the smallest value of t for which $s = 2.5$. [C]

10. A particle moves along a straight line and t seconds after starting its distance s m measured from O is given by $s = 12\cos 2t$. Find:
    (a) its distance from 0 after $\frac{\pi}{6}$ seconds
    (b) the smallest value of t for which $s = 3$.

11. Show that the equation $15\cos^2\theta = 13 + \sin\theta$ may be written as a quadratic equation in $\sin\theta$. Hence, solve the equation for $0° \leqslant \theta \leqslant 360°$. Give your answer correct to the nearest 0.1°. [C]

12. Sketch on a single diagram, the graphs of $y = \sin^{-1} x$, $y = \cos^{-1} x$ and $y = \tan^{-1} 2x$, where in each case $-1 \leqslant x \leqslant 1$ and principal values are referred to.
    The positive numbers p and q are such that $\sin^{-1} p = \tan^{-1} 2p$ and $\cos^{-1} q = \tan^{-1} 2q$. By using your sketch, or otherwise, show that $q < p$. [C]

13. By drawing appropriate sketch-graphs, find the number of solutions of the equation $\sin^{-1} x = \pi - 2x$, where $\sin^{-1} x$ is not restricted to the principal value. [C]

14. (a) Find all the angles between 0° and 360° which satisfy the equations
    (i) $\tan x = \tan 45°$
    (ii) $\sin 2y = -0.5$
    (b) Given that $\cos x = p$ and $180° < x < 360°$, find expressions in terms of p for
    (i) $\cos(-x)$
    (ii) $\sin x$ [C]

15. (a) Given that $\cos x = -0.5$ and $180° < x < 360°$, find the value of
    (i) x
    (ii) $\sin 5x$
    (b) Given that $\tan\alpha = p$, where p is acute, find, in terms of p
    (i) $\tan(-\alpha)$
    (ii) $\tan(\pi - \alpha)$
    (iii) $\tan\left(\frac{\pi}{2} - \alpha\right)$ [C]

16. (a) Find all angles, between 0° and 360°, which satisfy the equations:
    (i) $\sin x + \sin 60° = 0$
    (ii) $\tan 2y = -0.5$
    (b) Given that $\tan x = p$ and that x is acute, find an expression for $\sin x$ in terms of p. [C]

17. Find all the angles, between 0° and 360°, which satisfy the equations:
    (a) $\sin (2x + 10) = -0.5$
    (b) $\tan y = 3\cos y$ [C]

18. Find all angles, between 0° and 360°, which satisfy the equations:
    (a) $1 + \tan\left(\frac{3x}{2}\right) = 0$
    (b) $2\sin^2 z = 3\cos z$ [C]

19. Find all angles, between 0° and 360°, which satisfy the equations:
    (a) $2\sin 2x = 1$
    (b) $\sin^3 z = 8\cos^3 z$ [C]

20. Find all angles, between 0° and 360°, which satisfy the equations:
    (a) $\tan (x + 20°) = -0.8$
    (b) $2\cos^2 y + 3\sin y = 0$
    (c) $\tan z - 2\sin z = 0$ [C]

CHAPTER 6

> VECTORS 1

6.1 Operations on vectors

6.1.1 Addition of two vectors

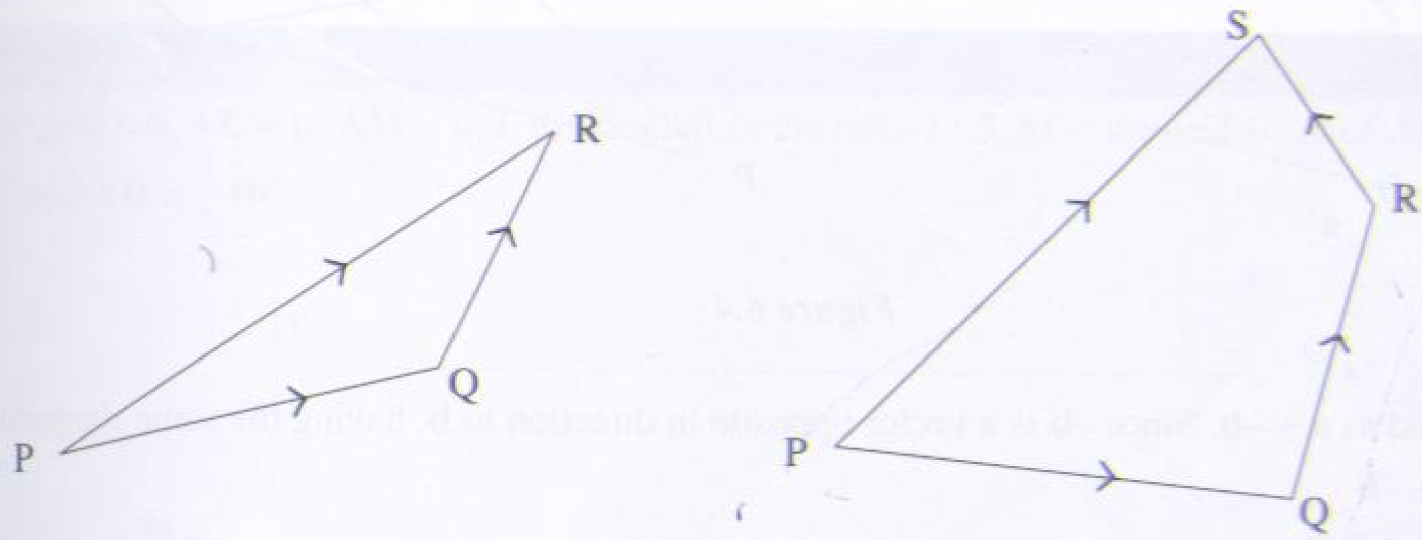


Figure 6.1

We recall that the sum of two vectors **PQ** and **QR** written **PQ** + **QR** is **PR**. i.e. **PQ** + **QR** = **PR** (Figure 6.1).

Similarly, **PQ** + **QR** + **RS** = **PS**

Any two vectors **a** and **b** can be added by considering two vectors **PQ** and **QR** equivalent to **a** and **b** (Figure 6.2).

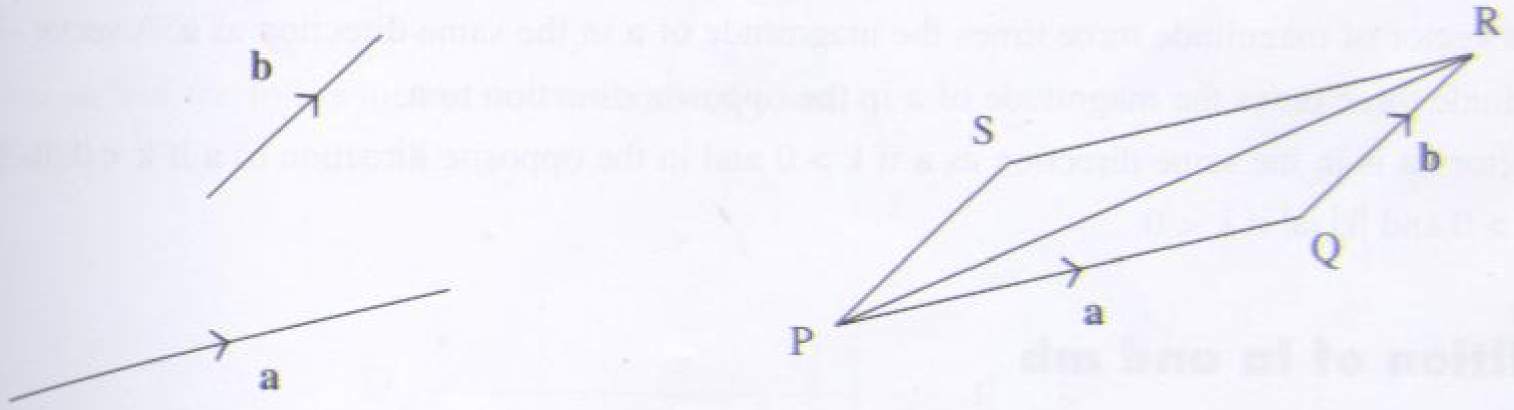


Figure 6.2

a + b = **PQ** + **QR**

= **PR**

So, a + b can be represented by **PR** where **PR** is the diagonal through P of the parallelogram PQRS.

To find **a** + **b** + **c**, we consider three vectors **PQ**, **QR** and **RS** equivalent to **a**, **b** and **c** respectively (Figure 6.3).

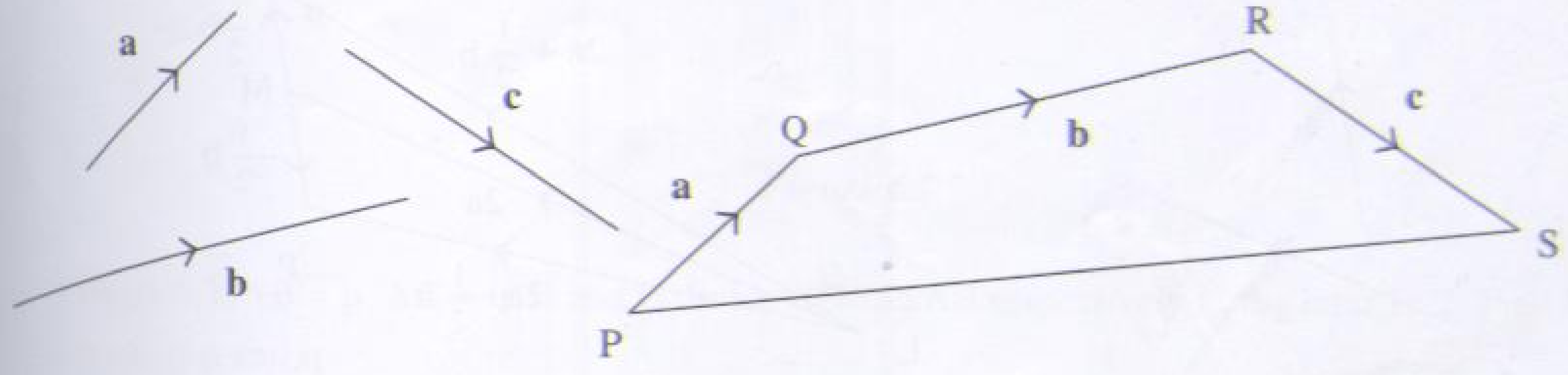


Figure 6.3

a + **b** + **c** = **PQ** + **QR** + **RS**
= **PS**

6.1.2 Subtraction of vectors

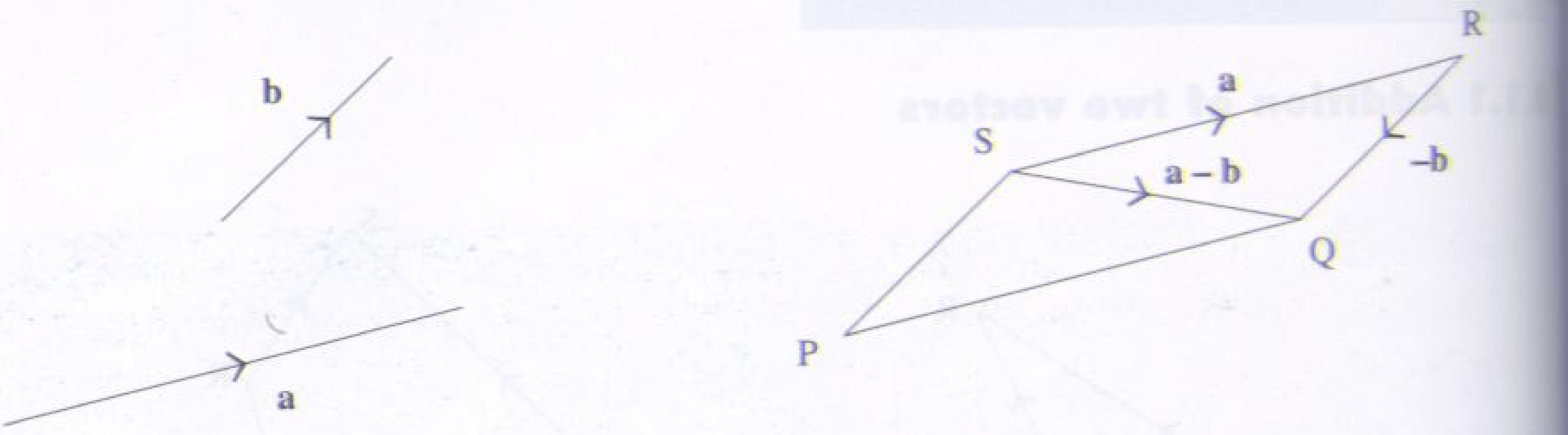


Figure 6.4

a – **b** can be regarded as **a** + –**b**. Since –**b** is a vector opposite in direction to **b**, having the same magnitude as **b**,

a – **b** = **a** + –**b**
= **SR** + **RQ**
= **SQ**

So, **a** – **b** is represented by **SQ** where SQ is the diagonal through S of the parallelogram PQRS.

6.1.3 Multiplication of a vector by a scalar

A vector 3**a** is a vector of magnitude three times the magnitude of **a** in the same direction as **a**. A vector –3**a** is a vector of magnitude three times the magnitude of **a** in the opposite direction to **a**.
Generally, a vector k**a** is in the same direction as **a** if $k > 0$ and in the opposite direction to **a** if $k < 0$. Its magnitude is $k|\mathbf{a}|$ if $k > 0$ and $|k|\,|\mathbf{a}|$ if $k < 0$.

6.1.4 Addition of la and mb

To add two vectors l$\underline{a}$ and m$\underline{b}$, we construct l**a** and m**b** first and add them as in (i) above.

Thus, in Figure 6.5, **LN** represents $2\mathbf{a} + \frac{1}{2}\mathbf{b}$.

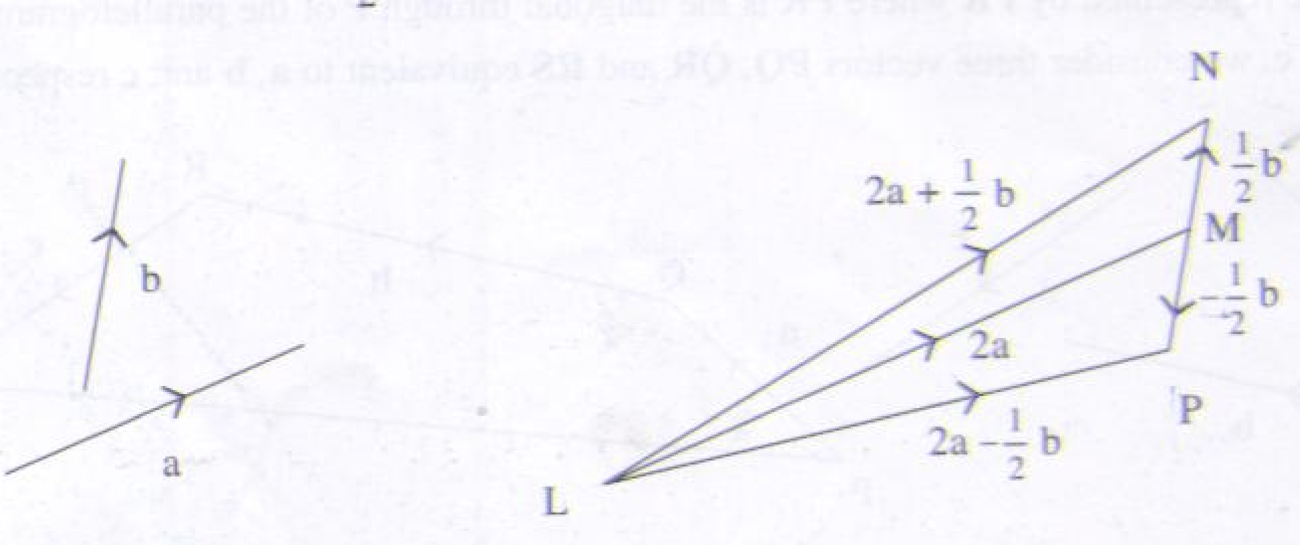


Figure 6.5

6.1.5 The vector la – mb

In Figure 6.5,

$$2a - \frac{1}{2}b = 2a + -\frac{1}{2}b$$

$$= LM + MP$$

$$= LP$$

Exercise 6 A

1. In Figure 6.6, **AL** = **p**, **AM** = **q**, L divides AB in the ratio 1 : 3, M is the mid-point of AD, AB is parallel to DC and **AB** = $\frac{1}{2}$**DC**.

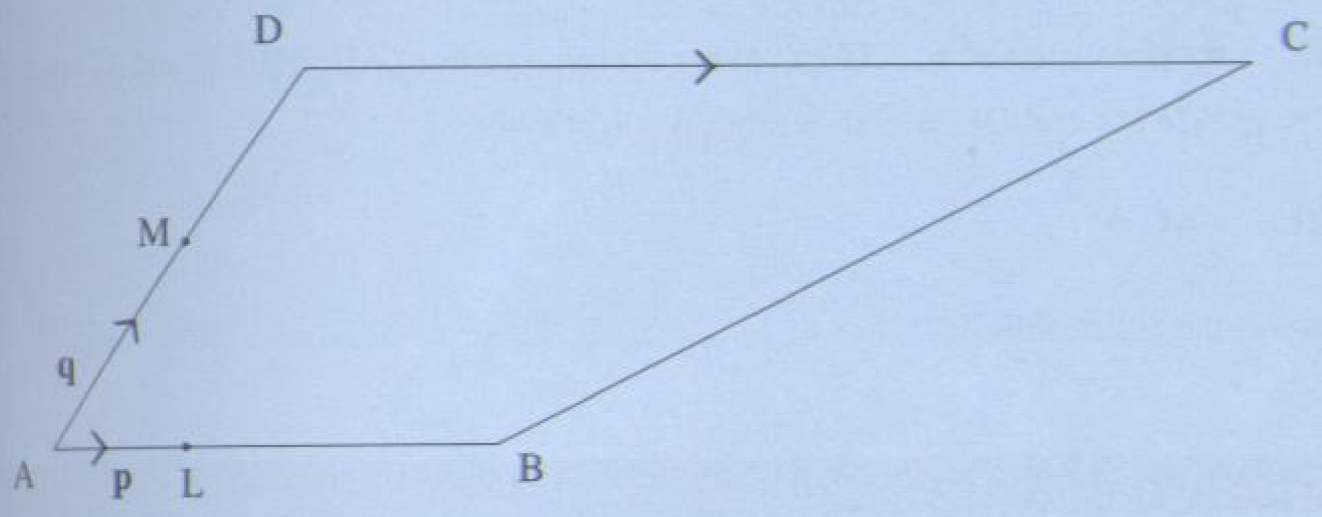


Figure 6.6

Find each of the following vectors in terms of **p** and **q**:

(i) **BD** (ii) **AC**

2.

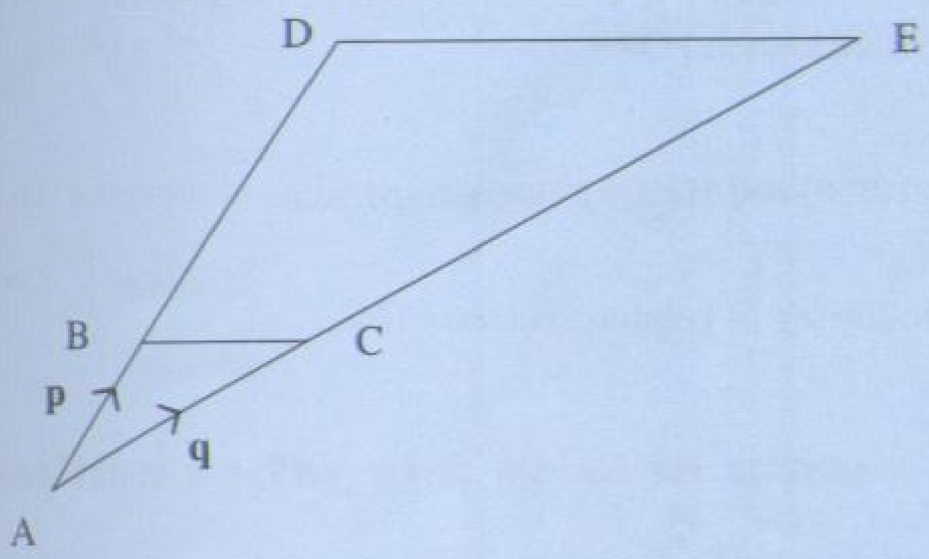


Figure 6.7

In Figure 6.7, **AB** = **p**, **AC** = **q**, B and C divide AD and AB respectively in the ratio 1 : 2. Find **BC** and **DE** in terms of **p** and **q**.

What can be deduced about the lengths and directions of **BC** and **DE**?

3.

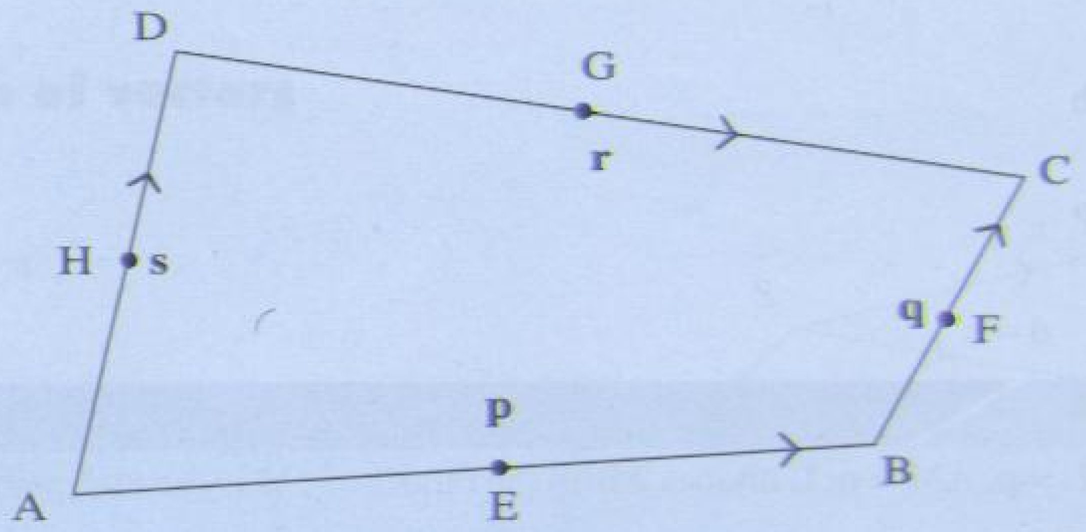


Figure 6.8

In Figure 6.8, ABCD is a quadrilateral and E, F, G and H are the mid-points of AB, BC, CD and AD respectively. Given **AB** = **p**, **BC** = **q**, **AD** = **s** and **DC** = **r**, show that **p** + **q** = **r** + **s**.

Show that **EF** = **HG** = $\frac{1}{2}$**AC**.

What be deduced about the figure EFGH?

4. Write down the displacement vector $\overrightarrow{XY}$ for each of the following pairs of points.
   (a) X (3, 5), Y (5, 9)
   (b) X (9, 7), Y (12, 4)
   (c) X (12, 5), Y (5, 4)
   (d) X (2, 3), Y (2, 5)
   (e) X (5, 1), Y (8, 1)

5. Given that P is the point (1, 3) and that and are $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are $\begin{pmatrix}4\\-1\end{pmatrix}$ and $\begin{pmatrix}2\\3\end{pmatrix}$ respectively, find the coordinates of the vertices Q, R and S of the parallelogram PQRS.

6. Show that the points E (1, 1), F (5, 4), G (8, 9) and H (0, 3) form a trapezium.

7. If $\mathbf{x} = \begin{pmatrix}3\\5\end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix}4\\6\end{pmatrix}$, write down the following as column vectors:

   (a) 3**x**

   (b) 2**y**

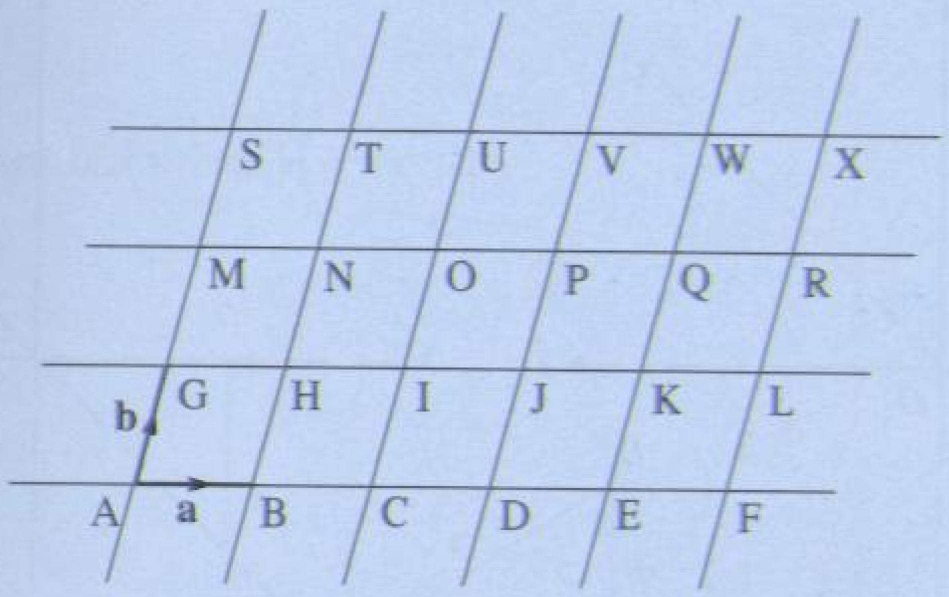
   (c) –**y**

   (d) $\frac{3}{2}\mathbf{y}$

   (e) **x** + 2**y**

   (f) 3**x** + 2**y**

8.



The figure shows sets of equally spaced parallel lines. It is given that $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AG} = \mathbf{b}$.

Express in terms of **a** and **b**:

(a) $\overrightarrow{OQ}$ (b) $\overrightarrow{QW}$ (c) $\overrightarrow{QR}$ (d) $\overrightarrow{EF}$ (e) $\overrightarrow{DK}$

(f) $\overrightarrow{DO}$ (g) $\overrightarrow{EW}$ (h) $\overrightarrow{LM}$ (i) $\overrightarrow{RL}$ (j) $\overrightarrow{BG}$

6.2 Position Vectors

6.2.1 Position vector of a point in a plane

The position vector of a point A is its position referred to an origin O. It is denoted by **OA** or $\overrightarrow{\mathrm{OA}}$.

If A has coordinates (3, 4), the position vector **OA** can be written as either $\begin{pmatrix}3\\4\end{pmatrix}$ or $3\mathbf{i} + 4\mathbf{j}$ where $\mathbf{i} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix}0\\1\end{pmatrix}$.

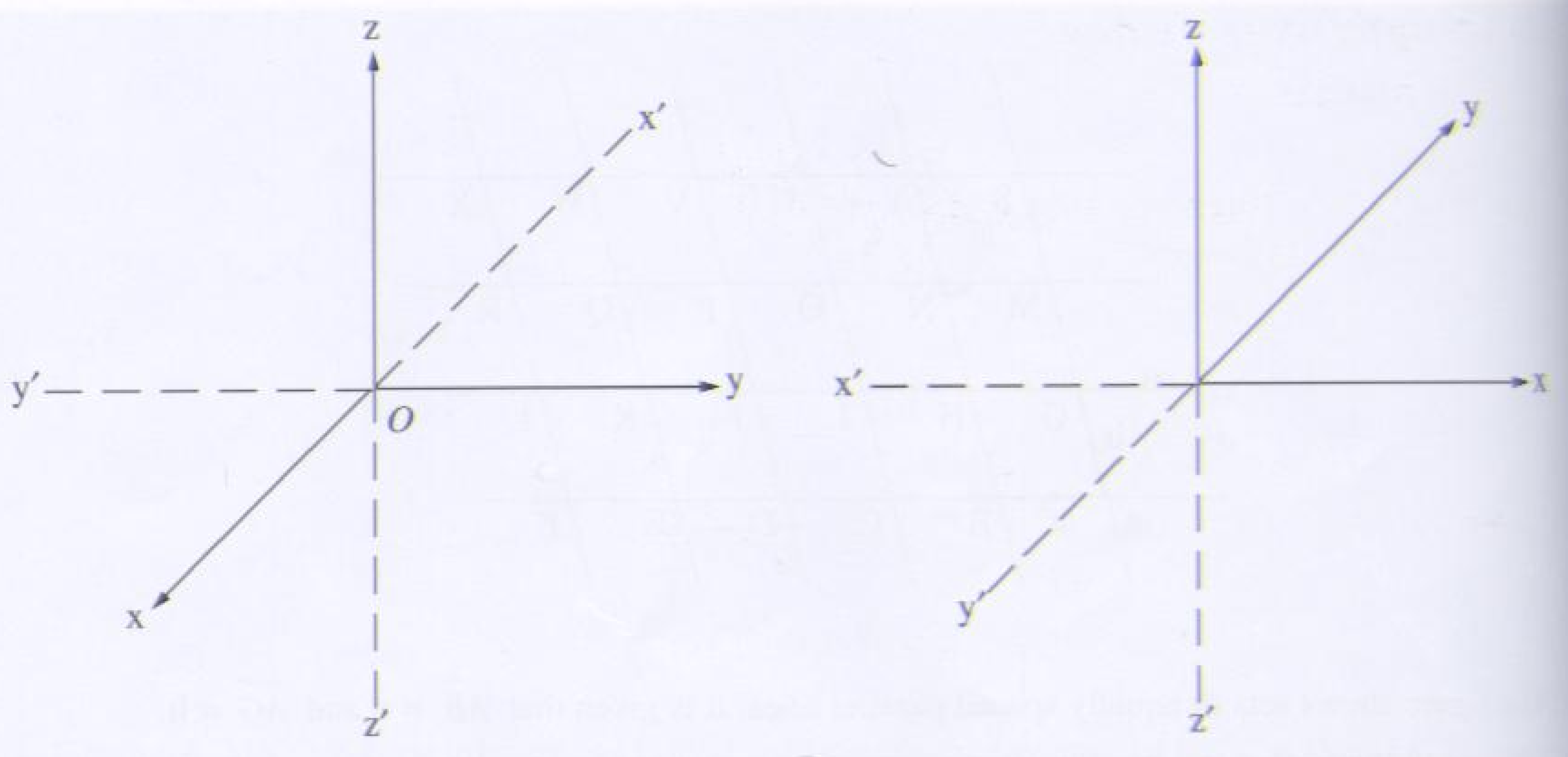
In general, if P is a point in a plane (2 dimensions), the position vector of the point A with coordinates (x, y) can be written as $\begin{pmatrix}x\\y\end{pmatrix}$ or $x\mathbf{i} + y\mathbf{j}$.

6.2.2 Position vector of a point in space

In a plane, the position of a point is determined with reference to two perpendicular lines called the x-axis and the y-axis.

In space, the position of a point is determined with reference to three mutually perpendicular lines, i.e. any two are perpendicular to each other. These lines are called the x-axis, the y-axis and the z-axis.

They may be represented as in Figure 6.9 (a) or Figure 6.9 (b).



(a) (b)

Figure 6.9

The broken lines show the negative directions of the axes. If a point P has coordinates (1, 2, 3), P is obtained b moving a unit along the positive direction of the x-axis, 2 units parallel to the positive direction of the y-axis an 3 units parallel to the positive direction of the z-axis.

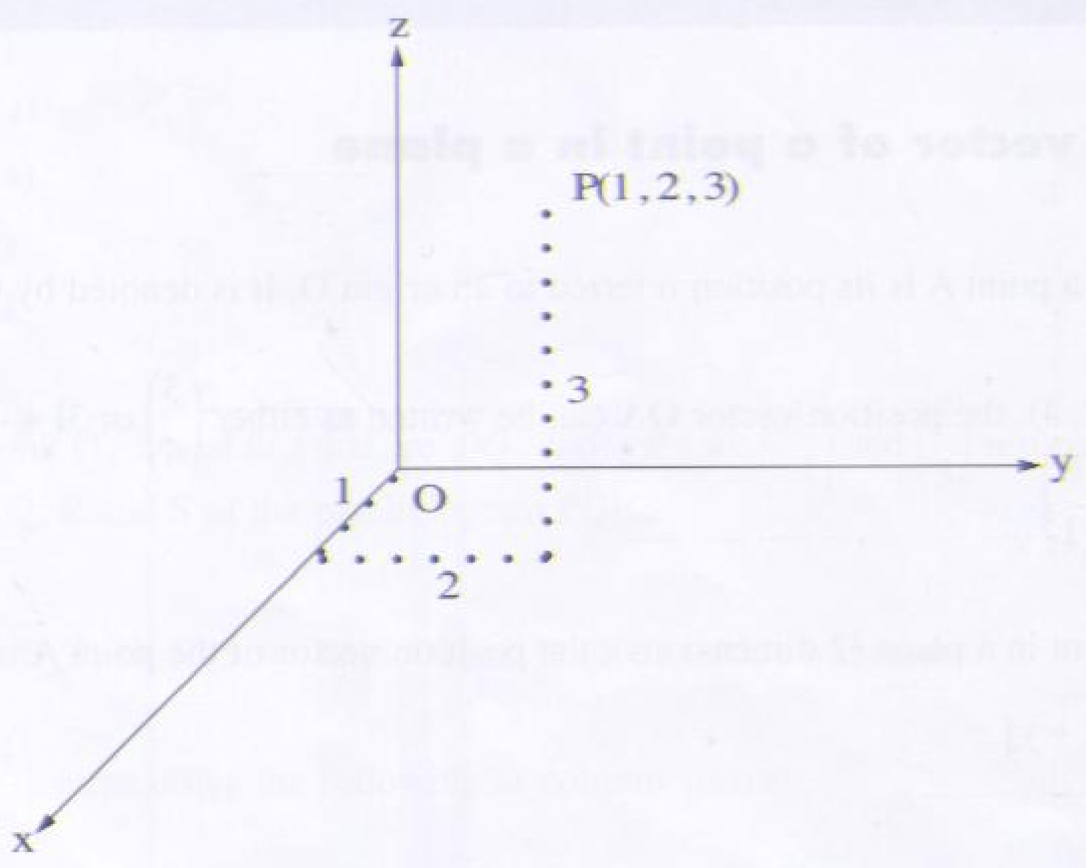


Figure 6.10

In a plane, unit vectors along Ox and Oy are **i** and **j** respectively.

In three dimensions, unit vectors along Ox, Oy and Oz are **i**, **j** and **k** respectively.

If P has coordinates (x, y, z), $\mathbf{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

6.2.3 Length of a vector

(i) *2 – dimensions*

If $\mathbf{r} = a\mathbf{i} + b\mathbf{j}$ or $\begin{pmatrix} a \\ b \end{pmatrix}$, the length of **r** written $|r| = \sqrt{a^2 + b^2}$.

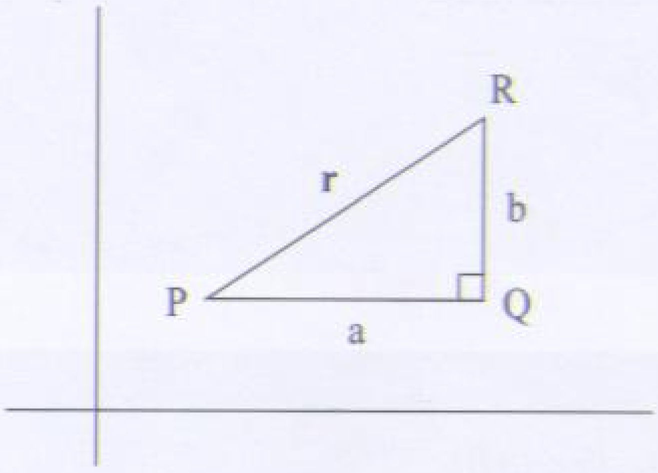


Figure 6.11

(ii) *3 – dimensions*

$\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ or $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ represented by **LP** in Figure 6.12.

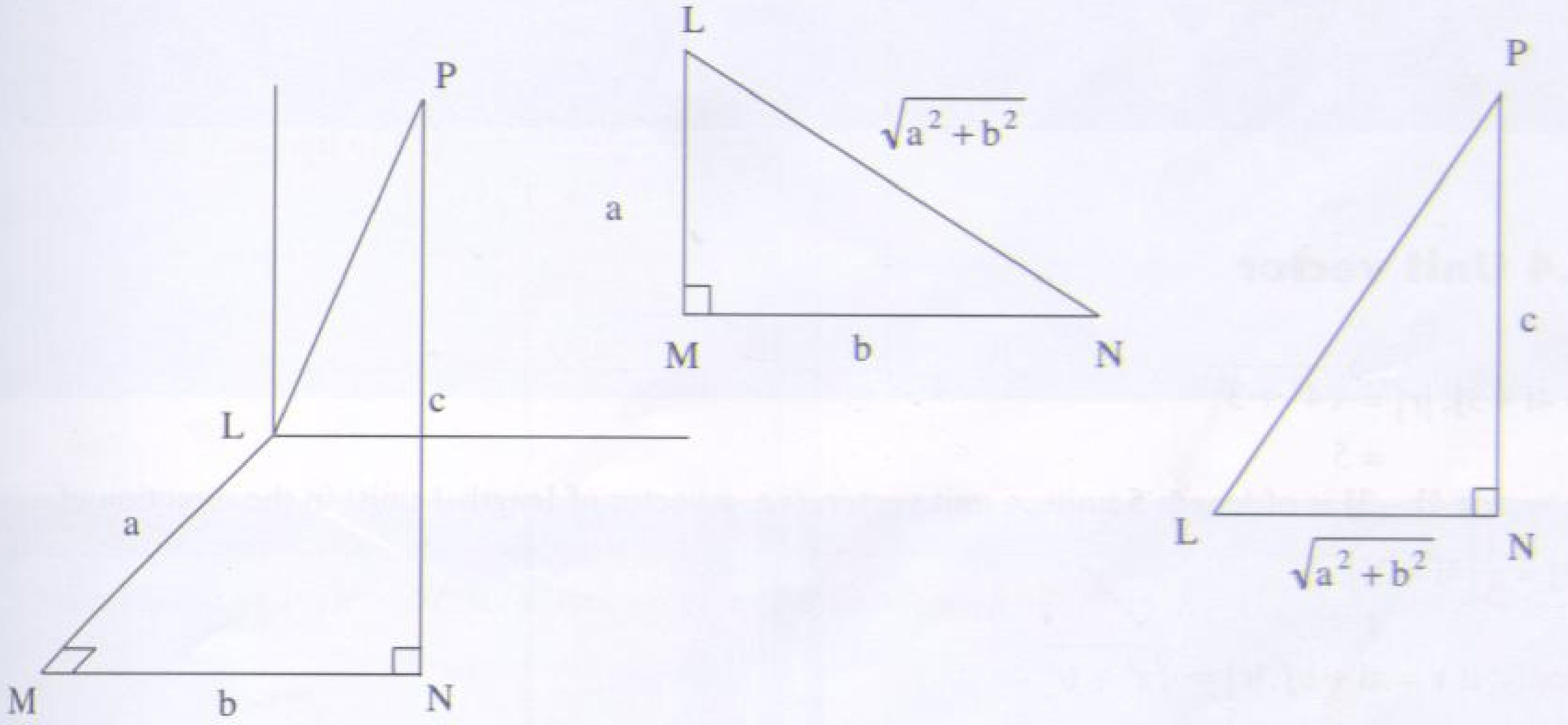


Figure 6.12

Angle LMN is a right angle as LM and MN are respectively parallel to the x-axis and y-axis. So, $LN = \sqrt{a^2 + b^2}$.
In this triangle PNL, angle PNL is a right angle as NP is parallel to the z-axis.

$$PL^2 = LN^2 + PN^2$$
$$= a^2 + b^2 + c^2$$

Hence, $|a\mathbf{i} + b\mathbf{j} + c\mathbf{k}| = \sqrt{a^2 + b^2 + c^2}$.

Example 1

Find the length of (a) $2i + 3j$ (b) $3i + 4j - 12k$

Solution

(a) $|2\mathbf{i} + 3\mathbf{j}| = \sqrt{2^2 + 3^2}$
$= \sqrt{13}$

(b) $|3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}| = \sqrt{3^2 + 4^2 + (-12)^2}$
$= \sqrt{169}$
$= 13$

Example 2

Given $A = (-1, 2, 3)$ and $B = (3, 1, -4)$, find $|\mathbf{AB}|$.

Solution

$\mathbf{AB} = \mathbf{b} - \mathbf{a}$
$= (3\mathbf{i} + \mathbf{j} - 4\mathbf{k}) - (-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$
$= 4\mathbf{i} - \mathbf{j} - 7\mathbf{k}$

$|\mathbf{AB}| = \sqrt{4^2 + (-1)^2 + (-7)^2}$
$= \sqrt{66}$

6.2.4 Unit vector

If $\mathbf{r} = 4\mathbf{i} - 3\mathbf{j}$, $|r| = \sqrt{4^2 + 3^2}$
$= 5$

Since vector $4\mathbf{i} - 3\mathbf{j}$ is of length 5 units, a unit vector, (i.e. a vector of length 1 unit) in the direction of $4\mathbf{i} - 3\mathbf{j} = \frac{1}{5}(4\mathbf{i} - 3\mathbf{j})$.

Generally, if $\mathbf{r} = \mathbf{ai} + \mathrm{b}\mathbf{j}$, $|r| = \sqrt{a^2 + b^2}$.

A unit vector in the direction of $\mathbf{ai} + \mathbf{bj}$ is therefore $\frac{\mathrm{ai} + \mathrm{bj}}{\sqrt{a^2 + b^2}}$.

If $\mathrm{r} = \mathbf{ai} + \mathbf{bj} + \mathrm{c}$, $|r| = \sqrt{a^2 + b^2 + c^2}$.

A unit vector in the direction of $\mathbf{ai} + \mathbf{bj} + \mathbf{ck}$ is therefore $\frac{\mathrm{ai} + \mathrm{bj} + \mathrm{ck}}{\sqrt{a^2 + b^2 + c^2}}$.

A unit vector in the direction of $\mathbf{r}$ is written $\hat{\mathbf{r}}$.

Example 3

Find the unit vector in the direction of (a) $\mathbf{i} - 3\mathbf{j}$ (b) $2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$.

Solution

(a) Unit vector in the direction of $\mathbf{i} - 3\mathbf{j} = \dfrac{\mathbf{i} - 3\mathbf{j}}{\sqrt{1^2 + (-3)^2}}$

$$= \frac{\mathbf{i} - 3\mathbf{j}}{\sqrt{10}}$$

(b) Unit vector in the direction of $2\mathbf{i} - \mathbf{j} - 3\mathbf{k} = \dfrac{2\mathbf{i} - \mathbf{j} - 3\mathbf{k}}{\sqrt{2^2 + (-1)^2 + (-3)^2}}$

$$= \frac{2\mathbf{i} - \mathbf{j} - 3\mathbf{k}}{\sqrt{14}}$$

Example 4

P and Q have coordinates (1, –3, 3) and (2, 1, 0) respectively. Find the unit vector in the direction of **PQ**.

Solution

$\mathrm{PQ} = \mathrm{q} - \mathbf{p}$

$= (2\mathbf{i} + \mathbf{j} + 0\mathbf{k}) - (\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

$= \mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$

Unit vector in the direction of $\mathbf{PQ} = \dfrac{\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}}{\sqrt{1^2 + 4^2 + (-3)^2}}$

$$= \frac{\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}}{\sqrt{26}}.$$

Example 5

Find the coordinates of P given **OP** *is the unit vector in the direction of* **AB** *where A = (1, –3, 3) and B = (3, 4, 5)*

Solution

$\mathrm{AB} = (3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}) - (\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

$= 2\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$

Unit vector in the direction of $\mathbf{AB} = \dfrac{2\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}}{\sqrt{4 + 49 + 4}}$

$$= \frac{2\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}}{\sqrt{57}}.$$

$$\mathbf{OP} = \frac{2}{\sqrt{57}}\mathbf{i} + \frac{7}{\sqrt{57}}\mathbf{j} + \frac{2}{\sqrt{57}}\mathbf{k}$$

$$\mathbf{P} = \left(\frac{2}{\sqrt{57}}, \frac{7}{\sqrt{57}}, \frac{2}{\sqrt{57}}\right)$$

Example 6

Find the coordinates of P given **AP** *is a vector of length 3 units in the direction of* **AB** *where $A = (1, -2, -1)$ an*
$B = (4, 2, -13)$.

Solution

$$\mathbf{AB} = (4\mathbf{i} + 2\mathbf{j} - 13\mathbf{k}) - (\mathbf{i} - 2\mathbf{j} - \mathbf{k})$$
$$= 3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}$$

Unit vector in the direction of **AB** $= \dfrac{3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}}{\sqrt{3^2 + 4^2 + (-12)^2}}$

$$= \frac{3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}}{13}$$

$$\mathbf{AP} = 3 \times \frac{3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}}{13}$$

$$= \frac{9}{13}\mathbf{i} + \frac{12}{13}\mathbf{j} - \frac{36}{13}\mathbf{k}$$

$$\mathbf{p} - \mathbf{a} = \frac{9}{13}\mathbf{i} + \frac{12}{13}\mathbf{j} - \frac{36}{13}\mathbf{k}$$

$$\mathbf{p} = \left(\frac{9}{13}\mathbf{i} + \frac{12}{13}\mathbf{j} - \frac{36}{13}\mathbf{k}\right) + (\mathbf{i} - 2\mathbf{j} - \mathbf{k})$$

$$= \frac{22}{13}\mathbf{i} - \frac{14}{13}\mathbf{j} - \frac{49}{13}\mathbf{k}$$

$$\mathbf{P} = \left(\frac{22}{13}, -\frac{14}{13}, -\frac{49}{13}\right)$$

Exercise 6 B

1. Find the unit vector in the direction of each of the following, leaving your answer in $\sqrt{\ }$ form, if not exact

(a) $3\mathbf{i} - 4\mathbf{j}$ (b) $5\mathbf{i} + 12\mathbf{j}$ (c) $8\mathbf{i} - 6\mathbf{j}$ (d) $\mathbf{i} - \mathbf{j}$ (e) $\mathbf{i} + \sqrt{3}\mathbf{j}$

(f) $6\mathbf{i} + \mathbf{j}$ (g) $\frac{3}{11}\mathbf{i} + \frac{4}{11}\mathbf{j}$ (h) $3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$ (i) $4\mathbf{i} - \mathbf{j} - \mathbf{k}$ (j) $\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$

(k) $2\mathbf{i} - \mathbf{j} - \mathbf{k}$ (l) $\begin{pmatrix} 5 \\ -12 \end{pmatrix}$ (m) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (n) $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ (o) $\begin{pmatrix} 4 \\ -3 \\ -5 \end{pmatrix}$

2. A, B, C and D have coordinates (2, –3), (5, 1), (–3, 9) and (–4, 5) respectively.
Find the unit vector in the direction of each of the following:
(a) **AB** (b) **AC** (c) **AD** (d) **BC** (e) **BD** (f) **CD**

3. A, B and C have coordinates (0, 1, 1), (2, –1, 0) and (–1, 2, 3) respectively.
Find the unit vector in the direction of:
(a) **AB** (b) **AC** (c) **BC**

4. Each of the following is a unit vector. In each case, find the value of p.
(a) $p\mathbf{i} - \frac{3}{5}\mathbf{j}$ (b) $(1 - p)\,\mathbf{i} + \frac{5}{13}\mathbf{j}$ (c) $p\mathbf{i} + \frac{4}{13}\mathbf{j} - \frac{12}{13}\mathbf{k}$
(d) $p\mathbf{i} + 2p\mathbf{j}$ (e) $p\mathbf{i} + 3p\mathbf{j} - 4p\mathbf{k}$ (f) $(1 - p)\mathbf{i} + \frac{3}{13}\mathbf{j} + \frac{4}{13}\mathbf{k}$

5. Find the coordinates of A given:
(a) **OA** is the unit vector in the direction of $6\mathbf{i} - 8\mathbf{j}$.
(b) **OA** is the unit vector in the direction of $3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$.

6. A and B have coordinates (–1, 0) and $\left(\mathbf{p}, \frac{3}{5}\right)$ respectively. Given that **AB** is a unit vector, find the possible values of p.

7. Three vectors **a**, **b** and **c** are such that $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.
Find unit vectors parallel to
(a) $\mathbf{a} + \mathbf{b}$ (b) $2\mathbf{a} - 3\mathbf{c}$

8. Three vectors **a**, **b** and **c** are such that $\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 0 \\ -2 \\ -3 \end{pmatrix}$.

Find unit vectors parallel to
(a) $2\mathbf{a} + 3\mathbf{b}$ (b) $3\mathbf{a} - 4\mathbf{c}$

9. Find the unit vector in the direction of $5\mathbf{i} - 12\mathbf{j}$. Hence, find the vector of magnitude 26 units in the direction of $5\mathrm{i} - 12\mathbf{j}$.

10. Find:
(a) the vector of magnitude 25 units in the direction of $3\mathbf{i} - 4\mathbf{j}$.
(b) the vector of magnitude $3\sqrt{2}$ units in the direction of $\mathbf{i} + \mathbf{j}$.
(c) the vector of magnitude 39 units in the direction of $3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}$.

(d) the vector of magnitude 6 units in the direction of $\begin{pmatrix}1\\2\\-1\end{pmatrix}$.

(e) the vector of magnitude 3 units in the direction of $\begin{pmatrix}\sqrt{2}\\1\\0\end{pmatrix}$.

11. Given **a** is the unit vector in the direction of $4\mathbf{i}-3\mathbf{j}$ and **b** is the unit vector in the direction of $5\mathbf{i}+12\mathbf{j}$, find
    (a) $10\mathbf{a}-26\mathbf{b}$
    (b) the unit vector in the direction of $10\mathbf{a}-26\mathbf{b}$.

12. Given **a** is a vector of magnitude 26 units in the direction of $4\mathbf{i}-12\mathbf{j}-3\mathbf{k}$ and **b** is a vector of magnitude 13 units in the direction of $3\mathbf{i}+12\mathbf{j}-4\mathbf{k}$, find the unit vector in the direction of $2\mathbf{a}+3\mathbf{b}$.

13. O is the origin and P, Q have coordinates (3, –2) and (–1, 1) respectively. Given **OR** is a vector of magnitude 10 units in the direction of **PQ**, find the coordinates of R.

14. P and Q have coordinates (1, –2, 0) and (13, 1, 4) respectively. Given that **OR** is a vector of magnitude 52 units in the direction of **QP,** find the coordinates of R.

15. A and B have coordinates (1, 3) and (6, –9) respectively. If C is a point on AB such that **AC** is a unit vector, find the coordinates of C.

16. A, P and Q have coordinates (1, –1, 2), (2, 1, –1) and (3, –1, 1) respectively. Given **AB** is a unit vector in the direction of **PQ**, find the coordinates of B.

17. A has position vector $8\mathbf{i}-6\mathbf{j}$ relative to the origin. Given B is a point on OA such that **OB** is a unit vector, find the position vector of B.

18. A has position vector $4\mathbf{i}+3\mathbf{j}-12\mathbf{k}$ and B is a point on OA such that **OB** is a unit vector. Find the position vector of B.

19. A and B have position vectors $\mathbf{i}+2\mathbf{j}$ and $9\mathbf{i}-4\mathbf{j}$ relative to the origin respectively. If C is a point on AB such that **AC** is a unit vector, find the coordinates of C.

20. A and B have position vectors $2\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $\mathbf{i}+2\mathbf{j}-4\mathbf{k}$ relative to the origin respectively. If C is a point on AB such that **AC** is a unit vector, find the coordinates of C.

6.3.1 Scalar (or dot) product of two vectors

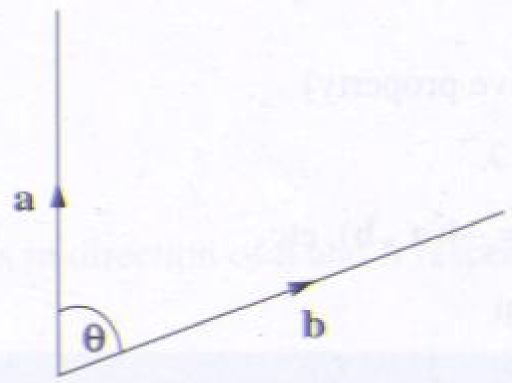


Figure 6.13

The scalar (or dot) product of two vectors **a** and **b** written

a . b is defined by

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \times \cos\theta$,

where θ is the angle between **a** and **b** (Figure 6.13).

If $|a| = 4$ units, $|b| = 5$ units and $\theta = 60°$,

$\mathbf{a} \cdot \mathbf{b} = 4 \times 5 \times \cos 60°$

$= 10.$

If two vectors **a** and **b** are parallel and in the same direction the angle between them is 0.

So, $\mathbf{a} \cdot \mathbf{b} = |a| \times |b| \times \cos 0°$

$= |a|\ |b|$

In particular, $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$

Also, $\mathbf{a} \cdot \mathbf{a} = |a|^2$

If two vectors **a** and **b** are parallel and opposite in direction

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos 180°$

$= -|\mathbf{a}||\mathbf{b}|$

If two vectors **a** and **b** are perpendicular

$\mathbf{a} \cdot \mathbf{b} = |a| \times |b| \times \cos 90°$

$= 0$

So, $\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0$

Conversely, if $\mathbf{a} \cdot \mathbf{b} = 0$

$|\mathbf{a}||\mathbf{b}| \times \cos\theta = 0$, where θ is the angle between **a** and **b**.

So either $|a| = 0$ or $|b| = 0$ or $\cos\theta = 0$.

If neither **a** nor **b** is a zero vector, $\cos\theta = 0, \theta = 90°$.

This is a very important result and is used to show that two vectors are perpendicular.

6.3.2 Properties of dot product

We list some important properties of dot product:

(i) $\mathbf{a} . \mathbf{b} = \mathbf{b} . \mathbf{a}$ (commutative property)

(ii) $\lambda\mathbf{a} . \mathbf{b} = \lambda(\mathbf{a} . \mathbf{b})$ for a scalar λ.

Thus, $3\,\mathbf{a} . \mathbf{b} = 3(\mathbf{a} . \mathbf{b})$, $-4\,\mathbf{a} . \mathbf{b} = -4(\mathbf{a} . \mathbf{b})$, etc.

(iii) $\mathbf{a} . \mu\mathbf{b} = \mu(\mathbf{a} . \mathbf{b})$ for a scalar μ

Thus, $\mathbf{a} . 4\mathbf{b} = 4(\mathbf{a} . \mathbf{b})$, $\mathbf{a} . -\frac{3}{4}\,\mathbf{b} = -\frac{3}{4}\,(\mathbf{a} . \mathbf{b})$, etc.

(iv) $\lambda\,\mathbf{a} . \mu\,\mathbf{b} = \lambda\mu\,(\mathbf{a} . \mathbf{b})$ for scalars λ and μ.

Thus, $2\mathbf{a} . 3\mathbf{b} = 6(\mathbf{a} . \mathbf{b})$, $5\mathbf{a} . -2\mathbf{b} = -10(\mathbf{a} . \mathbf{b})$, etc.

(v) $\mathbf{a} . (\mathbf{b} + \mathbf{c}) = \mathbf{a} . \mathbf{b} + \mathbf{a} . \mathbf{c}$ (associative property)

6.3.3 Finding a . b in terms of components of a and b

We consider first two dimensional vectors

For $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j}$, $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j}$,

$$\begin{aligned}
\mathbf{a} . \mathbf{b} &= \mathbf{a} . (x_2\mathbf{i} + y_2\mathbf{j}) \\
&= \mathbf{a} . x_2\mathbf{i} + \mathbf{a} . y_2\mathbf{j} && \text{Property (v)} \\
&= (x_1\mathbf{i} + y_1\mathbf{j}) . x_2\mathbf{i} + (x_1\mathbf{i} + y_1\mathbf{j}) . y_2\mathbf{j} \\
&= x_2\mathbf{i} . (x_1\mathbf{i} + y_1\mathbf{j}) + y_2\mathbf{j} . (x_1\mathbf{i} + y_1\mathbf{j}) && \text{Property (i)} \\
&= x_2\mathbf{i} . x_1\mathbf{i} + x_2\mathbf{i} . y_1\mathbf{j} + y_2\mathbf{j} . x_1\mathbf{i} + y_2\mathbf{j} . y_1\mathbf{j} && \text{Property (v)} \\
&= x_2\,x_1 + 0 + 0 + y_2\,y_1, \text{ as } \mathbf{i} . \mathbf{i} = 1, \mathbf{i} . \mathbf{j} = 0, \mathbf{j} . \mathbf{j} = 1 \\
&= x_1\,x_2 + y_1\,y_2
\end{aligned}$$

So, $(x_1\mathbf{i} + y_1\mathbf{j}) . (x_2\mathbf{i} + y_2\mathbf{j}) = x_1\,x_2 + y_1\,y_2$

or $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = x_1\,x_2 + y_1\,y_2$

For 3-dimensional vectors, if $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$, $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$,

$(x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}) . (x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) = x_1\,x_2 + y_1\,y_2 + z_1\,z_2$

or $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1\,x_2 + y_1\,y_2 + z_1\,z_2$

6.3.4 Angle between two vectors

Since $\mathbf{a} \cdot \mathbf{b} = |a| \times |b| \times \cos\theta$

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \times |\mathbf{b}|}$$

This result can also be written as

$\cos\theta = \hat{\mathbf{a}} \cdot \hat{\mathbf{b}}$ where $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are unit vectors in direction of **a** and **b** respectively.

Example 7

*Find the angle between 4**i** + 3j and 3**i** + 4j.*

Solution

$$\cos\theta = \frac{(4\mathbf{i}+3\mathbf{j}) \cdot (3\mathbf{i}+4\mathbf{j})}{|4\mathbf{i}+3\mathbf{j}| \times |3\mathbf{i}+4\mathbf{j}|}$$

$$= \frac{4 \times 3 + 3 \times 4}{\sqrt{16+9} \times \sqrt{9+16}}$$

$$= \frac{24}{25}$$

$$\theta = 16.3°$$

Example 8

*Show that the vectors 5**i** – 12j and 12**i** + 5j are perpendicular.*

Solution

As none of the vectors is a zero vector, it is sufficient to show that their scalar product is 0.

$$(5\mathbf{i} - 12\mathbf{j}) \cdot (12\mathbf{i} + 5\mathbf{j}) = 5 \times 12 + -12 \times 5$$
$$= 0$$

So, the two vectors are perpendicular.

Example 9

Find the smallest angle of the triangle ABC with A(–1, 2, 3), B(0, 1, 4) and C(4, –1, 5).

Solution

The smallest angle is opposite the shortest side.

$$AB = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$|AB| = \sqrt{1+1+1}$$
$$= \sqrt{3}$$

$$\mathbf{AC} = \begin{pmatrix}4\\-1\\5\end{pmatrix} - \begin{pmatrix}-1\\2\\3\end{pmatrix} = \begin{pmatrix}5\\-3\\2\end{pmatrix}$$

$$|AC| = \sqrt{25+9+4}$$

$$= \sqrt{38}$$

$$\mathbf{BC} = \begin{pmatrix}4\\-1\\5\end{pmatrix} - \begin{pmatrix}0\\1\\4\end{pmatrix} = \begin{pmatrix}4\\-2\\1\end{pmatrix}$$

$$|BC| = \sqrt{16+4+1}$$

$$= \sqrt{21}$$

AB being the shortest side, the smallest angle is opposite AB, i.e. $\angle C$. To find $\angle C$, we use the vectors away fro

C, i.e. **CA** and **CB**.

$$\text{Cos } \angle C = \frac{\mathbf{CA}\ .\ \mathbf{CB}}{|\ \mathbf{CA}\ | \times |\ \mathbf{CB}\ |}$$

$$= \frac{\begin{pmatrix}-5\\3\\-2\end{pmatrix} . \begin{pmatrix}-4\\2\\-1\end{pmatrix}}{\sqrt{25+9+4} \times \sqrt{16+4+1}}$$

$$= \frac{20+6+2}{\sqrt{38}\sqrt{21}}$$

$$\angle C \approx 7.6° \quad \text{(calculator)}$$

Example 10

The vectors ai + 2j – 4k and 4i – 2j – 5k are perpendicular. Find the value of a.

Solution

Since the vectors are perpendicular, their scalar product is 0.

$$a \times 4 + 2 \times -2 + -4 \times -5 = 0$$

$$4a - 4 + 20 = 0$$

$$4a = -16$$

$$a = -4$$

Exercise 6 C

1. Evaluate:

(a) $(2\mathbf{i} + \mathbf{j}) . (\mathbf{i} - \mathbf{j})$ (b) $(5\mathbf{i} - 3\mathbf{j}) . (4\mathbf{i} + \mathbf{j})$ (c) $(4\mathbf{i} - 2\mathbf{j}) . (6\mathbf{i} + 5\mathbf{j})$

(d) $\begin{pmatrix}4\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix}$ (e) $\begin{pmatrix}a\\b\end{pmatrix}\cdot\begin{pmatrix}b\\-a\end{pmatrix}$ (f) $\begin{pmatrix}a\\b\end{pmatrix}\cdot\begin{pmatrix}-b\\a\end{pmatrix}$

(g) $\begin{pmatrix}3\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}2\\3\\-1\end{pmatrix}$ (h) $\begin{pmatrix}5\\-2\\3\end{pmatrix}\cdot\begin{pmatrix}-3\\1\\4\end{pmatrix}$ (i) $\begin{pmatrix}-2\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\end{pmatrix}$

(j) $(\mathbf{i}+2\mathbf{j}+3\mathbf{k}).(2\mathbf{i}+4\mathbf{j}+5\mathbf{k})$ (k) $(4\mathbf{i}-2\mathbf{j}-3\mathbf{k}).(\mathbf{i}-\mathbf{j}-\mathbf{k})$

(l) $(6\mathbf{i}+\mathbf{j}).(\mathbf{i}-2\mathbf{j}+3\mathbf{k})$. (m) $(2\mathbf{i}+\mathbf{k}).(4\mathbf{i}-3\mathbf{j})$

2. Find the angle between each of the following pairs of vectors:

(a) $3\mathbf{i}-2\mathbf{j}$ and $2\mathbf{i}-3\mathbf{j}$ (b) $-3\mathbf{i}+4\mathbf{j}$ and $5\mathbf{i}-2\mathbf{j}$

(c) $\begin{pmatrix}6\\8\end{pmatrix}$ and $\begin{pmatrix}2\\-3\end{pmatrix}$ (d) $\begin{pmatrix}5\\-4\end{pmatrix}$ and $\begin{pmatrix}2\\3\end{pmatrix}$

(e) $\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and $2\mathbf{i}-\mathbf{j}+\mathbf{k}$ (f) $4\mathbf{i}-3\mathbf{j}+2\mathbf{k}$ and $3\mathbf{i}+2\mathbf{j}-\mathbf{k}$

(g) $\begin{pmatrix}2\\-1\\3\end{pmatrix}$ and $\begin{pmatrix}3\\1\\-2\end{pmatrix}$ (h) $\begin{pmatrix}3\\4\\12\end{pmatrix}$ and $\begin{pmatrix}4\\1\\-3\end{pmatrix}$

3. Find the angles of triangle ABC with:

(a) A(1, –1), B(3, 2) and C(6, 4) (b) A(6, 7), B(–1, 6) and C(3, 4)

(c) A(1, –1, 2), B(0, 2, 3) and C(4, 5, –3) (d) A(2, –1, 3), B(–1, 2, 3) and C(0, –3, 4)

4. (a) $3\mathbf{i}+a\mathbf{j}$ is perpendicular to $4\mathbf{i}-3\mathbf{j}$. Find the value of a.

(b) $a\mathbf{i}+b\mathbf{j}$ is perpendicular to $3\mathbf{i}-4\mathbf{j}$. Given $|a\mathbf{i}+b\mathbf{j}|=5$, find the possible values of a and of b.

(c) The position vectors of the points A and B relative to the origin are $3\mathbf{i}-2\mathbf{j}$ and $4\mathbf{i}+\mathbf{j}$ respectively. Find the value of p such that $\mathbf{i}+p\,\mathbf{j}$ is perpendicular to **AB**.

(d) Three vectors **a**, **b** and **c** are such that $\mathbf{a}+p\mathbf{b}$ is perpendicular to **c**. Given $\mathbf{a}=3\mathbf{i}-2\mathbf{j}$, $\mathbf{b}=2\mathbf{i}-\mathbf{j}$ and $\mathbf{c}=4\mathbf{i}+3\mathbf{j}$, find the value of p.

(e) The position vectors of the points **a**, **b** and **c** relative to the origin O are $2\mathbf{i}+\mathbf{j}$, $\mathbf{i}+2\mathbf{j}$ and $2\mathbf{i}-5\mathbf{j}$ respectively. Given that the angle ACB is θ, find the exact value of cos θ.

5. (a) $a\mathbf{i}+2\mathbf{j}-a\mathbf{k}$ is perpendicular to $2\mathbf{i}-3\mathbf{j}-\mathbf{k}$. Find the value of a.

(b) $a\mathbf{i}-3\mathbf{j}+b\mathbf{k}$ is perpendicular to $2\mathbf{i}+4\mathbf{j}+2\mathbf{k}$ and its length is of magnitude $\sqrt{29}$ units. Find the possible values of a and of b.

(c) The position vectors of two points A and B relative to the origin O are $a\mathbf{i}+2\mathbf{j}-3\mathbf{k}$ and $\mathbf{i}-2\mathbf{j}+4\mathbf{k}$. Given that $2\mathbf{i}-3\mathbf{j}+\mathbf{k}$ is perpendicular to **AB**, find the value of a.

(d) Three vectors **a**, **b**, **c** have position vectors $\mathbf{i}+2\mathbf{j}-\mathbf{k}$, $4\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $-2\mathbf{i}-3\mathbf{j}+7\mathbf{k}$ relative to the origin respectively. Find the value of p such that $p\mathbf{a}+\mathbf{b}$ is perpendicular to **BC**.

6. A and B have coordinates (1, –2) and (3, 4) respectively and the point P on AB is such that AP = λ AB
Find the position vector of P relative to the origin O.
Given C has coordinates (2, 2) and CP is perpendicular to AB, find the coordinates of P. Hence, find the
length of the perpendicular from C to AB.

7. A and B have coordinates (1, 2, –1) and (4, 3, –4) respectively and the point P on AB is such that AP = λ AB
Find the position vector of P relative to the origin O.
Find the coordinates of P such that **OP** is perpendicular to AB. Hence, find the length of the perpendicul
from O to AB.

8. A, B and C have coordinates (–1, 2, 3), (0, –1, 5) and (2, 3, –2) respectively. Find the coordinates of the fo
of the perpendicular from A to BC.
Hence, obtain the length of the perpendicular from A to BC.

9. If **a** and **b** are any two non-parallel (non-zero) vectors in a plane, write down any vector in the plane in term
of **a** and **b**. Show that if **p** is any vector perpendicular to both **a** and **b**, it is perpendicular to any vector in the
plane. What can be deduced about **p** and the plane?

10. Three vectors **a**, **b** and **c** are such that $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - k$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ and $\mathbf{c} = 4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$.
Find a vector $p\mathbf{i} + q\mathbf{j} + \mathbf{k}$ which is perpendicular to the plane ABC.

Miscellaneous Exercise 6

1. The position vectors of A and B, relative to the origin O, are $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ respective
The point C is given by OC = 2 OA. Find:
(a) the length of OC
(b) cos ∠AOB. [C

2. The position vector of a variable point P with respect to the origin O is $\mathbf{OP} = (3 + 2\lambda)\mathbf{i} + (1 - 2\lambda)\mathbf{j} + (2 + \lambda$
where λ is a scalar variable. Find an expression in terms of λ for $|\mathbf{OP}|^2$. Show that $|\mathbf{OP}|^2$ is least wh
$\lambda = -\frac{2}{3}$, and give the least value of $|\mathbf{OP}|$. [C

3. R is any point on the circumference of a circle, centre O, and AB is a diameter of the circle. The positi
vectors of A and R with respect to O are given by **OA** = **a** and **OR** = **r**. Write down expressions in terms of
and **r** for **AR** and **BR**, and evaluate the scalar product **AR . BR**. What is the corresponding geometrical resu
[C

4. Three points A, B and C have position vectors $6\mathbf{i} + 7\mathbf{j}$, $4\mathbf{j}$ and $4\mathbf{i} + 3\mathbf{j}$ respectively. P is the point lying on th
line BC between B and C such that $\frac{BP}{PC} = \frac{3}{2}$, and Q is the point on BC produced such that $\frac{BQ}{QC} = \frac{3}{2}$. Fi
position vectors of P and Q and evaluate the scalar product **AP . AQ**. Deduce, or prove otherwise, that AP
perpendicular to AQ. Show also that the ratio $\frac{|QC|}{|CP|}$ is equal to the ratio $\frac{|QB|}{|BP|}$. [C

5. The position vectors of the points A, B and C are given by $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{c} = 11\mathbf{i} + \lambda\mathbf{j} + 14\mathbf{k}$. Find:
   (a) the unit vector parallel to AB
   (b) the position vector of the point D such that ABCD is a parallelogram
   (c) the value of λ if A, B and C are collinear
   (d) the position vector of the point P on AB if AP : PB = 2 : 1. [C]

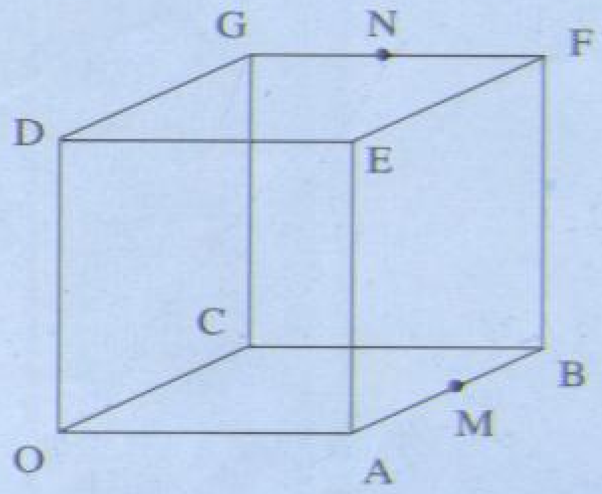
6. O is the origin, A is the point (1, 3), B is the point (–2, 1) and P is the variable point given by $\mathbf{OP} = \mathbf{OA} + t\,\mathbf{AB}$. Calculate:
   (a) the scalar product $\mathbf{OA} \cdot \mathbf{OB}$
   (b) the size, to the nearest degree of angle AOB
   (c) the value of t for which OP is perpendicular to AB
   (d) the value of t for which $|\mathbf{OP}| = |\mathbf{AB}|$. [C]

7. (a) $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors and $\mathbf{a} = (\mathbf{a} \cdot \mathbf{b})\mathbf{b}$. State the relation between the directions of $\mathbf{a}$ and $\mathbf{b}$ and find $|\mathbf{b}|$.
   (b) $\mathbf{a}$ is the position vector of a fixed point A relative to a fixed origin O. A variable point P has position vector $\mathbf{r}$ relative to O. Find the locus of P if $\mathbf{r} \cdot (\mathbf{r} - \mathbf{a}) = 0$. [C]

8. With respect to an origin O, the position vectors of points A, B and C are:
   $OA = 9\mathbf{i} + 7\mathbf{j} - \mathbf{k}$
   $OB = 3\mathbf{i} - 11\mathbf{j} + 5\mathbf{k}$
   $OC = 5\mathbf{i} - 5\mathbf{j} - \mathbf{k}$

   (a) Find the cosine of the angle CAB and prove that the area of the triangle ABC is $12\sqrt{10}$.
   (b) Find the position vectors of D and E, where D is the point of trisection of AB nearer to A and E is the mid-point of CD. [C]

9.



In the diagram, OABCDEFG is a cube in which the length of each edge is 2 units. Unit vectors **i**, **j**, **k** are parallel to **OA**, **OC**, **OD** respectively. The mid-points of AB and FG are M and N respectively.
(a) Express each of the vectors **ON** and **MG** in terms of **i**, **j**, **k**.
(b) Find the angle between the direction of **ON** and **MG**, correct to the nearest 0.1° [C]

10. Referred to an origin O, the position vectors of the points A and B are given respectively by $\mathbf{OA} = 3\mathbf{i} + \mathbf{j}$
$3\mathbf{k}$, $\mathrm{OB} = 5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$.

    Show that the cosine of angle AOB is equal to $\frac{4}{\sqrt{38}}$. Hence, or otherwise, find the position vectors of
point P on OB such that AP is perpendicular to OB.

    The reflection of A in the line OB is A′. Find the position vector of A′. [C

11. Referred to an origin O, points A and B have position vectors given respectively by $\mathbf{OA} = \mathbf{i} + 2\mathbf{j} -$
and $\mathbf{OB} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$. The point P on AB is such that $\mathrm{AP} : \mathrm{PB} = \lambda : 1 - \lambda$.

    Show that $\mathbf{OP} = (1 + \lambda)\mathbf{i} + (2 - 5\lambda)\mathbf{j} + (-2 + 8\lambda)\mathbf{k}$.

    (a) Find the value of λ for which **OP** is perpendicular to **AB**.

    (b) Find the value of λ for which angles AOP and POB are equal. [C

12. The position vectors of the points A, B, C with respect to an origin O are given by $\mathbf{a} = 3\mathbf{i}$, $\mathbf{b} = 2\mathbf{j}$, $\mathbf{c} =$
respectively.

    (a) Find the exact value of the cosine of angle ACB.

    (b) Show that the area of triangle ACB is $16\frac{1}{2}$ square units.

    (c) Write down the volume of the tetrahedron OABC, and hence, or otherwise, find the perpendicular
    from O to the plane ABC.

    [The volume of the tetrahedron is $\frac{1}{3} \times$ area of base $\times$ perpendicular height]. [C

13. The position vectors **a**, **b**, **c** of three points A, B, C respectively are given by $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} +$
$\mathbf{c} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

    Find:

    (a) a unit vector parallel to $\mathbf{a} + \mathbf{b} + \mathbf{c}$.

    (b) the cosine of the angle between $\mathbf{a} + \mathbf{b} + \mathbf{b}$ and the vector **a**.

    (c) the vector of the form $\mathbf{i} + \lambda\mathbf{j} + \mu\mathbf{k}$ perpendicular to both **a** and **b**.

    (d) the position vector of the point D which is such that ABCD is a parallelogram having BD as a

    [C

CHAPTER 7

> SERIES

7.1 Binomial expansion

7.1.1 The expansion of $(a + b)^n$ where n is a positive integer

$$(a+b)^0 = 1$$
$$(a+b)^1 = a + b = 1a + 1b$$
$$(a+b)^2 = a^2 + 2ab + b^2 = 1a^2 + 2ab + 1b^2$$
$$\begin{aligned}(a+b)^3 &= (a+b)\ (a+b)^2 \\ &= (a+b)\ (a^2 + 2ab + b^2) \\ &= a^3 + 3a^2b + 3ab^2 + b^3 \\ &= 1a^3 + 3a^2b + 3ab^2 + 1b^3\end{aligned}$$
$$\begin{aligned}(a+b)^4 &= (a+b)\ (a+b)^3 \\ &= (a+b)\ (a^3 + 3a^2b + 3ab^2 + b^3) \\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\ &= 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4\end{aligned}$$

We note that the expansion of $(a + b)^1$ contains 2 terms, that of $(a + b)^2$ 3 terms, that of $(a + b)^3$ 4 terms, etc.

So, $(a + b)^n$ will contain $(n + 1)$ terms if n is a positive integer.

Also, in $(a + b)^3$, the terms are a^3, a^2b, ab^2, b^3

In $(a + b)^4$, the terms are a^4, a^3b, a^2b^2, ab^3, b^4.

In $(a + b)^n$, the terms are $a^n, a^{n-1}b, a^{n-2}b^2, a^{n-3}b^3, \ldots\ldots, ab^{n-1}, b^n$.

We now have a closer look at the coefficients.

For power 1 the coefficients are 1, 1
2 the coefficients are 1, 2, 1
3 the coefficients are 1, 3, 3, 1
4 the coefficients are 1, 4, 6, 4, 1

These may be written for powers 1, 2, 3 and 4 as

$1, \frac{1}{1}$ for power 1

$1, \frac{2}{1}, \frac{2 \times 1}{1 \times 2}$ for power 2

$1, \frac{3}{1}, \frac{3 \times 2}{1 \times 2}, \frac{3 \times 2 \times 1}{1 \times 2 \times 3}$ for power 3

$1, \frac{4}{1}, \frac{4 \times 3}{1 \times 2}, \frac{4 \times 3 \times 2}{1 \times 2 \times 3}, \frac{4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4}$ for power 4

If this pattern holds for any power, the coefficients for powers 5 and 6 are respectively

$$1, \frac{5}{1}, \frac{5 \times 4}{1 \times 2}, \frac{5 \times 4 \times 3}{1 \times 2 \times 3}, \frac{5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4}, \frac{5 \times 4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4 \times 5}$$

$$1, \frac{6}{1}, \frac{6 \times 5}{1 \times 2}, \frac{6 \times 5 \times 4}{1 \times 2 \times 3}, \frac{6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4}, \frac{6 \times 5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4 \times 5}, \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4 \times 5 \times 6}$$

The coefficients for powers of n are

$$1, \frac{n}{1}, \frac{n(n-1)}{1 \times 2}, \frac{n(n-1)(n-2)}{1 \times 2 \times 3}, \frac{n(n-1)(n-2)(n-3)}{1 \times 2 \times 3 \times 4}, \ldots\ldots\ldots \frac{n(n-1)(n-2)\ldots\ldots 1}{1 \times 2 \times 3 \ldots\ldots n}$$

The coefficient $\frac{n}{1}$ is as written ${}^nC_1$ or $\binom{n}{1}$, $\frac{n(n-1)}{1 \times 2}$ is written as ${}^nC_2$ or $\binom{n}{2}$, $\frac{n(n-1)(n-2)}{1 \times 2 \times 3}$ is written

${}^nC_3$ or $\binom{n}{3}$.

So, the coefficients are 1, ${}^nC_1$ or $\binom{n}{1}$, ${}^nC_2$ or $\binom{n}{2}$, ${}^nC_3$ or $\binom{n}{3}$ ……… ${}^nC_{n-1}$, 1 which can also be written as ${}^nC_0$, ${}^nC$

${}^nC_2$, ${}^nC_3$ ……… ${}^nC_n$ or

$\binom{n}{0}, \binom{n}{1}, \binom{n}{2} \ldots\ldots\ldots \binom{n}{n}$ respectively.

So, $(a + b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + {}^nC_3 a^{n-3}b^3 + {}^nC_4 a^{n-4}b^4 \ldots\ldots + b^n$

This result is the *binomial theorem* for positive integral values of n. It can also be written as

$(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots\ldots + b^n$

Example 1

Obtain the expansion of $(3x + 2y)^5$ in descending powers of x.

Solution

The first term contains $(3x)^5$, the next $(3x)^4(2y)^1$, the next $(3x)^3(2y)^2$, etc.

The coefficients are 1, ${}^5C_1$, ${}^5C_2$, ${}^5C_3$, ${}^5C_4$, 1.

So, $(3x + 2y)^5 = (3x)^5 + {}^5C_1 (3x)^4(2y)^1 + {}^5C_2 (3x)^3(2y)^2 + {}^5C_3 (3x)^2(2y)^3 + {}^5C_4 (3x)^1(2y)^4 + (2y)^5$.

$= 243x^5 + 810x^4y + 1080x^3y^2 + 720x^2y^3 + 240xy^4 + 32y^5$

Note: ${}^nC_r$ is obtained on the calculator, e.g. ${}^5C_3$ is obtained by entering 5, pressing the key ${}^nC_r$, then enteri and pressing the = key.

Example 2

Obtain the first 4 terms of $\left(3 - \frac{1}{2}x\right)^7$ in ascending powers of x.

Solution

The first 4 terms contain $3^7, 3^6\left(-\frac{1}{2}x\right)^1, 3^5\left(-\frac{1}{2}x\right)^2, 3^4\left(-\frac{1}{2}x\right)^3$ and the coefficients are 1, ${}^7C_1$, ${}^7C_2$, ${}^7C_3$.

So, $\left(3-\frac{1}{2}x\right)^7 = 3^7 + {}^7C_1 3^6\left(-\frac{1}{2}x\right)^1 + {}^7C_2 3^5\left(-\frac{1}{2}x\right)^2 + {}^7C_3 3^4\left(-\frac{1}{2}x\right)^3$ + higher powers of x.

$= 2187 - \frac{729}{2}x + \frac{5103}{4}x^2 - \frac{2835}{8}x^3$ + higher powers of x.

The first 4 terms are therefore $2187 - \frac{729}{2}x + \frac{5103}{4}x^2 - \frac{2835}{8}x^3$.

7.1.2 Approximations

The binomial theorem may be used to find the approximate or exact values of powers of numbers as illustrated in the following examples.

Example 3

Find the value of 1.998^5 to six places of decimals.

Solution

$1.998^5 = (2 - 0.002)^5$

$= 2^5 + {}^5C_1 2^4(-0.002) + {}^5C_2 2^3(-0.002)^2 + {}^5C_3 2^2(-0.002)^3 + {}^5C_4 2(-0.002)^4 + (-0.002)^5$

As the value of 1.998^5 is required to six places of decimals, it is unnecessary to use all the terms.
We note $(0.002)^2$ contains 5 zeroes after the decimal place, $(0.002)^3$ contains eight zeroes after the decimal place.
It is therefore unnecessary to consider terms with higher powers than 3.

So, $1.998^5 \approx 2^5 + {}^5C_1 2^4(-0.002) + {}^5C_2 2^3(-0.002)^2 + {}^5C_3 2^2(-0.002)^3$

$= 31.840320$ to six places of decimals. (Calculator)

Example 4

Find the first four terms of $\left(2+\frac{1}{4}x\right)^8$ in ascending powers of x and hence obtain 2.0025^8 to 3 places of decimals.

Solution

$\left(2+\frac{1}{4}x\right)^8 = 2^8 + {}^8C_1 2^7\left(\frac{1}{4}x\right)^1 + {}^8C_2 2^6\left(\frac{1}{4}x\right)^2 + {}^8C_3 2^5\left(\frac{1}{4}x\right)^3$ + higher powers of x.

$\approx 256 + 256x + 112x^2 + 28x^3$

To obtain 2.0025^8, we put $2 + \frac{1}{4}x = 2.0025$:

$\frac{1}{4}x = 0.0025$

$x = 0.01$

So, $2.0025^8 \approx 256 + 256(0.01) + 112(0.01)^2 + 28(0.01)^3$

$= 258.571$ to 3 places of decimals.

Exercise 7 A

1. Find the binomial expansions of each of the following, simplifying the coefficients and giving them fractions where necessary:

   (a) $(2 + x)^4$ (b) $(3 - 2a)^5$ (c) $\left(3 + \frac{1}{2}x\right)^4$

   (d) $(2a - 3b)^3$ (e) $\left(2 - \frac{1}{3}x\right)^5$ (f) $\left(\frac{2}{3}a - \frac{3}{2}b\right)^4$

   (g) $\left(\frac{1}{2}a - \frac{4}{3}b\right)^3$ (h) $\left(\frac{3}{2}a - \frac{1}{3}b\right)^4$ (i) $\left(\frac{4}{3}x - \frac{3}{2}y\right)^3$

2. Find the first four terms in ascending powers of x of the binomial expansion of:

   (a) $(1 + 2x)^6$ (b) $(2 - 3x)^5$ (c) $\left(2 + \frac{1}{4}x\right)^6$

   (d) $\left(\frac{3}{2} - \frac{2}{3}x\right)^4$ (e) $\left(\frac{3}{4}y - 2x\right)^5$ (f) $\left(\frac{2}{5}y - \frac{5}{2}x\right)^5$

3. Obtain the binomial expansion of $(1 + 2x)^5$. Hence, find the value of 1.002^5 to six places of decimals.

4. Find the first four terms of $\left(2 + \frac{1}{8}\right)^6$ in ascending powers of x. Hence, find the value of 2.0125^6 to

   places of decimals.

5. Find the first four terms of $\left(2 - \frac{1}{3}x\right)^5$ in ascending powers of x. Hence, find the value of $\left(\frac{599}{300}\right)^5$ to

   places of decimals.

6. Find the first four terms of $(1 + 2x)^7$ in ascending powers of x. Hence, find the value of $\left(\frac{26}{25}\right)^7$ to places of decimals.

7.2.1 To find a given term or the coefficient of a team in the binomial expansion of (a +

Since $(a + b)^n = a^n + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}b^2 + {}^nC_3\, a^{n-3}b^3 + {}^nC_4\, a^{n-4}b^4 \cdots\cdots + b^n$

We note that

(a) the power of 'b' in any term is one less than the rank of the term. Thus, the 3^{rd} term contains b^{3-1}, the 4^{th} contains b^{4-1}, etc.

(b) the power of a + the power of b is equal to n for any term.

(c) the subscript r in ${}^nC_r$ in any term is equal to the power of b.

These results are used to find any term or the coefficient of any term in a given expansion as illustrated following examples:

Example 5

Find the 5^{th} term of the binomial expansion of $(3 - 2x)^8$ in ascending powers of x.

Solution

The 5^{th} term contains $(-2x)^4$, $(3)^{8-4}$ and its coefficient is ${}^8C_4$.

So, 5^{th} term $= {}^8C_4 (3)^4 (-2x)^4$

$= 90720\, x^4$.

Example 6

Find the 6th term of the binomial expansion of $\left(\frac{3x}{2}+\frac{2}{3x}\right)^9$ *in descending powers of* x *.*

Solution

The 6th term contains $\left(\frac{2}{3x}\right)^5, \left(\frac{3x}{2}\right)^{9-5}$ and its coefficient is ${}^9C_5$.

$$\text{So, 6}^{\text{th}}\text{ term} = {}^9C_5\left(\frac{3x}{2}\right)^4\left(\frac{2}{3x}\right)^5$$

$$= \frac{252}{3x}$$

Example 7

Find the coefficient of x^4 *in the expansion* $(1 + 2x)^7$*.*

Solution

The term in $x^4 = {}^7C_4\,(1)^3\,(2x)^4$

$= 560\,x^4$

Coefficient of $x^4 = 560$

Example 8

Find the coefficient of x^{10} *in the expansion* $(3 - 2x^2)^9$*.*

Solution

Term in x^{10} is obtained by considering $(-2x^2)^5$

Required term $= {}^9C_5\,3^4\,(-2x^2)^5$

$= -\,326\,592\,x^{10}$

Required coefficient $= -326\,592$.

Example 9

Find the coefficient of x^3 *in* $\left(x-\frac{2}{x}\right)^9$

Solution

In this case none of the 2 terms is a constant.

We rewrite $\left(x-\frac{2}{x}\right)^9$ as $\left(\frac{x^2-2}{x}\right)^9 = \frac{(x^2-2)^9}{x^9}$

As each term of $(x^2 - 2)^9$ is divided by x^9, to obtain x^3 we consider the term containing x^{12} in the expansion $(x^2 - 2)^9$.

$$\text{Required term} = \frac{{}^9C_3\left(x^2\right)^6\left(-2\right)^3}{x^9}$$

$$= -672\ x^3$$

Coefficient of $x^3 = -672$

Example 10

Find the term independent of x in the expansion of $\left(2x + \frac{3}{x}\right)^8$.

Solution

Rewriting $\left(2x + \frac{3}{x}\right)^8$ as $\frac{\left(2x^2 + 3\right)^8}{x^8}$, the term independent of x is obtained by considering the term containing x^8 i the expansion of $(2x^2 + 3)^8$.

$$\text{Required term} = \frac{{}^8C_4\left(2x^2\right)^4\left(3\right)^4}{x^8}$$

$$= 90\ 720.$$

7.2.2 To find a given term in a product of two binomials

To find a term containing a given power of x in a product containing at least one binomial expansion, it is un essary to obtain the complete expansion.

To find the term in x^3 in the expansion of $(2 + 3x)(1 + 3x)^4$, we consider only terms which give x^3.

We have to multiply 2 by a term in x^3 and 3x by a term in x^2. The only terms which are of interest to us are the fore the terms in x^2 and x^3.

To find the coefficient of x^4 in the expansion of $(1 + 2x - 3x^2)(2 + x)^7$, we multiply 1 by the term in x^4, 2x by t term in x^3, $-3x^2$ by the term in x^2. The three terms to be found in the expansion of $(2 + x)^7$ are therefore the te x^2, x^3 and x^4.

Example 11

Find the coefficient of x^3 *in* $(1+ 2x)(2 - x)^6$.

Solution

The term in x^3 is obtained by multiplying 1 by the term in x^3 and 2x by the term in x^2 in the expansion of (2 –

Term in $x^3 = {}^6C_3\ 2^3\ (-x)^3$

$= -160\ x^3$

Term in $x^2 = {}^6C_2\ 2^4\ (-x)^2$

$= 240\ x^2$

Required term = $1 \times -160x^3 + 2x \times 240x^2$

$= 320\ x^3$

Coefficient of $x^3 = 320$

Example 12

Find the first 4 terms in the expansion in ascending powers of x of $(2 + x)^4$ and $(1 - 3x)^6$. Hence, find the coefficient of x^3 in $(2 + x)^4\ (1 - 3x)^6$.

Solution

$(2+x)^4 = 2^4 + {}^4C_1\ 2^3\ (x) + {}^4C_2\ 2^2\ (x)^2 + {}^4C_3\ 2(x)^3$, ... etc.

$\approx 16 + 32x + 24x^2 + 8x^3$

$(1-3x)^6 = 1^6 + {}^6C_1\ (1)^5\ (-3x)^1 + {}^6C_2\ (1)^4\ (-3x)^2 + {}^6C_3\ (1)^3\ (-3x)^3$, ∴...

$\approx 1 - 18x + 135x^2 - 540x^3$.

Term in $x^3 = (16 \times -540 + 32 \times 135 + 24 \times -18 + 8 \times 1)\ x^3$

$= -4744\ x^3$

Coefficient of $x^3 = -4744$.

Exercise 7 B

1. Find: (a) the 4th term of the expansion of $(1 + 3x)^5$ in ascending powers of x.

   (b) the 5th term of the expansion of $(2 - x)^8$ in ascending powers of x.

   (c) the 6th term of the expansion of $(3 + 2x)^7$ in ascending powers of x.

   (d) the 5th term of the expansion of $\left(\frac{3x}{2} + \frac{2}{3}\right)^8$ in ascending powers of x.

   (e) the 8th term of the expansion of $\left(\frac{3x}{2} - \frac{2}{3x}\right)^{11}$ in descending powers of x.

2. Find the coefficient of:

   (a) x^2 in $(1 + 2x)^5$ (b) x^3 in $(2 - x)^6$ (c) x^{-3} in $\left(3 + \frac{1}{x}\right)^5$ (d) x^4 in $(1 - 2x^2)^5$

   (e) x^6 in $(2 + x^3)^7$ (f) x^{-8} in $\left(3 + \frac{4}{x^4}\right)^5$ (g) x^{-12} in $(x^{-4} + 2)^6$ (h) x^9 in $(x^3 - 2)^9$

3. (a) Find the coefficient of

   (i) x^2 in $\left(3x - \frac{2}{x}\right)^6$ (ii) x^5 in $\left(2x - \frac{1}{x}\right)^9$ (iii) x^{-4} in $\left(x - \frac{3}{x}\right)^6$ (iv) x^{-4} in $\left(2x + \frac{1}{x}\right)^{10}$

   (b) Find the term independent of x in

   (i) $\left(3x + \frac{2}{x}\right)^6$ (ii) $\left(2x - \frac{1}{x}\right)^{10}$ (iii) $\left(2x + \frac{3}{x}\right)^8$ (iv) $\left(3x^2 - \frac{1}{x}\right)^6$

4. Find the coefficient of
   (a) x^2 in $(1 + x)(2 - x)^6$ (b) x^3 in $(1 - 2x)(3 + x)^5$ (c) x^2 in $(1 - 2x + x^2)(1 + 2x$
   (d) x^3 in $(3 + x - 2x^2)(2 - x)^5$ (e) x^3 in $(1 + 2x^2)(3 - x)^4$

5. Given that the expansion of $(1 + ax)^3(1 + bx)^4$ in ascending powers of x is $1 + 5x + 3x^2$ + higher powers of find the value of a and of b.

6. Find the first three terms of $(1 + x)^5$ and $(2 - 3x)^5$. Hence or otherwise, find the coefficient of x^2 in $(2 - x - 3x$

7. The expansion of $(3 + 2x)(1 - x)^n$, in ascending powers of x as far as the term in x^2, is $3 - 10x + ax^2$. Fi the value of n and of a.

8. By considering the factors of quadratic expression or otherwise, find the coefficient of x^2 in $(2 + x - x^2)^4 - (3 - 2x - x^2)^5$.

7.3 Arithmetic progressions

7.3.1 Examples of arithmetic progressions

Consider the series $5 + 8 + 11 + 14 + \ldots\ldots$

Each term of the series is obtained from the preceding one by adding 3 to it.

Similarly for series $5 + 5\frac{3}{4} + 6\frac{1}{2} + 7\frac{1}{4} + \ldots\ldots$ each term is obtained from the preceding one by adding $\frac{3}{4}$ to it

For the series $10 + 8 + 6 + 4 + \ldots.$ each term is obtained from the preceding one by subtracting 2 from it or addin -2 to it.

All these series are called arithmetic series or arithmetic progressions.

$9 + 11 + 13 + 15 \ldots.$ is an arithmetic progression whereas

$3 + 5 + 7 + 10 \ldots.$ is not an arithmetic progression.

If U_r and U_{r+1} are any two consecutive terms of a series, the series is an arithmetic progression if $U_{r+1} - U_r$ is a const

This constant is called the *common difference*.

For the series $5 + 8 + 11 + 14\ldots.$ the common difference is 3 and for the series $10 + 8 + 6 + 4\ldots.$ it is -2.

If the terms increase, the common difference is positive and if they decrease, it is negative.

7.3.2 The n^th^ term of an arithmetic progression

We refer to an arithmetic progression as an A.P.

Consider the general arithmetic progression with first term a and common difference d. Denoting the first term T_1, the second term by T_2, etc. we have

$T_1 = a$

$T_2 = a + d$

$T_3 = a + d + d = a + 2d$

$T_4 = a + 2d + d = a + 3d$

From the pattern, we have $T_{10} = a + 9d$, $T_{25} = a + 24d$ and generally $T_n = a + (n - 1)d$.

Example 13

Find the 15th term of $4 + 9 + 14 + 19$ ....

Solution

$a = 4$

$d = 5$

$T_{15} = a + 14d$

$= 4 + 70 = 74$

Example 14

Find the first negative term of the progression $90 + 87 + 84 +$ ....

Solution

$a = 90.$

$d = -3$

$T_n = a + (n - 1)d$

$= 90 + (n - 1)(-3)$

$= 93 - 3n$

$93 - 3n < 0$

$-3n < -93$

$n > 31$

The first negative term is the 32nd term.

$T_{32} = a + 31d$

$= 90 + 31 \times -3$

$= -3.$

Example 15

The 5th term of an A.P is 13 and the 9th term is 25. Find the first term and the common difference.

Solution

$T_5 = a + 4d = 13$ (1)

$T_9 = a + 8d = 25$ (2)

$4d = 12$ (2) – (1)

$d = 3$

From (1) $a + 12 = 13$

$a = 1$

First term is 1 and the common difference 3.

Example 16

How many terms are there in the series 50 + 46 + 42 + ... + 2?

Solution

$a = 50$

$d = -4$

$T_n = 2$

$$\begin{aligned} a + (n-1)d &= 2 \\ 50 + (n-1)(-4) &= 2 \\ (n-1)(-4) &= -48 \\ n - 1 &= 12 \\ n &= 13 \end{aligned}$$

Number of terms = 13

7.3.3 The sum of the first n terms of an arithmetic progression

Consider the series 15 + 18 + 21 + ... + 87

The number of terms n is obtained by using

$$a + (n-1)d = T_n$$
$$15 + (n-1)3 = 87$$
$$n = 25$$

Denoting the sum by S

$S = 15 + 18 + 21 + ... + 81 + 84 + 87$ (1)

$S = 87 + 84 + 81 + ... + 21 + 18 + 15$ (2)

We have $2S = 102 + 102 + 102$ .... 25 times

$= 102 \times 25$

$S = 51 \times 25 = 1\,275.$

We use the same procedure to find the sum S of the first n terms of the A.P

$a + (a + d) + (a + 2d) + ... + l$

$S = a + (a + d) + (a + 2d) + ... + l$

$S = l + (l - d) + (l - 2d) + ... + a$

$2S = (a + l) + (a + l) + (a + l)$ n times

$= n(a + l)$

$$S = \frac{n}{2}(a + l)$$

As l is the n^{th} term, $l = a + (n-1)d$

$$S = \frac{n}{2}\{a + a + (n-1)d\}$$

$$= \frac{n}{2}\{2a + (n-1)d\}$$

To summarise, the three important formulae for an A.P are:

$T_n = a + (n - 1)d$

$S = \frac{n}{2}(a + l)$

$S = \frac{n}{2}\{2a + (n - 1)d\}$

Example 17

Find the sum of the first 25 terms of the arithmetic progression $25 + 22\frac{1}{2} + 20 + 17\frac{1}{2} + \ldots\ldots$

Solution

$a = 25$
$d = -2\frac{1}{2}$
$n = 25$

$$S = \frac{n}{2}\{2a + (n - 1)d\}$$

$$= \frac{25}{2}\left\{50 + 24 \times -2\frac{1}{2}\right\}$$

$$= \frac{25}{2} \times -10$$

$$= -125$$

Example 18

Find the sum of the progression $2.2 + 3.1 + 4.0 + 4.9 + \ldots\ldots. + 23.8$

Solution

We find n first

$$2.2 + (n - 1) \times 0.9 = 23.8$$
$$(n - 1) \times 0.9 = 21.6$$
$$n - 1 = 24$$
$$n = 25$$

$$S = \frac{n}{2}(a + l)$$

$$= \frac{25}{2}(2.2 + 23.8)$$

$$= 325.$$

Example 19

How many terms of the series 2 + 5 + 8 + ... are required to make a total of 1 365?

Solution

$a = 2$

$d = 3$

$S = 1\,365$

$$\text{Using } S = \frac{n}{2}\left\{2a + (n-1)d\right\}$$

$$1\,365 = \frac{n}{2}\left\{4 + (n-1)3\right\}$$

$$= \frac{n}{2}(3n + 1)$$

$$2\,730 = 3n^2 + n$$

$$3n^2 + n - 2\,730 = 0$$

$$n = \frac{-1 \pm \sqrt{1 - 4 \times 3 \times -2730}}{6}$$

$$= 30 \text{ or } -30.33$$

As n is a positive integer, n = 30

Number of terms = 30.

Exercise 7 C

1. Find:
   (a) the 19th term of $9 + 13 + 17 + 21$ ......
   (b) the 30th term of $21 + 20\frac{1}{2} + 20 + 19\frac{1}{2}$ ......
   (c) the 100th term of $194 + 191 + 188 + 185$ ......
   (d) the 25th term of $13.5 + 13.2 + 12.9 + 12.6$ ...
   (e) the nth term of $21 + 25 + 29 + 33$ ......
   (f) the nth term of $19 + 18\frac{1}{4} + 17\frac{1}{2} + 16\frac{3}{4}$ ......
   (g) the rth term of $1.2 + 2.5 + 3.8 + 5.1$ ......

2. Find the number of terms in each of the following progressions and hence obtain their sum:
   (a) $4 + 9 + 14 +$ .... $+ 129$
   (b) $8 + 7.8 + 7.6 +$ .... $+ 1.6$
   (c) $1 + 2 + 3 +$ .... $+ 100$
   (d) $-10 - 7 - 4 -$ .... $+ 35$
   (e) $3\frac{1}{2} + 3\frac{3}{4} + 4$.......... $+ 11\frac{3}{4}$

3. Find the first positive term of the progression $-100 - 96 - 92$ ......

4. The 4th term of an A.P is 31 and the 11th term is 3. Find the common difference, the first term and the first negative term.

   Find also the sum of the first 20 terms.

5. The 3rd term of an A.P is 11 and the 20th term is 45. Find the sum of the first 25 terms of the A.P.

6. Given $x + 1$, $2x - 3$ and $4x - 8$ are the consecutive terms of an A.P, find the value of x and of the common difference.

   Find the sum of the first 10 terms of the A.P.

7. Find the sum of:

   (a) $6 + 11 + 16 + ...$ to 25 terms

   (b) $5.2 + 5.6 + 6.0 + ...$ to 32 terms

   (c) $21 + 18 + 15 + ...$ to 40 terms

   (d) $8\frac{1}{4} + 9\frac{1}{2} + 10\frac{3}{4} + ...$ to 25 terms

   (e) $1 + 2 + 3 + ...$ to n terms

   (f) $x + 3x + 5x + ...$ to n terms

   (g) $2x + 4x + 6x + ...$ to n terms

   (h) $7 + 6.4 + 5.8 + ...$ to n terms.

8. The first term of an A.P is 4 and the last term is 79. Given that the sum of the A.P is 996, find the number of terms and the common difference.

9. Find the sum of the first n terms of the series $2 + 6 + 10 + ...$ Hence, find the least number of terms of the series which must be taken for the sum to exceed 20 000.

10. Find the sum of the first n terms of the series $100 + 96 + 92 + ...$

    Hence, find the least number of terms of the series which must be taken for the sum to be less than 0.

11. The sum of the second term and the third term of an A.P is 42 and the sum of the fourth term and the eighth term is 14. Find the first term, the common difference and the sum of the first 15 terms.

12. How many terms of the series $18 + 18.4 + 18.8 + 19.2 + ...$ must be taken for the sum to be 774.4?

13. The sum of the first ten terms of an A.P is 260 and the sum of the next ten terms is 660. Find the common difference and the first term.

14. The sum of the first n terms of an arithmetic progression is $\frac{7n - n^2}{4}$. Find the first term and the common difference. Find also the least number of terms required for the sum to be less than -6.

15. The sizes of the angles of a quadrilateral are in arithmetic progression and the largest angle exceeds the smallest angle by 120°. Find all the angles.

16. The lengths of the sides of a pentagon are in arithmetic progression and the perimeter of the pentagon is 45 cm. The largest side exceeds the smallest side by 8 cm. Find the length of each side.

17. A piece of string of 3.6 m is cut into 9 pieces. The lengths of the pieces are in arithmetic progression wi the largest piece measuring 60 cm. Find the length of the smallest piece.

18. (1), (4, 7), (10, 13, 16), (19, 22, 25, 28) ...... is a sequence of terms. Find:
    (a) the number of terms in the first n brackets.
    (b) the last term in the n^{th} bracket.
    (c) the least value of n for which the last term in the n^{th} bracket is to exceed 628.

7.4 Geometric progression

7.4.1 Examples of geometric progressions

Consider the series 2, 6, 18, 54 ......

Each term is obtained from the preceding one on multiplying by 3.

Similarly, for the series 24, 12, 6, 3 ...... each term is obtained from the preceding one on dividing by 2 multiplying by $\frac{1}{2}$.

For the series $-1 + 2 - 4 + 8$ ......each term is obtained from the preceding one on multiplying by −2.

All these series are geometric progressions.

$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$ ...... is a geometric progression whereas 3, −6, −12. −24 ...... is not a geometric progression.

If U_r and U_{r+1} are any two consecutive terms of a series, this series is a geometric progression if $\frac{U_{r+1}}{U_r}$ is constant. This constant is called the common ratio of the geometric progression.

7.4.2 Finding the n^{th} term of a geometric progression

We refer to a geometric progression as a G.P.

Consider the general geometric progression with first term a and the common ratio r.

$T_1 = a$

$T_2 = ar$

$T_3 = ar \times r = ar^2$

$T_4 = ar^2 \times r = ar^3$

We obtain $T_8 = ar^7$, $T_{20} = ar^{19}$ and generally the **n^{th} term $T_n = ar^{n-1}$**.

Example 20

Find the 8^{th} term of $2 - 4 + 8 + ...$

Solution

$a = 2$

$r = -2$

$T_8 = ar^7$

$= 2 \times (-2)^7$

$= -256$

Example 21

Find the number of terms in the series 2 + 6 + 18 + ... + 1 458.

Solution

$$\begin{aligned} a &= 2 \\ r &= 3 \\ T_n &= ar^{n-1} \\ &= 2 \times 3^{n-1} \\ 2 \times 3^{n-1} &= 1\,458 \\ 3^{n-1} &= 729 \\ &= 3^6 \\ n - 1 &= 6 \\ n &= 7 \end{aligned}$$

7.4.3 Sum of the first n terms of a G.P

For a G.P with first term a, common ratio r, the sum S is given by

$S = a + a^r + ar^2 + \ldots\ldots\ldots\ldots\ldots + ar^{n-1}$ (1)

Multiplying by r, we have

$rS = ar + ar^2 + \ldots\ldots\ldots\ldots\ldots + ar^{n-1} + ar^n$ (2)

We have
$$\begin{aligned} S - rS &= a - ar^n & \text{or} \quad rS - S &= ar^n - a \\ S(1 - r) &= a(1 - r^n) & \text{or} \quad S(r - 1) &= a(r^n - 1) \\ S &= \frac{a\left(1 - r^n\right)}{1 - r} & \text{or} \quad S &= \frac{a\left(r^n - 1\right)}{r - 1} \; (r \neq 1) \end{aligned}$$

We use $S = \frac{a\left(1 - r^n\right)}{1 - r}$ **preferably for $r < 1$ and $S = \frac{a\left(r^n - 1\right)}{r - 1}$ for $r > 1$.**

Example 22

Find the sum of the first 8 terms of the G.P $24 + 16 + \frac{32}{3} + \ldots\ldots$

Solution

$a = 24$,

$r = \frac{2}{3}$

As $r < 1$, we use $S = \frac{a\left(1 - r^n\right)}{1 - r}$

$$= \frac{24\left[1 - \left(\frac{2}{3}\right)^8\right]}{1 - \left(\frac{2}{3}\right)}$$

$$= 72\left[1 - \left(\frac{2}{3}\right)^8\right]$$

$$= 72\left[1 - \frac{1024}{6561}\right]$$

$$= \frac{72 \times 5537}{6561}$$

$$= 6\frac{556}{729}.$$

Example 23

Find the sum of the first 10 terms of the series 1 – 3 + 9 – 27 ...

Solution

$a = 1$
$r = -3$
As $r < 1$, we have $S = \frac{a(1 - r^n)}{1 - r}$

$$= \frac{1[1 - (-3)^{10}]}{1 - (-3)}$$

$$= \frac{1 - (59049)}{4}$$

$$= \frac{-59049}{4}$$

$$= -14\ 762.$$

7.4.4 Sum to infinity of a geometric progression

Consider the series $20 + 16 + \frac{64}{5} + \ldots\ldots$
The sum of the first n terms of the series S is given by

$$S = \frac{20\left[1 - \left(\frac{4}{5}\right)^n\right]}{1 - \frac{4}{5}}$$

$$= 100\left[1 - \left(\frac{4}{5}\right)^n\right]$$

As n increases, $\left(\frac{4}{5}\right)^n$ becomes smaller and smaller and gets nearer and nearer to 0.

We write $\left(\frac{4}{5}\right)^n \to 0$ as $n \to \infty$.

So, as n becomes very large written as $n \to \infty$, $S \to 100 \times 1 = 100$.

We say the sum to infinity is 100 and we write $S_\infty = 100$.

The number 100 is the sum to infinity of the geometric progression.

If $|r| < 1$, i.e. $-1 < r < 1$, $r^n \to 0$ as $n \to \infty$ the sum to infinity exists.

If $|r| > 1$, the series has no sum to infinity as r^n becomes larger and larger when n becomes larger and larger.

Thus, $8 - 12 + 18 - 27$ ...... has no sum to infinity as $|r| = \frac{3}{2}$.

We recall the important facts we have obtained for a G.P:

$T^n = ar^{n-1}$

$S = \frac{a(r^n - 1)}{r - 1}$ **for $r > 1$ or $\frac{a(1 - r^n)}{1 - r}$ for $r < 1$.**

$S_\infty = \frac{a}{r-1}$ **for $|r| < 1$.**

Example 24

Find the sum to infinity of $24 + 16 + \frac{32}{3} + \frac{64}{9}$......

Solution

$a = 1$

$r = \frac{2}{3}$

$S_\infty = \frac{a}{r-1}$

$= \frac{24}{1 - \frac{2}{3}}$

$= 72.$

Exercise 7 D

1. Find an expression for the required term of each of the following geometric progressions. Do not evaluate this expression but simplify it as much as possible:

(a) 8th term of $6 + 18 + 54$ ......

(b) 16th term of $32 + 16 + 8$ ......

(c) 30th term of $1 - 3 + 9$ ......

(d) 15th term of $8 - 6 + \frac{9}{2}$.......

(e) 21st term of $3 - 3 + 3$ ......

(f) $(r + 1)^{th}$ term of $3 - 6 + 12$ ......

(g) $(2x + 1)^{th}$ term of $1 - \frac{3}{4} + \frac{9}{16}$.......

2. Find the number of terms in each of the following geometric progressions:

   (a) $2 + 6 + 18 + ... + 1\,458$ (b) $3 - 6 + 12 + ... - 1\,536$

   (c) $0.03 + 0.06 + 0.12 + ... + 30.72$ (d) $\frac{1}{9} - \frac{1}{27} + \frac{1}{81} - ...... - \frac{1}{19683}$

   (e) $\frac{1}{9} - \frac{1}{6} + \frac{1}{4} - ...... - \frac{243}{128}$

3. The second term of a G.P is 30 and the fifth term is 810. Find the common ratio and the first term. Find an expression in its simplest form for the sum of the first ten terms but do not evaluate it. Explain why the G.P does not have a sum to infinity.

4. The third term of a G.P is 8 and the seventh term is $\frac{1}{2}$. Find the two possible values of the common and the sum to infinity of each progression.

5. n, 2n + 1 and 4n are the first three terms of a geometric progression. Find n and obtain the eighth term

6. 9n + 5, 2n and n–3 are the first three terms of a geometric progression. Find the two possible values of common ratio.

7. In each of the following, write down an expression for the sum to the required number of terms. Simpl but do not evaluate your answer.

   (a) $36 + 18 + 9$ ... to 10 terms (b) $3 - 6 + 12$ ... to 15 terms (c) $6 + 9 + 13.5$ ... to 8 terms

   (d) $80 - 20 + 5$ ... to 20 terms (e) $0.2 - 0.6 + 1.8$ ... to 25 terms (f) $\frac{3}{2} + 1 + \frac{2}{3}$...... to 9 terms

   (g) $9 - 12 + 16$ .... to 15 terms (h) $-\frac{3}{2} + \frac{9}{16} - \frac{27}{64}$...... to 10 terms.

8. Find an expression for the sum of each of the following geometric progressions. Simplify but do not ev ate your answer.

   (a) $2 + 4 + 8 + ... + 1024$ (b) $1 - 3 + 9 - .... + 729$ (c) $0.3 + 1.2 + 4.8 + ... + 3$

   (d) $\frac{1}{81} - \frac{1}{54} + \frac{1}{36} - ...... - \frac{27}{128}$ (e) $-\frac{32}{729} + \frac{16}{243} - \frac{8}{81} + ... + \frac{3}{4}$

9. In a geometric progression the sum of the second term and the fourth term is 120 and the sum of the term and the fifth term is 360. Find the common ratio, the first term and the sum of the first ten terms.

10. In each of the following examples, find whether the given progression has a sum to infinity. If it has a to infinity, find this sum:

    (a) $24 + 18 + 12 + ...$ (b) $12 - 6 + 3 + ...$ (c) $0.9 + 0.6 + 0.4 + ...$

    (d) $3 - 6 + 12 + ...$ (e) $-\frac{3}{2} + 1 - \frac{2}{3} + ...$

11. The first term of a G.P is 24 and its sum to infinity is 84. Find the common ratio.

12. The sum to infinity of a geometric progression is three times the first term and their sum is 48. Find the first term and the common ratio.

13. The sum of the first three terms of a geometric progression is 74 and its sum to infinity is 128. Find the first term of the progression.

14. A man borrows Rs. 500 000 from a bank which charges him 12% interest per annum on the outstanding amount due at the end of each year. He repays the bank in equal yearly instalments of p rupees. Find expressions for the amount due
    (a) after 1 year
    (b) after 2 years
    (c) after n years.
    Hence, find the value of p if he clears his debt in 15 years.

15. A businessman borrows Rs. 300 000 from a bank for 5 years. The bank charges him 15% interest per annum on the outstanding amount due at the end of each month. He repays the bank in equal monthly instalments of x rupees. Find, in terms of x, the outstanding amount due
    (a) after 1 month
    (b) after 2 months
    (c) after n months.
    Hence, find the value of x.

Miscellaneous Exercise 7

1. Find, in ascending powers of x, the first three terms in the expansion of $(1 + ax)^6$. Given that the first two non-zero terms in the expansion of $(1 + bx)(1 + ax)^6$ are 1 and $-\frac{21x^2}{4}$, find the possible values of a and of b.

2. (a) The sum of the first 8 terms of an arithmetic progression is 24 and the sum of the first 18 terms is 90. Calculate the value of the seventh term.
   (b) A geometric progression with a positive common ratio is such that the sum of the first 2 terms is $17\frac{1}{2}$ and the third term is $4\frac{2}{3}$. Calculate the value of the common ratio.
   [C]

3. Find the first three terms in the expansion, in ascending powers of x, of
   (a) $(1 - 3x)^5$ (b) $(2 + x)^4$
   Hence, find the coefficient of x^2 in the expansion of $(1 - 3x)^5 (2 + x)^4$. [C]

4. Obtain and simplify
   (a) the first four terms in the expansion of $(2 + x^2)^6$ in ascending powers of x
   (b) the coefficient of x^4 in the expansion of $(1 - x^2)(2 + x^2)^6$. [C]

5. (a) A length of 200 cm is divided into 25 sections whose lengths are in arithmetic progression. Given th sum of the lengths of the 3 smallest sections is 4.2 cm, find the length of the largest section.
   (b) An infinite geometric series has a finite sum. Given that the first term is 18 and that the sum of the first 3 terms is 38, calculate the value of:
   (i) the common ratio, (ii) the sum to infinity.

6. Evaluate the coefficients of x^5 and x^4 in the binomial expansion of $\left(\frac{x}{3}-3\right)^7$. Hence, evaluate the coeffic of x^5 in the expansion of $\left(\frac{x}{3}-3\right)^7(x+6)$.

7. (a) The first term of an arithmetic progression is 2. The sum of the first 8 terms is 58 and the sum of th whole series is 325. Calculate:
   (i) the common difference,
   (ii) the number of terms,
   (iii) the last term.
   (b) The first three terms of a geometric progression are x + 5, x + 1, x. Calculate:
   (i) the value of x,
   (ii) the common ratio,
   (iii) the sum to infinity.

8. (a) A geometric progression has non–zero first term a and common ratio r, where $0 < r < 1$. Given that sum of the first 8 terms of the progression is equal to half the sum to infinity, find the value of r, c rect to 3 decimal places. Given also that the 17th term of the progression is 10, find a.
   (b) An arithmetic progression has first term a and common difference 10. The sum of the first n terms the progression is 10 000. Express a in terms of n, and show that the nth term of the progression is

$$\frac{10000}{n}+5(n-1).$$

   Given that the nth term is less than 500, show that $n^2 - 101n + 2000 < 0$, and hence find the largest pos value of n.

9. (a) An arithmetic series has first term 7 and second term 6.8. Find the sum of the first 50 terms.
   (b) A funding body gives a grant to a sports organisation each year from 1995. The amount of grant in 1 is £10 000 and thereafter is 90% of the grant in each preceding year. Calculate:
   (i) the total amount paid in grants to the sports organisation during the years from 1995 to 2004 inclu
   (ii) the total amount paid in grants to the sports organisation during the years from 2005 to 2014 inclu

10. The expansion of $(2x+3x)\left(1-\frac{x}{2}\right)^n$, in ascending powers of x as far as the term in x^2, is $2-5x+ax^2$ Find the value of n and of a.

CHAPTER 8

> DIFFERENTIATION 1

8.1 Gradient of a curve

8.1.1 Definition

We have seen that the gradient of a straight line is the same for any two points chosen on the line.
We consider now the gradient of a curve and we define the gradient of a curve at a point as the gradient of the tangent to the curve at this point (Figure 8.1).

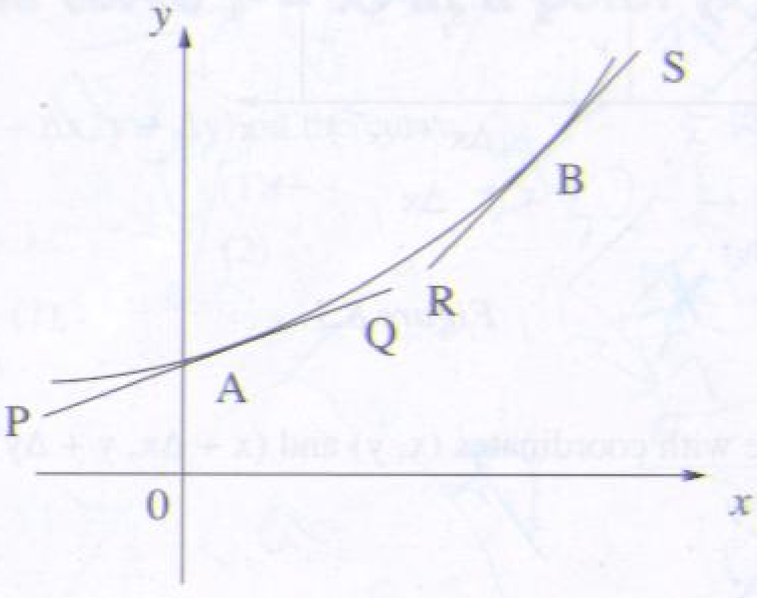


Figure 8.1

Thus, the gradient of the curve in Figure 8.1 at A is the gradient of PQ and that at B is the gradient of RS.
The gradient of the tangent to a curve at a point may therefore be obtained by drawing the curve and the tangent at that point. This method is inaccurate besides being lengthy. The next section looks at a theoretical way of obtaining the gradient of a curve.

8.1.2 A tangent as the limiting position of a chord

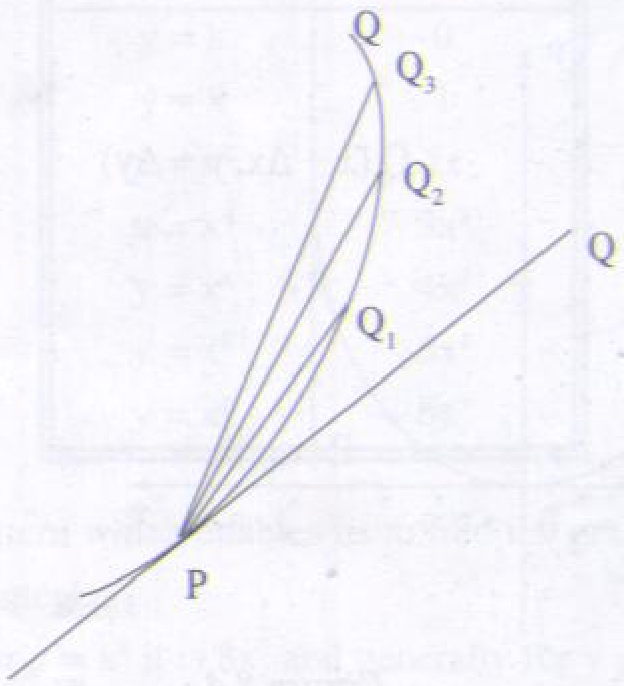


Figure 8.2

Figure 8.2 shows a chord PQ joining two points P and Q on the curve. By choosing points Q_1, Q_2, Q_3 nearer a nearer to P, we see that the chord PQ approaches the tangent at A, i.e. As Q → P, chord PQ → tangent at P whi is read 'as Q approaches P, chord PQ approaches the tangent at P'.

To find the gradient of a curve at point P, we therefore find the gradient of a chord PQ and find the value whi this gradient approaches as Q → P.

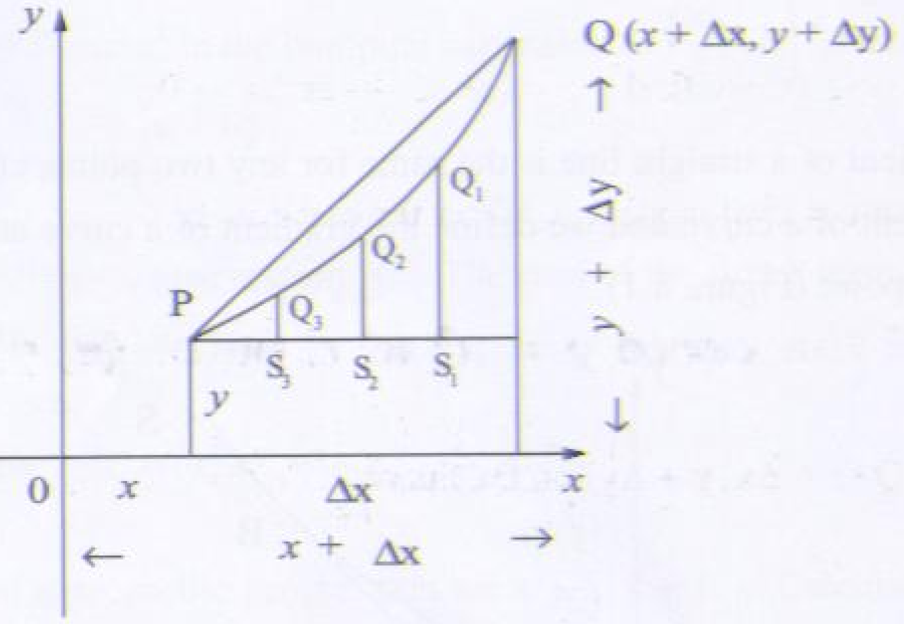


Figure 8.3

Consider two points P, Q on a curve with coordinates (x, y) and $(x + \Delta x, y + \Delta y)$ respectively. The gradient o the chord PQ is $\frac{y + \Delta y - y}{x + \Delta x - x}$ or $\frac{\Delta y}{\Delta x}$.

To obtain the gradient of the tangent at P, we have to find the value which $\frac{\Delta y}{\Delta x}$ approaches as Q → P. From Figure 8.3, we see that as Q → P, both $\Delta x \to 0$ and $\Delta y \to 0$.

The gradient of the curve at P is therefore obtained by finding the value of $\frac{\Delta y}{\Delta x}$ as $x \to 0$. This applies to any curve.

8.1.3 The gradient of the curve $y = x^2$ at a point (x, y) on the curve

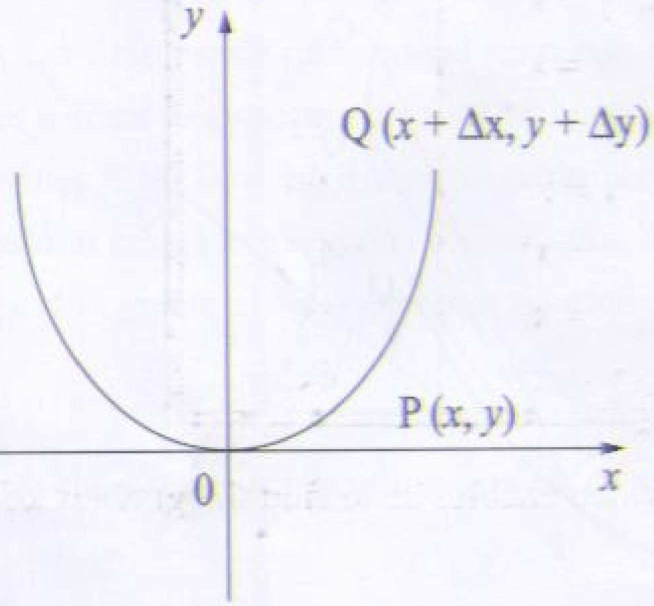


Figure 8.4

Since P and Q lie on the curve $y = x^2$

$$y = x^2 \qquad (1)$$

$$y + \Delta y = (x + \Delta x)^2 \qquad (2)$$

$$\Delta y = (x + \Delta x)^2 - x^2 \qquad (2) - (1)$$

$$= 2x\Delta x + \Delta x^2$$

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x$$

To find the gradient of the tangent, we find the value of $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$.

This value is $2x$.

So, the gradient of the curve $y = x^2$ at a point (x, y) on the curve is $2x$.

8.1.4 Gradient of the curve $y = x^3$ at a point (x, y) on the curve

Taking points $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ on the curve,

$$y = x^3 \qquad (1)$$

$$y + \Delta y = (x + \Delta x)^3 \qquad (2)$$

$$\Delta y = (x + \Delta x)^3 - x^3 \quad (2) - (1)$$

$$= 3x^2 \Delta x + 3x \Delta x^2 + \Delta x^3$$

$$\frac{\Delta y}{\Delta x} = 3x^2 + 3x\Delta x + \Delta x^2$$

As $Q \to P$, $\Delta x \to 0$, $\frac{\Delta y}{\Delta x} \to 3x^2$

The gradient of the curve $y = x^3$ at a point (x, y) on the curve is $3x^2$.

8.1.5 Gradient of the curve $y = x^n$ at a point (x, y) on the curve

If we find the gradient of the curves $y = x^4$, $y = x^5$, $y = x^6$, we obtain gradients $4x^3$, $5x^4$ and $6x^5$ respectively.

Equation	*Gradient*
$y = k$	0
$y = x$	1
$y = x^2$	$2x$
$y = x^3$	$3x^2$
$y = x^4$	$4x^3$
$y = x^5$	$5x^4$
$y = x^6$	$6x^5$

We see immediately that there is a pattern which enables us to find the gradient of a curve if we know its equation in the form $y = x^n$ where n is an integer.

Thus, for $y = x^7$ the gradient is $7x^6$, for $y = x^8$ it is $8x^7$ and generally for $y = x^n$, the gradient is nx^{n-1}. This result is true for all real values of n.

8.1.6 The $\frac{dy}{dx}$ symbol

Given y = f(x), we use the symbol $\frac{dy}{dx}$ for the gradient of the curve at the point (x, y) on the curve y = f(x).

Note:

1. *The gradient of a curve is also known as the derivative or differential coefficient or gradient function or derived function.*
2. *Given* $y = f(x)$*, the process of finding* $\frac{dy}{dx}$ *is called differentiation.*
3. *When the function is given in the form* $f{:}x \mapsto \ldots$ *the gradient of the curve is written* $f'{:}x \mapsto \ldots$ *Thus, if* $f{:}x \mapsto x^4, f'{:}x \mapsto 4x^3$
4. *Given y in terms of x, the gradient is written as* $\frac{dy}{dx}$*. Similarly, if A is given in terms of r, the gradient is* $\frac{dA}{dr}$

   if U is given in terms of f, gradient is $\frac{dU}{df}$ *etc.*

8.1.7 Gradient of y = kx^n for a constant k

$$\text{If } y = 2x^3, \frac{dy}{dx} = 2 \times \text{gradient of } x^3$$
$$= 2 \times 3x^2$$
$$= 6x^2$$

$$\text{If } y = 4x^6, \frac{dy}{dx} = 4 \times \text{gradient of } x^6$$
$$= 4 \times 6x^5$$
$$= 24x^5$$

Generally, if $y = kx^n, \frac{dy}{dx} = k \times nx^{n-1}$

$$= kn\, x^{n-1}$$

8.1.8 Gradient of a sum

If $y = 2x^4 + 3x^5$

$$\frac{dy}{dx} = 8x^3 + 15x^4$$

Generally, if $y = f(x) + g(x), \frac{dy}{dx} = f'(x) + g'(x)$

This result applies to the sum of two or more functions.

If $y = f(x) + g(x) + h(x), \frac{dy}{dx} = f'(x) + g'(x) + h'(x)$

8.1.9 Gradient of a difference

$y = 3x^6 - 2x^4$

$$\frac{dy}{dx} = 18x^5 - 8x^3$$

Generally, if $y = f(x) - g(x)$, $\frac{dy}{dx} = f'(x) - g'(x)$

Note: To find the gradient of a product or quotient of functions which are simple to multiply or divide, we multiply or divide first.

Example 1

Find $\frac{dy}{dx}$ given

(a) $y = 2x^3 + 7x^2 + 4x + 3$ (b) $y = 4x^3 - 5x - 8$

(c) $y = (3x^2 - 1)(2x + 3)$ (d) $y = \frac{5x^2 - 3x + 4}{x}$

(e) $y = \frac{(2x^2 + 1)(3x - 4)}{5x}$

Solution

(a)
$$y = 2x^3 + 7x^2 + 4x + 3$$
$$\frac{dy}{dx} = 6x^2 + 14x + 4 + 0$$
$$= 6x^2 + 14x + 4$$

(b)
$$y = 4x^3 - 5x - 8$$
$$\frac{dy}{dx} = 12x^2 - 5 - 0$$
$$= 12x^2 - 5$$

(c)
$$y = (3x^2 - 1)(2x + 3)$$
$$= 6x^3 + 9x^2 - 2x - 3 \quad \text{(multiplying first)}$$
$$\frac{dy}{dx} = 18x^2 + 18x - 2$$

(d)
$$y = \frac{5x^2 - 3x + 4}{x}$$
$$= \frac{5x^2}{x} - \frac{3x}{x} + \frac{4}{x} \quad \text{(dividing first)}$$
$$= 5x - 3 + 4x^{-1}$$
$$\frac{dy}{dx} = 5 - 4x^{-2}$$
$$= 5 - \frac{4}{x^2}$$

(e) $y = \frac{(2x^2 + 1)(3x - 4)}{5x}$

$= \frac{6x^3 - 8x^2 + 3x - 4}{5x}$

$= \frac{6}{5}x^2 - \frac{8}{5}x + \frac{3}{5} - \frac{4}{5}x^{-1}$

$\frac{dy}{dx} = \frac{12}{5}x - \frac{8}{5} + \frac{4}{5}x^{-2}$

$= \frac{12}{5}x - \frac{8}{5} + \frac{4}{5x^2}$

Example 2

Find the gradient of the curve $y = x^2 - 4x + 3$ at each of its points of intersection with the x-axis.

Solution

(a) $y = x^2 - 4x + 3$

$\frac{dy}{dx} = 2x - 4$

The curve intersects the x-axis where

$x^2 - 4x + 3 = 0$

$(x - 1)(x - 3) = 0$

$x = 1$ or 3.

At (1, 0), $\frac{dy}{dx} = 2 - 4$

$= -2$

At (3, 0), $\frac{dy}{dx} = 6 - 4$

$= 2$

Example 3

Find the gradient function of $f: r \mapsto 2r^3 + 3r^2 - 6r - 8$

Solution

$f: r \mapsto 2r^3 + 3r^2 - 6r - 8$

$f': r \mapsto 6r^2 + 6r - 6$

Exercise 8 A

1. Find $\frac{dy}{dx}$ given

(a) $y = 2x^5$ (b) $y = 6x^4$ (c) $y = 3x^{-5}$ (d) $y = 6x^{-\frac{1}{3}}$

(e) $y = 16x^{-\frac{1}{4}}$ (f) $y = \frac{8}{\sqrt{x}}$ (g) $y = \frac{8}{\sqrt[3]{x}}$ (h) $y = \frac{12}{\sqrt[4]{x}}$

2. Find $\frac{dy}{dx}$ given

(a) $y = 6x^4 + 3x^3$ (b) $y = 9x^5 + 2x^3 + 7x^2 + 1$

(c) $y = 5x^3 - 2x^2 - 3x$ (d) $y = (2x + 1)(3x - 5)$

(e) $y = (3x^2 - 1)(2x^3 + 3x)$ (f) $y = \frac{5x^4 - 3x^2 + 7x}{2x}$

(g) $y = \frac{(3x^2 - 1)(2x^3 - 5x)}{x^2}$ (h) $y = \frac{3}{7x^2}(2x + 1)(3x^2 - 4)$

3. Find

(a) $\frac{dA}{dr}$ given $A = 6r + 3r^2$

(b) $\frac{dV}{dt}$ given $V = 2t^3 - 3t^2 + t$

(c) $\frac{dU}{dx}$ given $U = (2x^2 + 1)(3x - 5)$

(d) $\frac{dS}{dt}$ given $S = \frac{(3t + 1)(2t - 3)}{t}$

(e) $\frac{dA}{dr}$ given $A = 2\pi r\,(h + r)$, π and h are constants.

4. Find f' given

(a) $f : x \mapsto 3x^3 - 6x^2 + 15x - 1$ (b) $f : t \mapsto (2t - 1)(t + 3)$ (c) $f : r \mapsto (r^2 - 1)(r + 2)$

(d) $f : s \mapsto \frac{2s^2 + 3s - 1}{5s^3}$ (e) $f : t \mapsto \frac{(2t - 1)(t^2 + 3)}{4t}$

5. Find the gradient of each of the following curves at the given point:

(a) $y = 2x^2 - x + 1$ at $x = -2$

(b) $y = (3x - 1)(2x + 5)$ at $x = \frac{1}{2}$

(c) $y = (x - 1)(x - 2)$ at its point of intersection with the y-axis.

(d) $y = (2x + 1)(x - 3)$ at its points of intersection with the x-axis.

(e) $y = \frac{(x - 1)(x + 2)}{x^2}$ at $x = 2$.

6. Find the coordinates of the points on the curve with the given equation at which the gradient has the
value:
   (a) $y = x^2 + 3x$, gradient 5
   (b) $y = x +$ [illegible], gradient –3
   (c) $y = x^3 - 6x^2 - 15x + 1$, gradient 0
   (d) $y = x^2 + \frac{16}{x}$, gradient 0

8.2 Equations of tangent and normal to a curve

To find the equation of the tangent to a curve at a given point on the curve, we find the gradient of the
i.e. the gradient of the curve at this point.

To find the equation of the tangent to the curve $y = x^3 - 2x^2 + x - 3$ at the point on the curve with x-coordinate
we therefore find the gradient to the curve at $x = -1$.

$$\frac{dy}{dx} = 3x^2 - 4x + 1$$

$$x = -1, \frac{dy}{dx} = 3 + 4 + 1$$
$$= 8$$

$$x = -1, y = (-1)^3 - 2(-1)^2 + (-1)\,) - 3$$
$$= -7$$

Equation of the tangent is $\frac{y - (-7)}{x - (-1)} = 8$

$$y + 7 = 8x + 8$$
$$y = 8x + 1$$

The normal is the line perpendicular to the tangent, passing through the point of contact.

Gradient of normal $= -\frac{1}{8}$

Equation of normal is $\frac{y + 7}{x + 1} = -\frac{1}{8}$

$$8y + 56 = -x - 1$$
$$8y + x + 57 = 0$$

Example 4

Find the equation of the tangent and the normal to the curve $y = x^3 - x^2 - x - 5$ at the point on the curve
x-coordinate 1.

Solution

$$y = x^3 - x^2 - x - 5$$
$$\frac{dy}{dx} = 3x^2 - 2x - 1$$

$x = 1,\ \frac{dy}{dx} = 3 - 2 - 1$

$= 0$

$x = 1,\quad y = 1 - 1 - 1 - 5$

$= -6$

As gradient of tangent is 0, tangent is a y-line.

Since it passes through $(1, -6)$ its equation is $y = -6$.

The normal is an x-line.

Its equation is $x = 1$.

Example 5

Find the coordinates of the point on the curve $y = x^2 - 2x + 3$ at which the tangent is parallel to the line $y + x = 5$. Find also the equation of the tangent.

Solution

Line $y + x = 5$ has gradient -1

$$y = x^2 - 2x + 3$$

$$\frac{dy}{dx} = 2x - 2$$

At the given point $2x - 2 = -1$

$$x = \frac{1}{2}$$

$$y = \left(\frac{1}{2}\right)^2 - 2 \times \frac{1}{2} + 3$$

$$= 2\frac{1}{4}$$

Coordinates of point $= \left(\frac{1}{2}, 2\frac{1}{4}\right)$

Since tangent is parallel to $y + x = 5$, its equation is $y + x = c$

$$c = 2\frac{1}{4} + \frac{1}{2} = 2\frac{3}{4}$$

Equation of tangent is $y + x = 2\frac{3}{4}$ or $4y + 4x = 11$.

Exercise 8 B

1. Find the equation of the tangent and normal to the given curve at the given point on the curve:

   (a) $y = 2x^2 - 3x - 1$ at $(1, -2)$

   (b) $y = 2x^3 + x^2 - 1$ at the point on the curve with x-coordinate -2.

   (c) $y = \frac{2}{x}$ at $\left(\frac{1}{2}, 4\right)$

(d) $y = \frac{1}{x^2}$ at the point on the curve with x-coordinate $\frac{1}{2}$.

(e) $y = \frac{3}{x} + \frac{2}{x^2}$ at the point on the curve with x-coordinate -1.

(f) $y = x^2 - 4x + 1$ at $(2, 1)$.

2. Find the coordinates of the point on the curve $y = 2x^2 - 6x + 3$ at which:
   (a) the tangent is parallel to the x-axis
   (b) the tangent is parallel to the line $y + 2x = 1$
   (c) the tangent is perpendicular to the line $y + 2x = 1$

3. Find the coordinates of the points on the curve $y = x^3 + 6x^2 + 9x - 1$ at which the tangents are paralle the x-axis. Hence, find the equations of the normals at these points.

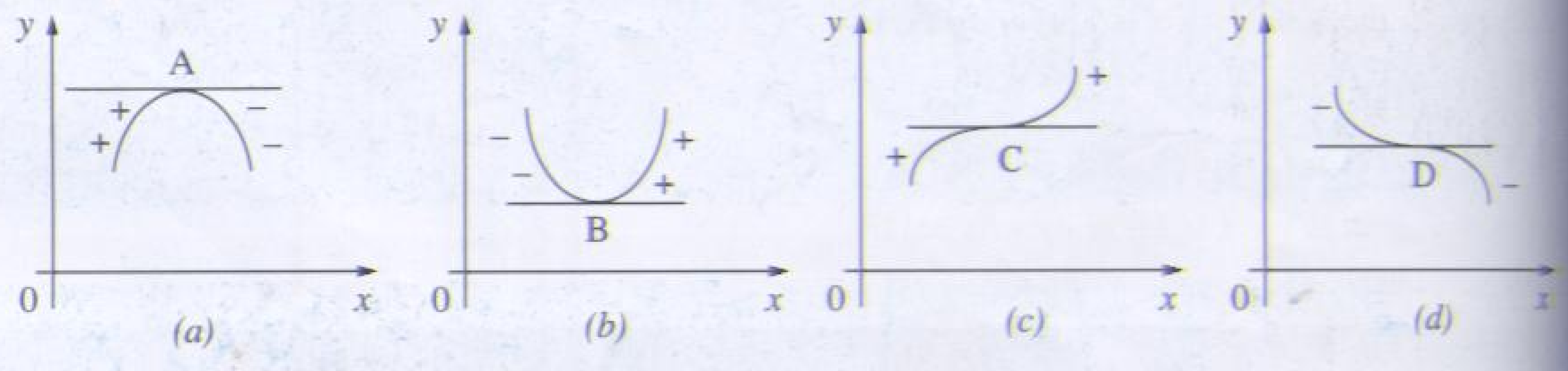
4. Find the coordinates of the points on the curve $y = x^3 + 2x^2 - x + 1$ at which the tangent is perpendicul the line $6y + x = 1$.

5. The normal to the curve $y = x^2 - 3x + 2$ at a point A on the curve is perpendicular to the line $y + x = 1$. Fi the coordinates of A.

6. The tangent at a point A on the curve $y = 3x^2 + 6x - 5$ is perpendicular to the line $y + 2x = 3$. Find coordinates of A.

8.3 Stationary points

Any point at which the tangent is parallel to the x-axis is called a stationary point.
A stationary point on a curve is therefore a point on the curve at which the gradient is 0.



(a) (b) (c) (d)

Figure 8.5

There are three types of stationary points as shown in Figure 8.5. In Figure 8.5(a), the gradient changes fro (just before A) to – (just after A). At A the gradient is 0.
A is called a maximum point.

In Figure 8.5(b), the gradient changes from – (just before B) to + (just after B). At B the gradient is 0.
B is called a minimum point.

In Figure 8.5(c) and (d), the gradient does not change sign (+ before and after C, – before and after D). At C and D the gradient is 0.
C and D are called stationary points of inflexion.

To summarise:

At a stationary point, the gradient is 0
For a maximum, the gradient changes from + to –
For a minimum, the gradient changes from – to +
For a stationary point of inflexion, the gradient does not change sign.

Example 6

Find the coordinates of the stationary point on the curve $y = x^2 + 2x - 3$ and determine its nature.

Solution

$y = x^2 + 2x - 3$

$\frac{dy}{dx} = 2x + 2$

At stationary point $2x + 2 = 0$

$x = -1$

$x < -1$, x near to -1, $\frac{dy}{dx} < 0$

$x > -1$, x near to -1, $\frac{dy}{dx} > 0$

So, the stationary point is a minimum.

$x = -1,$

$y = 1 - 2 - 3$

$= -4$

There is a minimum at $(-1, -4)$.

Example 7

Find the coordinates of the stationary points on the curve $y = x^3 - 6x^2 - 15x + 1$ and determine their nature.

Solution

$y = x^3 - 6x^2 - 15x + 1$

$\frac{dy}{dx} = 3x^2 - 12x - 15$

$= 3(x^2 - 4x - 5)$

$= 3(x + 1)(x - 5)$

$3(x + 1)(x - 5) = 0$

$x = -1$ or 5

$x < -1$, x near to -1, $\frac{dy}{dx} > 0$

$x > -1$, x near to -1, $\frac{dy}{dx} < 0$

So, there is a maximum at $x = -1$, $y = -1 - 6 + 15 + 1$

$= 9$

$x < 5$, x near to 5, $\frac{dy}{dx} < 0$

$x > 5$, x near to 5, $\frac{dy}{dx} > 0$

So, there is a minimum at $x = 5$, $y = 125 - 150 - 75 + 1$

$= -99$

Hence, there is a maximum at $(-1, 9)$ and a minimum at $(5, -99)$.

Example 8

Find the coordinates of the stationary point of $y = 8 - x^3$ and determine its nature.

Solution

$y = 8 - x^3$

$\frac{dy}{dx} = -3x^2$

$-3x^2 = 0$

$x = 0$

$x = 0$, $y = 8$

$x < 0$, x near to 0, $\frac{dy}{dx} < 0$

$x > 0$, x near to 0, $\frac{dy}{dx} < 0$

So, there is a stationary point of inflexion at $(0, 8)$.

8.3.2 Second order tests for maximum and minimum

If $y = x^3 - 2x^2 + 3x - 4$

$\frac{dy}{dx} = 3x^2 - 4x + 3$

$\frac{dy}{dx}$ is the first derivative of y.

If we now differentiate $\frac{dy}{dx}$, we obtain the second derivative of y which is written as $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = 6x - 4$$

If $y = 4x^3 - 3x + 1$

$$\frac{dy}{dx} = 12x^2 - 3$$

$$\frac{d^2y}{dx^2} = 24x$$

A useful test to determine whether y is a maximum or a minimum at $x = a$ is to find the algebraic sign of $\frac{d^2y}{dx^2}$ at $x = a$.

If $\frac{d^2y}{dx^2} > 0$ *at* $x = a$, *there is a minimum at* $x = a$.

If $\frac{d^2y}{dx^2} < 0$ *at* $x = a$, *there is a maximum at* $x = a$.

If $\frac{d^2y}{dx^2} = 0$, *the test cannot be used.*

Example 9

Find the x-coordinates of the stationary points of the curve $y = x^3 - 6x^2 - 15x - 1$ *and determine their nature.*

Solution

$$y = x^3 - 6x^2 - 15x - 1$$

$$\frac{dy}{dx} = 3x^2 - 12x - 15$$

$$\begin{aligned} 3x^2 - 12x - 15 &= 0 \\ x^2 - 4x - 5 &= 0 \\ (x+1)(x-5) &= 0 \\ x &= -1 \text{ or } 5 \\ \frac{d^2y}{dx^2} &= 6x - 12 \end{aligned}$$

At $x = -1$, $\frac{d^2y}{dx^2} < 0$. There is therefore a maximum at $x = -1$.

At $x = 5$, $\frac{d^2y}{dx^2} > 0$. There is therefore a minimum at $x = 5$.

8.3.3 Curve sketching

We use stationary points to sketch a curve with given equation. Points to remember when we consider sketchin of simple curves are:

- *At what points does the curve intersect the x-axis?*
- *At what points does it intersect the y-axis?*
- *Does it have stationary points?*
- *What happens to values of y as x becomes very large and positive?*
  Note that $x^4 > x^3 > x^2 > x$ for $x > 1$
- *What happens to values of y as x becomes very large and negative?*

Example 10

Sketch the curve $y = x^3 - 12x$.

Solution

1. $y = 0, x^3 - 12x = 0$

$$x(x^2 - 12) = 0$$
$$x = 0, \pm\sqrt{12}$$
$$= 0, -2\sqrt{3}, \ 2\sqrt{3}$$

So, the curve cuts the x–axis at $(0, 0)$, $(-2\sqrt{3}, 0)$, $(2\sqrt{3}, 0)$

2. $x = 0, y = 0$

   The curve cuts the y-axis at $(0, 0)$.

3. $\frac{dy}{dx} = 3x^2 - 12$

$$3x^2 - 12 = 0$$
$$x = +2 \text{ or } -2$$
$$\frac{d^2y}{dx^2} = 6x$$

$x = 2, \frac{d^2y}{dx^2} > 0$. So there is a minimum at $x = 2, y = -16$, i.e. there is a minimum at $(2, -16)$.

$x = -2, \frac{d^2y}{dx^2} < 0$. So there is a maximum at $x = -2, y = 16$, i.e. there is a maximum at $(-2, 16)$.

4. When x is large, x^3 is much larger than $12x$.

   So, when x becomes large and positive, y becomes large and positive.

   This is written $x \rightarrow +\infty, y \rightarrow +\infty$.

5. When x becomes large and negative, y becomes large and negative written $x \rightarrow -\infty, y \rightarrow -\infty$.

   The curve is therefore as shown in Figure 8.6.

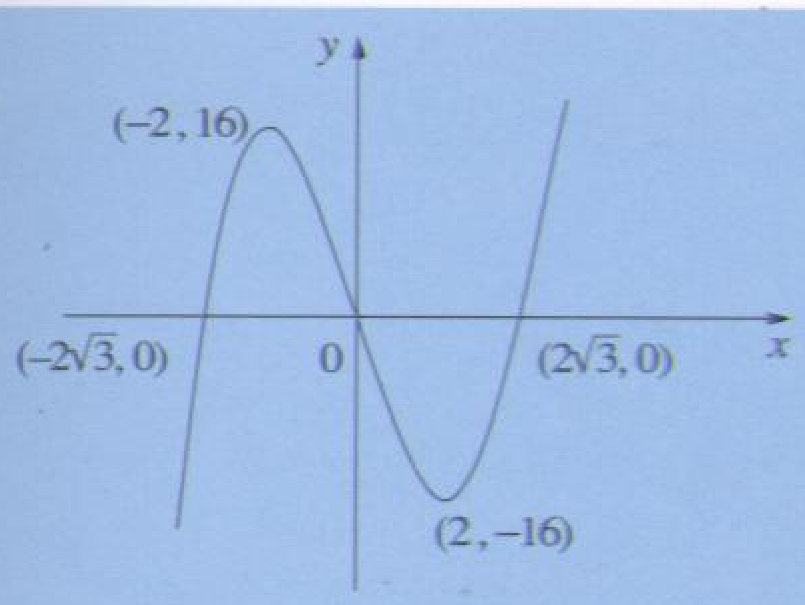


Figure 8.6

Exercise 8 C

1. Find the coordinates of the stationary points of each of the following curves and determine their nature:

(a) $y = x^2 - 4x + 5$
(b) $y = 6x - 2x^2$
(c) $y = x + \frac{1}{x}$
(d) $y = x^2 + \frac{54}{x}$
(e) $y = 2x^3 + 9x^2 + 12x - 1$
(f) $y = x^4 + 4x^3 - 8x^2 - 3$
(g) $y = x^2 - \frac{432}{x}$
(h) $y = 24x - 22x^2 + 4x^3$
(i) $y = x^3(1 - x)$
(j) $y = x^4(2x + 1)$
(k) $y = \frac{2}{x} + \frac{1}{2}\sqrt{x}$
(l) $y = x^3 - 15x + 7 - \frac{12}{x}$

2. Sketch the following curves:

(a) $y = x(x - 1)(x - 2)$
(b) $y = x^2(x - 1)$
(c) $y = x(x - 2)^2$
(d) $y = (x + 1)(x + 2)(x - 3)$
(e) $y = x(1 - x)^2$
(f) $y = x^4 - 32x$

8.4 Increasing and decreasing functions

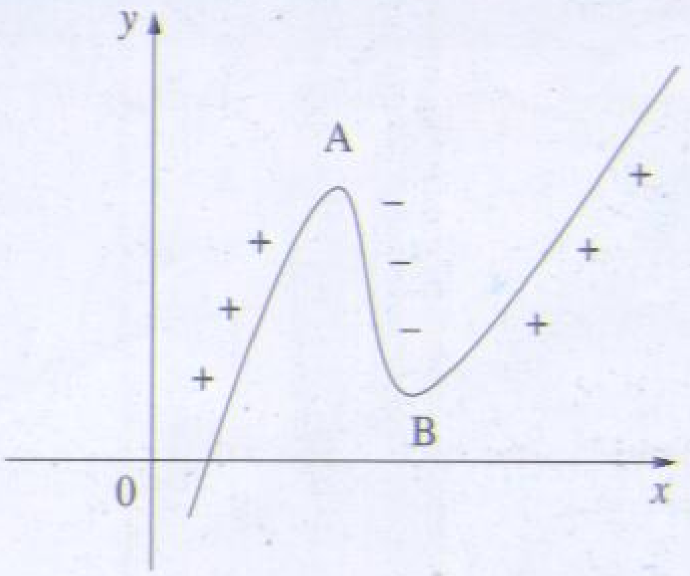


Figure 8.7

Figure 8.7 shows the graph of $y = f(x)$. We note that $f(x)$ increases as x increases when the gradient is positive and $f(x)$ decreases as x increases when the gradient is negative.

Example 11

Find the range of values of x for which $x^3 - 6x^2 - 15x + 1$ increases with x increasing.

Solution

$y = x^3 - 6x^2 - 15x + 1$

$\frac{dy}{dx} = 3x^2 - 12x - 15$

For y to increase as x increases $\frac{dy}{dx} > 0$

$3x^2 - 12x - 15 > 0$

$x^2 - 4x - 5 > 0$

Critical values of x when $x^2 - 4x - 5 = 0$

$(x - 5)(x + 1) = 0$

$x = 5 \text{ or } -1$

As the graph $y = x^2 - 4x - 5$ has a minimum, it is as shown in Figure 8.8.

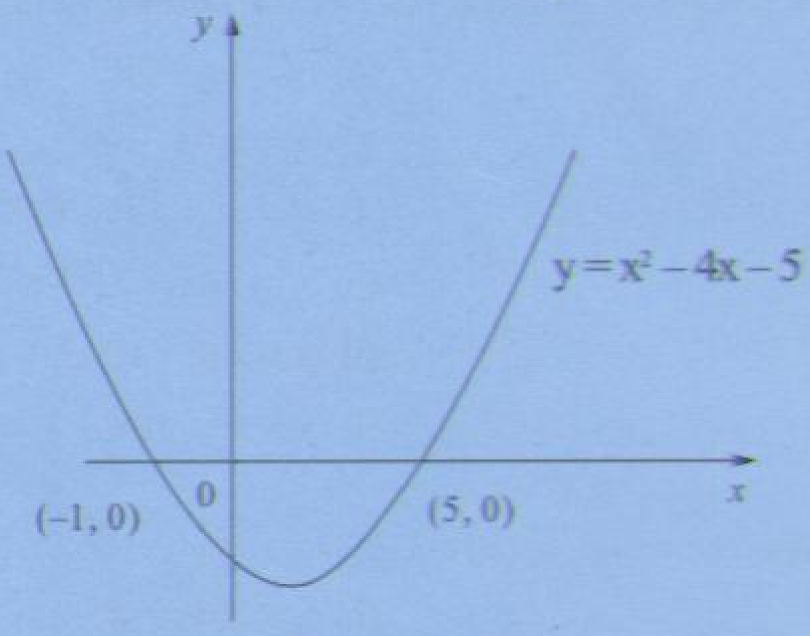


Figure 8.8

From the graph $x^2 - 4x - 5 > 0$ for $x < -1$ or $x > 5$.

So, $x^3 - 6x^2 - 15x + 1$ increases with x increasing for $x < -1$ or $x > 5$.

Example 12

Show that $x^3 - 3x^2 + 18x + 4$ is an increasing function for all real values of x.

Solution

$y = x^3 - 3x^2 + 18x + 4$

$\frac{dy}{dx} = 3x^2 - 6x + 18$

We show that $3x^2 - 6x + 18 > 0$ for all real values of x.

$$\begin{aligned} 3x^2 - 6x + 18 &= 3(x^2 - 2x + 6) \\ &= 3[x^2 - 2x + (1)^2 + 6 - (1)^2] \\ &= 3[(x - 1)^2 + 5] > 0 \text{ as minimum value of } (x - 1)^2 \text{ is } 0. \end{aligned}$$

Hence, $x^3 - 3x^2 + 18x + 4$ is an increasing function of x for all real values of x.

Example 13

Show that $2x^3 - 6x^2 + 9x + 1 > 6$ for $x > 1$.

Solution

Put $y = 2x^3 - 6x^2 + 9x + 1$

$$\frac{dy}{dx} = 6x^2 - 12x + 9$$

$$= 6\left(x^2 - 2x + \frac{3}{2}\right)$$

$$= 6\left[x^2 - 2x + (1)^2 + \frac{3}{2} - (1)^2\right]$$

$$= 6\left[(x-1)^2 + \frac{1}{2}\right]$$

$$= 6(x-1)^2 + 3$$

As $(x-1)^2 > 0$, $\frac{dy}{dx} > 3$ and so $\frac{dy}{dx} > 0$.

Hence, y is an increasing function of x.

$x = 1$, $2x^3 - 6x^2 + 9x + 1 = 6$

So, for $x > 1$, $2x^3 - 6x^2 + 9x + 1 > 6$.

8.4.2 Gradient of composite functions

Given $y = (x^3 + 2)^4$, we may find $\frac{dy}{dx}$ by expanding $(x^3 + 2)^4$ and differentiating term by term.

This method has the disadvantage that it is lengthy and the answer is obtained in a form which is not very useful.

It is better to write $y = u^4$ where $u = x^3 + 2$.

If x increases by Δx, let u increase by Δu and y by Δy.

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \times \frac{\Delta u}{\Delta x}$$

As $\Delta x \to 0$, $\Delta u \to 0$, $\Delta y \to 0$ and $\frac{\Delta u}{\Delta x} \to \frac{du}{dx}, \frac{\Delta y}{\Delta u} \to \frac{dy}{du}, \frac{\Delta y}{\Delta x} \to \frac{dy}{dx}$.

$$So, \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

This result is known as the chain rule.

In the given example $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= 4u^3 \times 3x^2$$

$$= 12x^2u^3$$

$$= 12x^2(x^3 + 2)^3$$

If $y = [f(x)]^n$ then putting $u = f(x)$, $y = u^n$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$= nu^{n-1} \times f'(x)$ where $f'(x)$ is the derivative of $f(x)$ with respect to x .

$$\frac{dy}{dx} = n\left[f(x)\right]^{n-1} \times f'(x)$$

So, *if* $y = \left[f(x)\right]^n$, $\frac{dy}{dx} = n\left[f(x)\right]^{n-1} \times f'(x)$.

This is a very important result and will be used very often.

Example 14

Find $\frac{dy}{dx}$ *given*

(a) $y = (3x^2 + 1)^6$ (b) $y = \frac{1}{\sqrt{4x+3}}$ (c) $y = \frac{3}{(5x-2)^3}$

Solution

(a)
$$y = (3x^2 + 1)^6$$
$$\frac{dy}{dx} = 6(3x^2 + 1)^5 \times 6x$$
$$= 36x(3x^2 + 1)^5$$

(b)
$$y = \frac{1}{\sqrt{4x+3}}$$
$$= (4x+3)^{-\frac{1}{2}}$$
$$\frac{dy}{dx} = -\frac{1}{2}(4x+3)^{-\frac{3}{2}} \times 4$$
$$= \frac{-2}{(4x+3)^{\frac{3}{2}}}$$

(c)
$$y = \frac{3}{(5x-2)^3}$$
$$y = 3(5x-2)^{-3}$$
$$\frac{dy}{dx} = -9(5x-2)^{-4} \times 5$$
$$= \frac{-45}{(5x-2)^4}$$

8.4.3 Rates of change

(a) Single variables

The rate of change of a variable x is written as $\frac{dx}{dt}$ where t is the symbol for time.

The rate of change of s is $\frac{ds}{dt}$, the rate of change of v is $\frac{dv}{dt}$, etc.

Familiar rates of change are $\frac{ds}{dt}$ where s is the displacement and $\frac{dv}{dt}$ where v is the velocity.

Example 15

Find the rate of change of s when (a) $t = 0$, (b) $s = 0$ for

(i) $s = (t-1)(t-2)$ (ii) $s = t(t-1)(t-2)$

Solution

(i) $s = (t-1)(t-2)$

$= t^2 - 3t + 2$

$\frac{ds}{dt} = 2t - 3$

(a) $t = 0, \frac{ds}{dt} = -3$

(b) $s = 0, (t-1)(t-2) = 0$

$t = 1$ or 2

$t = 1, \frac{ds}{dt} = -1$

$t = 2, \frac{ds}{dt} = 1$

(ii) $s = t(t-1)(t-2)$

$= t^3 - 3t^2 + 2t$

$\frac{ds}{dt} = 3t^2 - 6t + 2$

(a) $t = 0, \frac{ds}{dt} = 2$

(b) $s = 0, t = 0, 1$ or 2

$t = 0, \frac{ds}{dt} = 2$, obtained above

$t = 1, \frac{ds}{dt} = -1$

$t = 2, \frac{ds}{dt} = 2$

(b) Connected variables

Consider water flowing into a vessel with uniform cross section of area 10 square units. As the height h increases, the volume v increases and v, h are connected by the formula $v = 10h$ from which we obtain $\frac{dv}{dh} = 10$.

The rates of change of v, h are $\frac{dv}{dt}$ and $\frac{dh}{dt}$ respectively and we have $\frac{dv}{dt} = \frac{dv}{dh} \times \frac{dh}{dt}$ $\left(\text{compare with } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}\right)$.

So, $\frac{dv}{dt} = 10\frac{dh}{dt}$.

Example 16

The radius of a spherical balloon increases at a constant rate of 0.3 cms^{-1}. Find the rate at which the
increases when (a) the radius is 4 cm, (b) after 4 seconds, (c) when the volume is 36π cm^3.

Solution

Let r be the radius in cm, t the time in seconds, v the volume in cm^3.

$$v = 4\pi r^3$$

$$\frac{dv}{dr} = 4\pi r^2$$

$$\frac{dr}{dt} = 0.3$$

$$\frac{dv}{dt} = \frac{dv}{dr} \times \frac{dr}{dt}$$

(a) $r = 4, \frac{dv}{dt} = 1.2\pi \times 4^2$

$$= 19.2\pi$$

Volume increases at 19.2π cm^{3}s^{-1}

(b) $t = 4$, $r = 4 \times 0.3$

$$= 1.2$$

$$\frac{dv}{dt} = 1.2\pi \times 1.2^2$$

$$= 1.728\pi$$

Volume increases at 1.728π cm^{3}s^{-1}

(c) $v = 36\pi$

$$\frac{4}{3}\pi r^3 = 36\pi$$

$$r^3 = 27$$

$$r = 3$$

$$\frac{dv}{dt} = 1.2\pi \times 3^2$$

$$= 10.8\pi$$

Volume increases at 10.8π cm^{3}s^{-1}.

Example 17

A container is in the shape of an inverted circular cone with semi–vertical angle 30°. Water runs out of the vessel through a hole at the vertex of the cone, the water level decreasing at a constant rate of 0.1 cms^{-1}. Find the rate at which the water level in the vessel decreases when the volume of water in the vessel is 81π cm^3.

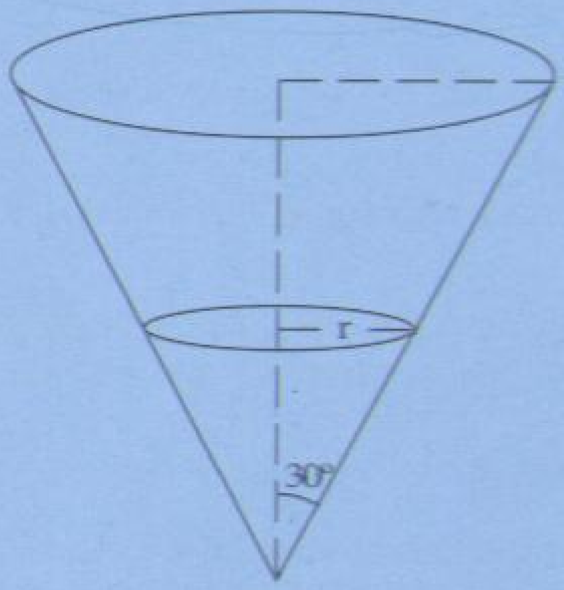


Figure 8.9

Solution

Let r be the radius in cm, h the height in cm, V the volume in cm^3, t the time in seconds.

$$\frac{r}{h} = \tan 30°$$
$$r = h\tan 30°$$
$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi h^3 \tan^2 30°$$
$$= \frac{1}{9}\pi h^3$$
$$\frac{dv}{dh} = \frac{1}{3}\pi h^2$$
$$\frac{dv}{dt} = \frac{dv}{dh} \times \frac{dh}{dt}$$
$$= \frac{1}{3}\pi h^2 \times -0.1 \text{ (the – sign as h decreases)}$$

When V = 81π

$$\frac{1}{9}\pi h^3 = 81\pi$$
$$h^3 = 729$$
$$h = 9$$
$$\frac{dv}{dt} = \frac{1}{3}\pi \times 9^2 \times -0.1$$
$$= -2.7\pi$$

So, volume decreases at the rate of 2.7π cm^{3}s^{-1}.

Exercise 8 D

1. Find the range of values of x for which the following functions of x increase with increasing values of x

   (a) $f(x) = x^2 - 4x + 3$

   (b) $f(x) = x^3 - 6x^2 + 9x + 1$

   (c) $f(x) = 4x^3 + 15x^2 - 18x - 3$

   (d) $f(x) = x + \frac{1}{x}, x \neq 0$

   (e) $f(x) = x^2 + \frac{16}{x}, x \neq 0$

2. Show that $x^3 - 3x^2 + 6x + 1$ increases with x for all real values of x.

3. Find the range of values of b for which the function $f:x \mapsto x^3 + 6x^2 + 3bx + 12$ is an increasing function.

4. Show that $x^3 + 3x^2 + 9x + 1 > -6$ if $x > -1$.

5. Show that $f:x \mapsto 2x^3 - 9x^2 + 12x + 1$ increases with x for $x > 2$ or $x < 1$.

6. Find the range of values of x for which the function $f:x \mapsto 2x^3 - 3x^2 - 12x + 1$ is a decreasing function.

7. Find the range of values of x for which $f(x) = x + \frac{1}{x}$ increases as x increases. Hence or otherwise show th the sum of a positive quantity and its reciprocal cannot be less than 2.

8. Find $\frac{dy}{dx}$ given:

   (a) $y = (4x + 2)^3$

   (b) $y = (4x + 1)^{\frac{1}{2}}$

   (c) $y = \sqrt[3]{5 - 4x}$

   (d) $y = 3(2x - 1)^{-5}$

   (e) $y = (x^2 + 4x + 1)^{\frac{3}{2}}$

   (f) $y = \frac{5}{(2x + 3)}$

   (g) $y = \frac{4}{(1 - 3x)^2}$

   (h) $y = \frac{3}{5 - 2x}$

   (i) $y = \frac{2}{\sqrt{x^2 + 4x - 1}}$

   (j) $y = \frac{4}{\sqrt[3]{2x^3 + 6x^2 - 1}}$

   (k) $y = \frac{4}{\sqrt[4]{3x^3 + 2x^2 - 1}}$

   (l) $y = \frac{5}{\sqrt{1 - 2x}\sqrt{1 + 3x}}$

9. (a) Given $s = 3(1 - 2t)^4$, find $\frac{ds}{dt}$ when $t = 1$.

   (b) Given $v = 3(2r - 1)^{\frac{1}{2}}$, find $\frac{dv}{dr}$ when $r = 5$.

   (c) Given $p = \frac{1}{\sqrt{3t + 1}}$, find $\frac{dp}{dt}$ when $t = 8$.

   (d) Given $T = 2\pi\sqrt{\frac{l}{g}}$, find $\frac{dT}{dl}$ when $\frac{1}{g} = 9$.

   (e) Given $s = 6\sqrt{9 + \left(\frac{r}{\pi}\right)^2}$, find $\frac{ds}{dr}$ when $r = 4\pi$

10. (a) Given $y = \left(x - \frac{1}{x}\right)^2$, find $\frac{dy}{dx}$.
(b) Given $A = 6\pi\sqrt{h^2 + 16}$, find $\frac{dA}{dh}$.

(c) Given $s = \left(2t - \frac{3}{t^2}\right)^{\frac{1}{2}}$, find $\frac{ds}{dt}$.
(d) Given $p = \frac{1}{\left(3x^2 - \frac{2}{x}\right)^4}$, find $\frac{dp}{dx}$.

(e) Given $u = 5\left(2r - \frac{1}{3r}\right)^{-\frac{3}{4}}$, find $\frac{du}{dr}$.

11. The volume of water in a vessel is $(4x^3 + 6x^2 + 3x)$ cm^3 when the depth of water is x cm. Water leaks from a tap into the vessel at a constant rate of 1 500 cm^{3}s^{-1}. Find the rate at which the water is rising when the depth of water in the vessel is 15 cm.

12. The radius of a spherical bubble of soap increases at a constant rate of 0.05 cms^{-1}. At what rate is its volume increasing (a) when its radius is 4 cm, (b) after 6 seconds, (c) when its volume is 288π cm^3?

13. The volume of a spherical balloon increases at a constant rate of 10 cm^{3}s^{-1}. At what rate is its radius increasing (a) when its radius is 2 cm; (b) after 3 seconds; (c) when its volume is $\frac{500\pi}{3}$ cm^3?

14. The surface area of a spherical balloon increases at a constant rate of 10 cm^{2}s^{-1}. At what rate is its radius increasing when the radius is 5 cm? Find the rate of increase of its volume at that instant.

15. The side of a cube increases at a constant rate of 0.02 cms^{-1}. Find the rate of increase of its volume (a) when the length of its side is 6 cm, (b) when its volume is 64 cm^3, (c) after 4 seconds.

16. The volume of an ice cube decreases at a constant rate of 3 cm^{3}s^{-1}. Find the rate of change of its side (a) when its length is 4 cm, (b) when the volume of the cube is 27 cm^3.

17. The volume of a cube decreases at a constant rate of 10 cm^{3}s^{-1}. Find the rate of change of its surface area when (a) the length of its side is 10 cm, (b) its volume is 512 cm^3, (c) its surface area is 150 cm^2.

18. A hollow inverted cone of semi vertical angle 30° contains water to a height of h cm. Show that the volume of water is $\frac{1}{9}\pi h^3$.
The water flows out through a small hole in the vertex. Find the rate at which the volume of water is decreasing when the depth is 6 cm, if the depth of water decreases at a constant rate of 0.2 cms^{-1}.

19. The height of a vessel in the form of an inverted cone is 16 cm and its radius is 6 cm. Find an expression for the volume of water in the vessel when the height is h cm.
The inverted vessel is originally full of water and the water leaks through a hole at the vertex at a constant rate of 1.2 cm^{3}s^{-1}.
Find at what rate (a) the water level is decreasing when the water reaches half–way up the cone,
(b) the area of the horizontal surface is decreasing then.

20.

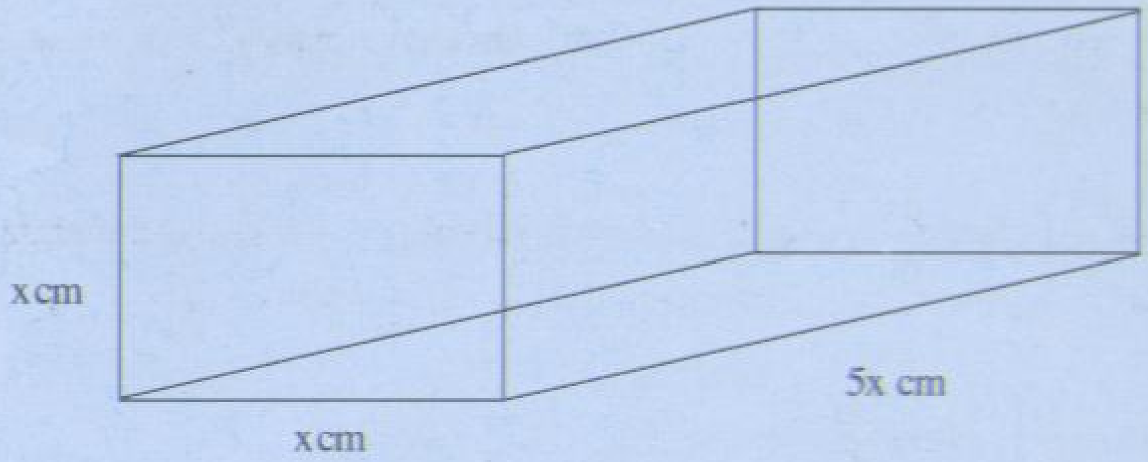


Figure 8.10

Figure 8.10 shows a rectangular block of ice, x cm by x cm and 5x cm.

(a) Find expressions for the total surface area, A cm^2 and the volume V cm^3 in terms of x.

(b) The ice is melting at a constant rate of 5 cm^3s^{-1}. Find the rate of change of the surface area when th
volume is 1000 cm^3.

Miscellaneous Exercise 8

1. Find the equation of the tangent to the curve $y = x^3 - 8x^2 + 15x$ at the point (4, –4). Calculate the coordina
of the point where the tangent meets the curve again. [C

2. The equation of a curve is $y = (3 - x^2)^6$. Find:
   (a) $\dfrac{dy}{dx}$
   (b) the equation of the normal at the point on the curve where x = 2. [C

3. Given that $y = \dfrac{16}{x^2} + \dfrac{x^3}{3}$, find the stationary value of y and determine whether it is a maximum or a minimu
[C

4. A container, initially empty, is being filled with liquid; t seconds after filling has begun, the depth of the liqui
in the container is x cm and the volume contained is V cm^3, where $V = \dfrac{\pi x}{2}(x + 2)$.
Given that V increases at a constant rate, and that x = 10 when t = 15, find:
   (a) $\dfrac{dV}{dt}$ in terms of π
   (b) the rate at which x is increasing at the moment when x = 7. [C

5. A and B are the points on the curve $y = 3x - \dfrac{8}{x}$ with x-coordinates 2 and 4 respectively. Find the x-coordina
of the point of intersection of the tangents at A and B. [C

6. Show that the equation of the normal to the curve $y = 2x + \dfrac{6}{x}$ at the point (2, 7) is y + 2x = 11. Given th
this normal meets the curve again at P, find the x-coordinate of P. [C

7. Find the maximum value of $4x - x^2 - 1$ and sketch the curve $y = 4x - x^2 - 1$ for $0 \leq x \leq 4$. Determine the equations of the tangents to the curve at the points whose x-coordinates are 1 and 3. Show that these tangents intersect at the point (2, 4). [C]

8. Liquid is poured into a container at a rate of $12\ cm^3s^{-1}$. The volume of liquid in the container is $V\ cm^3$, where $V = \frac{1}{2}(h^2 + 4h)$ and h cm is the height of liquid in the container. Find, when $V = 16$,
   (a) the value of h
   (b) the rate at which h is increasing. [C]

9. The equation of a curve is $y = 6x^2 - x^3$. Find the coordinates of the two stationary points on the curve and determine the nature of each of these stationary points.
   State the set of values of x for which $6x^2 - x^3$ is a decreasing function of x.
   The gradient at the point M on the curve is 12. Find the equation of the tangent to the curve at M. [C]

10. The tangent to the curve $y = 4x + \frac{8}{x}$ at the point (2, 12) meets the x-axis at A and the y-axis at B. Find the coordinates of the mid-point of AB. [C]

11. Find the gradient of the normal to the curve $y = (2x - 7)^5$ at the point on the curve where $x = 4$. [C]

12. The tangent to the curve $y = px^3$ at the point where $x = 2$ passes through the point $(1, -10)$. Find the value of p.
    [C]

13. Find the coordinates of the point on the curve $y = 3x^2 - 7x + 2$ at which the tangent is parallel to the line $y = 5x$. [C]

14. Find the coordinates of the stationary point on the curve $y = \frac{16x^3 + 4x^2 + 1}{2x^2}$ and determine the nature of this point. [C]

15. A curve has the equation $y = x + \frac{2}{x^2}$. Find:
    (a) an expression for $\frac{dy}{dx}$
    (b) the value of k for which $y + 2x = k$ is a normal to the curve. [C]

16. A particle moves in a straight line so that, at time t seconds after leaving a fixed point O, its velocity $v\ ms^{-1}$ is given by $v = \frac{27}{(2t + 1)^2} - 3$.
    Find:
    (a) the value of t for which the particle is at instantaneous rest.
    (b) the initial acceleration of the particle. [C]

17. Given that $y = \frac{16}{x^2} + \frac{x^3}{3}$, find the stationary value of y and determine whether it is a maximum or a minim

[C

18. The two variables x and y are related by the equation $y = 3x - \frac{4}{x}$.
    (a) Obtain an expression for $\frac{dy}{dx}$ in terms of x.
    (b) Hence, find the approximate increase in y as x increases from 2 to 2 + p, where p is small. [C

19. A spherical balloon is released from rest and expands as it rises. After rising for t seconds, its radius is r
and its surface area is A cm^2, where $A = 4\pi r^2$.
The initial radius of the balloon is 16 cm. Given that the rate of increase of its radius is constant and has
value of 0.8 cms^{-1}, find the rate of increase of A when t = 5. [C

20. Given that $y = 9x^{\frac{3}{2}}$, find:
    (a) the value of x when y = 36.
    (b) an expression for $\frac{dy}{dx}$.
Hence, find the approximate increase in x when y increases from 36.0 to 36.3. [C

CHAPTER 9

> INTEGRATION 1

9.1 Integration as the reverse of differentiation

9.1.1 Integral of $x^n (n \neq -1)$ with respect to x

Given y in terms of x, the process of finding $\frac{dy}{dx}$ is known as differentiation.

We now consider the reverse process. Given $\frac{dy}{dx}$ in terms of x, we are to find y. This process is known as *integration*.

Consider $y = x$, $y = x + 1$, $y = x + 2$, $y = x - 4$, $y = x - \frac{3}{4}$. In each case, $\frac{dy}{dx} = 1$.

So, if $\frac{dy}{dx} = 1$, $y = x + c$ where c is a constant.

Next consider $y = \frac{x^2}{2}$, $y = \frac{x^2}{2} + 5$, $y = \frac{x^2}{2} - 3$, $y = \frac{x^2}{2} + \frac{1}{5}$, etc.

In each case, $\frac{dy}{dx} = x$

So, if $\frac{dy}{dx} = x$, $y = \frac{x^2}{2} + c$.

Similarly, if $y = \frac{x^3}{3} + c$ where c is a constant, $\frac{dy}{dx} = x^2$.

So, if $\frac{dy}{dx} = x^2$, $y = \frac{x^3}{3} + c$.

Table 1 shows expressions for $\frac{dy}{dx}$ given y in terms of x.

y	c	$x + c$	$\frac{x^2}{2} + c$	$\frac{x^3}{3} + c$	$\frac{x^4}{4} + c$	$\frac{x^5}{5} + c$	$\frac{x^{n+1}}{n+1} + c$
$\frac{dy}{dx}$	0	1	x	x^2	x^3	x^4	x^n

Table 1

Table 2 shows expressions for y if $\frac{dy}{dx}$ is given in terms of x.

$\frac{dy}{dx}$	0	1	x	x^2	x^3	x^4	$x^n (n \neq -1)$
y	0	$x + c$	$\frac{x^2}{2} + c$	$\frac{x^3}{3} + c$	$\frac{x^4}{4} + c$	$\frac{x^5}{5} + c$	$\frac{x^{n+1}}{n+1} + c$

Table 2

It follows from this table that if $\frac{dy}{dx} = x^n$ $(n \neq -1)$, $y = \frac{x^{n+1}}{n+1} + c$.

We say the integral of x^n with respect to x is $\frac{x^{n+1}}{n+1} + c$.

Generally, if the gradient of $y = F(x)$ is given by $\frac{dy}{dx} = f(x)$, $F(x)$ is the integral of $f(x)$ with respect to x.

9.1.2 Notation for integral

Just as we used a notation for the gradient, we will use a notation for the integral.

If $\frac{dy}{dx} = f(x)$, we write $y = \int f(x)\,dx$ to be read as y is the integral of f(x) with respect to x.

So, if $\frac{dy}{dx} = x^2$, $y = \int x^2\,dx = \frac{1}{3}x^3 + c$

If $\frac{dy}{dx} = x^3$, $y = \int x^3\,dx = \frac{1}{4}x^4 + c$

So, $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ $(n \neq -1)$.

These integrals are indefinite integrals as they contain a constant.

9.1.3 $\int kx^n\,dx$ for a constant k

$$\int 2x^3\,dx = 2 \times \int x^3 dx$$

$$= 2\left[\frac{x^4}{4} + c\right]$$

$$= \frac{x^4}{2} + c$$

$$\int 3x^5\,dx = 3 \times \int x^5\,dx$$

$$= 3\left[\frac{x^6}{6} + c\right]$$

$$= \frac{x^6}{2} + c$$

Generally, $\int kx^n = \frac{kx^{n+1}}{n+1} + c$ $(n \neq -1)$.

9.1.4 Integral of a sum

The integral of a sum is the sum of integrals.

$$\int (2x^2 + 3)\,dx = \int 2x^2\,dx + \int 3\,dx$$

$$= \frac{2x^3}{3} + 3x + c$$

$$\int \left(3x^3 + 4x^2 + 5\right) dx = \int 3x^3\,dx + \int 4x^2\,dx + \int 5\,dx$$

$$= \frac{3x^4}{4} + \frac{4x^3}{3} + 5x + c$$

$$\int \left(2x^2 + 3x + 7\right) dx = \frac{2x^3}{3} + \frac{3x^2}{2} + 7x + c$$

$$\int \left(3x^3 + 4x^2 + 5x + 1\right) dx = \frac{3x^4}{4} + \frac{4x^3}{3} + \frac{5x^2}{2} + x + c$$

9.1.5 Integral of a difference

The integral of a difference is the difference of integrals.

Thus, $\int \left(2x^3 - 3x\right) dx = \int 2x^3\,dx - \int 3x\,dx$

$$= \frac{2x^4}{4} - \frac{3x^2}{2} + c$$

$$= \frac{x^4}{2} - \frac{3x^2}{2} + c$$

$$\int \left(5x^3 - 2x^2\right) dx = \frac{5x^4}{4} - \frac{2x^3}{3} + c$$

$$\int \left(2x^3 - 3x^2 - 2x\right) dx = \frac{2x^4}{4} - \frac{3x^3}{3} - \frac{2x^2}{2} + c$$

$$= \frac{x^4}{2} - x^3 - x^2 + c$$

It follows that if we have only sums or differences, we integrate term by term.

9.1.6 Integral of a product or a quotient

To integrate a product, we multiply first and to integrate a quotient, we divide first.

Thus, $\int (2x + 1)(3x - 4)\,dx = \int \left(6x^2 - 5x - 4\right) dx$

$$= \frac{6x^3}{3} - \frac{5x^2}{2} - 4x + c$$

$$= 2x^3 - \frac{5}{2}x^2 - 4x + c$$

$$\int \frac{3x^4 + 2x^2 - x^6}{4x}\,dx = \int\left(\frac{3}{4}x^3 + \frac{1}{2}x - \frac{1}{4}x^5\right)dx$$

$$= \frac{3}{4\times 4}x^4 + \frac{1}{2\times 2}x^2 - \frac{1}{4\times 6}x^6 + c$$

$$= \frac{3}{16}x^4 + \frac{1}{4}x^2 - \frac{1}{24}x^6 + c$$

Example 1

Find the following integrals:

(a) $\int\left(x^2 + 2x - 3\right)dx$ (b) $\int (2x + 1)(x - 2)\,dx$ (c) $\int\left(3t^2 - 1\right)\left(2t^3 + 5\right)dt$

Solution

(a) $$\int\left(x^2 + 2x - 3\right)dx = \frac{x^3}{3} + \frac{2x^2}{2} - 3x + c$$

$$= \frac{x^3}{3} + x^2 - 3x + c$$

(b) $$\int (2x + 1)(x - 2)\,dx = \int\left(2x^2 - 3x - 2\right)dx \quad \text{(multiplying first)}$$

$$= \frac{2x^3}{3} - \frac{3x^2}{2} - 2x + c$$

(c) $$\int\left(3t^2 - 1\right)\left(2t^3 + 5\right)dt = \int\left(6t^5 - 2t^3 + 15t^2 - 5\right)dt \quad \text{(multiplying first)}$$

$$= \frac{6t^6}{6} - \frac{2t^4}{4} + \frac{15t^3}{3} - 5t + c$$

$$= t^6 - \frac{1}{2}t^4 + 5t^3 - 5t + c$$

Example 2

Find the following integrals:

(a) $\int \frac{3t^5 + t^4 - 5}{2t^3}\,dt$ (b) $\int \frac{\left(2s^2 + 1\right)\left(3s^2 - 2\right)}{s^4}\,ds$

Solution

(a) $$\int \frac{3t^5 + t^4 - 5}{2t^3}\,dt = \int\left(\frac{3t^5}{2t^3} + \frac{t^4}{2t^3} - \frac{5}{2t^3}\right)dt \quad \text{(dividing first)}$$

$$= \int\left(\frac{3}{2}t^2 + \frac{t}{2} - \frac{5}{2}t^{-3}\right)dt$$

$$= \frac{3t^3}{2\times 3} + \frac{t^2}{2\times 2} - \frac{5t^{-2}}{2\times -2} + c$$

$$= \frac{1}{2}t^3 + \frac{1}{4}t^2 + \frac{5}{4t^2} + c$$

(b) $$\int \frac{(2s^2+1)(3s^2-2)}{s^4}\,ds = \int \frac{6s^4 - s^2 - 2}{s^4}\,ds$$

$$= \int\left(\frac{6s^4}{s^4} - \frac{s^2}{s^4} - \frac{2}{s^4}\right) ds$$

$$= \int\left(6 - s^{-2} - 2s^{-4}\right) ds$$

$$= 6s - \frac{s^{-1}}{-1} - \frac{2s^{-3}}{-3} + c$$

$$= 6s + \frac{1}{s} + \frac{2}{2s^3} + c$$

9.1.7 Finding the constant of integration

Given $\frac{dy}{dx}$ in terms of x, it is possible to find y completely in terms of x if we know a value of y corresponding to a given value of x.

Thus, if $\frac{dy}{dx} = 2x - 3$ and y = 2 when x = 1, it is possible to find y in terms of x.

For, if $\frac{dy}{dx} = 2x - 3$

$$y = \int (2x-3)\,dx$$

$$y = \frac{2x^2}{2} - 3x + c$$

$$y = x^2 - 3x + c$$

y = 2 when x = 1 gives $2 = 1 - 3 + c$

$$2 - 1 + 3 = c$$

$$4 = c$$

$$c = 4$$

$y = x^2 - 3x + 4$

Example 3

Given $\frac{ds}{dt} = (2t-1)(3t+2)$ and s = 1 when t = 0, find s in terms of t.

Solution

$$\frac{ds}{dt} = (2t-1)(3t+2)$$

$$= 6t^2 + t - 2$$

$$s = \int\left(6t^2 + t - 2\right) dt$$

$= 2t^3 + \frac{t^2}{2} - 2t + c$

$s = 1$ when $t = 0,\ 1 = 0 + 0 - 0 + c$

$1 = c$

$c = 1$

$s = 2t^3 + \frac{t^2}{2} - 2t + 1$

9.1.8 $\int (ax + b)^n\, dx,\ n \neq -1$

The integral of $(ax + b)^n$ with respect to x should contain $(ax + b)^{n+1}$.

But $\frac{d}{dx}(ax + b)^{n+1} = (n + 1)(ax + b)^n \times a$ (chain rule)

$= (n + 1)a\,(ax + b)^n$

It follows that $\int (n + 1)a\,(ax + b)^n\, dx = (ax + b)^{n+1}$ (excluding the constant of integration).

As $\int (n + 1)a\,(ax + b)^n\, dx = (n + 1)a \int (ax + b)^n\, dx$,

we have $(n + 1)a \int (ax + b)^n\, dx = (ax + b)^{n+1}$

$\int (ax + b)^n\, dx = \frac{(ax + b)^{n+1}}{(n + 1)a} + c$ (including the constant of integration).

This result holds *if $n \neq -1$*.

$$\text{Thus, } \int (2x + 3)^4\, dx = \frac{(2x + 3)^5}{5 \times 2} + c$$

$$= \frac{(2x + 3)^5}{10} + c$$

$$\int (3 - 4x)^6\, dx = \frac{(3 - 4x)^7}{-4 \times 7} + c$$

$$= -\frac{(3 - 4x)^7}{28} + c$$

Example 4

Given $\frac{ds}{dt} = 2(3t - 1)^4$ and $s = 8$ when $t = 1$, find s in terms of t.

Solution

$\frac{ds}{dt} = 2(3t - 1)^4$

$s = \frac{2(3t - 1)^5}{5 \times 3} + c$

$$= \frac{2}{15}(3t-1)^5 + c$$

$s = 8$ when $t = 1$, $8 = \frac{2}{15} \times 32 + c$

$$8 - \frac{64}{15} = c$$

$$\frac{56}{15} = c$$

$$c = \frac{56}{15}$$

$$S = \frac{2}{15}(3t-1)^5 + \frac{56}{15}$$

$$= \frac{2}{15}\left[(3t-1)^5 + 28\right]$$

Exercise 9 A

1. Find y in terms of x given $\frac{dy}{dx}$ is:

   (a) $2x - 3$
   (b) $3x^2 - 4x + 2$
   (c) $2x^3 + x^2 - x - 5$
   (d) $(2x-1)(x^2-3)$
   (e) $(x^2 - 2x + 5)(3x + 1)$
   (f) $\frac{3}{x^2} + \frac{4}{x^3}$
   (g) $\frac{1}{x^2} - \frac{2}{x^3} + \frac{5}{x^4}$
   (h) $\frac{2x^3 + 3x^2 - 7x + 9}{3x^5}$
   (i) $\frac{(3x-1)(2x^2+7x-3)}{4x^5}$

2. Given $\frac{dy}{dx} = 2x^2 - x + 3$ and $y = -1$ when $x = 1$, find y in terms of x.

3. Given $\frac{dA}{dr} = r^3 - 2r^2 + r - 3$ and $A = 1$ when $r = 0$, find A in terms of r.

4. Given $\frac{d^2y}{dx^2} = 3x + 1$. If $\frac{dy}{dx} = 2$ and $y = 1$ when $x = 1$, find y in terms of x.

5. Given $f'(x) = 2x^2 - x + 3$ and $f(-2) = 3$, find f(x).

6. Given $\frac{dA}{dx} = \frac{(2x+1)(x-1)}{x^4}$ and $A = 1$ when $x = 1$, find A in terms of x.

7. Find:

   (a) $\int 2x^3\,dx$
   (b) $\int 3x^5\,dx$
   (c) $\int \frac{4}{x^6}\,dx$
   (d) $\int \frac{9}{x^2} + \frac{7}{x^3}\,dx$
   (e) $\int (3+r)(2-r)\,dr$
   (f) $\int (1-t^2)(1+t^2)\,dt$
   (g) $\int (2+3s)(1-2s)\,ds$
   (h) $\int \frac{3t^2 + 4t - 1}{2t^4}\,dt$
   (i) $\int \frac{(4s-1)(2s+3)}{3s^4}\,ds$

8. Find:

(a) $\int (3x + 2)^3\,dx$ (b) $\int 4(2x - 1)^5\,dx$ (c) $\int \sqrt{4x + 1}\,dx$

(d) $\int \frac{2}{(3x - 2)^2}\,dx$ (e) $\int \frac{5}{(1 - 2x)^3}\,dx$ (f) $\int \frac{6}{\sqrt{3x + 1}}\,dx$

(g) $\int \frac{2}{\sqrt[3]{5 - 3x}}\,dx$ (h) $\int \frac{4}{(5 - 2r)^3}\,dr$ (i) $\int \frac{6}{(3t - 5)^5}\,dt$

9. Given $\frac{ds}{dt} = \frac{4}{(2t - 1)^3}$ and s = 1 when t = 0, find s in terms of t.

10. Given $\frac{dA}{dr} = \frac{3}{(2r + 1)^3} - 5r + 1$ and A = 0 when r = 1, find A in terms of r.

11. The gradient of a curve at any point (x, y) on the curve is $2x^2 - x + 3$. Given that the curve passes thro
the point (–3, 0), find its equation.

12. Find the equation of the curve which has gradient $2 - 3x - x^2$ at any point (x, y) on the curve and wh
passes through the point (1, –2).

13. The gradient of a curve at any point (x, y) on the curve is directly proportional to $\sqrt{x}$. Given that the cu
passes through the points (1, 4) and (4, 1), find its equation.

14. The gradient of a curve at any point (x, y) on the curve is a + bx. The gradient of the curve at the point (0,
on the curve is 3 and the curve passes through the point (1, 2). Find the equation of the curve.

9.2 Definite integrals

9.2.1 The notation $[F(x)]_a^b$

$[F(x)]_a^b$ is an abbreviated form of F(b) – F(a).

Thus, $[2x^2 + x]_1^2 = [8 + 2] - [2 + 1]$

$= 10 - 3$

$= 7$

$[x^3 + x - 2]_0^{-1} = [-1 - 1 - 2] - [0 + 0 - 2]$

$= -4 + 2$

$= -2$

9.2.2 The notation $\int_a^b f(x)\,dx$ and its evaluation

If $\int f(x)\,dx = F(x) + c$

then $\int_a^b f(x)\,dx = [F(x) + c]_a^b$

$$= [F(b) + c] - [F(a) + c]$$

$$= F(b) - F(a).$$

It is therefore unnecessary to write down the constant of integration. This type of integral is known as a definite integral.

$$\int_1^3 (2x + 1)\,dx = \left[x^2 + x\right]_1^3$$

$$= [9 + 3] - [1 + 1]$$

$$= 10$$

$$\int_{-2}^0 \left(x^2 - 2x + 5\right) dx = \left[\frac{x^3}{3} - x^2 + 5x\right]_{-2}^0$$

$$= 0 - \left[\frac{-8}{3} - 4 - 10\right]$$

$$= \frac{8}{3} + 14$$

$$= 16\frac{2}{3}$$

Example 5

Find (a) $\int_0^1 (1 - 2x)^3\,dx$ (b) $\int_1^\infty \frac{3}{x^2}\,dx$

Solution

(a) $\int_0^1 (1 - 2x)^3\,dx = \left[\frac{(1 - 2x)^4}{-8}\right]_0^1$

$$= \left[-\frac{1}{8}\right] - \left[-\frac{1}{8}\right]$$

$$= 0$$

(b) $\int_1^\infty \frac{3}{x^2}\,dx = \left[-\frac{3}{x}\right]_1^\infty$

$= 0 - [3]$, as $\frac{1}{x} \to 0$ when $x \to \infty$

$= 3$

Exercise 9 B

1. Evaluate :

(a) $\int_{-3}^{-1} x \, dx$ (b) $\int_{1}^{2} 2x^3 \, dx$ (c) $\int_{-2}^{-1} (3x^2 + 2x) \, dx$

(d) $\int_{1}^{2} (2x^2 - 3) \, dx$ (e) $\int_{1}^{3} (2x^2 - 1)(x + 1) \, dx$ (f) $\int_{1}^{2} \frac{x^3 - 2}{3x^2} \, dx$

(g) $\int_{1}^{2} \frac{(2x - 1)(x - 1)}{x^4} \, dx$ (h) $\int_{1}^{4} \frac{3}{\sqrt{x}} \, dx$ (i) $\int_{4}^{9} \frac{2 - x}{\sqrt{x}} \, dx$

2. Evaluate :

(a) $\int_{0}^{1} (x + 1)^3 \, dx$ (b) $\int_{-2}^{-1} (2x + 3)^3 \, dx$ (c) $\int_{1}^{2} (2x + 3)^4 \, dx$ (d) $\int_{1}^{2} \sqrt{5x - 1} \, dx$

(e) $\int_{0}^{5} \frac{2}{\sqrt{3x + 1}} \, dx$ (f) $\int_{-\infty}^{-1} \frac{1}{x^3} \, dx$ (g) $\int_{0}^{2} \frac{5}{\sqrt{4x + 1}} \, dx$ (h) $\int_{-2}^{0} \frac{3}{\sqrt{1 - 4x}} \, dx$

9.3 Area under a curve

(a) Area enclosed by a curve, the x-axis and lines parallel to the y-axis

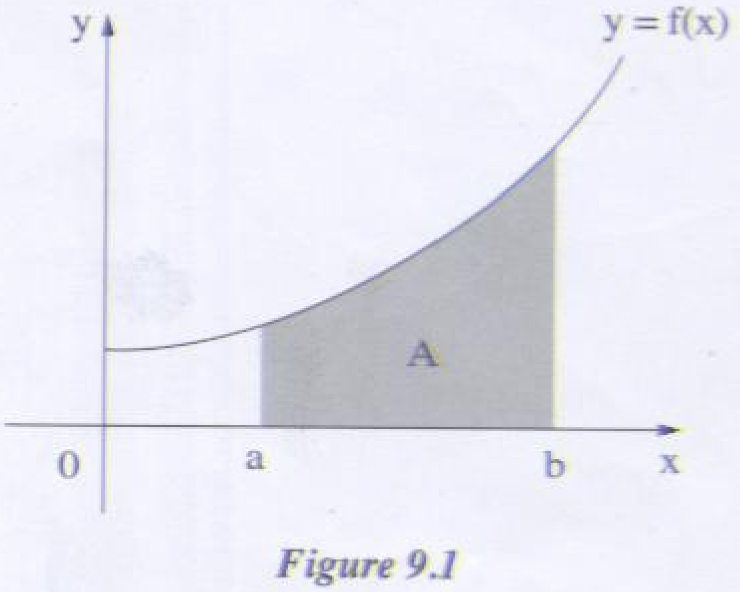


Figure 9.1

The shaded region shown is enclosed by the curve $y = f(x)$, lines $x = a$, $x = b$ and $y = 0$ (i.e. the x-axis). The
of this region is denoted by A.

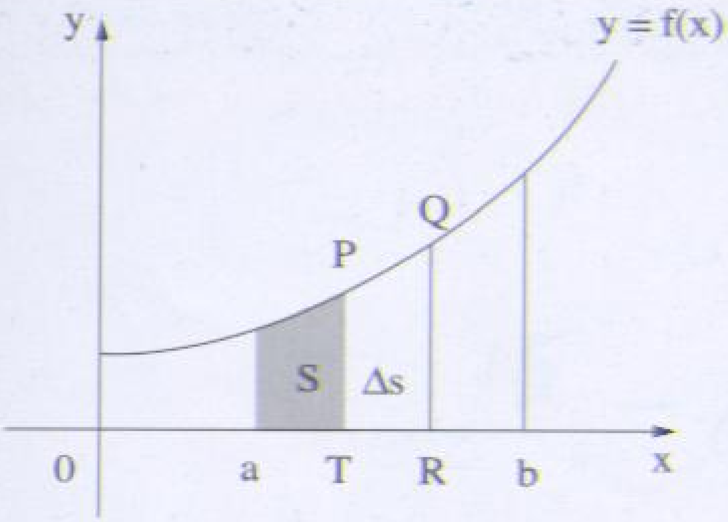


Figure 9.2

Let P, Q be points on the curve with coordinates (x, y), $(x + \Delta x\ , y + \Delta y\ )$ respectively (Figure 9.2).

If the area bounded by the curve, line $x = a$, line PT and x-axis is denoted by S, then as x increases by Δx, y by Δy, s increases by Δs .

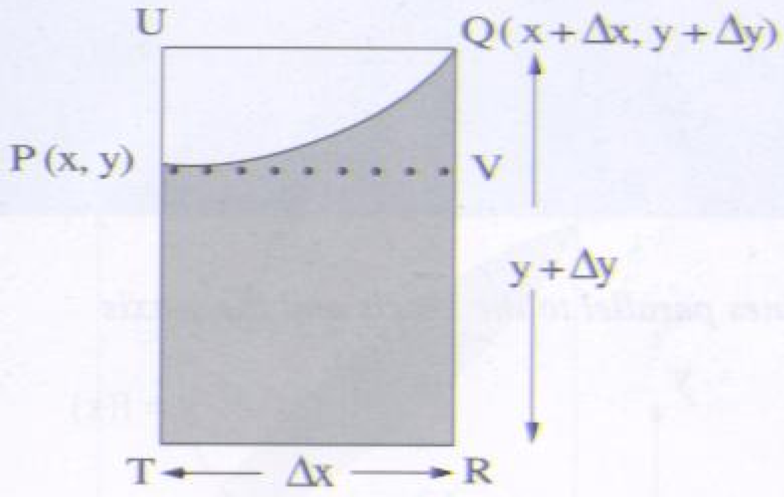


Figure 9.3

From Figure 9.3, we see that area PTRV < Δs < area TRQU.

$$y\,\Delta x \quad < \Delta s \quad < (y + \Delta y\ )\,\Delta x$$

$$y \quad < \frac{\Delta s}{\Delta x} \quad < y + \Delta y$$

as $\Delta x \to 0, \Delta s \to 0, \dfrac{\Delta s}{\Delta x} \to \dfrac{ds}{dx}$ and $\Delta y \to 0$

$$\frac{ds}{dx} = y$$

$$s = \int y\,dx = \int f(x)\,dx = \int F(x) + c, \text{ say}$$

when $x = a, s = 0, c = -F(a)$

when $x = a, s = A$

$A\ = F(b) - F(a)$

$$= \int_a^b f(x)\,dx$$

$$= \int_a^b y\,dx$$

This result holds irrespective of the shape of the curve for all curves continuous in [a, b].

Example 6

Find the area enclosed by the curve $y = x^2 + 1$, lines $x = 1, x = 2$ and the x-axis.

Solution

$$\text{Required area} = \int_1^2 \left(x^2 + 1\right) dx$$

$$= \left[\frac{x^3}{3} + x\right]_1^2$$

$$= \left(\frac{8}{3} + 2\right) - \left(\frac{1}{3} + 1\right)$$

$$= 4\frac{2}{3} - 1\frac{1}{3}$$

$$= 3\frac{1}{3}$$

So, area $= 3\frac{1}{3}$ square units.

(b) Area enclosed by a curve, lines parallel to the x-axis and the y-axis

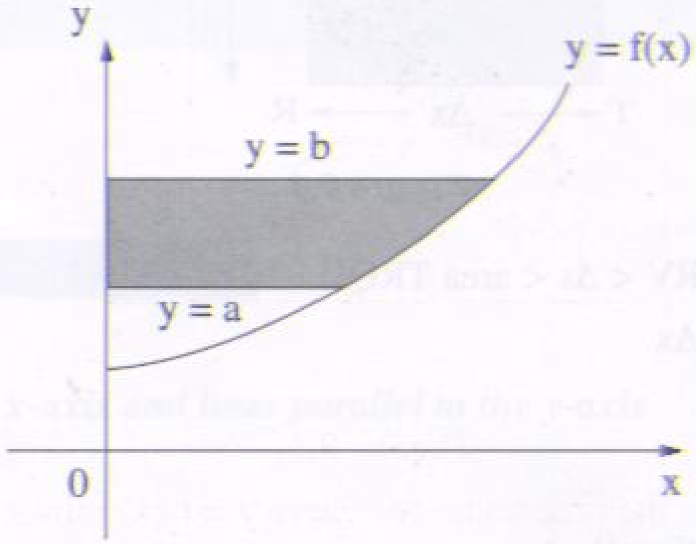


Figure 9.4

The area of the region bounded by a curve, the lines $y = a$, $y = b$ and the y-axis is $\int_a^b x\, dy$.

Example 7

Find the area enclosed by the curve $y = x^2 + 1$, lines $y = 5$, $y = 10$ and the y-axis.

Solution

$$\text{Required area} = \int_5^{10} x\, dy$$

$$= \int_5^{10} (y - 1)^{\frac{1}{2}}\, dy$$

$$= \left[\frac{2}{3}(y-1)^{\frac{3}{2}}\right]_5^{10}$$

$$= \frac{2}{3} \times 9^{\frac{3}{2}} - \frac{2}{3} \times 4^{\frac{3}{2}}$$

$$= 18 - \frac{16}{3}$$

$$= 12\frac{2}{3}$$

Area = $12\frac{2}{3}$ square units.

(c) Area enclosed by two curves

Consider curves $y = f_1(x)$ and $y = f_2(x)$ shown in Figure 9.5.

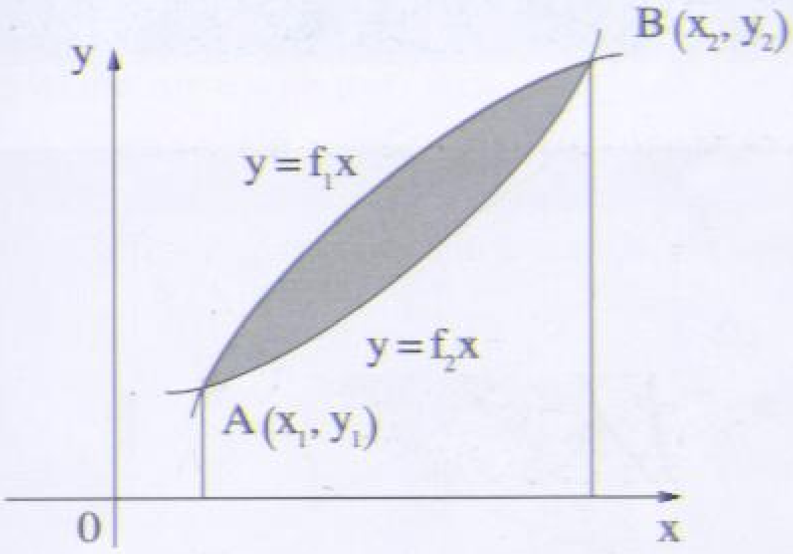


Figure 9.5

The area enclosed by the two curves is the area under the curve $y = f_1(x)$ minus the area under the curve $y = f_2(x)$.

Required area = $\int_{x_1}^{x_2} [f_1(x) - f_2(x)]\, dx$

Example 8

Find the area enclosed by the curve $y = 2x - x^2$ and $y = x^2 - x$.

Solution

We find the points of intersection of the two curves.

$2x - x^2 = x^2 - x$

$-2x^2 + 3x = 0$

$x(-2x + 3) = 0$

$x = 0$ or $\frac{3}{2}$

$$\text{Required area} = \int_0^{\frac{3}{2}} \left[\left(2x - x^2\right) - \left(x^2 - x\right)\right] dx$$

$$= \int_0^{\frac{3}{2}} \left(3x - 2x^2\right) dx$$

$$= \left[\frac{3x^2}{2} - \frac{2x^3}{3}\right]_0^{\frac{3}{2}}$$

$$= \left[\frac{3}{2} \times \frac{9}{4} - \frac{2}{3} \times \frac{27}{8}\right] - [0 - 0]$$

$$= \frac{27}{8} - \frac{9}{4}$$

$$= 1\frac{1}{8}$$

Area = $1\frac{1}{8}$ square units.

Example 9

Find the area enclosed by the curve $y = x^2 - x - 2$ and the x-axis.

Solution

The curve cuts the x-axis at $x^2 - x - 2 = 0$

$$(x - 2)(x + 1) = 0$$

$$x = 2 \text{ or } -1$$

$$\int_{-1}^{2} \left(x^2 - x - 2\right) dx = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^{2}$$

$$= \left[\frac{8}{3} - 2 - 4\right] - \left[-\frac{1}{3} - \frac{1}{2} + 2\right]$$

$$= -\frac{10}{3} - \left[\frac{7}{6}\right]$$

$$= -4\frac{1}{2}$$

We note that the answer obtained is negative. This is because the region is below the x-axis. We ignore the negative sign and write the area as $4\frac{1}{2}$ square units.

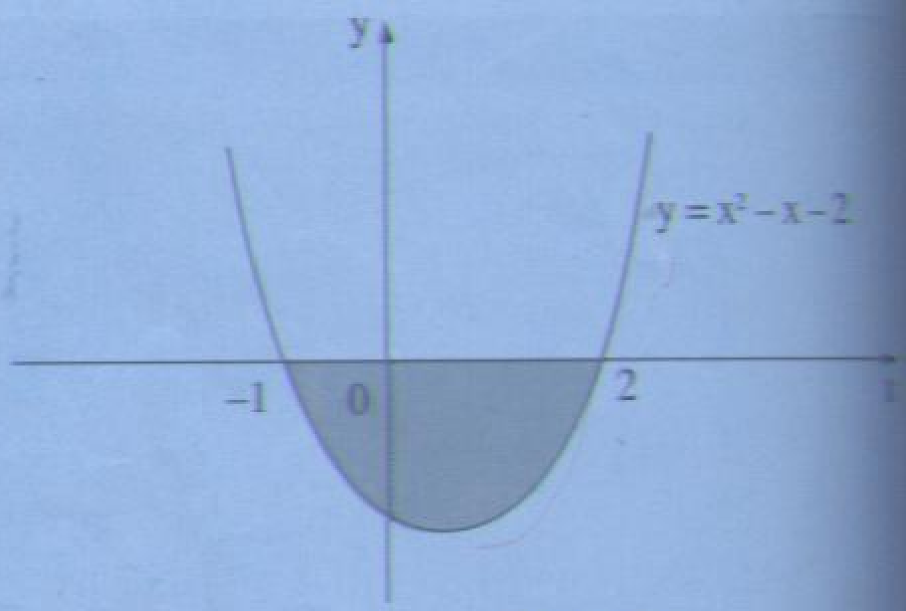


Figure 9.6

Note: If a region lies partly below the x-axis and partly above the x-axis, we find the two areas separately.

This is illustrated in Example 10.

Example 10

Find the area enclosed by the curve $y = x^2 - 4x + 3$ shown in Figure 9.7, the line $x = 2$, $x = 4$ and the x-axis.

Solution

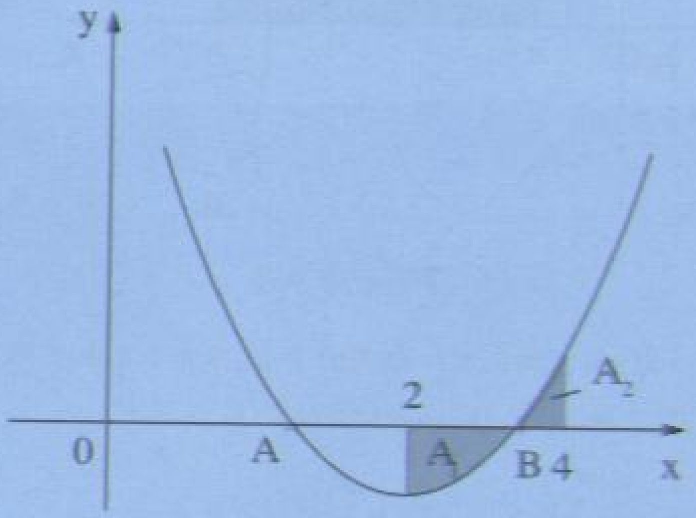


Figure 9.7

We find the points of intersection of the curve with the x-axis.

$$x^2 - 4x + 3 = 0$$

$$(x - 1)(x - 3) = 0$$

$$x = 1 \text{ or } 3$$

So, A has coordinates (1, 0) and B (3, 0)

$$A_1 = \int_2^3 (x^2 - 4x + 3)\,dx$$

$$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_2^3$$

$$= [9 - 18 + 9] - \left[\frac{8}{3} - 8 + 6\right]$$

$$= -\frac{2}{3}$$

$$A_2 = \int_3^4 (x^2 - 4x + 3)\,dx$$

$$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_3^4$$

$$= \left[\frac{64}{3} - 32 + 12\right] - [9 - 18 + 9]$$

$$= 1\frac{1}{3}$$

Required area $= \left(\frac{2}{3} + 1\frac{1}{3}\right)$ square units

= 2 square units.

9.3.2 Volumes of solids of revolution

(a) Rotation about the x-axis of the region enclosed by $y = f(x)$, $x = a$, $x = b$ and x-axis.

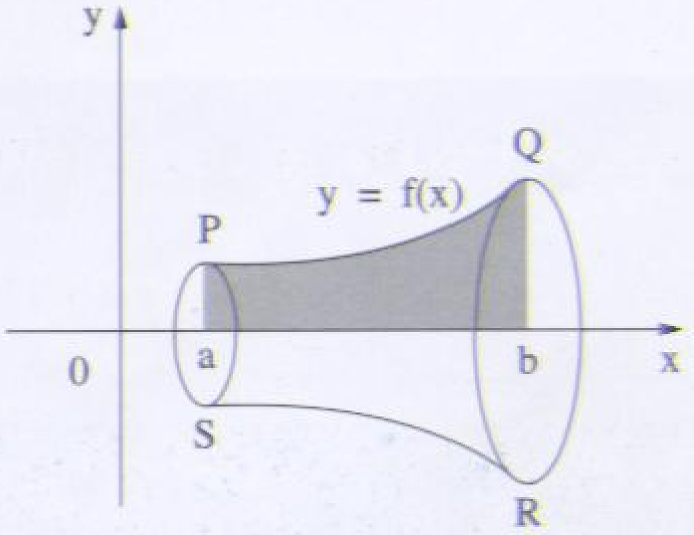


Figure 9.8

If the region enclosed by the curve $y = f(x)$, $x = a$, $x = b$ and x-axis is rotated through 4 right angles about the
we obtain the solid of revolution PQRS shown (Figure 9.8).

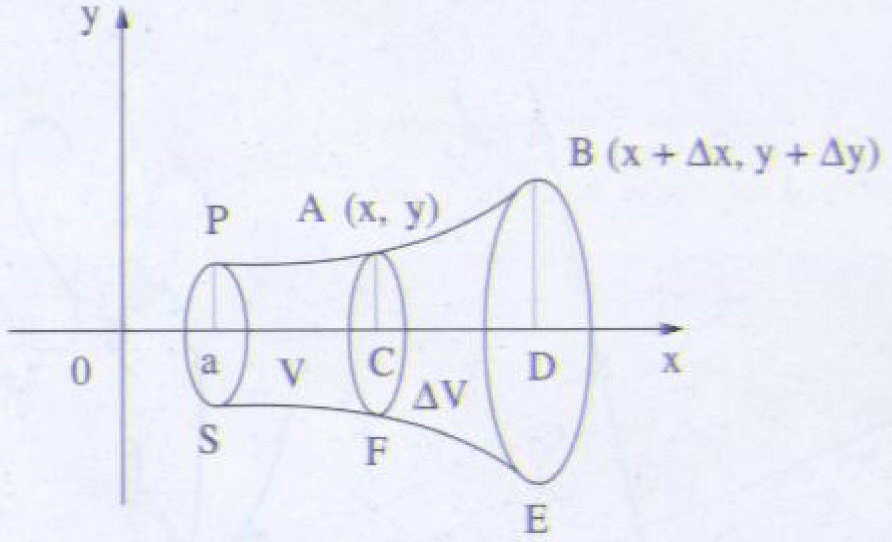


Figure 9.9

Let v be the volume of the solid of revolution obtained by rotating the region enclosed by the curve, line $x =$
line AC and the x-axis through 4-right angles about the x-axis.
If B has coordinates $(x + \Delta x, y + \Delta y)$, the region enclosed by the curve, lines AC, BD and the x-axis
solid ABEF. As the x-coordinate increases by Δx and the y-coordinate increases by Δy from A to B, let volume

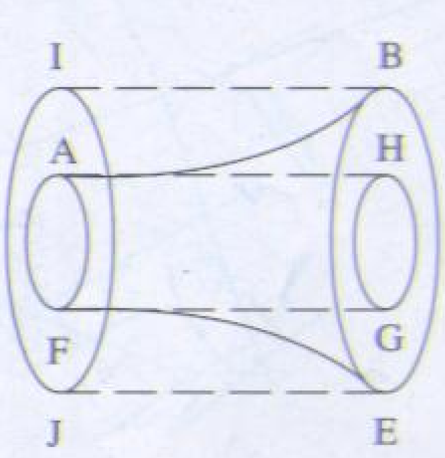


Figure 9.10

Volume of AFGH < Δv < Volume of IJEB

$$\pi y^2\ \Delta x < \Delta v < \pi(y + \Delta y)^2\ \Delta x$$

$$\pi y^2 < \frac{\Delta v}{\Delta x} < \pi(y + \Delta y)^2$$

As $\Delta x \to 0, \Delta y \to 0, \frac{\Delta v}{\Delta x} \to \frac{dv}{dx}$

We obtain $\frac{dv}{dx} = \pi y^2$

The volume v obtained by rotating the region enclosed by the curve y = f(x), lines x = a, x = b and the x-axis is

$\pi\int_a^b y^2\,dx.$

Example 11

Find the volume obtained by rotating the region enclosed by the curve $y = x^2 + 1$, $x = 1$, $x = 2$ and the x-axis through 4 right angles about the x-axis.

Solution

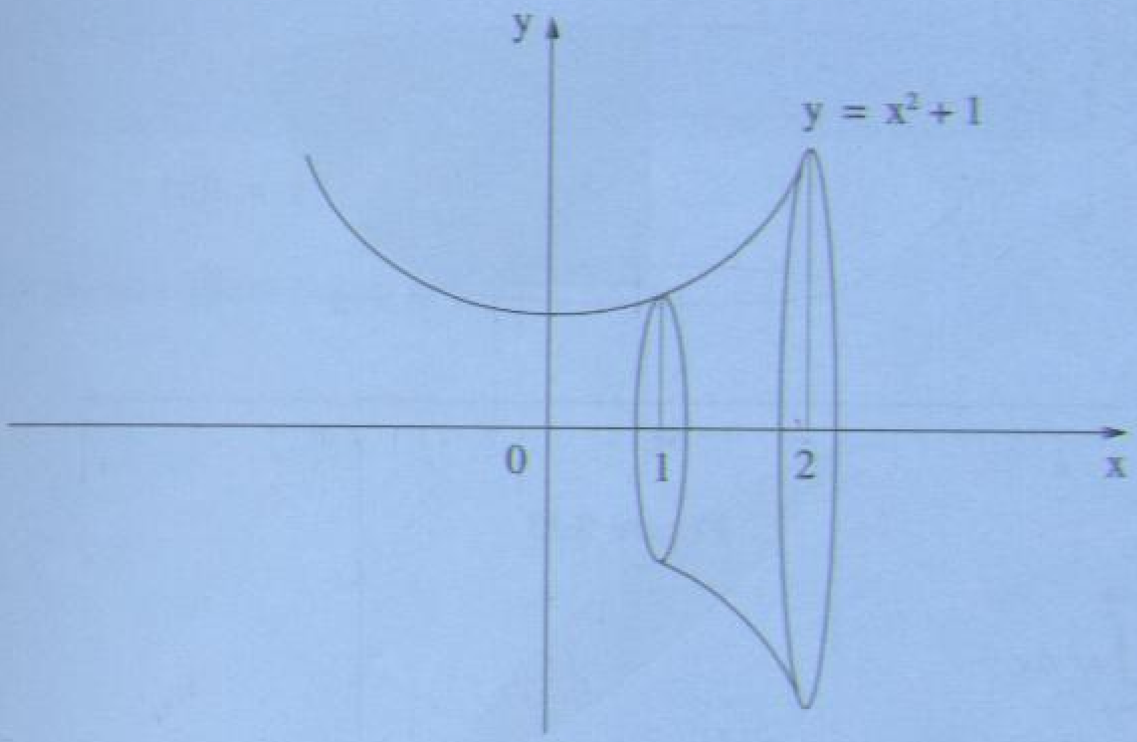


Figure 9.11

$$\text{Required volume} = \pi\int_1^2 y^2\,dx$$

$$= \pi\int_1^2 (x^2+1)^2\,dx$$

$$= \pi\int_1^2 (x^4+2x^2+1)\,dx$$

$$= \pi\left[\frac{x^5}{5}+\frac{2x^3}{3}+x\right]_1^2$$

$$= \pi\left[\frac{32}{5}+\frac{16}{3}+2\right]-\pi\left[\frac{1}{5}+\frac{2}{3}+1\right]$$

$$= \pi\left[\frac{96+80+30}{15}\right]-\pi\left[\frac{3+10+15}{15}\right]$$

$$= \frac{206\pi}{15} - \frac{28\pi}{15}$$

$$= \frac{178}{15}\pi$$

$$= 11\frac{13}{15}\pi$$

Volume $= 11\frac{13}{15}\pi$ cubic units.

(b) Rotation about the y-axis of region enclosed by $y = f(x)$, $y = a$, $y = b$ and y-axis.

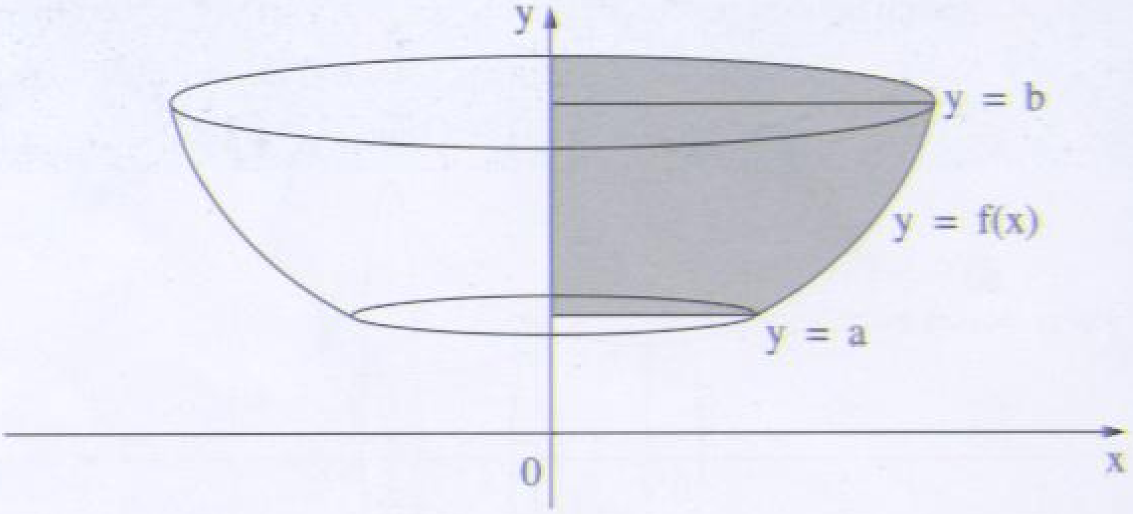


Figure 9.12

The volume obtained is $\pi \int_{y=a}^{y=b} x^2 \, dy$.

Example 12

Find the volume obtained by rotating the region enclosed by the curve $y = x^2 + 1$, lines $y = 2$, $y = 3$ and
y-axis through 4 right angles about the y-axis.

Solution

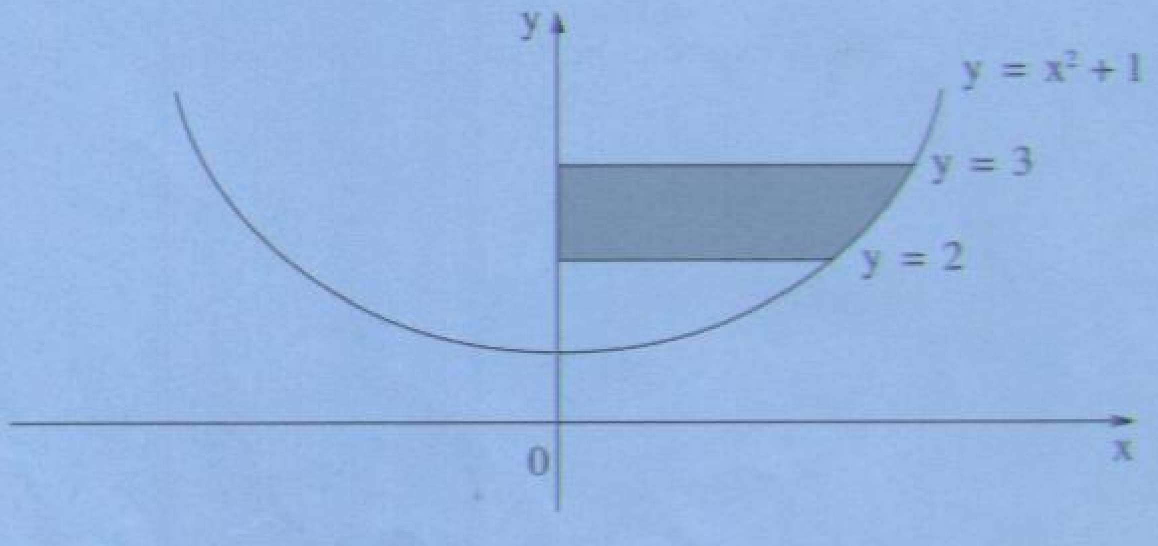


Figure 9.13

$$\text{Required volume} = \pi \int_{2}^{3} x^2 \, dy$$

$$= \pi \int_{2}^{3} (y - 1) \, dy$$

$$= \pi \left[\frac{y^2}{2} - y\right]_2^3$$

$$= \pi\left(\frac{9}{2} - 3\right) - \pi(2 - 2)$$

$$= \frac{3\pi}{2}$$

Volume = $\frac{3\pi}{2}$ cubic units.

(c) Rotation of other regions about x-axis or y-axis

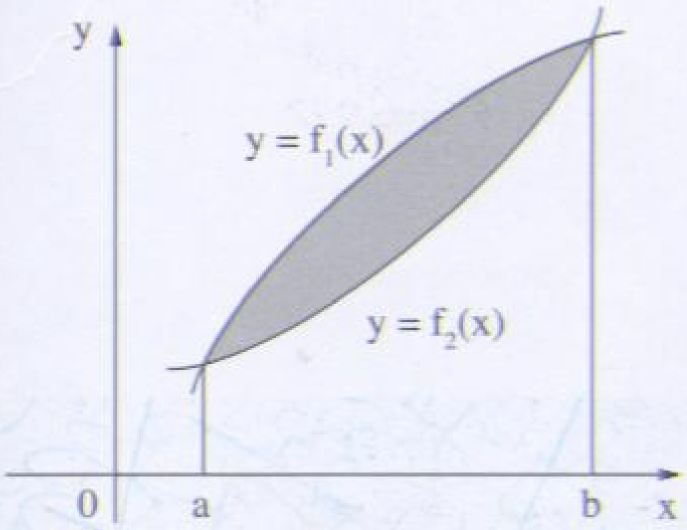


Figure 9.14

If the shaded region is rotated through 4-right angles about the x-axis, the volume is

$$\pi \int_{a}^{b} [f_1(x)]^2 \, dx - \pi \int_{a}^{b} [f_2(x)]^2 \, dx$$

This can be simply written as $\pi \int_{a}^{b} [f_1(x)]^2 - [f_2(x)]^2 \, dx$

Example 13

The region enclosed by the parabolas $y^2 = 9x$ and $x^2 = 9y$ is rotated through 4 right angles about the x-axis. F the volume of the solid of revolution obtained.

Solution

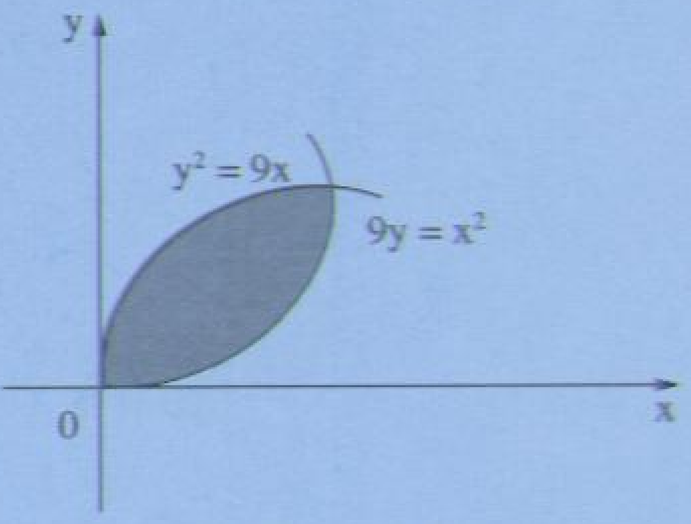


Figure 9.15

To find the points of intersection, we solve the equations:

$y^2 = 9x$ (1)

$x^2 = 9y$ (2)

Squaring (1) $y^4 = 81x^2$ (A)

From (2) & (A), $y^4 = 81 \times 9y$

$$y^4 = 729y$$

$$y^4 - 729y = 0$$

$$y(y^3 - 729) = 0$$

$$y = 0 \text{ or } y^3 - 729 = 0$$

$$y = 0 \text{ or } 9$$

$$y = 0, x = 0$$

$$y = 9, x = 9$$

Points of intersection are (0, 0) and (9, 9).

$$\text{Volume } = \pi \int_0^9 \left[(9x) - \left(\frac{x^2}{9}\right)^2\right]$$

$$= \pi\left[\left(\frac{9x^2}{2}\right) - \left(\frac{x^5}{405}\right)\right]_0^9$$

$$= \pi\left[\frac{729}{2} - \frac{729}{405}\right]$$

$$= 218.7\,\pi$$

Required volume = 218.7 π cubic units.

Exercise 9 C

1. Calculate the area bounded by each of the following curves, the x-axis and the given lines:
   (a) $y = x^2 + 3$, $x = 2$ and $x = 4$
   (b) $y = 2x^3 + 3$, $x = 1$ and $x = 2$
   (c) $y = x^2 - 4x + 9$, $x = -1$ and $x = 1$
   (d) $y = x^2 + x - 6$, $x = -3$ and $x = 2$
   (e) $y = x^2 + 2x + 7$, $x = -3$ and $x = 0$
   (f) $y = (2x - 1)^2$, $x = 0$ and $x = 2$.

2. Calculate the area bounded by each of the following curves, the y-axis and the given lines:
   (a) $y = x^2 + 3$, $y = 3$ and $y = 4$
   (b) $y^2 = 9x$, $y = 1$ and $y = 3$
   (c) $y = 2x^3$, $y = 2$ and $y = 16$
   (d) $y^2 = x - 2$, $y = -1$ and $y = 1$
   (e) $y = (2x - 1)^2$, $y = 4$ and $y = 9$.

3. Sketch each of the following curves and find in each case the area enclosed by the curve, the x-axis and the given lines:
   (a) $y = x^2 - 2x$, $x = 0$ and $x = 2$
   (b) $y = x^2 + 2x - 3$, $x = -3$ and $x = 1$
   (c) $y = 6 - x - x^2$, $x = -4$ and $x = 1$
   (d) $y = 2x^2 + x - 3$, $x = -3$ and $x = 0$.

4. Calculate the area enclosed by each of the following pairs of curves:
   (a) $y = x^2 + x - 2$, $y = 2x^2 - x - 5$
   (b) $y = 6 + x - x^2$, $y = 4 - 2x - 2x^2$
   (c) $y^2 = 4x$ and $x^2 = 4y$

5.

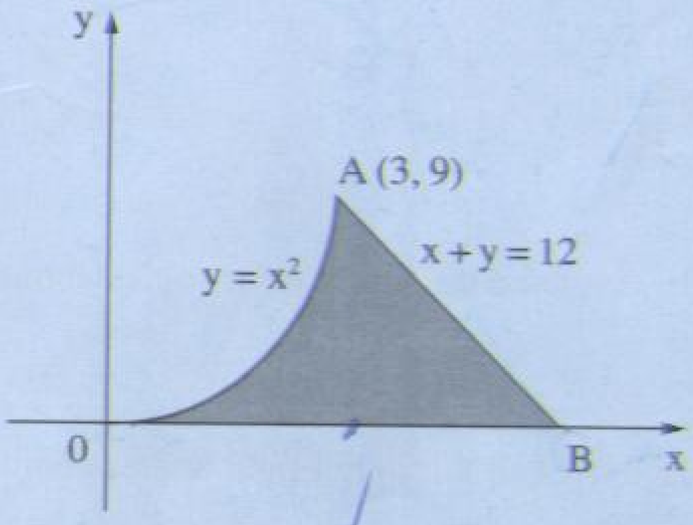


Figure 9.16

In Figure 9.16, OA is part of the curve $y = x^2$. A has coordinates (3, 9) and line AB has equation $x + y = 12$. Find the area of the shaded region.

6.

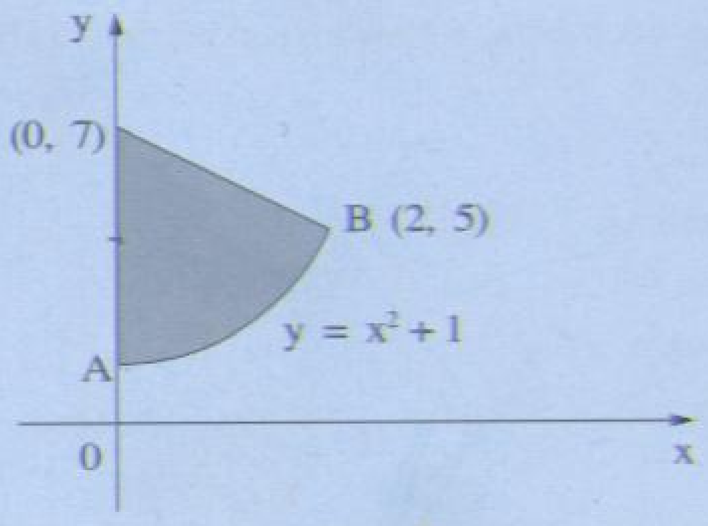


Figure 9.17

In Figure 9.17 AB is an arc of the curve $y = x^2 + 1$, A lies on the y-axis and BC is a straight line.
area of the shaded region.

7. Find the volume of the solid of revolution obtained by rotating the region bounded by each of the
curves and the given lines through 4 right angles about the x-axis:
   (a) $y = x^2, x = 1, x = 2$ and $y = 0$
   (b) $y = x^2 + 1, x = 0, x = 3$ and $y = 0$
   (c) $y^2 = x, x = 1, x = 4$ and $y = 0$
   (d) $y^2 = 9x + 1, x = 0, x = 1$ and $y = 0$
   (e) $y = (x - 2)^2, x = -1, x = 1$ and $y = 0$
   (f) $y = x^3, x = -1, x = 2$ and $y = 0$
   (g) $y = (2x - 1)^2, x = -1, x = 1$ and $y = 0$.

8. Find the volume of the solid of revolution obtained by the rotation of the region bounded by each
given curves and the given lines through 4 right angles about the y-axis:
   (a) $y = x^2, y = 1, y = 3$ and $x = 0$
   (b) $y = x^2 + 1, y = 2, y = 5$ and $x = 0$
   (c) $y^2 = x, y = 1, y = 2$ and $x = 0$
   (d) $y = 9x^2 + 1, y = 1, y = 10$ and $x = 0$
   (e) $y = x^3, y = -8, y = -1$ and $x = 0$
   (f) $y^2 = -x, y = 2, y = 3$ and $x = 0$.

9. The shaded region in Figure 9.16 is rotated through 4 right angles about the x-axis. Find the volume
solid of revolution obtained.

10. The shaded region in Figure 9.17 is rotated through 4 right angles about the y-axis. Find the volume
solid of revolution obtained.

11. Find the volume obtained when the region bounded by the curve $y = 3x + \frac{4}{x}$, the x-axis, the lines $x =$
$x = 2$ is rotated through 4 right angles about the x-axis.

12.

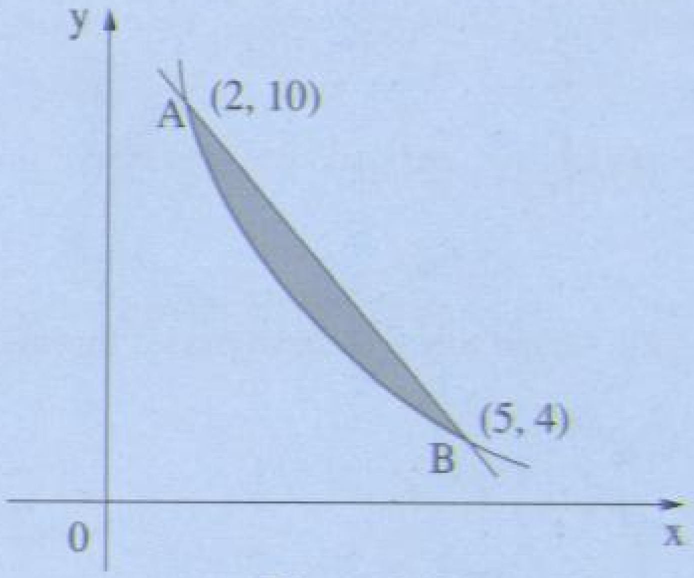


Figure 9.18

Figure 9.18 shows part of the curve $y = \frac{20}{x}$. The points A and B with coordinates (2, 10) and (5, 4) lie on the curve.

Find:

(a) the equation of AB

(b) the volume obtained when the shaded region is rotated through 4 right angles about the x-axis.

(c) the volume obtained when the shaded region is rotated through 4 right angles about the y-axis.

13.

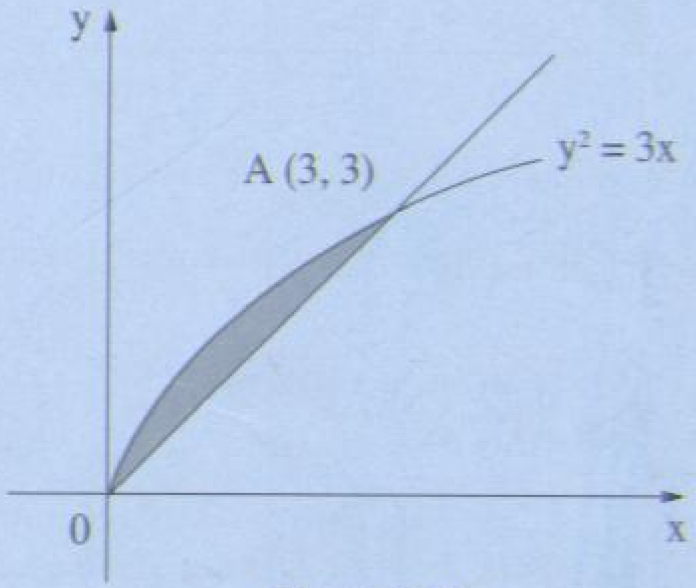


Figure 9.19

Figure 9.19 shows part of the curve $y^2 = 3x$ and the line OA where O is the origin and A has coordinates (3, 3). Find the volume obtained when the shaded region is rotated through 4 right angles about (a) the x-axis, (b) the y-axis.

14. Sketch the curve $y = x^2 - 4x + 3$ for $-3 \leqslant x \leqslant 6$. Find:

(a) the area enclosed by the curve, the lines $x = 2$, $x = 4$ and the x-axis

(b) the volume obtained by rotating the region enclosed by the lines $x = 2$, $x = 3$, the x-axis and the curve through 4 right angles about the x-axis.

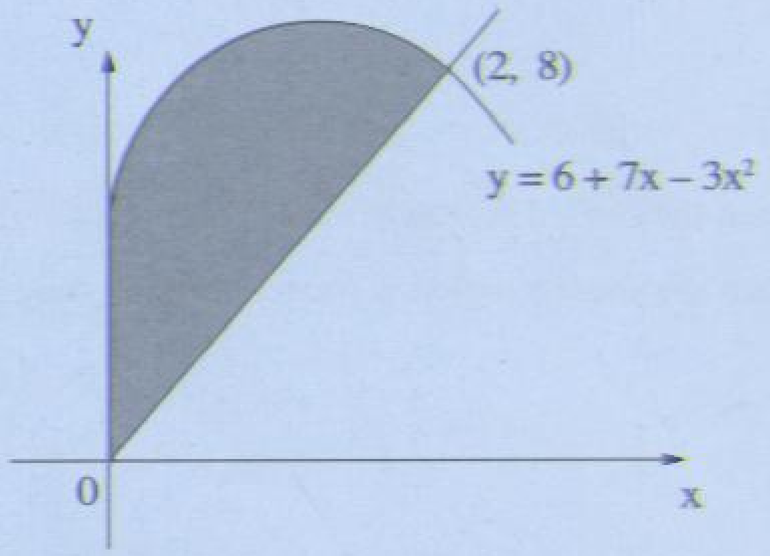
Miscellaneous Exercise 9

1. Evaluate: (a) $\int_1^{\frac{4}{3}} (3x - 2)\, dx$ (b) $\int_1^{\frac{4}{3}} (3x - 2)^5\, dx$

2. A curve is such that $\frac{dy}{dx} = (3x - 2)^2$. Given that the curve passes through (1, 2), find its equation.

3. Find: (a) $\int \frac{1}{\sqrt{4x + 3}}\, dx$ (b) $\int_0^1 (2x - 1)^3\, dx$

4. The diagram shows part of the curve $y = 6 + 7x - 3x^2$. Calculate the area of the shaded region.

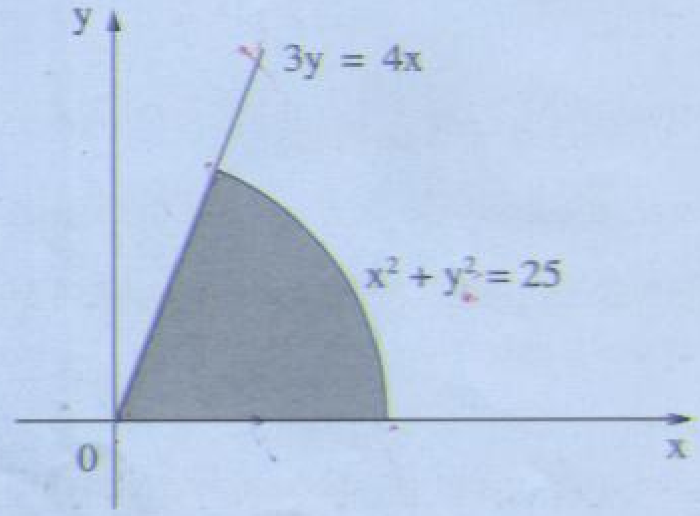


5. Find: (a) $\int (2x + 1)^{10}\, dx$ (b) $\int \sqrt{2x + 1}\, dx$

6. Find: (a) $\int_1^{\infty} \frac{1}{x^2}\, dx$ (b) $\int_1^2 (2x - 1)^{-5}\, dx$

7. Find the area of the region bounded by the curve $y = \frac{1}{\sqrt{2x + 1}}$, the x-axis and the lines $x = 1$ and $x = 5$

8.



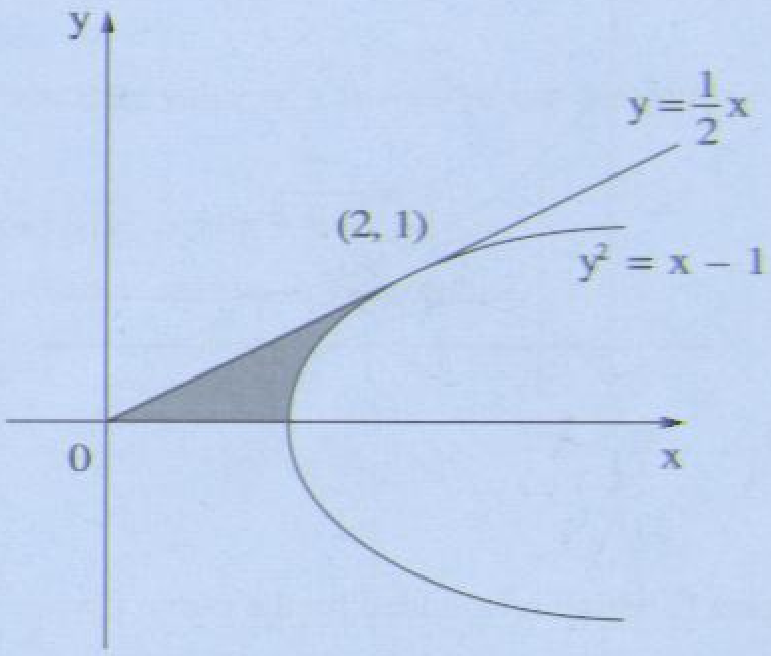
Calculate the volume generated when the region shown is rotated through 360° about the x-axis.

9. Given $\frac{dy}{dx} = \frac{a}{x^2} + 1$ and that when $x = 1$, $\frac{dy}{dx} = 3$ and $y = 3$, find the value of y when $x = 2$. [C]

10. The curve for which $\frac{dy}{dx} = kx - 5$, where k is a constant, passes through the points (1, 0) and (0, 6). Find:
    (a) the equation of the curve
    (b) the x-coordinate of the stationary point on the curve. [C]

11. (a) Find $\int \left(\frac{1}{x^2} - \frac{1}{x^3}\right) dx$

    (b)

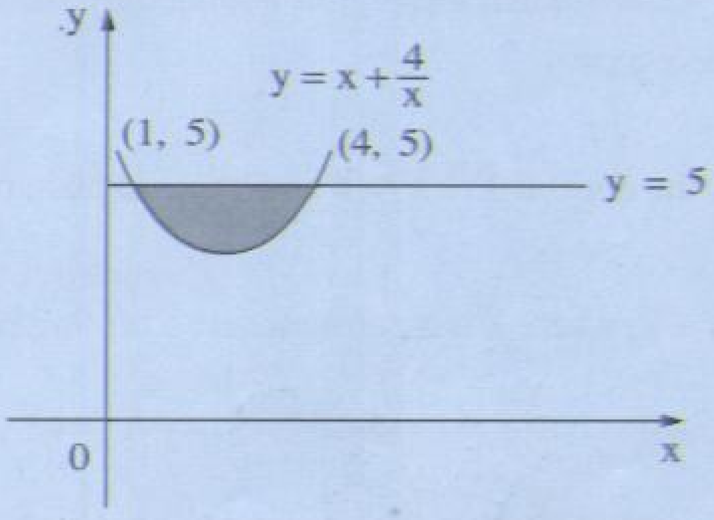


The line $y = \frac{1}{2}x$ is the tangent to the curve $y^2 = x - 1$ at the point (2, 1). Calculate the volume swept out when the shaded region shown in the diagram is rotated through 360° about the x-axis. [C]

12. The curve $y = (x - 2)^2$ and $y = 4x - x^2 - 2$ intersect at the points (1, 1) and (3, 1). Find the area enclosed by the two curves. [C]

13. (a) Given that $\frac{dy}{dx} = 2x + 3$ and that $y = 3$ when $x = -1$, express y in terms of x.

    (b)

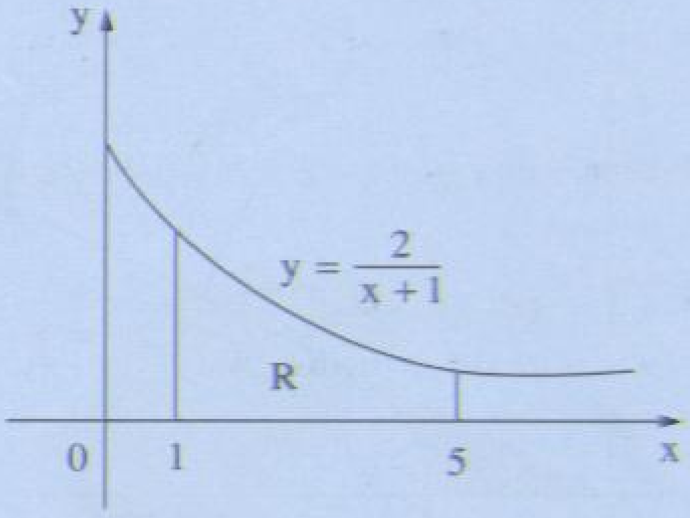


The diagram shows the line $y = 5$ intersecting the curve $y = x + \frac{4}{x}$ at (1, 5) and (4, 5). Calculate the volume generated when the shaded region is rotated through 360° aboutthe x-axis. [C]

14. Determine the volume of the solid of revolution formed when the region in the first quadrant bounded by the y-axis, the lines $y = 2$ and $y = 4$ and the curve $x^2 = y^3$ is rotated about the y-axis.

15. The curve for which $\frac{dy}{dx} = 2x + k$ where k is a constant has the tangent at (3, 6) passing through the
State the gradient of this tangent and determine:
   (a) the value of k
   (b) the equation of the curve.

16.



The diagram shows the region R which is bounded by the curve $y = \frac{2}{x+1}$, the x-axis, and the lines x =

and $x = 5$. Show that the volume of the solid when R is rotated completely about the x-axis is $\frac{4}{3}\pi$.

17. (a) Given that the curve $y = ax^2 + \frac{b}{x}$ has a gradient of –5 at the point (2, –2), find the value of a and

   (b) Given the curve $y = 2x - \frac{8}{x^2}$,
      (i) Find the area of the region enclosed by the curve, the x-axis and the lines $x = 2$ and $x = 4$.
      (ii) Show that when x is positive, y increases as x increases.

18. Find the volume of the solid formed when the region bounded by the curve $y^2 = 3x$ and the line y = rotated through 360° about the x-axis.

19. Sketch the graph of $y = \frac{12}{x}$ for the domain $2 \leqslant x \leqslant 4$ and hence explain briefly why $6 < \int_2^4 \frac{12}{x}\,dx < 12$.

20. The curve for which $\frac{dy}{dx} = 4x + k$, where k is a constant, has a turning point at (–2, –1).
Find:
   (a) the value of k
   (b) the coordinates of the point at which the curve meets the y-axis.

CHAPTER 10

> ALGEBRA

10.1 The absolute value of x

10.1.1 Definition of |x|

The absolute value of –3 is 3, that of $-1\frac{1}{2}$ is $1\frac{1}{2}$ and that of –2.4 is 2.4, etc.

If x is negative, it follows that the absolute value of x is –x. We use the symbol $|x|$ to stand for the absolute value of x. If x is positive or 0, $|x| = x$.
If follows that $|x| = x$ (for $x \geqslant 0$) and $|x| = -x$ (for $x < 0$).
The graph of $y = |x|$ or $f{:}x \mapsto |x|$ is then as shown in Figure 10.1.

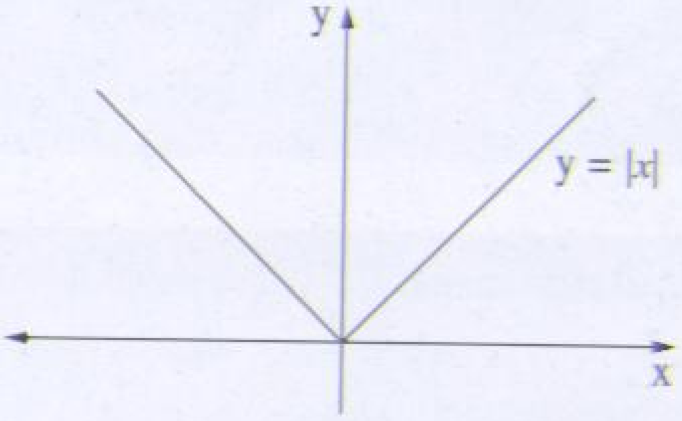


Figure 10.1

10.1.2 Properties of |x|

(a) If $|a| = |b|$, either a = b or a = –b
i.e. $a^2 = b^2$
If $a^2 = b^2$, $a = \pm b$, i.e. $|a| = |b|$.

(b) If $|x|. < 3$, $-3 < x < 3$ as shown in Figure 10.2.

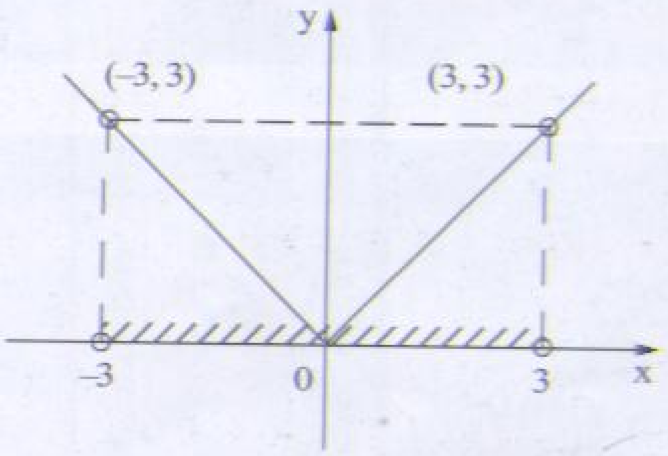


Figure 10.2

If $|x|. < b, -b < x < b$

It follows that if $|x - a| < b$

$$-b < x - a < b$$

$$a - b < x < a + b$$

If $|x|. > b, x < -b$ or $x > b$.

10.1.3 Equations involving |x|

We consider solving equations involving $|x|$.

Example 1

Solve $|x - 2| = 3$.

Solution

If $|x - 2| = 3$, either $x - 2 = -3$ or $x - 2 = 3$

$$x = -1 \qquad x = 5$$

Solution check: Both values satisfy $|x - 2| = 3$

Example 2

Solve $|3x - 2| = |2x + 1|$

Solution

If $|a| = |b|$, $a^2 = b^2$

So, if $|3x - 2| = |2x + 1|$

$$(3x - 2)^2 = (2x + 1)^2$$

$$(3x - 2)^2 - (2x + 1)^2 = 0$$

$$(5x - 1)(x - 3) = 0$$

$$x = \frac{1}{5} \text{ or } 3$$

Solution check: Both values satisfy $|3x - 2| = |2x + 1|$

Example 3

Solve $\left|\frac{2x - 1}{x + 2}\right| = 4$.

Solution

Method 1 $\quad \frac{2x - 1}{x + 2} = 4$ or $\frac{2x - 1}{x + 2} = -4$

$$2x - 1 = 4x + 8 \quad \text{or} \quad 2x - 1 = -4x - 8$$

$$x = -\frac{9}{2} \quad \text{or} \quad x = -\frac{7}{6}$$

Method 2 $$\frac{(2x-1)^2}{(x+2)^2} = 16$$

Solve this equation to obtain $x = -\frac{9}{2}$ or $-\frac{7}{6}$.

Solution check: Both values satisfy $\left|\frac{2x-1}{x+2}\right| = 4$

10.1.4 Inequalities involving |x|

We consider solutions of inequalities involving $|x|$

Example 4

Solve $|2x + 3| < 5$.

Solution

If $|2x + 3| < 5$

$-5 < 2x + 3 < 5$

$-8 < 2x < 2$

$-4 < x < 1$

Example 5

Solve $|3x + 1| > 4$.

Solution

If $|3x + 1| > 4$ either $3x + 1 < -4$ or $3x + 1 > 4$

$3x < -5$ or $3x > 3$

$x < -\frac{5}{3}$ or $x > 1$

Example 6

Solve $\left|\frac{2x+1}{x+2}\right| < 2$.

Solution

We use $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

$$\left|\frac{2x+1}{x+2}\right| < 2$$

$$\frac{|2x+1|}{|x+2|} < 2$$

$$\frac{(2x+1)^2}{(x+2)^2} < 4$$

$$\begin{aligned} (2x+1)^2 &< 4\,(x+2)^2 \\ 4x^2+4x+1 &< 4x^2+16x+16 \\ -12x &< 15 \\ x &> -\frac{5}{4} \end{aligned}$$

Exercise 10 A

1. Solve the equations:

(a) $|x-2| = 3$ (b) $|32-x| = 5$ (c) $|3x-1| = 2x$

(d) $|5x+1| = 3x$ (e) $|2x+1| = x-1$ (f) $|3x-2| = |x+1|$

2. Solve the inequalities:

(a) $|2x+1| \leqslant 5$ (b) $|4x-1| \geqslant 3$ (c) $\left|\frac{3x+1}{2x-1}\right| < 2$

(d) $\left|\frac{4x-1}{3x+1}\right| \geqslant 2$ (e) $|2x+1| \geqslant 3|x-1|$ (f) $|3x-2| < 2|x+1|$

10.2 Polynomials

Expressions of the type $3x^4 + 2x^3 - x^2 + x - 7$, $2x^3 - 7x + 1$, $3x^2 - 4x - 2$, $2x + 1$ are called *polynomials* in x. $2x + 1$ is a linear polynomial in x and $3x^2 - 4x - 2$ is a quadratic polynomial in x.

A polynomial in x is of the form $a_0 + a_1x + a_2x^2 + \ldots\ldots a_nx^n$ (a_n is a constant and $n \in Z^+$). The highest power of x in a polynomial is called the degree of the polynomial.

Thus, the degree of $3x^4 + 2x^3 - x^2 + x - 7$ is 4, $2x^3 - 7x + 1$ has degree 3, etc.

10.2.2 Division of a polynomial by another polynomial

To divide $3x^4 + 2x^3 - x^2 + x - 7$ by $x + 2$, we proceed as shown

$$\begin{array}{r|l} & 3x^3 \\ \hline x+2 & 3x^4+2x^3-x^2+x-7 \end{array}$$

The terms of the polynomial to be divided and of the divisor are arranged in descending powers of x.

We divide the highest power of x in the dividend by the highest power of x in the divisor. This gives $3x^3$.

We multiply this answer by the divisor which gives $3x^4 + 6x^3$.

$$\begin{array}{r|l}
 & 3x^3-4x^2 \\ \hline
x+2 & 3x^4+2x^3-x^2+x-7 \\
 & \underline{3x^4+6x^3} \\
 & \quad -4x^3-x^2
\end{array}$$

We do the subtraction as shown and lower the next term not appearing in this expression, i.e. the term in x^2. We now divide the next highest power of x in the remainder by the division, this gives $-4x^2$. We multiply $-4x^2$ by the divisor.

$$\begin{array}{r|l}
 & 3x^3-4x^2+7x-13 \\ \hline
x+2 & 3x^4+2x^3-x^2+x-7 \\
 & \underline{3x^4+6x^3} \\
 & \quad -4x^3-x^2 \\
 & \quad \underline{-4x^3-8x^2} \\
 & \qquad\qquad 7x^2+x \\
 & \qquad\qquad \underline{7x^2+14x} \\
 & \qquad\qquad\quad -13x-7 \\
 & \qquad\qquad\quad \underline{-13x-26} \\
 & \qquad\qquad\qquad\quad 19
\end{array}$$

This process continues until we obtain a remainder of lesser degree than that of the divisor. In this case, the quotient is $3x^3 - 4x^2 + 7x - 13$ and the remainder is 19.

Example 7

Divide $3x^4 - 6x^3 + 9x^2 - 5x + 8$ by $x^2 + x - 3$.

Solution

$$\begin{array}{r|l}
 & 3x^2-9x \\ \hline
x^2+x-3 & 3x^4-6x^3+9x^2-5x+8 \\
 & \underline{3x^4+3x^3-9x^2} \\
 & \quad -9x^3+18x^2
\end{array}$$

We divide the highest power of x in the dividend by the highest power of x in the divisor. This gives $3x^2$ which we multiply by the divisor giving $3x^4 + 3x^3 - 9x^2$ which we subtract from the dividend to give $-9x^3 + 18x^2$. We next lower $-5x$.

$$\begin{array}{r|l}
 & 3x^2-9x \\ \hline
x^2+x-3 & 3x^4-6x^3+9x^2-5x+8 \\
 & \underline{3x^4+3x^3-9x^2} \\
 & \quad -9x^3+18x^2-5x \\
 & \quad -9x^3-9x^2+27x
\end{array}$$

We next divide $-9x^3$ by x^2 to give $-9x$. Multiplying $-9x$ by the divisor, we obtain $-9x^3 - 9x^2 + 27x$. We continue the process

$$
\begin{array}{r|l}
 & 3x^2 - 9x + 27 \\
\hline
x^2 + x - 3 & 3x^4 - 6x^3 + 9x^2 - 5x + 8 \\
 & \underline{3x^4 + 3x^3 - 9x^2} \\
 & \quad - 9x^3 + 18x^2 - 5x \\
 & \quad \underline{- 9x^3 - 9x^2 + 27x} \\
 & \qquad\qquad 27x^2 - 32x + 8 \\
 & \qquad\qquad \underline{27x^2 + 27x - 81} \\
 & \qquad\qquad\qquad - 59x + 89
\end{array}
$$

We obtain a quotient of $3x^2 - 9x + 27$ and a remainder of $-59x + 89$.

Exercise 10 B

1. Divide $2x^3 + 3x^2 - 5x + 2$ by $x - 3$
2. Divide $6x^4 + 2x^3 - 6x^2 + 8x - 10$ by $x + 3$
3. Divide $8x^4 - 6x^3 + 3x^2 - 4x - 6$ by $2x - 1$
4. Divide $6x^3 + 2x^2 - 7x + 8$ by $x + 1$
5. Divide $10x^4 + 6x^3 - 6x^2 + 2$ by $x - 1$
6. Divide $6x^3 + 5x^2 - 3x + 7$ by $2x^2 + x + 1$
7. Divide $8x^3 - 6x^2 + 2x - 5$ by $4x^2 - x - 1$
8. Divide $10x^4 + 8x^3 - 9x^2 + 7x - 3$ by $x^2 + x - 3$

10.3.1 Remainder theorem

Consider the division of the polynomial $2x^3 - x^2 + 3x - 7$ by $x - 2$. The quotient and the remainder can be obtai
by using the method in 10.2
If we are interested in finding the remainder which is a constant; we write
$2x^3 - x^2 + 3x - 7 \equiv (x - 2)\,Q(x) + R$ where $Q(x)$ is the quotient and R the remainder.
To obtain R, we put $(x - 2) = 0$, i.e. $x = 2$ and obtain $16 - 4 + 6 - 7 = R$ or $R = 11$. This method can be used
find the remainder when a polynomial $f(x)$ is divided by a linear polynomial $x + a$.
Writing $f(x) \equiv (x + a)\,Q(x) + R$, we have $R = f(-a)$.
So, if $f(x)$ is divided by $x + a$, the remainder is $f(-a)$.

Example 8

Find the remainder when $2x^3 + x^2 - x - 12$ is divided by $x + 2$.

Solution

Writing $f(x) \equiv 2x^3 + x^2 - x - 12$, the remainder is

$$
\begin{aligned}
f(-2) &= -16 + 4 + 2 - 2 \\
&= -12
\end{aligned}
$$

Example 9

$3x^3 + 2x^2 - 7x + a$ leaves a remainder of 2 when divided by $x - 1$. Find the value of a.

Solution

$f(x) \equiv 3x^3 + 2x^2 - 7x + a$

Remainder $= f(1) = 3 + 2 - 7 + a$

$= a - 2$

As remainder $= 2$

$a - 2 = 2$

$a = 4$

10.3.2 Factor theorem

If f(x) is divisible by $x + a$ or $x + a$ is a factor of f(x), it follows that the remainder $f(-a) = 0$.

Example 10

Show $x - 2$ is a factor of $2x^3 + x^2 - 9x - 2$.

Solution

$f(x) \equiv 2x^3 + x^2 - 9x - 2$

Remainder $= f(2) = 16 + 4 - 18 - 2$

$= 0$

So, $x - 2$ is a factor of $2x^3 + x^2 - 9x - 2$.

Example 11

$ax^3 - 6x^2 + 2x - 12$ is divisible by $x + 2$. Find the value of a.

Solution

$f(x) \equiv ax^3 - 6x^2 + 2x - 12$

$f(-2) = -8a - 24 - 4 - 12$

$= -8a - 40$

$-8a - 40 = 0$

$a = -5$

10.3.3 Factors of a cubic polynomial

The factor theorem can be used to find the factors of a cubic polynomial.

To find the factors of $x^3 - x^2 - 10x - 8$, we obtain a factor by trial and error.

$f(x) \equiv x^3 - x^2 - 10x - 8$

$f(1) = 1 - 1 - 10 - 8 \neq 0$

$f(-1) = -1 - 1 + 10 - 8 = 0$

So, $x + 1$ is a factor of $x^3 - x^2 - 10x - 8$

To find the remaining factors, we divide $x^3 - x^2 - 10x - 8$ by $x + 1$.

$$
\begin{array}{r|l}
 & x^2-2x-8 \\ \cline{2-2}
\underline{x+1} & x^3-x^2-10x-8 \\
 & x^3+x^2 \\ \cline{2-2}
 & \quad -2x^2-10x \\
 & \quad \underline{-2x^2-2x} \\
 & \qquad\quad -8x-8 \\
 & \qquad\quad -8x-8
\end{array}
$$

$$
\begin{aligned}
\text{So, } x^3 - x^2 - 10x - 8 &\equiv (x + 1)(x^2 - 2x - 8) \\
&\equiv (x + 1)(x - 4)(x + 2)
\end{aligned}
$$

Example 12

Find the factors of $2x^3 - 3x^2 - 11x + 6$.

Hence, solve the equation $2x^3 - 3x^2 - 11x + 6 = 0$.

Solution

$$
\begin{aligned}
f(x) &= 2x^3 - 3x^2 - 11x + 6 \\
f(1) &= 2 - 3 - 11 + 6 \neq 0 \\
f(-1) &= -2 - 3 + 11 + 6 \neq 0 \\
f(2) &= 16 - 12 - 22 + 6 \neq 0 \\
f(-2) &= -16 - 12 + 22 + 6 = 0
\end{aligned}
$$

So, $x + 2$ is factor.

To obtain remaining factors, we divide $2x^3 - 3x^2 - 11x + 6$ by $x + 2$.

$$
\begin{array}{r|l}
 & 2x^2-7x+3 \\ \cline{2-2}
\underline{x+2} & 2x^3-3x^2-11x+6 \\
 & 2x^3+4x^2 \\ \cline{2-2}
 & \quad -7x^2-11x \\
 & \quad \underline{-7x^2-14x} \\
 & \qquad\qquad 3x+6 \\
 & \qquad\qquad 3x+6
\end{array}
$$

$$
\begin{aligned}
\text{So, } 2x^3 - 3x^2 - 11x + 6 &\equiv (x + 2)(2x^2 - 7x + 3) \\
&\equiv (x + 2)(2x - 1)(x - 3)
\end{aligned}
$$

$$2x^3 - 3x^2 - 11x + 6 = 0$$

$$(x + 2)(2x - 1)(x - 3) = 0$$

$$x = -2 \text{ or } \frac{1}{2} \text{ or } 3$$

Example 13

$ax^3 + bx^2 + 4x - 8$ leaves remainder $15x + 2$ when divided by $x^2 - x - 2$. Find the value of a and of b.

Solution

$ax^3 + bx^2 + 4x - 8 - (15x + 2)$, i.e. $ax^3 + bx^2 - 11x - 10$ is divisible by $x^2 - x - 2$, i.e. $(x + 1)(x - 2)$.

Since $ax^3 + bx^2 - 11x - 10$ is divisible by $(x + 1)(x - 2)$, it is divisible by $x + 1$ and by $x - 2$.

Putting $f(x) \equiv ax^3 + bx^2 - 11x - 10$

$$f(-1) = -a + b + 11 - 10 = 0$$

$$-a + b = -1 \quad (1)$$

$$f(2) = 8a + 4b - 22 - 10 = 0$$

$$8a + 4b = 32$$

$$2a + b = 8 \quad (2)$$

$$3a = 9 \quad (2) - (1)$$

$$a = 3$$

Substituting $a = 3$ in (1), $b = 2$.

Exercise 10 C

1. Find the remainder when

   (a) $2x^3 + x^2 - x - 6$ is divided by $x + 1$ (b) $3x^3 + 2x - 8$ is divided by $x - 2$

   (c) $x^3 - x^2 + x + 10$ is divided by $x + 1$ (d) $2x^4 + x^2 - 20x - 8$ is divided by $x - 3$

   (e) $3x^4 - 2x^3 + x^2 + 7x - 10$ is divided by $x + 2$.

2. Find the value of a, given

   (a) $ax^3 + 6x^2 - 3x + 8$ is divisible by $x - 2$ (b) $2x^3 + 5x^2 - 3x + a$ is divisible by $x + 3$

   (c) $3x^3 + 3x^2 + ax + 10$ leaves remainder 2 when divided by $x + 1$.

3. Find the factors of

   (a) $x^3 + 4x^2 + x - 6$ (b) $2x^3 + 5x^2 + x - 2$ (c) $2x^3 - x^2 - 13x - 6$

   (d) $6x^3 + 4x^2 - 15x + 2$ (e) $2x^3 + 5x^2 - x - 1$

4. Solve the following equations:

   (a) $x^3 - 7x - 6 = 0$ (b) $2x^3 + x^2 - 5x + 2 = 0$ (c) $4x^3 + 12x^2 + 5x - 6 = 0$

   (d) $x^3 + 5x^2 + 2x - 2 = 0$ (e) $2x^3 + 5x^2 - 4x - 3 = 0$

5. $ax^3 + bx^2 - x - 6$ is divisible by $x^2 - x - 2$. Find the value of a and of b.

6. The expression $ax^3 + bx^2 - 53x - 8$ leaves remainder $48x - 86$ when divided by $x^2 - 5x + 6$. Find the value of a and of b.

10.4 Partial fractions

Two or more fractions can be combined to give a single fraction.

Thus, $\frac{3}{x-1}+\frac{2}{x-2} \equiv \frac{3(x-2)+2(x-1)}{(x-1)(x-2)}$

$$\equiv \frac{5x-8}{(x-1)(x-2)}$$

$$\frac{1}{(x+2)}+\frac{2x}{(x-1)^2} \equiv \frac{(x-1)^2+2x(x+2)}{(x+2)(x-1)^2}$$

$$\equiv \frac{3x^2+2x+1}{(x+2)(x-1)^2}$$

Conversely, it is possible to write $\frac{f(x)}{g(x)}$ where f(x) and g(x) are both polynomials in x as combinations of fractions called partial fractions.

10.4.1 Degree of f(x) < degree of g(x)

Type 1

Denominator consists of linear distinct factors, e.g.

$\frac{2x-1}{(x-1)(x-2)}, \frac{x+1}{(x-2)(x-3)(x+4)}$, etc

Consider $\frac{2x-1}{(x-1)(x-2)}$. We assume that it can be written as $\frac{A}{(x-1)}+\frac{B}{(x-2)}$, i.e.

$$\frac{2x-1}{(x-1)(x-2)} \equiv \frac{A}{(x-1)}+\frac{B}{(x-2)}$$

$$\equiv \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

$$\begin{aligned} 2x-1 &\equiv A(x-2)+B(x-1) \\ x=1,\ 1 &= -A \\ A &= -1 \\ x=2,\ 3 &= B \end{aligned}$$

So, $\frac{2x-1}{(x-1)(x-2)} \equiv \frac{-1}{x-1}+\frac{3}{x-2}$

Similarly,

$$\frac{x+1}{(x-2)(x-3)(x+4)} \equiv \frac{A}{(x-2)}+\frac{B}{(x-3)}+\frac{C}{(x+4)}$$

$$\equiv \frac{A(x-3)(x+4)+B(x-2)(x+4)+C(x-2)(x-3)}{(x-2)(x-3)(x+4)}$$

So, $x+1 \equiv A(x-3)(x+4)+B(x-2)(x+4)+C(x-2)(x-3)$

Put x = 2, $3 = -6A$

$$A = -\frac{1}{2}$$

x = 3, $4 = 7B$

$$B = \frac{4}{7}$$

x = −4, $-3 = 42C$

$$C = -\frac{1}{14}$$

So, $\dfrac{x+1}{(x-2)(x-3)(x+4)} \equiv \dfrac{1}{2(x-2)} + \dfrac{4}{7(x-3)} - \dfrac{1}{14(x+4)}$

Type 2

Denominator contains repeated factors, e.g. $\dfrac{2x-1}{(x-1)(x-2)^2}$ or $\dfrac{3x}{x^2(x+1)}$.

We write $\dfrac{2x-1}{(x-1)(x-2)^2} \equiv \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2}$

Note : The denominator has degree 3 and we take three constants A, B and C.

$$\frac{2x-1}{(x-1)(x-2)^2} \equiv \frac{A(x-2)^2 + B(x-1)(x-2) + C(x-1)}{(x-1)(x-2)^2}$$

$$2x-1 = A(x-2)^2 + B(x-1)(x-2) + C(x-1)$$

Put x = 1, $1 = A$

x = 2, $3 = C$

Equating coefficients of x^2, $0 = A + B$

$$B = -1$$

So, $\dfrac{2x-1}{(x-1)(x-2)^2} = \dfrac{1}{x-1} + \dfrac{1}{x-2} + \dfrac{3}{(x-2)^2}$

Similarly, $\dfrac{3x}{x^2(x+1)}$ can be written as $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$

Generally, $\dfrac{ax+b}{(x-\alpha)(x-\beta)^2} \equiv \dfrac{A}{x-\alpha} + \dfrac{B}{x-\beta} + \dfrac{C}{(x-\beta)^2}$

Type 3

Denominator contains a linear factor and a non factorisable quadratic, e.g.

$$\frac{2x+3}{(x-1)(x^2+9)}, \frac{3x-1}{(x+1)(x^2-2x+3)}$$

We write $\dfrac{2x+3}{(x-1)(x^2+9)} \equiv \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+9}$

Note : Again there are 3 constants as degree of denominator is 3.

$$\frac{2x+3}{(x-1)(x^2+9)} \equiv \frac{A(x^2+9)+Bx(x-1)+C(x-1)}{(x-1)(x^2+9)}$$

$$2x+3 \equiv A(x^2+9)+Bx(x-1)+C(x-1)$$

Put x = 1, $5 = 10A$

$$A = \frac{1}{2}$$

Put x = 0, $3 = 9A - C$

$$C = \frac{9}{2} - 3$$

$$= \frac{3}{2}$$

Equating coefficients of x^2, $0 = A + B$

$$B = -\frac{1}{2}$$

So, $$\frac{2x+3}{(x-1)(x^2+9)} \equiv \frac{1}{2(x-1)} + \frac{-\frac{1}{2}x+\frac{3}{2}}{x^2+9}$$

$$\equiv \frac{1}{2(x-1)} + \frac{3-x}{2(x^2+9)}$$

Similarly,

$$\frac{3x-1}{(x+1)(x^2-2x+3)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2-2x+3}$$

$$3x-1 \equiv A(x^2-2x+3)+Bx(x+1)+C(x+1)$$

Put x = –1, $-4 = 6A$

$$A = -\frac{2}{3}$$

x = 0, $-1 = 3A + C$

$$C = -1 + 2$$

$$= 1$$

Equating coefficients of x^2, $0 = A + B$

$$B = \frac{2}{3}$$

So, $$\frac{3x-1}{(x+1)(x^2-2x+3)} \equiv -\frac{2}{3(x+1)} + \frac{\frac{2}{3}x+1}{x^2-2x+3}$$

$$\equiv -\frac{2}{3(x+1)} + \frac{2x+3}{3(x^2-2x+3)}$$

10.4.2 Degree of numerator > degree of denominator

If the degree of the numerator is greater than or equal to the degree of the denominator, we perform a division to obtain a remainder with degree less than the degree of the denominator.

Example 14

Find the partial fractions of $\dfrac{x^3 - x^2 + x - 5}{(x - 1)(x + 2)(x - 3)}$

Solution

As degree of numerator is equal to the degree of the denominator, we perform a division first

$$\begin{aligned}(x - 1)(x + 2)(x - 3) &= (x - 1)(x^2 - x - 6)\\ &= x^3 - 2x^2 - 5x + 6\end{aligned}$$

$$\begin{array}{r|l} & \quad\quad\quad 1 \\ \hline x^3 - 2x^2 - 5x + 6 & x^3 - x^2 + x - 5 \\ & \underline{x^3 - 2x^2 - 5x + 6} \\ & \quad\quad x^2 + 6x - 11 \end{array}$$

$$\frac{x^3 - x^2 + x - 5}{(x-1)(x+2)(x-3)} \equiv 1 + \frac{x^2 + 6x - 11}{(x-1)(x+2)(x-3)}$$

$$\frac{x^2 + 6x - 11}{(x-1)(x+2)(x-3)} \equiv \frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{C}{(x-3)}$$

$$\equiv \frac{A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2)}{(x-1)(x+2)(x-3)}$$

$$x^2 + 6x - 11 \equiv A(x + 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x + 2)$$

$$x = 1, \quad 1 + 6 - 11 = A \times 3 \times -2$$

$$A = \frac{-4}{-6}$$

$$= \frac{2}{3}$$

$$x = -2, \quad 4 - 12 - 11 = B \times -3 \times -5$$

$$B = \frac{-19}{15}$$

$$= -\frac{19}{15}$$

$$x = 3, \quad 9 + 18 - 11 = C \times 2 \times 5$$

$$16 = 10\,C$$

$$C = \frac{8}{5}$$

So, $\dfrac{x^2 + 6x - 11}{(x-1)(x+2)(x-3)} \equiv \dfrac{2}{3(x-1)} - \dfrac{19}{15(x+2)} + \dfrac{8}{5(x-3)}$

Exercise 10 D

Find the partial fractions of:

1. $\dfrac{2x}{(x+1)(x-2)}$

2. $\dfrac{3x-4}{(2x-1)(3x+2)}$

3. $\dfrac{5x-1}{(x-1)(x-2)(x-3)}$

4. $\dfrac{2x^2+2x-3}{(x-1)(x+2)(x-3)}$

5. $\dfrac{2x^2+3}{(x+1)(2x-1)(3x+2)}$

6. $\dfrac{x^2+5}{(x-1)(x-2)^2}$

7. $\dfrac{2x-1}{x^2(x+3)}$

8. $\dfrac{x^2-x+1}{(x+1)(x+2)^2}$

9. $\dfrac{3x-4}{(x+2)(x^2+9)}$

10. $\dfrac{x^2-x+1}{(x+2)(2x^2+3)}$

11. $\dfrac{x^3-x^2+2x-1}{(x-1)(x-2)(x-3)}$

12. $\dfrac{x^3+x^2-3}{(x-1)(x-2)^2}$

13. $\dfrac{x^2+2x-1}{x^2(x-3)}$

14. $\dfrac{x^3+x^2-3x+5}{(x-1)(x-3)}$

15. $\dfrac{x^4+2x^2-3x+4}{(x-1)(x+2)(x-3)}$

10.5 Expansion of $(1 + x)^n$ where n is not a positive integer

If n is a positive integer, we know already that

$$(1+x)^n = 1^n + {}^nC_1\, 1^{n-1}x + {}^nC_2\, 1^{n-2}x^2 + {}^nC_3\, 1^{n-3}x^3 + \ldots\ldots\ldots\ldots + x^n$$
$$= 1 + nx + {}^nC_2\, x^2 + {}^nC_3\, x^3 + \ldots\ldots\ldots\ldots + x^n$$

This series contains n + 1 terms and is valid for all values of x.

Consider next $(1 + x)^{-1} = \dfrac{1}{1+x}$

If we perform a long division, we obtain

$$\begin{array}{r|l} & 1 - x + x^2 - x^3 \\ \hline 1 + x & 1 \\ & \underline{1 + x} \\ & -x \\ & \underline{-x - x^2} \\ & +x^2 \\ & \underline{x^2 + x^3} \\ & -x^3 \\ & \underline{-x^3 - x^4} \\ & x^4 \ \ldots \end{array}$$

So, $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots\ldots\ldots$

We obtain in this case an infinite series.

Also, $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots\ldots\ldots$

$$= 1 + (-1)x + \frac{(-1)(-2)}{1\times 2}x^2 + \frac{(-1)(-2)(-3)}{1\times 2\times 3}x^3 + \frac{(-1)(-2)(-3)(-4)}{1\times 2\times 3\times 4}x^4 + \ldots$$

$$= 1 + \binom{-1}{1}x + \binom{-1}{2}x^2 + \binom{-1}{3}x^3 + \binom{-1}{4}x^4 + \binom{-1}{5}x^5 + \ldots$$

$$\text{where } \begin{pmatrix}-1\\ r\end{pmatrix} = \frac{(-1)(-2)(-3)....(-r)}{1\times 2\times 3....\times r}$$

More generally, $(1 + x)^n = 1+\begin{pmatrix}n\\ 1\end{pmatrix}x+\begin{pmatrix}n\\ 2\end{pmatrix}x^2+\begin{pmatrix}n\\ 3\end{pmatrix}x^3+ ....$

$$\text{where } \begin{pmatrix}n\\ r\end{pmatrix} = \frac{n(n-1)(n-2)....(n-r+1)}{1\times 2\times 3....\times r}$$

This series is infinite and is valid only for $|x| < 1$.

Example 15

Find the expansion of $(1 + 2x)^{-2}$ up to the term in x^3 stating the range of values of x for which it is valid.

Solution

$$(1+2x)^{-2} = 1+\begin{pmatrix}-2\\ 1\end{pmatrix}(2x)+\begin{pmatrix}-2\\ 2\end{pmatrix}(2x^2)+\begin{pmatrix}-2\\ 3\end{pmatrix}(2x^3)+ ....$$

$$= 1+-2\times 2x+\frac{-2\times -3}{1\times 2}4x^2+\frac{-2\times -3\times -4}{1\times 2\times 3}8x^3+ ....$$

$$= 1-4x+12x^2-32x^3 ....$$

The expansion is valid for $|2x| < 1$, i.e. $|x| < \frac{1}{2}$.

Example 16

Find the expansion of $(8 + 3x)^{-\frac{1}{3}}$ up to the term in x^3 stating the range of values of x for which it is valid.

Solution

$$(8+3x)^{-\frac{1}{3}} = 8^{-\frac{1}{3}}\left(1+\frac{3}{8}x\right)^{-\frac{1}{3}}$$

$$= \frac{1}{2}\left(1+\frac{3}{8}x\right)^{-\frac{1}{3}}$$

$$\left(1+\frac{3}{8}x\right)^{-\frac{1}{3}} = 1+\begin{pmatrix}-\frac{1}{3}\\ 1\end{pmatrix}\left(\frac{3}{8}x\right)+\begin{pmatrix}-\frac{1}{3}\\ 2\end{pmatrix}\left(\frac{3}{8}x\right)^2+\begin{pmatrix}-\frac{1}{3}\\ 3\end{pmatrix}\left(\frac{3}{8}x\right)^3+ ......$$

$$= 1-\frac{1}{8}x+\frac{5}{128}x^2-\frac{5}{384}x^3+ ....$$

$$(8+3x)^{-\frac{1}{3}} = \frac{1}{2}-\frac{1}{16}x+\frac{5}{256}x^2-\frac{5}{768}x^3 ....$$

Range of values of x for which expansion is valid is $\left|\frac{3}{8}x\right| < 1$, i.e. $|x| < \frac{8}{3}$.

Example 17

Find the expansion of $(9+x)^{\frac{1}{2}}$ *as far as the term in* x^3*. Hence, obtain the value of* $\sqrt{9.05}$ *to four places of decimals.*

Solution

$$(9+x)^{\frac{1}{2}} = 9^{\frac{1}{2}}\left(1+\frac{x}{9}\right)^{\frac{1}{2}}$$

$$= 3\left(1+\frac{x}{9}\right)^{\frac{1}{2}}$$

$$\left(1+\frac{x}{9}\right)^{\frac{1}{2}} = 1+\binom{\frac{1}{2}}{1}\left(\frac{x}{9}\right)+\binom{\frac{1}{2}}{2}\left(\frac{x}{9}\right)^2+\binom{\frac{1}{2}}{3}\left(\frac{x}{9}\right)^3+\ldots\ldots$$

$$= 1+\frac{x}{18}+\frac{x^2}{324}-\frac{x^3}{11664}+\ldots\ldots$$

This expansion is valid for $|x| < 9$.

$$(9+x)^{\frac{1}{2}} \approx 3+\frac{x}{6}-\frac{x^2}{108}+\frac{x^3}{3888}$$

$$\sqrt{9.05} = (9+0.05)^{\frac{1}{2}}$$

$$= 3+\frac{0.05}{6}+\frac{(0.05)^2}{108}-\frac{(0.05)^3}{3888}$$

$= 3.0083$ (calculator)

10.5.2 Use of $(1 \pm x)^{-1}$, $(1 \pm x)^{-2}$

It can be checked that $(1+x)^{-1} = 1 - x + x^2 - x^3 \ldots\ldots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 \ldots\ldots$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 \ldots\ldots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 \ldots\ldots$

So, $(1+2x)^{-1} = 1 - 2x + (2x)^2 - (2x)^3 \ldots\ldots$

$= 1 - 2x + 4x^2 - 8x^3 \ldots\ldots$

$(1-3x)^{-2} = 1 + 2(-3x) + 3(-3x)^2 + 4(-3x)^3 \ldots\ldots$

$= 1 - 6x + 27x^2 - 108x^3 \ldots\ldots$

Example 18

Find the partial fractions of $\frac{x-17}{(x-2)(x+3)}$ *. Hence, obtain the expansion of* $\frac{x-17}{(x-2)(x+3)}$ *as a power series in x as far as the term in* x^3 *and find the range of values of x for which the expansion is valid.*

Solution

$$\frac{x-17}{(x-2)(x+3)} \equiv \frac{A}{x-2} + \frac{B}{x+3}$$

$$\equiv \frac{A(x+3)+B(x-2)}{(x-2)(x+3)}$$

$$x-17 \equiv A(x+3)+B(x-2)$$

$$x=2, -15 = 5A$$

$$A = -3$$

$$x=-3, -20 = -5B$$

$$B = 4$$

$$\frac{x-17}{(x-2)(x+3)} \equiv \frac{-3}{x-2} + \frac{4}{x+3}$$

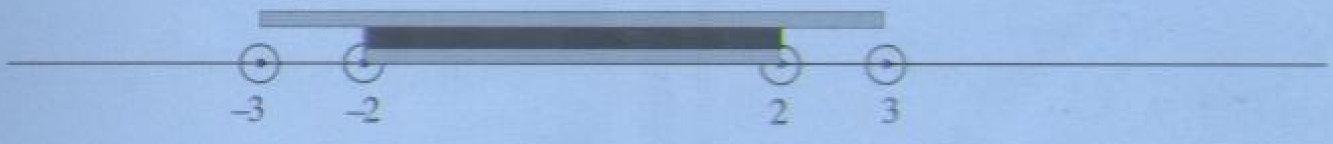
$$\equiv \frac{3}{2-x} + \frac{4}{3+x}$$

$$\equiv \frac{3}{2\left(1-\frac{x}{2}\right)} + \frac{4}{3\left(1+\frac{x}{3}\right)}$$

$$\equiv \frac{3}{2}\left(1-\frac{x}{2}\right)^{-1} + \frac{4}{3}\left(1+\frac{x}{3}\right)^{-1}$$

$$\equiv \frac{3}{2}\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}....\right) + \frac{4}{3}\left(1-\frac{x}{3}+\frac{x^2}{9}-\frac{x^3}{27}....\right)$$

$$\equiv \frac{17}{6} + \frac{11}{36}x + \frac{113}{216}x^2 + \frac{179}{1296}x^3......$$



The expansion of $(2-x)^{-1}$ is valid for $|x| < 2$.

The expansion of $(3+x)^{-1}$ is valid for $|x| < 3$.

Both expansions are valid for $|x| < 2$ as shown on the number line.

Exercise 10 E

1. Find the expansion of each of the following as far as the term in x^3 stating the range of values of x for which it is valid:

   (a) $(1-2x)^{-3}$ (b) $\sqrt{1+4x}$ (c) $\dfrac{1}{\sqrt{1+2x}}$ (d) $(2+3x)^{-2}$

   (e) $(4-x)^{\frac{1}{2}}$ (f) $(8-x)^{\frac{1}{3}}$ (g) $\dfrac{1}{\sqrt{9+2x}}$ (h) $\dfrac{1}{\sqrt[3]{8+x}}$

2. Expand $(1+3x)^{\frac{1}{3}}$ in ascending powers of x as far as the term in x^3. By substituting $x = \frac{1}{8}$, evaluate $\sqrt[3]{11}$ correct to two decimal places.

3. Write down the first three terms in the binomial expansion of $(1-x)^{\frac{1}{3}}$. By substituting $x = \frac{1}{1000}$, evaluate $\sqrt[3]{37}$ to four decimal places.

4. Express $\dfrac{7-x}{(2-x)(3+x)}$ in partial fractions. Hence, obtain $\dfrac{7-x}{(2-x)(3+x)}$ as a power series in x as far as the term in x^3. Find the range of values of x for which this expansion is valid.

5. Express $\dfrac{x(1+3x)}{(1-x)(1+x^2)}$ in partial fractions. Hence, obtain the expansion of $\dfrac{x(1+3x)}{(1-x)(1+x^2)}$ as a power series in x as far as the term in x^5. Find the range of values of x for which the expansion is valid.

Miscellaneous Exercise 10

1. Given that $(x+2)$ is a factor of $x^4 + ax^2 + 3x + 2$, find the value of a. [C]

2. Solve the equation $4|x| = |x-1|$. On the same diagram, sketch the graphs of $y = 4|x|$ and $y = |x-1|$ and hence, or otherwise, solve the inequality $4|x| > |x-1|$. [C]

3. The cubic polynomial $x^3 + ax^2 + bx - 8$, where a and b are constants, has factors $(x+1)$ and $(x+2)$. Find the value of a and of b. [C]

4. Express $\dfrac{2x^2-6x+7}{(2x+3)(x^2+1)}$ in the form $\dfrac{A}{(2x+3)} + \dfrac{Bx+C}{(x^2+1)}$, where A, B, C are numerical constants which are to be found. [C]

5. Use the binomial expansion to find $(1.0006)^{\frac{1}{3}}$ correct to eight places of decimals. [C]

6. Express $f(x) \equiv \dfrac{2}{2 - 3x + x^2}$ in partial fractions and hence, or otherwise, obtain f (x) as a series of ascending powers of x, giving the first four non-zero terms of this expansion. State the set of values of x for which this expansion is valid, and find the coefficient of x^n in this expansion. [C]

7. Write down and simplify the series expansion of $\dfrac{1}{\sqrt{1+x}}$ where $|x| < 1$, up to and including the term in x^3.

   Show that using just three terms of the series with $x = 0.4$ gives a value for $\dfrac{1}{\sqrt{1.4}}$ which differs from the value obtained by using a calculator by less than 0.7%.

   [C]

8. Express $f(x) \equiv \dfrac{2x + 4}{(x - 1)(x + 3)}$ in partial fractions.
   (a) If x is small, obtain an expansion of f(x) in ascending powers of x as far as the term in x^2.
   (b) If x is large, obtain an expansion of f(x) in ascending powers of $\dfrac{1}{x}$ as far as the term in $\dfrac{1}{x^3}$ [C]

9. Let $f(x) \equiv \dfrac{x^2 + 5x}{(1 + x)(1 - x)^2}$. Express f(x) in the form $\dfrac{A}{1+x} + \dfrac{B}{1-x} + \dfrac{C}{(1-x)^2}$ where A, B, C are constants. The expansion of f(x), in ascending powers of x, is $C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots\dots + C_r x^r + \dots\dots$ Find C_0, C_1, C_2 and show that $C_3 = 11$. [C]

10. Find the value of a for which $(x - 2)$ is a factor of $3x^3 + ax^2 + x - 2$. Show that, for this value of a, the cubic equation $3x^3 + ax^2 + x - 2 = 0$ has only one real root. [C]

11. The polynomial $x^5 - 3x^4 + 2x^3 - 2x^2 + 3x + 1$ is denoted by f(x).
    (a) Show that neither $(x - 1)$ nor $(x + 1)$ is a factor of f(x).
    (b) By substituting $x = 1$ and $x = -1$ in the identity $f(x) \equiv (x^2 - 1)\, q(x) + ax + b$ where q(x) is a polynomial and a and b are constants, or otherwise, find the remainder when f(x) is divided by $(x^2 - 1)$.
    (c) Show, by carrying out the division, or otherwise, that when f(x) is divided by $(x^2 + 1)$, the remainder is 2x.
    (d) Find all the real roots of the equation $f(x) = 2x$. [C]

12. Express f(x) in partial fractions, where $f(x) = \dfrac{2 + x + x^2}{(2 + x)(1 + x)^2}$. Hence, or otherwise, show that if x is sufficiently small for x^4 and higher powers to be neglected, then $f(x) \approx 1 - \dfrac{1}{2}x^3$.

    Hence, estimate the value of $\displaystyle\int_{-0.1}^{0.1} f(x)\, dx$. [C]

13. Give the binomial expansion, for small x, of $(1 + x)^{\frac{1}{4}}$ up to and including the term in x^2, and simplify the coefficients. By putting $x = \dfrac{1}{16}$ in your expansion, show that $\sqrt[4]{17} \approx \dfrac{8317}{4096}$. [C]

14. Given that $y = \dfrac{1}{\sqrt{1+2x} + \sqrt{1+x}}$ where $x > -\dfrac{1}{2}$, show that, provided $x \neq 0$, $y = \dfrac{1}{x}\left[\sqrt{1+2x} - \sqrt{1+x}\right]$.

    Using this second form for y, express y as a series of ascending powers of x up to and including the term in

    Hence show, by putting $x = \dfrac{1}{100}$, that $\dfrac{10}{\sqrt{102} + \sqrt{101}} \approx \dfrac{79407}{160000}$ [C]

15. The polynomials P(x) and Q(x) are defined by

    $P(x) = x^8 - 1$

    $Q(x) = x^4 + 4x^3 + ax^2 + bx + 5$

    (a) Show that $x - 1$ and $x + 1$ are factors of P(x).

    (b) It is known that when Q(x) is divided by $x^2 - 1$, a remainder $2x + 3$ is obtained. Find the value of a and of b.

    (c) With these values of a and b, find the remainder when the polynomial [3 P(x) + 4 Q(x)] is divided by $x^2 - 1$

    [C]

16. The polynomial P(x), where $P(x) = x^4 + ax^3 + bx^2 - 2x - 4$ has factor $(x - 1)$ and $(x + 2)$.

    (a) Show that $a = 3$ and $b = 2$.

    (b) Find the third real factor of P(x) and show that this factor is positive for all real values of x.

    (c) Find the set of values of x for which P(x) is positive. [C]

17. The polynomial $2x^3 - 3ax^2 + ax + b$ has a factor $x - 1$ and, when divided by $x + 2$, a remainder of -54 is obtained. Find the value of a and of b.

    With these values of a and b, factorise the polynomial completely. Hence, or otherwise, find all the real factors of

    (a) $2x^6 - 9x^4 + 3x^2 + 4$ (b) $4x^3 + 3x^2 - 9x + 2$ [C]

CHAPTER 11

> LOGARITHMIC & EXPONENTIAL FUNCTION

11.1 Definition of logarithm

Consider 3^4, its value is 81. So the power of 3 which gives 81 is 4.

We say the logarithm of 81 to base 3 is 4 and we write $\log_3 81 = 4$ which we read as logarithm of 81 to base 3 is equal to 4. Similarly, $2^5 = 32$ and we write $\log_2 32 = 5$ which is read as logarithm of 32 to base 2 is equal to 5. Generally, if $a^m = b$ $(a > 0, a \neq 1)$ logarithm of b to base a is m and we write $\log_a b = m$.

Example 1

Find: (i) $\log_7 49$ *(ii)* $\log_8 2$ *(iii)* $\log_{\frac{1}{32}}\left(\frac{1}{16}\right)$

Solution

(a) $\log_7 49 = 2$ as $7^2 = 49$

(b) It is difficult to write 2 directly as a power of 8.

Put $\log_8 2 = x$

$$2 = 8^x$$
$$= 2^{3x}$$
$$3^x = 1$$
$$x = \frac{1}{3}$$

(c) $\log_{\frac{1}{32}}\left(\frac{1}{16}\right) = x$

$$\left(\frac{1}{32}\right)^x = \left(\frac{1}{16}\right)$$
$$(2^{-5})^x = 2^{-4}$$
$$2^{-5x} = 2^{-4}$$
$$-5x = -4$$
$$x = \frac{4}{5}$$

Example 2

Convert to index form: (a) $\log_a x = p$ (b) $\log_b \sqrt{xy} = \frac{1}{2}$

Solution

(a) $\log_a x = p$

$a^p = x$

(b) $\log_b \sqrt{xy} = \frac{1}{2}$

$b^{\frac{1}{2}} = \sqrt{xy}$

$b = xy$

11.1.2 Laws of logarithms

Law 1: $\log_x (a \times b) = \log_x a + \log_x b$

Consider $\log_x a = m$, $\log_x b = n$

$x^m = a$ and $x^n = b$

$ab = x^m \times x^n$

$= x^{m+n}$

Hence, $\log_x(ab) = \log_x x^{m+n}$

$= m + n$ (by definition)

$= \log_x a + \log_x b$

Law 2: $\log_x \left(\frac{a}{b}\right) = \log_x a - \log_x b$

$\log_x a = m$, $a = x^m$

$\log_x b = n$, $b = x^n$

$\frac{a}{b} = \frac{x^m}{x^n}$

$= x^{m-n}$

Hence, $\log_x \left(\frac{a}{b}\right) = m - n$ (by definition)

$= \log_x a - \log_x b$

Law 3: $\log_x a^b = b \log_x a$

$\log_x a = m, a = x^n$

$a^b = \left(x^m\right)^b$

$$= x^{mb}$$

$$\log_x a^b = mb$$

$$= bm$$

$$= b \log_x a$$

We now consider applications of these rules.

Example 3

Express $\log_{10}(a^2b^3)$ in terms of $\log_{10} a$ and $\log_{10} b$.

Solution

$$\log_{10}(a^2b^3) = \log_{10} a^2 + \log_{10} b^3 \quad \text{(Law 1)}$$

$$= 2\log_{10} a + 3\log_{10} b \quad \text{(Law 3)}$$

Convention: *Henceforth, $\log_{10} x$ will be written as $\lg x$.*

Example 4

Express $\lg\left(\frac{x\sqrt[3]{y}}{z^2}\right)$ in terms of $\lg x$, $\lg y$ and $\lg z$.

Solution

$$\lg\left(\frac{x\sqrt[3]{y}}{z^2}\right) = \lg x\sqrt[3]{y} - \lg z^2 \quad \text{(Law 2)}$$

$$= \lg x + \lg \sqrt[3]{y} - \lg z^2 \quad \text{(Law 1)}$$

$$= \lg x + \frac{1}{3}\lg y - 2\lg z \quad \text{(Law 3)}$$

Example 5

Express $\frac{1}{2}\lg 9 + 2\lg 3 - 3\lg 2$ as a single logarithm.

Solution

$$\frac{1}{2}\lg 9 + 2\lg 3 - 3\lg 2 = \lg 9^{\frac{1}{2}} + \lg 3^2 - \lg 2^3 \quad \text{(converse of law 3)}$$

$$= \lg\left(9^{\frac{1}{2}} \times 3^2\right) - \lg 2^3 \quad \text{(converse of law 1)}$$

$$= \lg \frac{9^{\frac{1}{2}} \times 3^2}{2^3} \quad \text{(converse of law 2)}$$

$$= \lg \frac{27}{8}$$

Example 6

Simplify $\frac{\lg 64}{\lg 32}$.

Solution

$$\frac{\lg 64}{\lg 32} = \frac{\lg 2^6}{\lg 2^5}$$

$$= \frac{6 \lg 2}{5 \lg 2}$$

$$= \frac{6}{5}$$

Note: Confusion should not be made between $\lg \frac{64}{32}$ *(which is* $\lg 2$*) and* $\frac{\lg 64}{\lg 32}$.

Example 7

Given $\lg 2 = 0.3010$ *and* $\lg 3 = 0.4771$*, find without use of calculator:*

(a) $\lg 12$ *(b)* $\lg \frac{9}{8}$ *(c)* $\lg 5$ *(d)* $\lg 75$

Solution

(a) $\lg 12 = \lg (2^2 \times 3)$

$= \lg 2^2 + \lg 3$

$= 2\lg 2 + \lg 3$

$= 0.6020 + 0.4771$

$= 1.0791.$

(b) $\lg \frac{9}{8} = \lg \frac{3^2}{2^3}$

$= \lg 3^2 - \lg 2^3$

$= 2\lg 3 - 3\lg 2$

$= 0.9542 - 0.9030$

$= 0.0512$

(c) $\lg 5 = \lg \frac{10}{2}$

$= \lg 10 - \lg 2$

$= 1 - 0.3010$

$= 0.6990$

(d) $\lg 75 = \lg (3 \times 5^2)$

$= \lg 3 + 2\lg 5$

$= 0.4771 + 1.3980$

$= 1.8751$

Exercise 11 A

1. Rewrite in index form:

(a) $\log_5 25 = 2$ (b) $\log_7 343 = 3$ (c) $\log_{16} 4 = \frac{1}{2}$ (d) $\log_{27}\left(\frac{1}{3}\right) = -\frac{1}{3}$

(e) $\log_3\left(\frac{1}{9}\right) = -2$ (f) $\log_e x = y$ (g) $\log_{10} x = y$ (h) $\log_y x = z$

2. Rewrite in logarithmic form

(a) $3^2 = 9$ (b) $4^3 = 64$ (c) $3^5 = 243$ (d) $7^3 = 343$ (e) $11^3 = 1331$

3. Evaluate:

(a) $\log_3 27$ (b) $\log_5 125$ (c) $\log_4 1$ (d) $\log_{27} 3$

(e) $\log_{81} 27$ (f) $\log_{16} 4$ (g) $\log_{100} 10\,000$ (h) $\log_4\left(\frac{1}{16}\right)$

(i) $\log_8\left(\frac{1}{4}\right)$ (j) $\log_{\frac{5}{2}}\left(\frac{125}{8}\right)$ (k) $\log_{\frac{8}{27}}\left(\frac{2}{3}\right)$ (l) $\log_{x^2} x^3$

(m) $\log_{e^3} \frac{1}{e^5}$ (n) $\log_{10^{-3}} 10^{-5}$ (o) $\log_{\frac{1}{a^m}} a^n$

4. Express in terms of lg a, lg b and/or lg c:

(a) $\lg ab^2c^3$ (b) $\lg \frac{c}{ab}$ (c) $\lg \frac{a}{b^3}$ (d) $\lg \frac{c}{ab^2}$

(e) $\lg \frac{a^3b^2}{c^5}$ (f) $\lg \frac{100c}{ab^2}$ (g) $\lg \sqrt{ab^4}$ (h) $\lg \frac{10b}{a\sqrt{c}}$

(i) $\lg \frac{a^3b^5}{\sqrt[3]{c^2}}$ (j) $\lg \sqrt{\frac{a^3}{b^2c^4}}$ (k) $\lg \sqrt[3]{\frac{1000a^2}{bc^3}}$ (l) $\lg \sqrt{\frac{10}{a^2b^3c}}$

5. Express as a single logarithm:

(a) $\lg 2 + \lg 7$ (b) $\lg 28 - \lg 49$ (c) $\lg 2 + \lg 3 - \lg 5$

(d) $2\lg 4 + 3\lg 2$ (e) $2\lg 4 + 4\lg 3 - \lg 27$ (f) $\lg a + \lg b - \lg c$

(g) $2\lg a + \frac{1}{3}\lg b + \frac{1}{2}\lg c$ (h) $2 + 3\lg 2$ (i) $1 - 2\lg 5$

(j) $\frac{1}{3}\lg a + \frac{1}{2}\lg b - \frac{4}{5}\lg c$ (k) $2\lg a - 3\lg b - \frac{1}{2}\lg c$ (l) $3\lg a + 2\lg b - 3$

6. Evaluate:

(a) $\lg 1000$ (b) $\lg \frac{1}{100}$ (c) $\frac{\lg 27}{\lg 9}$ (d) $\frac{\lg 16}{\lg 4}$

(e) $\frac{\lg 49}{\lg 343}$ (f) $\frac{\lg 125}{\lg 625}$ (g) $\frac{\lg 64}{\lg 1024}$ (h) $\frac{\lg 25}{\lg 3125}$

7. Given lg 2 = 0.3010 and lg 3 = 0.4771, find, without use of a calculator, the value of:
   (a) lg 32 (b) lg 27 (c) lg 108 (d) lg 2.25
   (e) lg 5 (f) lg 225 (g) $\lg\sqrt{45}$ (h) $\lg\sqrt{120}$
   (i) lg 2.4 (j) $\lg\frac{4}{27}$

11.2 The number represented by e

The sum of the infinite series $1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}$...... is denoted by the letter e. We note that the value of e cannot be found exactly. However, it can be obtained to as many decimal places as required. Its value is 2.718 correct to three places of decimal and 2.7183 to four places.

It has the important property that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$...... This result can be obtained by considering the binomial expansion $\left(1 + \frac{1}{n}\right)^{nx}$ of and will be dealt with later in the course.

11.2.2 Logarithm to base e

From the definition of logarithm $\log_e e^x = x$. So, $\log_e e^2 = 2$, $\log_e \frac{1}{e} = -1$, etc.

Convention: *Henceforth, we will use ln x to stand for $\log_e x$.*

Note
$\log_{10} = \lg$
$\log_e = \ln$

So, $\ln e^3 = 3$, $\ln\sqrt{e} = \ln e^{\frac{1}{2}} = \frac{1}{2}$.

Logarithm of any number to base e is obtained on using the ln function key of the calculator.

If $y = \ln x$, i.e. $y = \log_e x$, $x = e^y$.

If we write $f{:}x \mapsto \ln x$, the inverse function is obtained by writing

$$\ln x = y$$
$$x = e^y$$

So, the inverse of $f{:}x \mapsto \ln x$ is $f^{-1}{:}y \mapsto e^y$ or $f^{-1}{:}x \mapsto e^x$.

Similarly, the inverse of $f{:}x \mapsto \lg x$ is $f^{-1}{:}x \mapsto 10^x$.

More generally, the inverse of $f{:}x \mapsto \log_a x$ is $f^{-1}{:}x \mapsto a^x$.

Example 8

Find x given lg x = 1.7.

Solution

$\lg x = 1.7$
$x = 10^{1.7}$
≈ 50.1 (calculator)

Example 9

Find x given ln x = –2.46.

Solution

$$\begin{aligned} \ln x &= -2.46 \\ x &= e^{-2.46} \\ &\approx 0.0854 \end{aligned}$$

11.2.3 Solving equations involving e^x

(a) **Simple equations**

If $e^x = a, x = \ln a$

So, if $e^x = 2.8, x = \ln 2.8$

$$\approx 1.03$$

Similarly, if $e^x = 0.234$

$$\begin{aligned} x &= \ln 0.234 \\ &\approx -1.45 \end{aligned}$$

(b) **Harder equations**

Equations of the type $ae^x + be^{-x} + c = o$ can be solved by using the substitution $y = e^x$.

Thus, $2e^x + 3e^{-x} - 5 = 0$ reduces to the form

$$\begin{aligned} 2y + \frac{3}{y} - 5 &= 0 \\ 2y^2 + 3 - 5y &= 0 \\ 2y^2 - 5y + 3 &= 0 \\ (2y - 3)(y - 1) &= 0 \\ y &= 1.5 \text{ or } 1 \\ e^x &= 1.5 \text{ or } 1 \\ x &= \ln 1.5 \text{ or } \ln 1 \\ &= 0.405 \text{ or } 0. \end{aligned}$$

Note: $e^x > 0$ for all real values of x

11.2.4 Solving equations of the form $a^x = b$

To solve the equation $5^x = 125$, we write 125 as 5^3.

$$\begin{aligned} 5^x &= 5^3 \\ x &= 3 \end{aligned}$$

However, this method cannot be used to solve $3^x = 7$ as 7 cannot be written as an exact power of 3.

Instead, we proceed as follows:

$$\begin{aligned} 3^x &= 7 \\ \ln 3^x &= \ln 7 \\ x \ln 3 &= \ln 7 \\ x &= \frac{\ln 17}{\ln 3} \\ &\approx 1.77 \text{ (calculator)} \end{aligned}$$

Note
$\lg 10 = 1$
$\log_y y = 1$

Example 10

Solve $\left(\frac{2}{7}\right)^x = \frac{1}{11}$.

Solution

$$\left(\frac{2}{7}\right)^x = \frac{1}{11}$$

$$\ln\left(\frac{2}{7}\right)^x = \ln\left(\frac{1}{11}\right)$$

$$x\ln\left(\frac{2}{7}\right) = \ln\left(\frac{1}{11}\right)$$

$$x = \frac{\ln\frac{1}{11}}{\ln\frac{2}{7}}$$

$$\approx 1.91 \text{ (calculator)}$$

Note
log 1 = 0
to any base

Note:

1. We may use lg instead of ln.

2. If a number is between 0 and 1, its logarithm is negative.

11.2.5 Solving exponential inequalities

To solve $a^x \leqslant b$, we take logarithms to base 10 or to base e.

Example 11

Solve $\left(\frac{3}{4}\right)^x < \frac{4}{315}$.

Solution

$$\left(\frac{3}{4}\right)^x < \frac{4}{315}$$

$$x\ln\frac{3}{4} < \ln\frac{4}{315}$$

$$x > \frac{\ln\frac{4}{315}}{\ln\frac{3}{4}}, \text{ as } \ln\frac{3}{4} < 0$$

$$x > 15.2 \quad \text{(calculator)}$$

Example 12

Solve $5 \times \left(\frac{3}{7}\right)^{x-1} \geqslant 2.$

Solution

$$5 \times \left(\frac{3}{7}\right)^{x-1} \geqslant 2$$

$$\left(\frac{3}{7}\right)^{x-1} \geqslant \frac{2}{5}$$

$$(x-1)\ln\frac{3}{7} \geqslant \ln\frac{2}{5}$$

$$(x-1) \leqslant \frac{\ln\frac{2}{5}}{\ln\frac{3}{7}}, \text{ as } \ln\frac{3}{7} < 0$$

$$x-1 \leqslant 1.08 \quad \text{(calculator)}$$

$$x \leqslant 2.08$$

11.2.6 Graph of $y = e^x$

As $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots\ldots$

If $x > 0$, e^x consists only of positive terms and so $e^x > 0$.

$x = 0, e^x = 1$

If $x < 0$, put $x = -a$ where $a > 0$.

$e^x = e^{-a} = \frac{1}{e^a} > 0$ as $e^a > 0$ for $a > 0$.

It follows that $e^x > 0$ for all values of x.

Also, as $|x|$ becomes larger and larger, if $x > 0$, e^x becomes larger and larger, i.e. $x \to +\infty, e^x \to +\infty$.

If $|x|$ becomes larger and larger and $x < 0$, e^x becomes smaller and smaller, i.e. $x \to -\infty, e^x \to 0$.

The graph of $y = e^x$ or $f{:}x \mapsto e^x$ is then as shown in Figure 11.1.

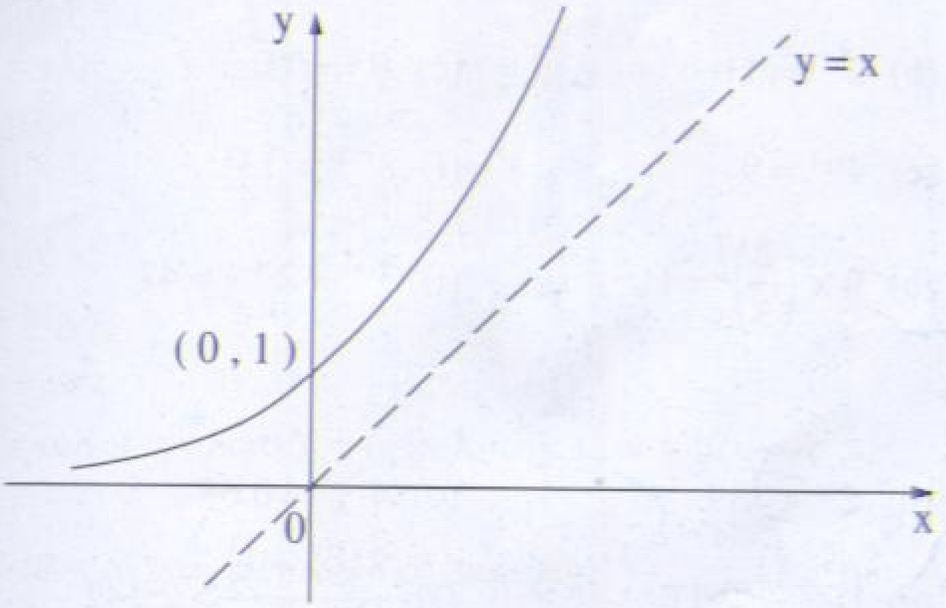


Figure 11.1

As the inverse of $f:x \mapsto e^x$ is $f^{-1}:x \mapsto \ln x$, the graph of $y = \ln x$ is obtained by reflecting the graph of $y = e^x$ in the line $y = x$.

Exercise 11 B

1. Find the value of each of the following to three places of decimal:

(a) $e^{-2.5}$ (b) $e^{0.7}$ (c) $e^{-0.93}$ (d) $e^{-\frac{3}{7}}$

(e) $e^{\sqrt{2}}$ (f) $e^{-1.83}$ (g) $e^{\sqrt{\frac{3}{7}}}$ (h) $e^{-\frac{2\sqrt{3}}{9}}$

2. Find the value of each of the following to three places of decimal:

(a) $\ln 2.713$ (b) $\ln 1.94$ (c) $\ln 0.76$ (d) $\ln 25.8$

(e) $\ln\sqrt{3}$ (f) $\ln\left(\sqrt{3}+1\right)$ (g) $\ln\left(2\sqrt{3}-1\right)$ (h) $\ln\left(2\sqrt{3}+3\sqrt{5}\right)$

3. Find x to three places of decimal given:

(a) $\lg x = -1.414$ (b) $\lg x = -2.12$ (c) $\lg x = 1.82$ (d) $\lg x = \sqrt{2}$

(e) $\lg x = \frac{9}{7}$ (f) $\lg x = \frac{2}{\sqrt{3}}$ (g) $\ln x = 1.2$ (h) $\ln x = -2.8$

(i) $\ln x = \sqrt{5}$ (j) $\ln x = \frac{2}{9}$ (k) $\ln x = -\frac{3}{7}$ (l) $\ln x = \frac{2\sqrt{3}}{5}$

4. Find x to three places of decimal given:

(a) $e^x = 5$ (b) $e^{\sqrt{x}} = 0.8$ (c) $e^{2x-1} = 1.632$

(d) $e^{2-x} = 0.64$ (e) $e^{2\sqrt{x+1}} = 1.41$ (f) $e^{\frac{1}{x}} = 0.5324$

5. Solve the following equations:

(a) $3e^x + \frac{1}{e^x} - 4 = 0$ (b) $e^{2x} - 4e^x - 5 = 0$ (c) $3e^x - \frac{28}{e^x} - 11 = 0$

(d) $5e^x - 2e^{-x} - 2 = 0$ (e) $5e^x + 2e^{-x} - 11 = 0$

6. Solve the equations:

(a) $3^x = 11$ (b) $2^{x-1} = 11$ (c) $5^{-x} = 11$

(d) $\left(\frac{5}{6}\right)^x = 0.93$ (e) $4^{x+1} = 9$ (f) $3^{1-2x} = 13$

(g) $3^x \times 3^{2x-4} = 13$ (h) $9 \times \left(\frac{2}{3}\right)^x = 11$ (i) $3^{x+1} \times 2^{2x-5} = 47$

7. Solve the inequations:

(a) $3^x > 1\,843$ (b) $2^{3x} < 789$ (c) $4^{-3x} \leqslant 0.63$

(d) $\left(\frac{3}{7}\right)^x \leqslant \frac{4}{9}$ (e) $\left(\frac{2}{3}\right)^{-x} \geqslant 11$ (f) $\frac{2\left(3^n - 1\right)}{3-1} > 2\,142$

(g) $\dfrac{3\left[1-\left(\frac{2}{3}\right)^n\right]}{1-\frac{2}{3}} \leqslant 8$ (h) $2 \times 3^{2-3x} < 1$ (i) $7 \times \left(\frac{2}{3}\right)^{n-1} \leqslant 0.8$

8. Find the n^th^ term of the series 2 + 6 + 18 + ... Find the least value of n for which this term exceeds 2000 and hence obtain the first term of the series exceeding 2000.

9. Find the first term of the series 32 + 24 + 18 + ... which is less than $\frac{1}{10}$.

10. How many terms of the series 9 + 12 + 16 + ... must be taken for the sum to exceed 25 000?

11. The first term of a geometric progression is 8 and the common ratio is 0.2. Find:
    (a) the sum to infinity
    (b) the least value of n for which the sum to n terms differs from the sum to infinity by less than 0.008.

12. The first term of a geometric progression is 5 and the common ratio is 0.9. S_n represents the sum of the first n terms of the progression and S_∞ its sum to infinity. Find the least value of n for which $|S_n - S_\infty| < 0.03$.

11.3 Reduction of $y = ab^x$ and $y = ax^b$ to linear forms

We know already that the equation of a straight line is of the form $y = mx + c$ and conversely that $y = mx + c$ represents a straight line of gradient m and making a y–intercept of c.

The equations $y = ab^x$ and $y = ax^b$ are non–linear equations which can however be transformed to linear forms by taking logarithm.

Consider $y = ab^x$

$$\begin{aligned} \lg y &= \lg (ab^x) \\ &= \lg a + \lg b^x \\ &= \lg a + x\lg b \\ &= \lg a + (\lg b)\, x \\ &= (\lg b)\, x + \lg a \end{aligned}$$

which is of the form $Y = mX + c$ where $Y = \lg y$, $m = \lg b$ and $c = \lg a$.

Similarly for $y = ax^b$

$$\begin{aligned} \lg y &= \lg (ax^b) \\ &= \lg a + \lg x^b \\ &= \lg a + b\lg x \\ &= b\lg x + \lg a \end{aligned}$$

which is of the form $Y = mX + c$ where $Y = \lg y$, $X = \lg x$, $m = b$, $c = \lg a$.

Note: We can take logarithm to base e instead of base 10.

Example 13

Figure 11.2 shows the graph of lg y against x. Obtain y in terms of x.

Solution

Using $Y = mX + c$, i.e. $\lg y = mx + c$

gradient $m = \frac{8-2}{4-1}$

$= 2$

Using $Y = mX + c$

$2 = 2 \times 1 + c$

$c = 0$

So, $\lg y = 2x$

$y = 10^{2x}$

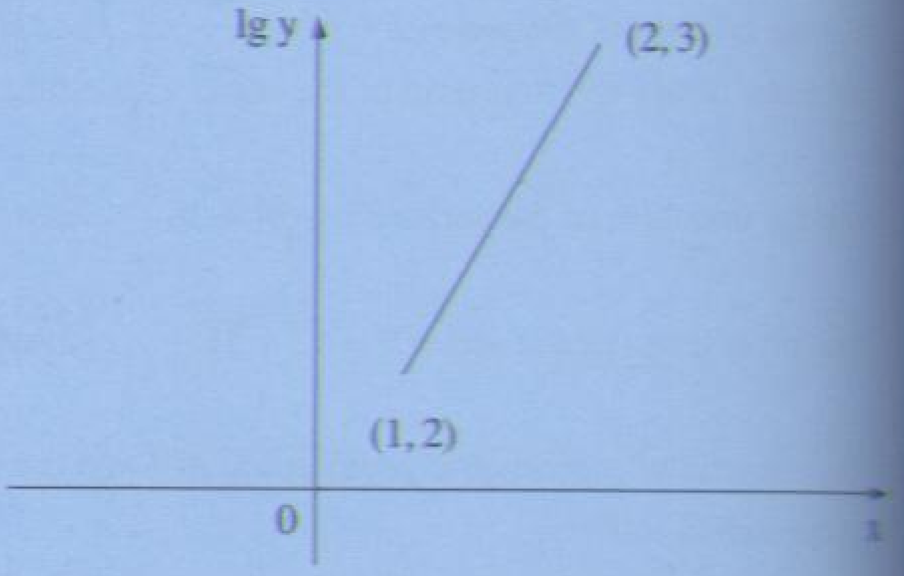


Figure 11.2

Example 14

Figure 11.3 shows the graph of ln y against ln x. Obtain y in terms of x.

Solution

Using $Y = mX + c$, i.e. $\ln y = m(\ln x) + c$

$m = \frac{15-3}{4-1}$

$= 4$

Also, $3 = 4 \times 1 + c$

$c = -1$

So, $\ln y = 4\ln x - 1$

$= \ln x^4 - \ln e$

$= \ln \frac{x^4}{e}$

$y = \frac{x^4}{e}$

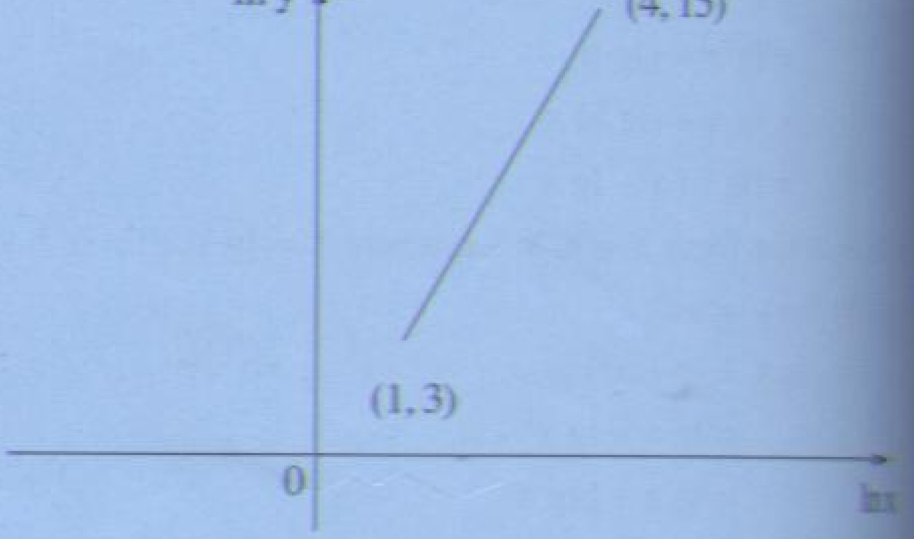


Figure 11.3

Exercise 11 C

1. The variables x and y are related in such a way that the graph of ln y against x is a straight line through (2, 2.48) and (4, 5.26). Find y in terms of x.

2. The variables x and y are related in such a way that the graph of ln y against x is a straight line through (0.693, –0.2877) and (1.39, –1.674). Find y in terms of x.

3. The table shows experimental values of two variables x and y.

x	1	1.5	2	2.5	3
y	3	0.39	0.20	0.08	0.04

It is thought that x and y are related by the equation $y = ax^b$ where a and b are constants. By reducing $y = ax^b$ to a suitable linear form, draw a straight line graph and obtain the value of a and of b.

4. The table shows experimental values of x and y.

x	1	3	5	7	9
y	2	6.8	15.4	34.3	47.7

It is known that x and y are related by the equation $y = ab^x$ where a and b are constants. By reducing $y = ab^x$ to a linear form, draw a straight line graph and obtain the value of a and of b.

5. The table shows experimental values of two variables t and s.

t	1.5	3.5	5.5	7.5	9.5
s	2.08	2.51	3.04	3.70	4.90

It is known that t and s are related by one of the two equations $s = at^b$ or $s = ab^t$. By reducing each of the two equations to linear form and drawing appropriate graphs, find out which of the two relations hold. Find also the value of a and of b.

Miscellaneous Exercise 11

1. (a) Given that $2^x4^y = 128$ and that $\ln(4x - y) = \ln 2 + \ln 5$, calculate the value of x and of y.

   (b) Solve the equations:

   (i) $\lg(1 - 2x) - 2\lg x = 1 - \lg(2 - 5x)$

   (ii) $3^{y+1} = 4^y$

(c) Using graph paper, draw the curve $y = \ln(x + 1)$ for $0 \leqslant x \leqslant 4$, taking values of x at unit intervals. By drawing an appropriate straight line, obtain an approximate value for the positive root of the equation $x - 2\ln(x + 1) = 0$. [C]

2. (a) Solve $5^x = 10$

(b) Solve the simultaneous equations

$$3^p = 9(27)^q$$
$$\log_2 7 - \log_2(11q - 2p) = 1$$

[C]

3. (a) Given that $y = 120(1.08)^x$, find

(i) the value of y when $x = 4.6$

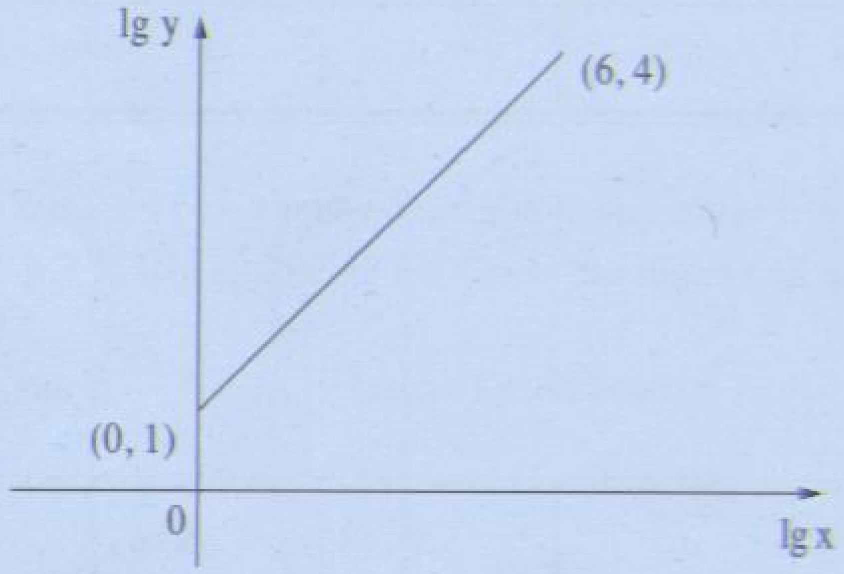
(ii) the value of x when $y = 450$.

(b) Solve the equation $\lg(2x) - 3\lg 2 = \frac{1}{2}\lg(x - 3)$.

(c) Sketch the graph of the function $f:x \mapsto 3e^{2x}$ for real values of x, showing on your diagram the coordinates of any points of intersection with the axes.

State the range of f and obtain an expression for f^{-1} in terms of x. [C]

4.



The variables x and y are related in such a way that when lg y is plotted against lg x, a straight line is obtained as shown in the figure. Given that this line passes through (0, 1) and (6, 4). Find:

(a) the value of x when $\lg y = 3$

(b) the value of a and of n when the relationship between x and y is expressed in the form $y = ax^n$. [C]

5. (a) Solve the equations:

(i) $2\lg(x - 1) = \lg x$

(ii) $\lg(20y) - \lg(y - 8) = 2$

(b) By using the substitution $y = e^{2x}$, solve the equation $e^{2x} + 4e^{-2x} = 4$.

(c) (i) Sketch the graph of $y = \ln x$ for $x > 0$.

(ii) Express $x^2 = e^{x-2}$ in the form $\ln x = ax + b$

(iii) Insert on your sketch the additional graph required to obtain a graphical solution of $x^2 = e^{x-2}$. [C]

6. (a) Given the simultaneous equations $2^x = 3^y$, $x + y = 1$, show that $x = \frac{\ln 3}{\ln 6}$.

   (b) Show that $e^{\frac{x}{2}}$ can be written in the form k^x for a suitable value of k, which should be found correct to three significant figures. [C]

7. The function f is given by $f:x \mapsto \ln(1 + x)$, $x \in \mathbb{R}$, $x > -1$. Express the definition of f^{-1} in a similar form. [C]

8. Solve each of the following equations, to find x in terms of a, where $a > 0$, and $a \neq e^2$.

   (a) $a^x = e^{2x+1}$

   (b) $2\ln 2x = 1 + \ln a$ [C]

9. The table below shows experimental values of two variables x and y.

x	2	4	6	8
y	5.3	12.3	27.4	61.5

It is known that x and y are related by an equation of the form $y = ab^x$. Express this equation in a form suitable for drawing a straight line graph. Draw this graph for the given data and use it to evaluate a and b. [C]

10. Find the sum to infinity S_∞ of the series $24 + 16 + 10\frac{2}{3} + \ldots\ldots$ and an expression for S_n, the sum of the first n terms of the series. Find the least number of terms which must be taken for $|S_n - S_\infty|$ to be less than 0.5.

11. (a) The curve with equation $py = q^x$ passes through the points $(1, -12)$ and $\left(-2, \frac{3}{16}\right)$. Find the value of p and of q.

    (b) Solve the simultaneous equations

$$\lg x + 2\lg y = 3$$
$$x^2 y = 125$$

[C]

12. The table below shows experimental values of two variables x and y.

x	1	2	2.5	3
y	2.12	2.86	3.34	4.18

It is known that x and y are connected by the equation $ay^2 - bx^3 = 1$. By plotting y^2 against x^3, obtain a straight line to estimate the value of a and of b.

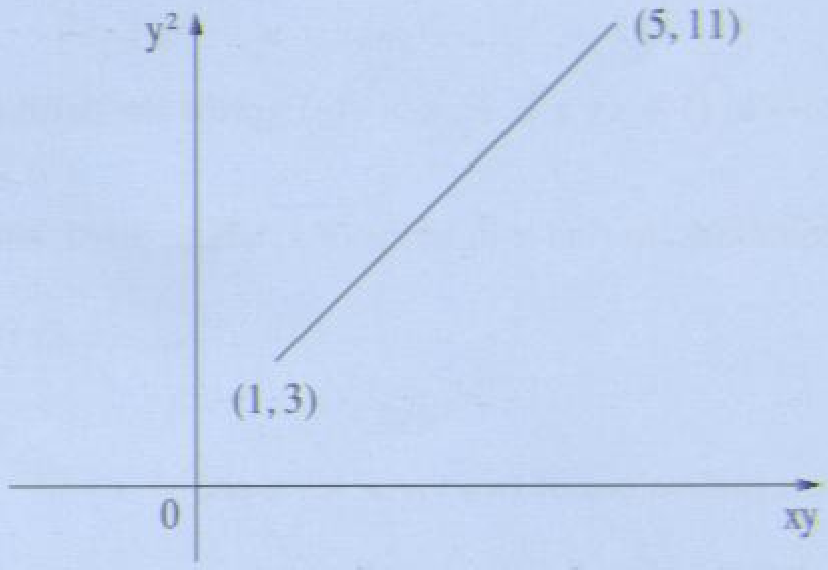
By drawing a suitable straight line, find the value of x and of y which satisfy the simultaneous equations

$$ay^2 - bx^3 = 1$$
$$y^2 - x^3 = 1$$

[C]

13. (a) Variables x and y are related by the equation $y^2 = px^q$. When the graph of lg y against lg x is drawn, the resulting straight line has a gradient of –2 and an intercept of 0.5 on the axis of lg y. Calculate the value of p and of q.

(b)



The variables x and y are related in such a way that, when y^2 is plotted against xy, a straight line is obtained which passes through the points (1, 3) and (5, 11). Find the value of x when y = 3. [C

14. (a) Solve the equation $\lg(x - 4) + 2\lg 3 = 1 + \lg\left(\frac{x}{2}\right)$.

(b) The first three terms of an arithmetic progression are $\log_2 32$, $\log_2 p$ and $\log_2 q$. The common difference is –3. Evaluate p and q. [C

15. The table shows experimental values of two variables x and y.

t	5	10	15	20	25
s	57.5	37.0	23.7	15.2	9.7

It is known that x and y are related by the equation $y = Ae^{-kx}$, where A and k are constants. Using graph paper, plot ln y against x for the above data and use your graph to estimate the value of A and of k. [C

CHAPTER 12

> TRIGONOMETRY 2

12.1 The reciprocal ratios sec x, cosec x and cot x

We are already familiar with the three trigonometric ratios sine, cosine and tangent. We now define three new trigonometric ratios:

- Secant of an angle α abbreviated as sec α is defined as $\frac{1}{\cos\alpha}$.
- Cosecant of an angle α abbreviated as cosec α is defined as $\frac{1}{\sin\alpha}$.
- Cotangent of an angle α abbreviated as cot α is defined as $\frac{1}{\tan\alpha}$.

So, $\sec\alpha = \frac{1}{\cos\alpha}$, $\operatorname{cosec}\alpha = \frac{1}{\sin\alpha}$, $\cot\alpha = \frac{1}{\tan\alpha}$

From these definitions, we may find each of the trigonometric ratios of α. The algebraic sign of each of these ratios will be the same as the algebraic sign of the inverse ratio.

$$\sec 300^\circ = \frac{1}{\cos 300^\circ} = \frac{1}{\cos 60^\circ} = 2$$

$$\operatorname{cosec} 210^\circ = \frac{1}{\sin 210^\circ} = \frac{1}{-\sin 30^\circ} = -2$$

$$\cot 300^\circ = \frac{1}{\tan 300^\circ} = \frac{1}{-\tan 60^\circ} = \frac{-1}{\sqrt{3}}$$

Example 1

Find the exact value of:

(a) sec 225° *(b) cosec 120°* *(c) cot 135°*

Solution

(a) $\sec 225^\circ = \frac{1}{\cos 225^\circ}$

$= \frac{1}{-\cos 45^\circ}$

$= 1 \div \frac{-1}{\sqrt{2}}$

$= -\sqrt{2}$

(b) $\operatorname{cosec} 120° = \dfrac{1}{\sin 120°}$

$= \dfrac{1}{\sin 60°}$

$= 1 \div \dfrac{\sqrt{3}}{2}$

$= \dfrac{2}{\sqrt{3}}$

(c) $\cot 135° = \dfrac{1}{\tan 135°}$

$= \dfrac{1}{-\tan 45°}$

$= \dfrac{-1}{1}$

$= -1$

If non-exact values are required, calculators may be used to give these values to the required accuracy.

Example 2

Find the value to three decimal places of:

(a) sec 200° *(b) cosec 326°* *(c) cot 208°*

Solution

(a) $\sec 200° = \dfrac{1}{\cos 200°}$

≈ -1.064

(b) $\operatorname{cosec} 326° = \dfrac{1}{\sin 326°}$

≈ -1.788

(c) $\cot 208° = \dfrac{1}{\tan 208°}$

≈ 1.881

Example 3

Solve the following equations for $0° \leqslant x \leqslant 360°$:

(a) $\sec x = 2.32$ (b) $\operatorname{cosec} x = -6.342$ (c) $\cot x = -3.526$

Solution

(a) $\sec x = 2.32$

$$\cos x = \frac{1}{2.32}$$

Key angle ≈ 64.47°

x Q1 64.47 °

Q4 295.53°

$x \approx 64.5°, 295.5°$ (to one tenth of a degree)

(b) $\operatorname{cosec} x = -6.342$

$$\sin x = \frac{1}{-6.342}$$

Key angle ≈ 9.1°

x Q3 189.1°

Q4 350.9°

$x \approx 189.1°, 350.9°$

(c) $\cot x = -9.526$

$$\tan x = \frac{1}{-9.526}$$

Key angle ≈ 6.0°

x Q2 174.0°

Q4 354.0°

$x \approx 174.0°, 354.0°$

12.1.2 Graph of y = cosec x

Since the value of sin x is the same as sin (360° + x), it is sufficient to draw the graph of $y = \operatorname{cosec} x$ for $0° \leqslant x \leqslant 360°$.

We note that since, $\operatorname{cosec} x = \dfrac{1}{\sin x}$ is not defined for $\sin x = 0$, i.e. for $x = 0°, 180°, 360°$, etc.

We compile a table of values of cosec x for $0° \leqslant x \leqslant 360°$.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
cosec x	–	2	1.15	1	1.15	2	–	–2	–1.15	–1	–1.15	–2

The graph of y = cosec x is then as shown for 0° ⩽ x ⩽ 360°.

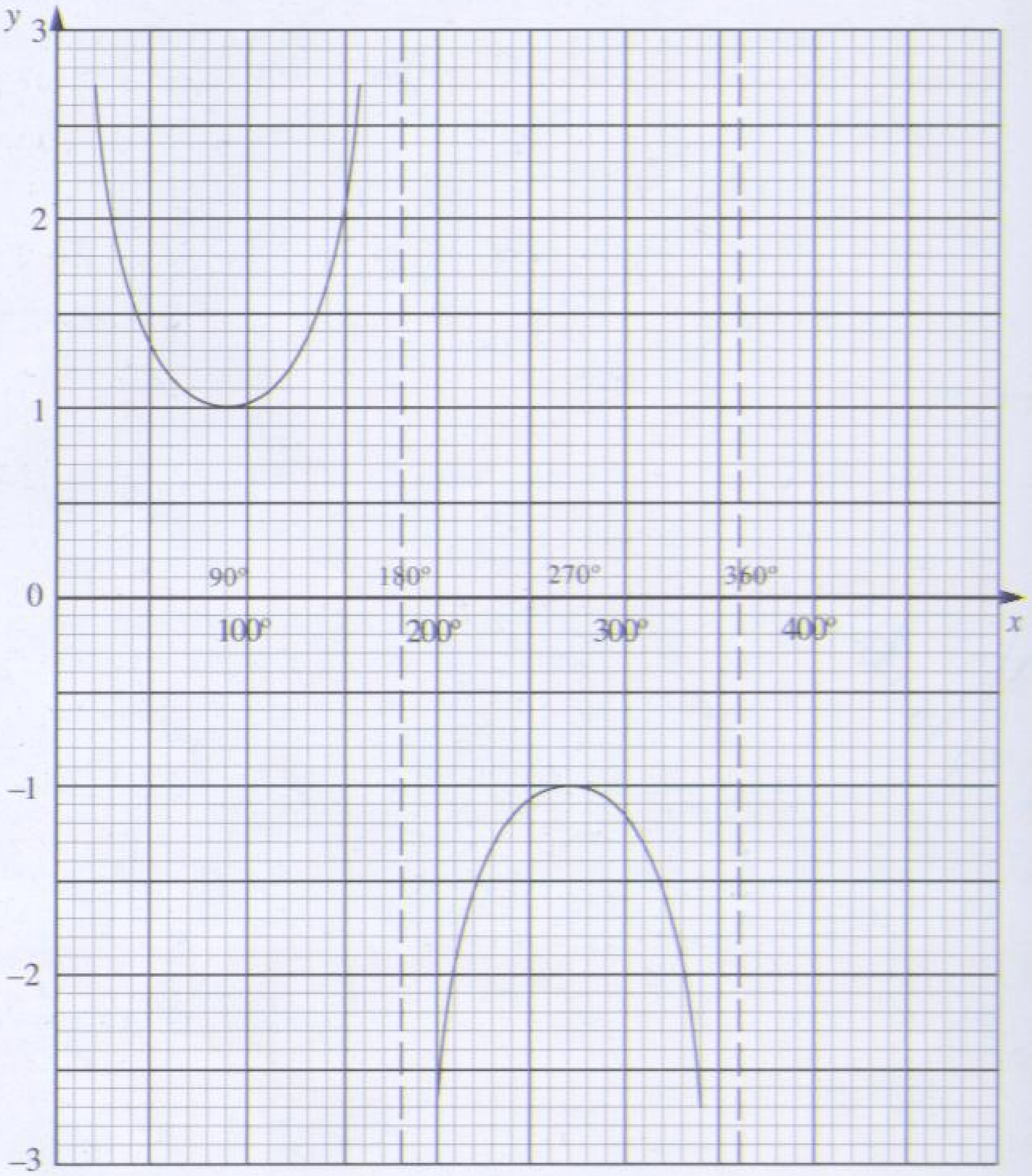


Figure 12.1

The graph for –360° ⩽ x ⩽ 720° is as shown.

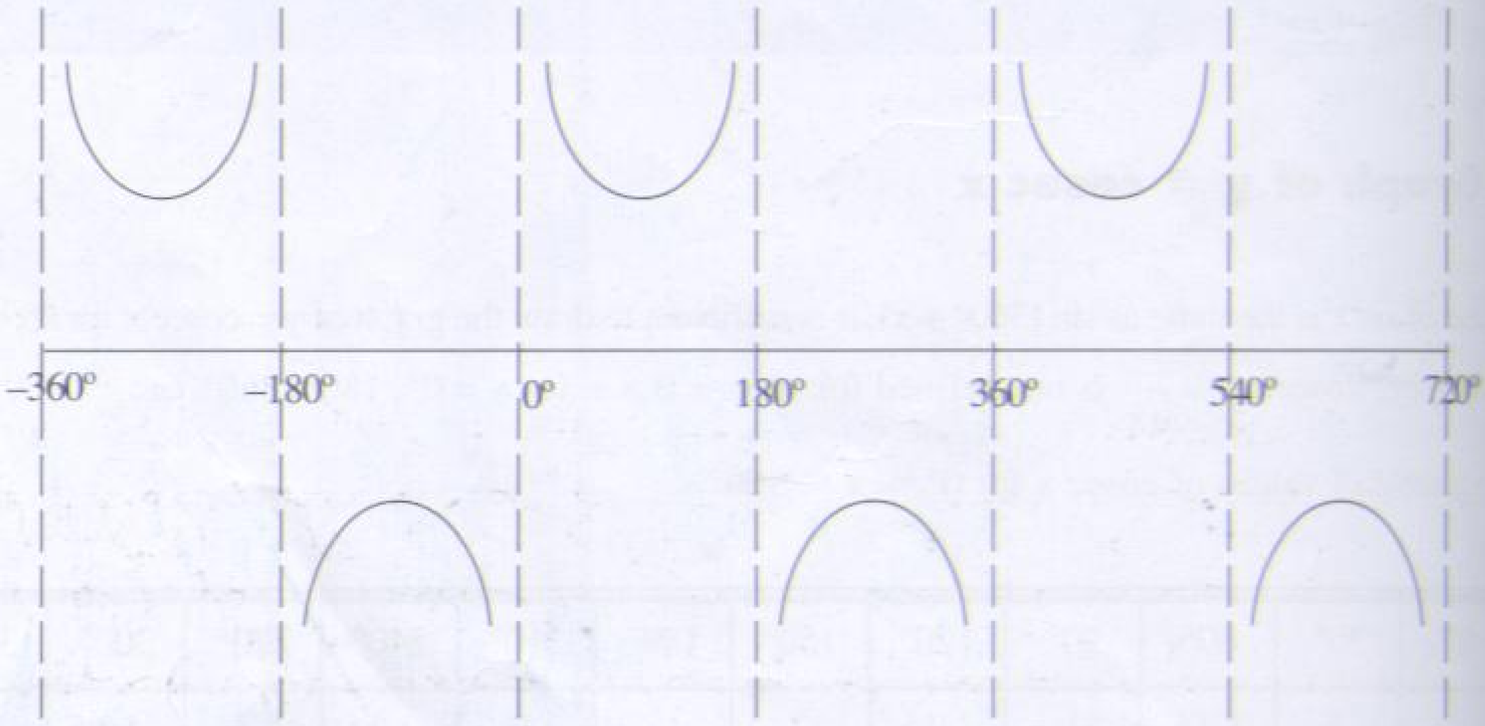


Figure 12.2

Exercise 12 A

1. Find the exact value of:

(a) sec 120°
(b) cosec 135°
(c) cot 210°
(d) sec 330°
(e) cosec 240°
(f) cot 330°
(g) sec 210°
(h) cosec 330°
(i) cot 120°
(j) sec 150°
(k) cosec 210°
(l) cot 315°
(m) sec 225°
(n) cosec 315°
(o) cot 240°
(p) sec 315°
(q) cosec 150°
(r) cot 150°

2. Find the value to 3 places of decimals of:

(a) sec 132°
(b) cosec 146°
(c) cot 228°
(d) sec 346°
(e) cosec 248°
(f) cot 308°
(g) sec 226°
(h) cosec 346°
(i) cot 130°
(j) sec 173°
(k) cosec 226°
(l) cot 346°
(m) sec 266°
(n) cosec 308°
(o) cot 266°
(p) sec 327°
(q) cosec 130°
(r) cot 173°

3. Solve for 0° ≤ x ≤ 360° each of the following equations:

(a) $\sec x = \frac{2}{\sqrt{3}}$
(b) $\operatorname{cosec} x = -2$
(c) $\cot x = \sqrt{3}$
(d) $\sec x = -2$
(e) $\operatorname{cosec} x = \frac{2}{\sqrt{3}}$
(f) $\cot x = \frac{-1}{\sqrt{3}}$
(g) $\sec x = \sqrt{2}$
(h) $\operatorname{cosec} x = -\sqrt{2}$
(i) $\cot x = 1$
(j) $\sec 2x = \frac{-2}{\sqrt{3}}$
(k) $\operatorname{cosec} \frac{1}{2}x = \frac{-2}{\sqrt{3}}$
(l) $\cot 3x = -\sqrt{3}$
(m) $\sec x = 2.743$
(n) $\operatorname{cosec} x = 2.31$
(o) $\cot x = 2.643$
(p) $\sec x = -1.75$
(q) $\operatorname{cosec} x = -2.31$
(r) $\cot x = -0.452$
(s) $\sec 2x = 3.91$
(t) $\operatorname{cosec} \frac{1}{2}x = -2.32$
(u) $\cot 3x = -0.8660$

4. Complete the table below and draw the graph of y = sec x for 0° ≤ x ≤ 360°.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
sec x	1		2	undefined	−2		−1			undefined			

Hence, sketch the graph of y = sec x for −360° ≤ x ≤ 720°.

5. Complete the table below and draw the graph of y = cot x for $0° \leqslant x \leqslant 360°$.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
cot x		1.73	0.867										

Hence, sketch the graph of y = cot x for $-360° \leqslant x \leqslant 720°$.

12.2 Identities

We know already that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cos^2\theta + \sin^2\theta \equiv 1$.

Since $\cot\theta = \dfrac{1}{\tan\theta}$, we obtain $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ (A)

Also, since $\cos^2\theta + \sin^2\theta = 1$

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$1 + \tan^2\theta = \sec^2\theta \qquad \text{(B)}$$

Also, $$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta \qquad \text{(C)}$$

We obtain therefore the identities:

$$\boxed{\begin{aligned}&\cos^2\theta + \sin^2\theta = 1\\ &\sec^2\theta = 1 + \tan^2\theta\\ &\operatorname{cosec}^2\theta = 1 + \cot^2\theta\end{aligned}}$$

We consider the use of these identities.

Example 4

Given $\sin\theta = -\frac{3}{5}$, find the exact values of each of the other trigonometrical ratios.

Solution

$\sin\theta = -\dfrac{3}{5}$

$\operatorname{cosec}\theta = -\dfrac{5}{3}$

Using $\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta = 1 - \dfrac{9}{25}$

$= \dfrac{16}{25}$

$\cos\theta = \pm\frac{4}{5}$

$\sec\theta = \pm\frac{5}{4}$

$$\begin{aligned}\tan\theta &= \frac{\sin\theta}{\cos\theta}\\ &= -\frac{3}{5} \div \pm\frac{4}{5}\\ &= \pm\frac{3}{4}\end{aligned}$$

$\cot\theta = \pm\frac{4}{3}$.

Example 5

Given $\tan\theta = \frac{3}{4}$, *find the exact value(s) of each of the other trigonometrical ratios.*

Solution

$\tan\theta = \frac{3}{4}$

$\cot\theta = \frac{4}{3}$

$$\begin{aligned}\sec^2\theta &= 1 + \tan^2\theta\\ &= 1 + \frac{9}{16}\\ &= \frac{25}{16}\end{aligned}$$

$\sec\theta = \pm\frac{5}{4}$

$\cos\theta = \pm\frac{4}{5}$

Using $\frac{\sin\theta}{\cos\theta} = \tan\theta$

$$\begin{aligned}\sin\theta &= \cos\theta\tan\theta\\ &= \pm\frac{4}{5} \times \frac{3}{4}\\ &= \pm\frac{3}{5}\end{aligned}$$

$\text{cosec}\,\theta = \pm\frac{5}{3}$

Example 6

Prove the identity $\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \equiv \sec\theta \operatorname{cosec}\theta.$

Solution

$$\begin{aligned}
\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} &= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} \\
&= \frac{1}{\cos\theta\sin\theta} \\
&= \frac{1}{\cos\theta} \times \frac{1}{\sin\theta} \\
&= \sec\theta \operatorname{cosec}\theta
\end{aligned}$$

Example 7

Prove the identity $\cot\theta + \tan\theta \equiv \sec^2\theta\cot\theta.$

Solution

$$\begin{aligned}
\cot\theta + \tan\theta &= \frac{1}{\tan\theta} + \tan\theta \\
&= \frac{1 + \tan^2\theta}{\tan\theta} \\
&= \frac{\sec^2\theta}{\tan\theta} \\
&= \sec^2\theta\cot\theta
\end{aligned}$$

Exercise 12 B

1. For each of the following, find the exact value of each of the remaining trigonometrical ratios:

(a) $\sin\theta = \frac{1}{\sqrt{3}}$ (b) $\cos\theta = \frac{2}{\sqrt{5}}$ (c) $\tan\theta = -\frac{2}{3}$

(d) $\operatorname{cosec}\theta = -\frac{5}{3}$ (e) $\sec\theta = -\frac{4}{3}$ (f) $\cot\theta = \frac{5}{3}$

(g) $\sin\theta = -\frac{3}{5}$, $180° < \theta < 270°$ (h) $\tan\theta = -\frac{4}{3}$, $90° < \theta < 180°$

(i) $\sec\theta = \frac{5}{3}$, $270° < \theta < 360°$ (j) $\cot\theta = \frac{4}{3}$, $180° < \theta < 270°$

2. Prove the following identities:

(a) $\sec^2\theta - \operatorname{cosec}^2\theta \equiv \tan^2\theta - \cot^2\theta$ (b) $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) \equiv 1$

(c) $\sec^2\theta \operatorname{cosec}^2\theta \equiv \sec^2\theta + \operatorname{cosec}^2\theta$ (d) $(\sec^2\theta - 1)(\operatorname{cosec}^2\theta - 1) \equiv 1$

(e) $\frac{1-\tan^2\theta}{\sec^2\theta} \equiv 1 - 2\sin^2\theta$

(f) $\cos^4\theta - \sin^4\theta \equiv 1 - 2\sin^2\theta$

(g) $\frac{2\tan\theta}{1+\tan^2\theta} \equiv 2\sin\theta\cos\theta$

(h) $\frac{(\sin\theta + \cos\theta)^2}{\sin\theta\cos\theta} - 2 \equiv \sec\theta \operatorname{cosec}\theta$

(i) $\sec\theta \operatorname{cosec}\theta - \cot\theta \equiv \tan\theta$

(j) $\frac{(\sin\theta + \cos\theta)^2}{\sin\theta\cos\theta} \equiv 2 + \sec\theta \operatorname{cosec}\theta$

(k) $(\sec\theta - \tan\theta)(\operatorname{cosec}\theta + 1) \equiv \cot\theta$

12.3 Expansions of sin (A ± B), cos (A ± B) and tan (A ± B)

(i)

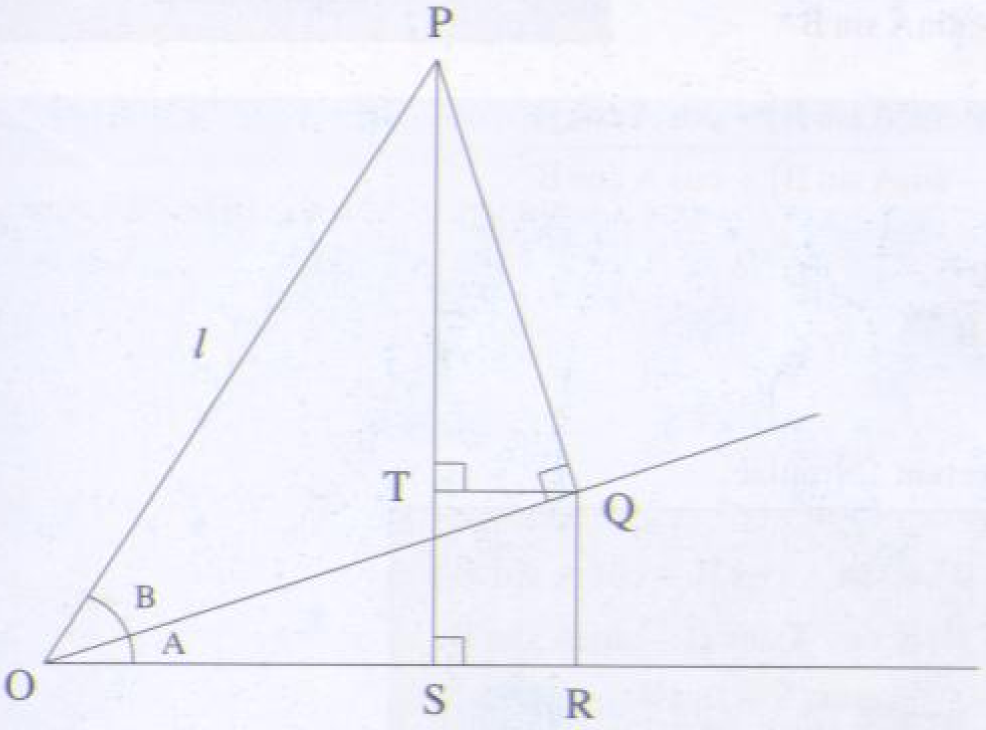


Figure 12.3

In Figure 12.3, OP is of length l and makes an angle of $(A + B)$ with OR, PS and QR are perpendicular to OR, PQ is perpendicular to OQ and TQ is perpendicular to PS. In triangle POS, $PS = l\sin(A + B)$ and $OS = l\cos(A + B)$. In triangle POQ, $PQ = l\sin B$ and $OQ = l\cos B$.

In triangle PTQ, $PT = PQ\cos A$ as $\angle QPT = A$

$= l\sin B\cos A$

$TQ = PQ\sin A$

$= l\sin B\sin A$

$= l\sin A\sin B$

In triangle OQR, $QR = OQ\sin A$

$= l\cos B\sin A$

$= l\sin A\cos B$

$OR = OQ\cos A$

$= l\cos B\cos A$

$= l\cos A\cos B$

Using $PS = PT + TS$

$= PT + QR$

We have $l\sin(A+B) = l\sin B\cos A + l\sin A\cos B$

$$\sin(A+B) = \sin B\cos A + \sin A\cos B$$
$$= \sin A\cos B + \sin B\cos A$$
$$= \sin A\cos B + \cos A\sin B$$

Using $OS = OR - SR$

$$= OR - TQ$$

We have $l\cos(A+B) = l\cos A\cos B - l\sin A\sin B$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

$$= \frac{(\sin A\cos B + \cos A\sin B) \div \cos A\cos B}{(\cos A\cos B - \sin A\sin B) \div \cos A\cos B}$$

$$= \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

We have derived three important formulae:

$$\mathbf{\sin(A+B) \equiv \sin A\cos B + \cos A\sin B}$$
$$\mathbf{\cos(A+B) \equiv \cos A\cos B - \sin A\sin B}$$
$$\mathbf{\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A\tan B}}$$

(ii)

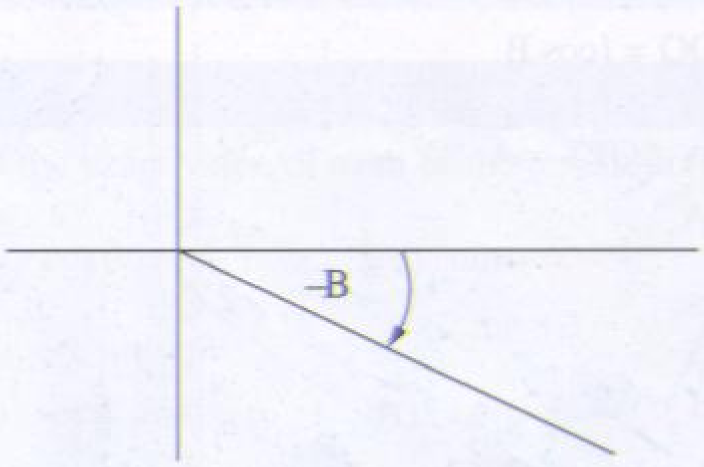


Figure 12.4

Figure 13.4 shows an angle –B which is equivalent to (360° – B).

$\sin(-B) = \sin(360° - B) = -\sin B$

$\cos(-B) = \cos(360° - B) = \cos B$

$\tan(-B) = \tan(360° - B) = -\tan B$

So, $\sin(A-B) = \sin(A+(-B))$

$$= \sin A \cos(-B) + \cos A \sin(-B)$$
$$= \sin A \cos B - \cos A \sin B$$

Similarly, $\cos(A-B) = \cos A \cos B + \sin A \sin B$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

We have the six important formulae grouped as:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \pm \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \pm \tan A \tan B}$$

Example 8

Find the exact value of: (a) sin 75° (b) cos 165° (c) tan 15°

Solution

(a) $\sin 75° = \sin(45° + 30°)$

$$= \sin 45° \cos 30° + \cos 45° \sin 30°$$
$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$$
$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$
$$= \frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}$$
$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

(b) $\cos 165° = \cos(120° + 45°)$

$$= \cos 120° \cos 45° - \sin 120° \sin 45°$$
$$= -\cos 60° \cos 45° - \sin 60° \sin 45°$$
$$= -\frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$$
$$= \frac{-1-\sqrt{3}}{2\sqrt{2}}$$
$$= \frac{-\sqrt{2}\left(1+\sqrt{3}\right)}{4}$$
$$= \frac{-\left(\sqrt{2}+\sqrt{6}\right)}{4}$$

(c) $\tan 15° = \tan(45° - 30°)$

$$= \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

$$= \frac{(\sqrt{3} - 1)(\sqrt{3} - 1)}{(\sqrt{3} + 1)(\sqrt{3} - 1)}$$

$$= \frac{3 + 1 - 2\sqrt{3}}{3 - 1}$$

$$= \frac{4 - 2\sqrt{3}}{2}$$

$$= 2 - \sqrt{3}$$

Example 9

Given $\sin A = \frac{3}{5}$ *and A is acute,* $\cos B = \frac{12}{13}$ *and B is reflex, find the exact value of:*

(a) $\sin(A - B)$ *(b)* $\sec(A + B)$ *(c)* $\cot(A - B)$

Solution

$\sin A = \frac{3}{5}$

Using $\cos^2 A + \sin^2 A = 1$

$$\cos A = \pm\frac{4}{5}$$

As A is acute, $\cos A = \frac{4}{5}$, $\tan A = \frac{\sin A}{\cos A} = \frac{3}{4}$

$\cos B = \frac{12}{13}$

Using $\cos^2 B + \sin^2 B = 1$

$$\sin B = \pm\frac{5}{13}$$

As B is reflex, $\sin B = -\frac{5}{13}$, $\tan B = \frac{\sin B}{\cos B} = -\frac{5}{12}$

(a) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

$$= \frac{3}{5} \times \frac{12}{13} - \frac{4}{5} \times \frac{-5}{13}$$

$$= \frac{56}{65}$$

(b) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$= \frac{4}{5} \times \frac{12}{13} - \frac{3}{5} \times \frac{-5}{13}$$

$$= \frac{63}{65}$$

$$\sec(A+B) = \frac{65}{63}$$

(c) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$$= \frac{3}{4} - \left(-\frac{5}{12}\right)$$

$$= 1 + \frac{3}{4} \times \frac{-5}{12}$$

$$= \frac{36+20}{48-15},$$ multiplying numerator and denominator by 48

$$= \frac{56}{33}$$

$$\cot(A-B) = \frac{33}{56}$$

Example 10

Prove the identity $\dfrac{\sin(A+B)+\sin(A-B)}{\cos(A+B)-\cos(A-B)} \equiv -\cot B$

Solution

$$\frac{\sin(A+B)+\sin(A-B)}{\cos(A+B)-\cos(A-B)} = \frac{\sin A\cos B + \cos A\sin B + \sin A\cos B - \cos A\sin B}{\cos A\cos B - \sin A\sin B - (\cos A\cos B + \sin A\sin B)}$$

$$= \frac{2\sin A\cos B}{-2\sin A\sin B}$$

$$= -\cot B$$

Example 11

Write as a single trigonometrical function:

(a) $\sin 35° \cos 15° + \cos 35° \sin 15°$ *(b)* $\dfrac{\sqrt{3}}{2}\cos x° - \dfrac{1}{2}\sin x°$

Solution

(a) $\sin 35° \cos 15° + \cos 35° \sin 15° = \sin(35° + 15°)$

$$= \sin 50°$$

(b) $\dfrac{\sqrt{3}}{2}\cos x° - \dfrac{1}{2}\sin x° = \sin 60° \cos x° - \cos 60° \sin x°$

$$= \sin(60 - x)°$$

Alternatively, $\dfrac{\sqrt{3}}{2}\cos x° - \dfrac{1}{2}\sin x° = \cos 30° \cos x° - \sin 30° \sin x°$

$$= \cos(30 + x)°$$

Exercise 12 C

1. Find the exact value of each of the following:
   (a) $\sin 105°$ (b) $\cos 75°$ (c) $\tan 75°$
   (d) $\operatorname{cosec} 15°$ (e) $\sec 105°$ (f) $\cot 165°$

2. Given $\sin A = \frac{12}{13}$ and $\sin B = -\frac{3}{5}$ where A is acute and B is reflex, find the exact value of:
   (a) $\sin(A - B)$ (b) $\sec(A + B)$ (c) $\cot(A + B)$

3. Given $\cos A = \frac{3}{5}$ and $\tan B = \frac{15}{8}$ where both A and B are reflex, find the exact value of:
   (a) $\operatorname{cosec}(A + B)$ (b) $\cos(A - B)$ (c) $\tan(A - B)$

4. Given $\operatorname{cosec} A = \frac{13}{5}$ and $\cot B = -\frac{8}{15}$ where both A and B are obtuse, find the exact value of:
   (a) $\sin(A - B)$ (b) $\sec(A + B)$ (c) $\cot(A + B)$

5. Given $\sec A = \frac{17}{15}$ and $\tan B = -\frac{12}{5}$ where both A and B are reflex, find the exact value of:
   (a) $\operatorname{cosec}(A + B)$ (b) $\sec(A - B)$ (c) $\cot(A + B)$

6. Prove the identities:
   (a) $\frac{\sin(A+B)+\sin(A-B)}{\sin(A+B)-\sin(A-B)} \equiv \tan A \cot B$ (b) $\frac{\cos(A+B)+\cos(A-B)}{\cos(A+B)-\cos(A-B)} \equiv -\cot A \cot B$
   (c) $\tan A + \tan B \equiv \sin(A + B) \sec A \sec B$ (d) $\frac{\cos(A+B)+\cos(A-B)}{\sin(A+B)-\sin(A-B)} \equiv \cot B$
   (e) $\tan(A + 45°)\tan(A - 45°) \equiv -1$ (f) $\sin(45° + A) + \sin(45° - A) \equiv \sqrt{2}\cos A$
   (g) $\tan(A + B) - \tan A \equiv \sin B \sec A \sec(A + B)$ (h) $\operatorname{cosec}(A + B) \equiv \frac{\sec A \sec B}{\tan A + \tan B}$
   H/W * (i) $\sec(A - B) \equiv \frac{\sec A \sec B}{1 + \tan A \tan B}$

7. Given $\tan A = \frac{3}{5}$ and $\tan(A + B) = \frac{9}{4}$, find the exact value of tan B.

8. Given $\tan(A + 45°) = \frac{5}{3}$, find the exact value of tan A.

9. Express as single trigonometrical expressions:
   (a) $\sin 35° \cos 18° + \cos 35° \sin 18°$ (b) $\sin 47° \cos 29° - \cos 47° \sin 29°$
   (c) $\cos 67° \cos 31° - \sin 67° \sin 31°$ (d) $\cos 82° \cos 18° + \sin 82° \sin 18°$
   (e) $\frac{\sqrt{3}}{2}\cos x° + \frac{1}{2}\sin x°$ (f) $\frac{1}{\sqrt{2}}\sin x° - \frac{1}{\sqrt{2}}\cos x°$
   (g) $-\frac{1}{2}\sin x° + \frac{\sqrt{3}}{2}\cos x°$ (h) $\frac{\sqrt{3} + \tan x°}{1 - \sqrt{3}\tan x°}$

(i) $\dfrac{1+\sqrt{3}\tan x^\circ}{\sqrt{3}-\tan x^\circ}$ (j) $\dfrac{1+\tan x^\circ}{1-\tan x^\circ}$

10. Prove the identities:

(a) $\sin(A+B)+\sin(A-B)\equiv 2\sin A\cos B$ (b) $\sin(A+B)-\sin(A-B)\equiv 2\sin B\cos A$

Use the substitutions $A+B=x, A-B=y$ to show that :

$\sin x+\sin y = 2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$ and $\sin x-\sin y = 2\sin\dfrac{x-y}{2}\cos\dfrac{x+y}{2}$.

Hence, find the exact value of sin 75° + sin 15° and of sin 105° – sin 15°.

11. Prove the identities:

(a) $\cos(A+B)+\cos(A-B)\equiv 2\cos A\cos B$ (b) $\cos(A+B)-\cos(A-B)\equiv -2\sin A\sin B$

Use these two identities to show that :

$\cos x+\cos y = 2\cos\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$ and $\cos x-\cos y = -2\sin\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$.

Hence, find the exact value of cos 75° + cos 15° and of cos 105° – cos 15°.

12. Use the formulae $\sin(A+B)+\sin(A-B)=2\sin A\cos B$ and $\sin(A+B)-\sin(A-B)=2\sin B\cos A$ to find acute angles A and B such that $\sin A\cos B=\dfrac{1}{3}$ and $\cos A\sin B=\dfrac{1}{6}$.

12.4 Double angle formulae

$$\begin{aligned}\sin 2A &= \sin(A+A)\\ &= \sin A\cos A+\cos A\sin A\\ &= 2\sin A\cos A \qquad (1A)\end{aligned}$$

$$\begin{aligned}\cos 2A &= \cos(A+A)\\ &= \cos A\cos A-\sin A\sin A\\ &= \cos^2 A-\sin^2 A \qquad (2A)\\ &= 1-\sin^2 A-\sin^2 A\\ &= 1-2\sin^2 A \qquad (2B)\\ &= 1-2(1-\cos^2 A)\\ &= 2\cos^2 A-1 \qquad (2C)\end{aligned}$$

$$\begin{aligned}\tan 2A &= \tan(A+A)\\ &= \frac{\tan A+\tan A}{1-\tan A\tan A}\\ &= \frac{2\tan A}{1-\tan^2 A} \qquad (3)\end{aligned}$$

Also, $\sin 2A = 2\sin A\cos A$

$$= 2\frac{\sin A}{\cos A}\cos^2 A$$

$$= \frac{2\tan A}{\sec^2 A}$$

$$= \frac{2\tan A}{\sec^2 A} \qquad \text{(1B)}$$

$$\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A \\ &= \cos^2 A\,(1 - \tan^2 A) \\ &= \frac{1}{\sec^2 A}\,(1 - \tan^2 A) \\ &= \frac{1 - \tan^2 A}{1 + \tan^2 A} \qquad \text{(2D)} \end{aligned}$$

We have obtained the following formulae:

$$\boxed{\begin{aligned} \sin 2A &= 2\sin A\cos A && \text{(1A)} \\ &= \frac{2\tan A}{1+\tan^2 A} && \text{(1B)} \\ \cos 2A &= \cos^2 A - \sin^2 A && \text{(2A)} \\ &= 2\cos^2 A - 1 && \text{(2B)} \\ &= 1 - 2\sin^2 A && \text{(2C)} \\ &= \frac{1-\tan^2 A}{1+\tan^2 A} && \text{(2D)} \\ \tan 2A &= \frac{2\tan A}{1-\tan^2 A} && \text{(3)} \end{aligned}}$$

We note that $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\sin 4x = 2\sin 2x\cos 2x = \frac{2\tan 2x}{1+\tan^2 2x}$$

$$\begin{aligned} \cos x &= 2\cos^2\frac{x}{2} - 1 \\ &= 1 - 2\sin^2\frac{x}{2} \\ &= \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} \text{ etc.} \end{aligned}$$

In particular, $\cos^2\frac{x}{2} = \frac{1}{2}(1+\cos x)$ **and** $\sin^2\frac{x}{2} = \frac{1}{2}(1-\cos x)$.

Example 12

Given $\sin A = \frac{1}{3}$*, find the exact value of:*

(a) $\sin 2A$ *(b)* $\cos 4A$ *(c)* $\tan 2A$

Solution

$$\sin A = \frac{1}{3}$$

$$\cos^2 A = 1 - \sin^2 A$$

$$= \frac{8}{9}$$

$$\cos A = \pm\frac{2\sqrt{2}}{3}$$

$$\tan A = \frac{\sin A}{\cos A}$$

$$= \pm\frac{1}{2\sqrt{2}}$$

(a) $\sin 2A = 2 \sin A \cos A$

$$= 2 \times \frac{1}{3} \times \pm \frac{2\sqrt{2}}{3}$$

$$= \pm \frac{4\sqrt{2}}{9}$$

(b) $\cos 4A = 1 - 2 \sin^2 2A$

$$= 1 - 2 \times \left(\pm\frac{4\sqrt{2}}{9}\right)^2$$

$$= 1 - \frac{64}{81}$$

$$= \frac{17}{81}$$

(c) $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$

$$= \frac{2 \times \pm\frac{1}{2\sqrt{2}}}{1 - \left(\pm\frac{1}{2\sqrt{2}}\right)^2}$$

$$= \pm\frac{1}{\sqrt{2}} \div \frac{7}{8}$$

$$= \pm\frac{8}{7\sqrt{2}}$$

$$= \pm\frac{4\sqrt{2}}{7}$$

Example 13

Given $\sin A = \frac{3}{5}$ *and A is acute, find the exact value of* $\cos\frac{A}{2}$ *and of* $\sin\frac{A}{2}$.

Solution

We find cos A first

$$\cos^2 A = 1 - \sin^2 A$$
$$= 1 - \frac{9}{25}$$
$$= \frac{16}{25}$$
$$\cos A = \frac{4}{5} \qquad \text{(A is acute)}$$
$$\cos^2\frac{A}{2} = \frac{1}{2}(1 + \cos A)$$
$$= \frac{1}{2}\left(1 + \frac{4}{5}\right)$$
$$= \frac{9}{10}$$
$$\cos\frac{A}{2} = \frac{3}{\sqrt{10}} \qquad \left(\frac{A}{2}\text{ is acute as A is acute}\right)$$
$$= \frac{3\sqrt{10}}{10}$$
$$\sin^2\frac{A}{2} = \frac{1}{2}(1 - \cos A)$$
$$= \frac{1}{2}\left(1 - \frac{4}{5}\right)$$
$$= \frac{1}{10}$$
$$\sin\frac{A}{2} = \frac{1}{\sqrt{10}} \qquad \left(\frac{A}{2}\text{ is acute}\right)$$
$$= \frac{\sqrt{10}}{10}$$

Example 14

Given $\tan A = \frac{3}{4}$, find the exact values of $\tan\frac{A}{2}$.

Solution

$$\tan A = \frac{2\tan\frac{A}{2}}{1 - \tan^2\frac{A}{2}}$$

$$= \frac{2t}{1-t^2} \quad \left(t = \tan\frac{A}{2}\right)$$

$$\frac{2t}{1-t^2} = \frac{3}{4}$$

$$8t = 3 - 3t^2$$

$$3t^2 + 8t - 3 = 0$$

$$(3t-1)(t+3) = 0$$

$$t = \frac{1}{3} \text{ or } -3$$

$$\tan\frac{A}{2} = \frac{1}{3} \text{ or } -3$$

Example 15

Find the exact value of:

(a) $2\sin 15° \cos 15°$ (b) $2\cos^2 22\frac{1}{2}° - 1$ (c) $\frac{2\tan 30°}{1 - \tan^2 30°}$

Solution

(a) $2\sin 15° \cos 15° = \sin 30°$

$$= \frac{1}{2}$$

(b) $2\cos^2 22\frac{1}{2}° - 1 = \cos 45°$

$$= \frac{1}{\sqrt{2}}$$

(c) $\frac{2\tan 30°}{1 - \tan^2 30°} = \tan 60°$

$$= \sqrt{3}$$

Example 16

Given $x = 2\cos\theta$ and $y = 1 - 3\cos 2\theta$, find y in terms of x.

Solution

$\cos\theta = \frac{x}{2}$

$y = 1 - 3\cos 2\theta$

$= 1 - 3(2\cos^2\theta - 1)$

$= 1 - 6\cos^2\theta + 3$

$= 4 - 6\left(\frac{x^2}{4}\right)$

$= 4 - \frac{3x^2}{2}$

Example 17

Prove the identity $\frac{\sin 2\theta}{1 + \cos 2\theta} \equiv \tan\theta$

Solution

Method 1

$$\frac{\sin 2\theta}{1+\cos 2\theta} \equiv \frac{2\sin\theta\cos\theta}{1+2\cos^2\theta-1}$$

$$\equiv \frac{2\sin\theta\cos\theta}{2\cos^2\theta}$$

$$\equiv \tan\theta$$

Method 2

$$\frac{\sin 2\theta}{1+\cos 2\theta} \equiv \frac{\dfrac{2\tan\theta}{1+\tan^2\theta}}{1+\dfrac{1-\tan^2\theta}{1+\tan^2\theta}}$$

$$\equiv \frac{2\tan\theta}{1+\tan^2\theta+1-\tan^2\theta}$$

$$\equiv \tan\theta$$

Exercise 12 D

1. A is an obtuse angle and $\sin A = \frac{3}{5}$. Find the exact value of:

   (a) $\sin 2A$ (b) $\cos 4A$ (c) $\tan 2A$ (d) $\cos\frac{A}{2}$ (e) $\tan\frac{A}{2}$

2. A is a reflex angle and $\cos A = \frac{12}{13}$. Find the exact value of:

   (a) $\operatorname{cosec} 2A$ (b) $\sec 4A$ (c) $\tan 2A$ (d) $\sin\frac{A}{2}$ (e) $\tan\frac{A}{2}$

3. A is a reflex angle and $\tan A = \frac{12}{5}$. Find the exact value of:

   (a) $\operatorname{cosec} 2A$ (b) $\sec 4A$ (c) $\tan 2A$ (d) $\cos\frac{A}{2}$ (e) $\tan\frac{A}{2}$

4. Find the value(s) of $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ for each of the following:

   (a) $\cos A = -\frac{12}{13}$ (b) $\sin A = \frac{15}{17}$ and A is acute

   (c) $\cos A = \frac{3}{5}$ and A is reflex (d) $\sin A = -\frac{5}{13}$

5. Find the exact value of:

(a) $\operatorname{cosec} 22\frac{1}{2}^{\circ} \sec 22\frac{1}{2}^{\circ}$ (b) $1 - 2\sin^2 15°$ (c) $\dfrac{2\tan 22\frac{1}{2}}{1 - \tan^2 22\frac{1}{2}}$

(d) $\dfrac{1 - \tan^2 30°}{\tan 30°}$ (e) $\dfrac{2\sin^2 30° - 1}{2\cos^2 15° - 1}$ (f) $\dfrac{\cot 60°}{1 - \cot^2 60°}$

6. Find the relation between x and y only if:

(a) $x = 1 - \cos\theta, y = 1 + \cos 2\theta$ (b) $x = 2\sin\theta - 1, \quad y = \cos 2\theta + 3$

(c) $x = 2\tan\theta, y = 2 - \tan 2\theta$ (d) $x = 1 + \sec\theta, \quad y = 2 + \sec 2\theta$

(e) $x = 2 - 3\operatorname{cosec}\theta, y = 1 + \sec 2\theta$ (f) $x = 3\cot\theta - 1, \quad y = 1 + \cot 2\theta$

7. Prove the identities:

(a) $\dfrac{1 + \cot 2\theta}{\sin 2\theta} = \cot\theta$ (b) $\operatorname{cosec} 2\theta - \cot 2\theta = \tan\theta$

(c) $\operatorname{cosec} 2\theta + \cot 2\theta = \cot\theta$ (d) $1 + \tan\frac{\theta}{2}\tan\frac{\theta}{4} = \sec\frac{\theta}{2}$

(e) $\dfrac{\sin\theta + \sin 2\theta}{1 + \cos\theta + \cos 2\theta} = \tan\theta$ (f) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(g) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ (h) $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

(i) $\cos 4\theta = 1 - 8\sin^2\theta + 8\sin^4\theta$

12.5 Reduction of a cos θ + b sin θ to the form R cos (θ ± α) or R sin (θ ± α), R > 0, α acute

Consider $3\cos\theta + 4\sin\theta$

It can be reduced to the form $R\cos(\theta - \alpha)$

$$\begin{aligned} 3\cos\theta + 4\sin\theta &\equiv R\cos(\theta - \alpha) \\ &\equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha \\ &\equiv R\cos\alpha\cos\theta + R\sin\alpha\sin\theta \end{aligned}$$

Equating coefficients of $\cos\theta$ and $\sin\theta$, we have:

$R\cos\alpha = 3$ (1)

$R\sin\alpha = 4$ (2)

$R^2(\cos^2\alpha + \sin^2\alpha) = 25$

$R^2 = 25$

$R = 5$ $(R > 0)$

$\tan\alpha = \frac{4}{3}$ (2) ÷ (1)

$\alpha = 53.1°$ (α acute)

So, $3\cos\theta + 4\sin\theta \equiv 5\cos(\theta - 53.1°)$

It can also be reduced to the form $R\sin(\theta + \alpha)$.

$3\cos\theta + 4\sin\theta \equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

$4\sin\theta + 3\cos\theta \equiv R\cos\alpha\sin\theta + R\sin\alpha\cos\theta$

Equating coefficients of $\sin\theta$ and $\cos\theta$

we have $R\cos\alpha = 4$ (1)

$R\sin\alpha = 3$ (2)

$R^2 = 25$

$R = 5$ (for $R > 0$)

$\tan\alpha = \frac{3}{4}$ (2) ÷ (1)

$\alpha = 36.9°$ (α acute)

$4\sin\theta + 3\cos\theta \equiv 5\sin(\theta + 36.9°)$

Note: $3\cos\theta - 4\sin\theta$ can be reduced to the form $R\cos(\theta + \alpha)$ and $3\sin\theta - 4\cos\theta$ can be reduced to the
$R\sin(\theta - \alpha)$.

12.5.2 Solutions of equations of the type acos θ + bsin θ = c

We first reduce $a\cos\theta + b\sin\theta$ to the appropriate form.

For example, we solve $2\cos\theta + 3\sin\theta = 1$ by reducing $2\cos\theta + 3\sin\theta$ to the form $R\cos(\theta - \alpha)$ or Rsin (θ +

To solve $3\sin\theta - 2\cos\theta = 1$, we reduce $3\sin\theta - 2\cos\theta$ to the form $R\sin(\theta - \alpha)$ etc.

Example 18

Solve the equation $3\cos x + 2\sin x \equiv 1$ for $0° \leq x \leq 360°$.

Solution

We reduce $3\cos x + 2\sin x$ to the form $R\cos(\theta - \alpha)$

$3\cos x + 2\sin x \equiv R\cos x\cos\alpha + R\sin x\sin\alpha$

$\equiv R\cos\alpha\cos x + R\sin\alpha\sin x$

Equating coefficients of cos x and sin x, we have:

$R\cos\alpha = 3$ (1)

$R\sin\alpha = 2$ (2)

$R^2 = 3^2 + 2^2$

$= 13$

$R = \sqrt{13}$

$\tan\alpha = \frac{2}{3}$ (2) ÷ (1)

$\alpha = 33.69°$

$3\cos x + 2\sin x = 1$

$\sqrt{13}\cos(x - 33.69°) = 1$

$\cos(x - 33.69°) = \frac{1}{\sqrt{13}}$

Key angle = 73.90°

$x - 33.69° =$ Q1 73.90°

Q4 286.10°

$x = 107.6°$ or $319.8°$

12.5.3 Maximum value and minimum value of acos x + bsin x

We know that sin x has a maximum value of 1 when $x = 90° + 360n°$ where n is an integer. Also, sin x has a minimum value of –1 when $x = 270° + 360n°$.

Similarly, cos x has a maximum value of 1 when $x = 360n°$ and a minimum value of –1 when $x = 180° + 360n°$.

To find the maximum or minimum of acos x + bsin x, we reduce it to one of the appropriate forms $R\cos(x \pm \alpha)$ or $R\sin(x \pm \alpha)$ as illustrated in the following example:

Example 19

Reduce the expression 3cos x + 4sin x to the form Rcos (x – α). Hence, obtain the maximum and the minimum values of 3cos x + 4sin x and a value of x in the range 0° ≤ x ≤ 360° for which (a) the maximum value, (b) the minimum value occurs.

Solution

$$3\cos x + 4\sin x \equiv R\cos(x - \alpha)$$
$$\equiv R\cos x \cos\alpha + R\sin x \sin\alpha$$
$$\equiv R\cos\alpha \cos x + R\sin\alpha \sin x$$

$3 = R\cos\alpha$ (1)

$4 = R\sin\alpha$ (2)

$R^2 = 9 + 16$

$= 25$

$R = 5$

$\tan\alpha = \frac{4}{3}$ (2) ÷ (1)

$\alpha = 53.1°$

So, $3\cos x + 4\sin x \equiv 5\cos(x - 53.1°)$

Maximum value = 5 when $x - 53.1° = 360n°$

Minimum value = –5 when $x - 53.1° = 180° + 360n°$

A value of x for which the maximum occurs is 53.1° when n = 0 and a value of x for which the minimum occurs is 233.1°.

Note: We can also reduce the expression 3cos x + 4sin x to the form Rsin (x + α) to obtain the maximum value and the minimum value.

Exercise 12 E

1. Solve the following equations for $0° \leqslant \theta \leqslant 360°$.

(a) $\cos\theta + \sqrt{3}\sin\theta = 1$

(b) $12\cos\theta - 5\sin\theta = 6.5$

(c) $3\sin\theta + 2\cos\theta = 2$

(d) $5\sin\theta - 3\cos\theta = 1$

(e) $2\cos 2\theta + 3\sin 2\theta = 3$

(f) $8\sin 2\theta - 6\cos 2\theta = 3$

2. Find the maximum value and the minimum value of each of the following expressions and the values of x for which these values occur in the range $0° \leqslant x \leqslant 360°$.

(a) $2\cos x + 3\sin x$

(b) $4\cos x - 3\sin x$

(c) $\sqrt{3}\sin x - \cos x$

(d) $3\sin x + 2\cos x$

(e) $4 + 3\cos x - 2\sin x$

(f) $2\cos x + \sin x - 1.$

Miscellaneous Exercise 12

1. Prove the identity $\sec^2 x + \operatorname{cosec}^2 x \equiv \sec^2 x \operatorname{cosec}^2 x$.

2. (a) Find all the angles between 0° and 360° which satisfy the equation $4\sin x + 6\cos x = 5$.

(b) Given that A, B and C are the angles of a triangle, show that $\tan C = \dfrac{\tan A + \tan B}{\tan A \tan B - 1}$.

Given further that $\tan B = 3$ and that $\tan C = 2\tan A$, find, without using tables or a calculator:

(i) the angle A

(ii) $\tan(B - C)$

3. (a) Find all the angles between 0° and 360° which satisfy the equation $4\cos x - 3\sin x = 1$.

(b) Given that $\cos\theta = c$ and that θ is acute, express in terms of c:

(i) $\operatorname{cosec}\theta$ (ii) $\cot\theta$ (iii) $\sin 2\theta$ (iv) $\tan(\theta + 45°)$

4. Prove the identity $\dfrac{1 + \sin x}{\cos x} + \dfrac{\cos x}{1 + \sin x} \equiv 2\sec x$

5. (a) Find all the angles between 0° and 360° which satisfy the equation $4\cos\theta + 2\sin\theta = 1$.

(b) Given that $\dfrac{\cos(A + B)}{\cos(A - B)} = \dfrac{3}{4}$, prove that $\cos A\cos B = 7\sin A\sin B$ and deduce a relationship between $\tan A$ and $\tan B$.

Given further that $A + B = 45°$, calculate the value of $\tan A + \tan B$.

6. Find all the angles between 0° and 180° which satisfy the equation:

(a) $\cos\dfrac{2}{3}x = \dfrac{2}{3}$

(b) $3\cot y - 4\cos y = 0$

(c) $3\sec^2 z = 7 + 4\tan z$

7. (a) Given that $\sin\alpha = \frac{4}{5}$, where $90° < \alpha < 180°$ and that $\cos\beta = -\frac{5}{13}$, where $180° < \beta < 270°$, calculate, without using tables or calculator:

   (i) $\sin(\alpha - \beta)$ (ii) $\cos 2\alpha$ (iii) $\sin 2\beta$

   (b) Given that $3\cos\theta + \sin\theta\ R\cos(\theta - \alpha)$, where R is positive and α is acute, evaluate R and α. Hence solve the equation $3\cos\theta + \sin\theta = 2$ for $0° < \theta < 360°$. [C]

8. Prove the identity $\sec x \operatorname{cosec} x - \cot x \equiv \tan x$ [C]

9. (a) Find all the angles between 0° and 360° which satisfy the equation $24\cos x - 7\sin x = 12.5$.

   (b) Given that tan = t and that θ is acute, express in terms of t:

   (i) $\sin\theta$ (ii) $\sin 2\theta$ (iii) $\cot(\theta + 45°)$ [C]

10. Prove the identity $\frac{(\sin x + \cos x)^2}{\sin x \cos x} \equiv 2 + \sec x \operatorname{cosec} x$. [C]

11. Find all the angles between 0° and 360° inclusive which satisfy the equation:

    (a) $\operatorname{cosec}(2x - 30°) = 2$

    (b) $2\cos y = 3\cos^2 y$

    (c) $2\cot^2 z = 4 - \operatorname{cosec} z$ [C]

12. (a) Find all the angles between 0° and 360° which satisfy the equation $8\sin x + 15\cos x = 10$.

    (b) Two acute angles α and β are such that $\tan\alpha = \frac{4}{3}$ and $\tan(\alpha + \beta) = -1$. Without evaluating α or β

    (i) show that $\tan\beta = 7$

    (ii) evaluate $\sin\alpha$ and $\sin\beta$

    (iii) evaluate $\sin^2 2\alpha + \sin^2 2\beta$. [C]

13. Prove the identity $\cos x \operatorname{cosec} x + \sin x \sec x \equiv \operatorname{cosec} x \sec x$. [C]

14. (a) Express $12\cos x + 9\sin x$ in the form $R\cos(x - \alpha)$ where R is positive and αis acute. Hence, find:

    (i) the acute angle x for which $12\cos x + 9\sin x = 11$

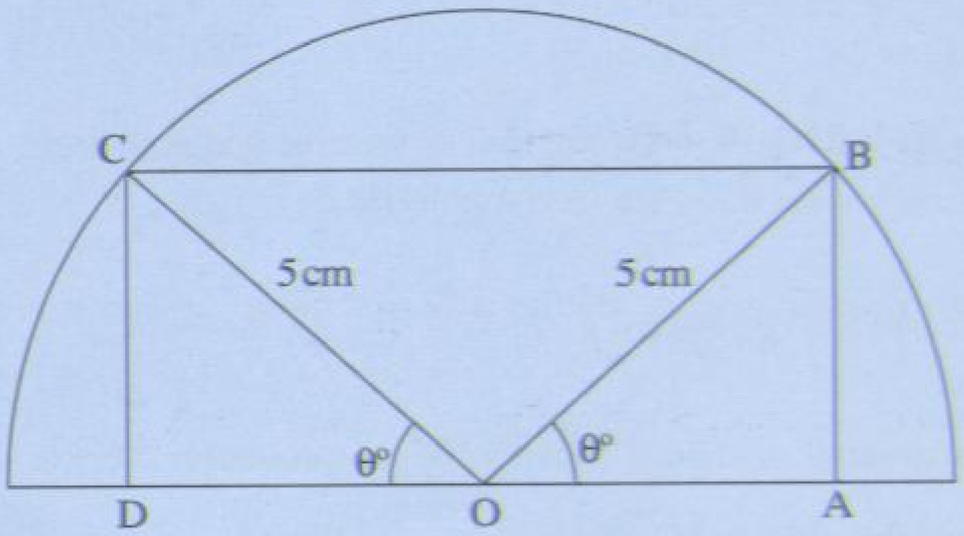
    (ii) the maximum value of $12\cos x + 9\sin x$.

    (b) By expressing $\cos 3x = \cos(2x + x)$, show that $\cos 3x = 4\cos^3 x - 3\cos x$. Hence, find all the angles between 0° and 180° which satisfy the equation $\cos 3x + 2\cos x = 0$. [C]

15. (a) Given that $\sin A = \frac{1}{3}$, find without using tables or calculator, the value of

    (i) $\cos 2A$ (ii) $\cos 4A$

(b) The diagram shows a rectangle ABCD inside a semi–circle centre O and radius 5 cm, such that $\angle BOA = \angle COD = \theta°$.



(i) Show that the perimeter, P, of the rectangle is given by $P = 20\cos\theta + 10\sin\theta$.

(ii) Express P in the form $r\cos(\theta - \alpha)$ and hence find the value of for which $P = 16$.

(iii) Find the value of k for which the area of the rectangle is $k\sin 2\theta$ cm^2.

CHAPTER 13

> DIFFERENTIATION 2

13.1 Derivatives of e^x and $e^{f(x)}$

We recall that $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$...... and $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$......

The function $f:x \mapsto e^x$ has the important property that $f':x \mapsto e^x$, i.e. if $y = e^x$, $\frac{dy}{dx} = e^x$.

Consider $y = e^{f(x)}$

Writing $f(x) = u$ and using $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$,

$$\begin{aligned} \text{we have } \frac{dy}{dx} &= e^u \times f'(x) \\ &= e^{f(x)} \times f'(x) \end{aligned}$$

So,

$$\boxed{\begin{aligned} \frac{d}{dx}(e^x) &= e^x \\ \frac{d}{dx}(e^{f(x)}) &= f'(x)\, e^{f(x)} \end{aligned}}$$

$$\begin{aligned} \text{Hence, } \frac{d}{dx}\left(e^{x^2}\right) &= 2xe^{x^2} \\ \frac{d}{dx}\left(e^{\sqrt{x+1}}\right) &= \frac{1}{2}(x+1)^{-\frac{1}{2}} e^{\sqrt{x+1}} \\ &= \frac{1}{2\sqrt{x+1}} e^{\sqrt{x+1}} \\ \frac{d}{dx}\left(e^{\frac{1}{x+2}}\right) &= -\frac{1}{(x+2)^2} e^{\frac{1}{x+2}} \end{aligned}$$

13.1.2 Derivative of ln x and ln f(x)

$$\begin{aligned} \text{If } \quad y &= \ln x \\ x &= e^y \\ \text{So, } \frac{dx}{dy} &= e^y \\ &= x \\ \frac{dy}{dx} &= \frac{1}{x} \end{aligned}$$

$$\text{Hence, } \frac{d}{dx}(\ln x) = \frac{1}{x}$$

If y = ln f(x)

Writing f(x) = u, we have y = ln u

Using $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

We have $\frac{dy}{dx} = \frac{1}{u} \times f'(x)$

$$= \frac{f'(x)}{f(x)}$$

So,

$$\boxed{\begin{aligned} \frac{d}{dx}(\ln x) &= \frac{1}{x} \\ \frac{d}{dx}(\ln f(x)) &= \frac{f'(x)}{f(x)} \end{aligned}}$$

If $y = \ln(x^2 + 2), \frac{dy}{dx} = \frac{2x}{x^2 + 2}$

$y = \ln(x^2 + 3x - 1), \frac{dy}{dx} = \frac{2x + 3}{x^2 + 3x - 1}$

The formulae ln ab = ln a + ln b, $\ln \frac{a}{b}$ = ln a – ln b and ln a^b = bln a are useful to differentiate certain types of functions as illustrated in the following examples:

Example 1

Find $\frac{dy}{dx}$ given:

(a) $y = \ln \frac{2}{x}$ *(b)* $y = \ln \sqrt{2x + 1}$ *(c)* $\ln \left[\frac{(x+1)\sqrt{2x-3}}{x-2}\right]$

Solution

(a) $y = \ln \frac{2}{x}$

$= \ln 2 - \ln x$

$\frac{dy}{dx} = 0 - \frac{1}{x}$

$= -\frac{1}{x}$

(b) $y = \ln \sqrt{2x + 1}$

$= \frac{1}{2} \ln(2x + 1)$

$\frac{dy}{dx} = \frac{1}{2} \times \frac{2}{2x + 1}$

$= \frac{1}{2x + 1}$

(c) $y = \ln\left[\frac{(x+1)\sqrt{2x-3}}{x-2}\right]$

$$= \ln(x+1) + \frac{1}{2}\ln(2x-3) - \ln(x-2)$$

$$\frac{dy}{dx} = \frac{1}{x+1} + \frac{1}{2} \times \frac{2}{2x-3} - \frac{1}{x-2}$$

$$= \frac{1}{x+1} + \frac{1}{2x-3} - \frac{1}{x-2}$$

$$= \frac{(2x-3)(x-2) + (x+1)(x-2) - (x+1)(2x-3)}{(x+1)(2x-3)(x-2)}$$

$$= \frac{x^2 - 7x + 7}{(x+1)(2x-3)(x-2)}$$

13.1.3 Derivatives of sin x, cos x, tan x, sin f(x), cos f(x), tan f(x)

If angle x is in radians, then we will accept without proof that $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$ and $\frac{d}{dx}(\tan x) = \sec^2 x$.

Consider $y = \sin f(x)$

Writing $u = f(x)$, $y = \sin u$

Using $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$, we have

$$\frac{dy}{dx} = \cos u \times f'(x)$$

$$= \cos x \times f'(x)$$

Similarly, if $y = \cos f(x)$, $\frac{dy}{dx} = -\sin f(x) \times f'(x)$ and if $y = \tan f(x)$, $\frac{dy}{dx} = \sec^2 f(x) \times f'(x)$.

Hence,

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sin f(x)) = f'(x)\cos f(x)$$

$$\frac{d}{dx}(\cos f(x)) = -f'(x)\sin f(x)$$

$$\frac{d}{dx}(\tan f(x)) = -f'(x)\sec^2 f(x)$$

Example 2

Find $\frac{dy}{dx}$ *given:* (a) $y = \sin 2x$ (b) $y = \cos\sqrt{x}$ (c) $y = \tan\left(\frac{1}{4}x + \pi\right)$

Solution

(a)
$$y = \sin 2x$$
$$\frac{dy}{dx} = \cos 2x \times 2$$
$$= 2\cos 2x$$

(b)
$$y = \cos\sqrt{x}$$
$$\frac{dy}{dx} = -\sin\sqrt{x} \times \frac{1}{2}x^{-\frac{1}{2}}$$
$$= -\frac{\sin\sqrt{x}}{2\sqrt{x}}$$

(c)
$$y = \tan\left(\frac{1}{4}x + \pi\right)$$
$$\frac{dy}{dx} = \sec^2\left(\frac{1}{4}x + \pi\right) \times \frac{1}{4}$$
$$= \frac{1}{4}\sec^2\left(\frac{1}{4}x + \pi\right)$$

Example 3

Find $\frac{dy}{dx}$ *given:* (a) $y = e^{\sin 3x}$ (b) $y = \ln\cos 2x$ (c) $y = \ln\tan\sqrt{x}$

Solution

(a)
$$y = e^{\sin 3x}$$
$$\frac{dy}{dx} = e^{\sin 3x} \times \cos 3x \times 3$$
$$= 3\cos 3x\, e^{\sin 3x}$$

(b)
$$y = \ln\cos 2x$$
$$\frac{dy}{dx} = \frac{-\sin 2x \times 2}{\cos 2x}$$
$$= -2\tan 2x$$

(c)
$$y = \ln\tan\sqrt{x}$$
$$\frac{dy}{dx} = \frac{\sec^2\sqrt{x} \times \frac{1}{2\sqrt{x}}}{\tan\sqrt{x}}$$
$$= \frac{1}{2\sqrt{x}\cos^2\sqrt{x}} \times \frac{\cos\sqrt{x}}{\sin\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}\sin\sqrt{x}\cos\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}\sin 2\sqrt{x}}$$

$$= \frac{\text{cosec } 2\sqrt{x}}{\sqrt{x}}$$

Example 4

Find $\frac{dy}{dx}$ *given:* *(a)* $y = 2\sin^3 x$ *(b)* $y = 3\cos^4 3x$ *(c)* $y = 2\tan^3\left(4x + \frac{\pi}{4}\right)$

Solution

(a) We write $y = 2\sin^3 x$ as $y = 2(\sin x)^3$

$$\frac{dy}{dx} = 2 \times 3(\sin x)^2 \times \cos x \left[\text{using } \frac{d}{dx}(f(x))^n = n(f(x))^{n-1} \times f'(x)\right]$$

$$= 6\sin^2 x \cos x$$

(b) $y = 3\cos^4 3x$

$$= 3(\cos 3x)^4$$

$$\frac{dy}{dx} = 4 \times 3(\cos 3x)^3 \times -\sin 3x \times 3$$

$$= -36\sin 3x \cos^3 3x$$

(c) $y = 2\tan^3\left(4x + \frac{\pi}{4}\right)$

$$= 2\left[\tan\left(4x + \frac{\pi}{4}\right)\right]^3$$

$$\frac{dy}{dx} = 3 \times 2\left[\tan\left(4x + \frac{\pi}{4}\right)\right]^2 \times \sec^2\left(4x + \frac{\pi}{4}\right) \times 4$$

$$= 24\sec^2\left(4x + \frac{\pi}{4}\right)\tan^2\left(4x + \frac{\pi}{4}\right)$$

Exercise 13 A

1. Find $\frac{dy}{dx}$ given:

(a) $y = e^{2x}$

(b) $y = e^{-3x}$

(c) $y = e^{\sqrt{x+2}}$

(d) $y = e^{\sin x}$

(e) $y = e^{\frac{1}{x}}$

(f) $y = e^{\cos x}$

(g) $y = 3e^{5x}$

(h) $y = 4e^{-\frac{1}{2}x}$

(i) $y = 2e^{\tan x}$

(j) $y = e^{(2x+1)^2}$

(k) $y = 5e^{x(x-1)}$

(l) $y = 2e^{\sin\sqrt{x}}$

(m) $y = e^{\sin^2 x}$

(n) $y = e^{\cos^3 2x}$

2. Find $\frac{dy}{dx}$ given:

(a) $y = \ln \frac{1}{x}$
(b) $y = \ln \frac{x}{3}$
(c) $y = \ln (-x)$
(d) $y = \ln \sqrt{x}$
(e) $y = 3\ln (2x + 5)$
(f) $y = 4\ln \cos 3x$
(g) $y = 3\ln \left(e^{2x} + 1\right)$
(h) $y = \ln \tan x$
(i) $y = \ln \frac{2x+1}{x-3}$
(j) $y = \ln \frac{2x+1}{\sqrt{x^2+3}}$
(k) $y = 3\ln \frac{2}{x}$
(l) $y = \ln \left[\frac{(2x+1)^3}{\sqrt{4x-3}}\right]$
(m) $y = \ln (\sin x + \cos x)$
(n) $y = \ln \frac{e^x - 1}{e^x + 1}$
(o) $y = \ln \frac{\sin x + \cos x}{\sin x - \cos x}$

3. Find $\frac{dy}{dx}$ given:

(a) $y = 3\sin 2x$
(b) $y = 4\cos 3x$
(c) $y = 5\tan \left(\frac{1}{3}x\right)$
(d) $y = 8\sin \left(\frac{1}{4}x + \frac{\pi}{6}\right)$
(e) $y = 12\cos \left(\frac{1}{3}x - \frac{\pi}{4}\right)$
(f) $y = 6\tan (2x - \pi)$
(g) $y = -3\sin^4 x$
(h) $y = 4\cos^2 5x$
(i) $y = 3\sqrt{\tan 4x}$
(j) $y = (1 + 2\cos 3x)^4$
(k) $y = 3\tan^4 \left(2x - \frac{\pi}{6}\right)$
(l) $y = 5\sqrt{\sin\left(3x + \frac{\pi}{6}\right)}$

4. Find the equation of the tangent to the curve $y = e^{3x}$ at the point on the curve with x-coordinate 1.

5. Find the gradient of the curve $y = e^{2x} - 4$ at its point of intersection with the x-axis.

6. A particle moves in a straight line so that, at time t seconds after leaving a fixed point O, its displacement s m is given by $s = 10 - 10e^{-t} - \frac{1}{5}t$.
Calculate:
(a) the initial velocity of the particle
(b) the value of s when the particle is instantaneously at rest
(c) the acceleration of the particle at this instant.

7. Find the gradient of the tangent to the curve $y = \ln (4 - 3x)$ at the point on the curve with x-coordinate 1

8. Find the equation of the tangent to the curve $y = \ln (6 + 5x)$ at the point on the curve with x-coordinate -1

9. Find the coordinates of the point on the curve $y = \ln (6 - 3x)$ at which the normal is parallel to $2y = 3x - 5$

10. The equation of a curve is $y = \ln (x^2 + 4x)$ where $x > 0$. Find the x-coordinate of the point on the curve which the tangent to the curve is parallel to the line $3y = 2x + 1$.

13.2.1 Derivative of a product

If the product is simple, we multiply first. For example:

If $y = (x^2 + 2)(3x - 4)$

$y = 3x^3 - 4x^2 + 6x - 8$

$\frac{dy}{dx} = 9x^2 - 8x + 6$

However, this method is not useful if we have to find $\frac{dy}{dx}$ for $y = (3x - 2)^4(5x + 1)^3$ say.

In this case, put $(3x - 2)^4 = u$ and $(5x + 1)^3 = v$

So, $y = uv$ (1)

As x increases by a small amount Δx, u increases by a small amount Δu, v by a small amount Δv and y by a small amount Δy.

$$\begin{aligned} \text{So, } y + \Delta y &= (u + \Delta u)(v + \Delta v) \\ &= uv + u\Delta v + v\Delta v + \Delta u\Delta v \quad (2) \\ \Delta y &= u\Delta v + v\Delta u + \Delta u\Delta v \ (2) - (1) \\ \frac{\Delta y}{\Delta x} &= u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x} \end{aligned}$$

As $\Delta x \to 0, \Delta u \to 0, \Delta v \to 0, \Delta y \to 0, \frac{\Delta u}{\Delta x} \to \frac{du}{dx}, \frac{\Delta v}{\Delta x} \to \frac{dv}{dx}$ and $\frac{\Delta y}{\Delta x} \to \frac{dy}{dx}$.

We therefore have:

if $y = uv$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Applying the formula to $y = (3x - 2)^4(5x + 1)^3$

$y = uv$ where $u = (3x - 2)^4$ and $v = (5x + 1)^3$

$\frac{du}{dx} = 4(3x - 2)^3 \times 3 = 12(3x - 2)^3$

$\frac{dv}{dx} = 3(5x + 1)^2 \times 5 = 15(5x + 1)^2$

$$\begin{aligned} \frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= (3x - 2)^4 \times 15(5x + 1)^2 + (5x + 1)^3 \times 12(3x - 2)^3 \\ &= 15(3x - 2)^4(5x + 1)^2 + 12(3x - 2)^3(5x + 1)^3 \\ &= 3(3x - 2)^3(5x + 1)^2\,[5(3x - 2) + 4(5x + 1)] \\ &= 3(3x - 2)^3(5x + 1)^2\,(35x - 6) \end{aligned}$$

Example 5

Find $\frac{dy}{dx}$ *given* $y = (2x + 3)\sqrt{5x + 1}$

Solution

Put $y = uv$ where $u = 2x + 3$, $v = \sqrt{5x + 1} = (5x + 1)^{\frac{1}{2}}$

$$\frac{du}{dx} = 2,\ \frac{dv}{dx} = \frac{1}{2}(5x + 1)^{-\frac{1}{2}} \times 5$$

$$= \frac{5}{2(5x + 1)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= \frac{(2x + 3) \times 5}{2(5x + 1)^{\frac{1}{2}}} + (5x + 1)^{\frac{1}{2}} \times 2$$

$$= \frac{10x + 15 + 4(5x + 1)}{2(5x + 1)^{\frac{1}{2}}}$$

$$= \frac{30x + 19}{2\sqrt{5x + 1}}$$

Example 6

Find $\frac{dy}{dx}$ *given:* *(a)* $y = x^2e^{3x}$ *(b)* $y = x^2\ln(2x - 1)$ *(c)* $y = x^2\sin 3x$

Solution

(a) $y = x^2e^{3x}$

$$\frac{dy}{dx} = x^2 \times e^{3x} \times 3 + e^{3x} \times 2x$$

$$= 3x^2e^{3x} + 2xe^{3x}$$

$$= xe^{3x}(3x + 2)$$

(b) $y = x^2\ln(2x - 1)$

$$\frac{dy}{dx} = x^2 \times \frac{2}{2x - 1} + \ln(2x - 1) \times 2x$$

$$= \frac{2x^2}{2x - 1} + 2x\ln(2x - 1)$$

(c) $y = x^2\sin 3x$

$$\frac{dy}{dx} = x^2 \times \cos 3x \times 3 + 2x\sin 3x$$

$$= 3x^2\cos 3x + 2x\sin 3x$$

$$= x(3x\cos 3x + 2\sin 3x)$$

13.2.2 Derivative of a quotient

Whenever it is possible, we perform the division first.

Thus, if $y = \frac{2x^3 + 3x^2 + 4}{3x^2}$

we write $y = \frac{2}{3}x + 1 + \frac{4}{3}x^{-1}$

$$\frac{dy}{dx} = \frac{2}{3} - \frac{4}{3}x^{-2}$$

However, it is not always possible to perform the division. Thus, if $y = \frac{2x + 3}{\sqrt{5x - 4}}$, we may write $y = (2x + 3)(5x - 4)^{-\frac{1}{2}}$ and use the product rule.

However, it is better to write $\qquad y = \frac{u}{v} \qquad (1)$

With the usual notation $\qquad y + \Delta y = \frac{u + \Delta u}{v + \Delta v} \qquad (2)$

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} \qquad (2) - (1)$$

$$= \frac{uv + v\Delta u - uv - u\Delta v}{v(v + \Delta v)}$$

$$= \frac{v\Delta u - u\Delta v}{v(v + \Delta v)}$$

$$\frac{\Delta y}{\Delta x} = \frac{v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}}{v(v + \Delta v)}$$

As $\Delta x \to 0$, $\Delta u \to 0$, $\Delta v \to 0$, $\Delta y \to 0$, etc.

We have:

$$\boxed{\begin{array}{l} \text{If } y = \frac{u}{v} \\ \\ \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \end{array}}$$

This formula is better remembered as :

Gradient of a quotient = $\dfrac{\textit{Dominator} \times \textit{gradient of numerator} - \textit{numerator} \times \textit{gradient of dominator}}{\textit{square of dominator}}$

If $y = \frac{2x + 3}{5x - 4}$

$$\frac{dy}{dx} = \frac{(5x - 4) \times 2 - (2x + 3) \times 5}{(5x - 4)^2}$$

$$= \frac{-23}{(5x - 4)^2}$$

Example 7

Find $\frac{dy}{dx}$ *given* $y = \frac{4x-1}{(2x+3)^2}$

Solution

$$y = \frac{4x-1}{(2x+3)^2}$$

$$\frac{dy}{dx} = \frac{(2x+3)^2 \times 4 - (4x-1) \times 2(2x+3) \times 2}{(2x+3)^4}$$

$$= \frac{4(2x+3)^2 - 4(2x+3)(4x-1)}{(2x+3)^4}$$

$$= \frac{4(2x+3)[(2x+3)-(4x-1)]}{(2x+3)^4}$$

$$= \frac{4(2x+3)(4-2x)}{(2x+3)^2}$$

$$= \frac{8(2-x)}{(2x+3)^3}$$

Example 8

Find $\frac{dy}{dx}$ *given:* (a) $y = \frac{\sin 2x}{x}$ (b) $y = \frac{x}{e^x - 1}$ (c) $y = \frac{\ln x}{x}$

Solution

(a)
$$y = \frac{\sin 2x}{x}$$

$$\frac{dy}{dx} = \frac{x \times 2\cos 2x - \sin 2x}{x^2}$$

$$= \frac{2x\cos 2x - \sin 2x}{x^2}$$

(b)
$$y = \frac{x}{e^x - 1}$$

$$\frac{dy}{dx} = \frac{(e^x - 1) \times 1 - xe^x}{(e^x - 1)^2}$$

$$= \frac{e^x - xe^x - 1}{(e^x - 1)^2}$$

(c)
$$y = \frac{\ln x}{x}$$

$$\frac{dy}{dx} = \frac{x \times \frac{1}{x} - \ln x \times 1}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

Exercise 13 B

1. Find $\frac{dy}{dx}$ given:

(a) $y = (2x + 1)^2(x - 3)$
(b) $y = (2x + 3)^3(3x - 1)^4$
(c) $y = (3x - 1)^2(2 - 3x)^3$
(d) $y = (3 - 2x)^5(4 - 3x)^4$
(e) $y = (2x^2 + 1)(3x - 2)^2$
(f) $y = (2x + 3)\sqrt{4x + 1}$
(g) $y = (3x^2 - 4)\sqrt{6x + 1}$
(h) $y = (3x - 2)\sqrt{5 + 7x}$

2. Find $\frac{dy}{dx}$ given:

(a) $y = e^{x\sin x}$
(b) $y = 3e^{x\cos 4x}$
(c) $y = e^{x\cos x}$
(d) $y = e^{2x\sin x}$
(e) $y = e^{x\tan x}$
(f) $y = 3e^{4x\sin 3x}$
(g) $y = 4e^{-3x}\cos\frac{1}{3}x$
(h) $y = 3e^{-4x}\tan 3x$
(i) $y = x\ln x$
(j) $y = x^2e^{3x}$
(k) $y = xe^{\sqrt{x}}$
(l) $y = e^{x\ln x}$
(m) $y = x\sin 3x$
(n) $y = x^2\cos x$
(o) $y = x^2\tan\frac{1}{2}x$

3. Find $\frac{dy}{dx}$ given:

(a) $y = \frac{2x - 1}{3x - 4}$
(b) $y = \frac{4 - 3x}{5 + 2x}$
(c) $y = \frac{\sqrt{3x + 1}}{2x - 1}$
(d) $y = \frac{(2x + 3)^2}{3x - 5}$
(e) $y = \frac{5 + 2x}{(2 - 3x)^3}$
(f) $y = \frac{3x}{(2x - 1)^2}$

4. Find $\frac{dy}{dx}$ given:

(a) $y = \frac{\sin x}{3 + \cos x}$
(b) $y = \frac{e^x}{x}$
(c) $y = \frac{\sin x}{e^x}$
(d) $y = \frac{e^x}{\sin x}$
(e) $y = \frac{\ln x}{\sin x}$
(f) $y = \frac{\sin x + \cos x}{\sin x - \cos x}$
(g) $y = \frac{1 - e^x}{1 + e^x}$
(h) $y = \frac{e^x}{e^x + 1}$
(i) $y = \cot x$

5. Find the coordinates of the stationary point of the curve $y = x\ln x$ and determine its nature.

6. Find $\frac{dy}{dx}$ given $y = e^x(\sin x - \cos x)$.

7. Find the x-coordinates of the stationary points of the curve $y = x^2e^{3x}$ and determine their nature.

8. Find the x-coordinate of the stationary point of the curve $y = e^x\sin x$ for $0 \leq x \leq \pi$ and determine its nature.

9. Find the x-coordinate of the stationary point of the curve $y = e^{2x}\cos x$ for $0 \leq x \leq \pi$ and determine its nature.

10. Find the x-coordinate of the stationary point of the curve $y = \frac{\ln x}{x}$ and determine its nature.

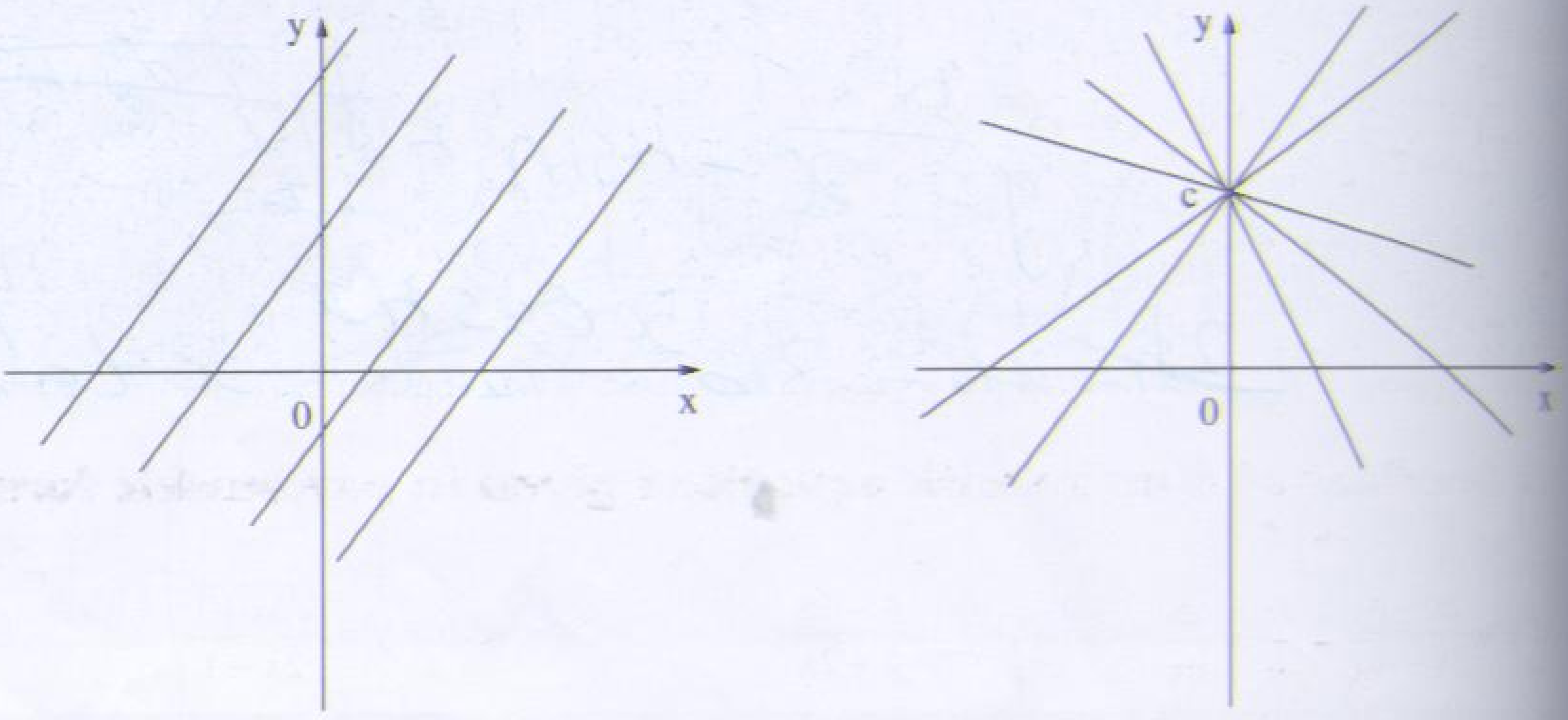
11. Given $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$, show that $\frac{d}{dx}(\tan x) = \sec^2 x$.

13.3.1 Parametric equations

If y = mx + c and both m and c are constants, the graph of y = mx + c is a straight line of gradient m through (0, c).

If m is a constant and c is a variable, y = mx + c represents a set of parallel straight lines, c is called a parameter (Figure 13.1 (a)).

If c is a constant and m is a variable, y = mx + c represents a set of straight lines through (0, c). In this case, m is the parameter (Figure 13.1 (b)).



(a) *(b)*

Figure 13.1

Usually, the equation of a curve is given in cartesian form : $y = x^2$, $y^2 = x$, $\frac{x^2}{9} + \frac{y^2}{16} = 1$ are all equations in cartesian form.

It is sometimes more useful to give the equation of a curve in parametric form, e.g. $x = at^2$, $y = 2at$; $x = t - 1$, $y = 3t^2 + 2$, etc.

From the parametric equations of a curve we can obtain the cartesian equation as illustrated in the following examples.

Example 9

The parametric equations of a curve are x = 2t – 1, y = t² + 5. Find its cartesian equation.

Solution

$x = 2t - 1 \Rightarrow t = \frac{x+1}{2}$ (1)

$y = t^2 + 5 \Rightarrow t^2 = y - 5$ (2)

From (1) & (2)

$$\begin{aligned} y - 5 &= \left(\frac{x+1}{2}\right)^2 \\ &= \frac{(x+1)^2}{4} \\ y &= \frac{(x+1)^2}{4} + 5 \\ &= \frac{(x+1)^2 + 20}{4} \\ 4y &= x^2 + 2x + 21 \end{aligned}$$

Example 10

Find the cartesian equation of a curve with parametric equations $x = 2\sin\theta$, $y = 3\cos 2\theta + 1$.

Solution

Note that $-2 \leq x \leq 2$ as $-1 \leq \sin\theta \leq 1$

$x = 2\sin\theta \Rightarrow \sin\theta = \frac{x}{2}$ (1)

$y = 3\cos 2\theta + 1 \Rightarrow \cos 2\theta = \frac{y-1}{3}$ (2)

As $\cos 2\theta = 1 - 2\sin^2\theta$

$$\frac{y-1}{3} = 1 - \frac{x^2}{2}$$

$$2y - 2 = 6 - 3x^2$$

$$2y = 8 - 3x^2, -2 \leq x \leq 2$$

13.3.2 Gradient of a curve with equations given in parametric form

Consider $x = 3t^2$, $y = 6t$

To find $\frac{dy}{dx}$, we use $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$$= \frac{dy}{dt} \div \frac{dx}{dt}$$

$$= \frac{6}{6t}$$

$$= \frac{1}{t}$$

If we need to find the gradient at (12, –12), we find the value of t at this point.

At (12, –12), $3t^2 = 12$ and $6t = -12$ which are both satisfied by $t = -2$.

Example 11

Find the gradient of the tangent to the curve with parametric equations $x = \frac{1}{t}$, $y = 2t^2$ at $\left(\frac{1}{p}, 2p^2\right)$.

Solution

$x = \frac{1}{t}$, $\frac{dx}{dt} = -\frac{1}{t^2}$

$y = 2t^2$, $\frac{dy}{dt} = 4t$

$\frac{dy}{dx} = 4t \div -\frac{1}{t^2}$

$= -4t^3$

At $\left(\frac{1}{p}, 2p^2\right)$, $\frac{1}{t} = \frac{1}{p}$ and $2t^2 = 2p^2$

$t = p$

$\frac{dy}{dx} = -4p^3$

13.3.3 Equations of tangent and normal to a curve with equations given in parametric form

The equations of the tangent and normal are obtained by finding the gradient of the tangent and the gradient of the normal as illustrated in the following example.

Example 12

Find the equations of the tangent and the normal to the curve with parametric equations $x = \frac{1}{t^2}$, $y = 2t$ *at the point (1, –2) on the curve.*

Solution

$x = \frac{1}{t^2} = t^{-2}, \frac{dx}{dt} = -2t^{-3}$

$y = 2t, \frac{dy}{dt} = 2$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$= 2 \div -\frac{2}{t^3}$$

$$= -t^3$$

At (1, –2), $\frac{1}{t^2} = 1$ and $2t = -2$, i.e. $t = -1$

$\frac{dy}{dx} = 1$

Equation of tangent at (1, –2) is $\frac{y+2}{x-1} = 1$

$$y = x - 3$$

Gradient of normal = –1

Equation of normal at (1, –2) is $\frac{y+2}{x-1} = -1$

$$y + x + 1 = 0$$

13.3.4 Gradients of implicit functions

Usually, the cartesian equation of a curve is given by y in terms of x, e.g. $y = 2x^2$, $y = (3x + 4)^2$, $y = \frac{2}{(x-1)^2}, x \neq 1$. y is said to be an explicit function of x.

Next, consider $x^3y + y^5 = 35$, $2x^2 - 4xy - y^3 = 15$. In each case, we may not write y completely in terms of x. In these two cases, we say that y is an implicit function of x.

Before differentiating implicit functions, it is important to note the following:

$$\frac{d}{dx}(y) = \frac{dy}{dx}$$

$$\frac{d}{dx}(y^2) = 2y\,\frac{dy}{dx}$$

$$\frac{d}{dx}(y^3) = 3y^2\,\frac{dy}{dx}$$

More generally, $\frac{d}{dx}(y^n) = ny^{n-1}\,\frac{dy}{dx}$

Also, from the product rule,

$$\frac{d}{dx}(xy) = x\,\frac{dy}{dx} + y$$

$$\frac{d}{dx}(xy^2) = x \times 2y\,\frac{dy}{dx} + y^2$$

$$= 2xy\,\frac{dy}{dx} + y^2$$

$$\frac{d}{dx}(xy^3) = x \times 3y^2\,\frac{dy}{dx} + y^3$$

$$= 3xy^2\,\frac{dy}{dx} + y^3.$$

Example 13

Find $\frac{dy}{dx}$ in terms of x and y given $x^2 + y^2 - 3x^2y + 4x = 9$.

Solution

$$\frac{d}{dx}(x^2 + y^2 - 3x^2y + 4x) = \frac{d}{dx}(9)$$

$$2x + 2y\,\frac{dy}{dx} - \left(3x^2\,\frac{dy}{dx} + 6xy\right) + 4 = 0$$

$$(2y - 3x^2)\,\frac{dy}{dx} = 6xy - 2x - 4$$

$$\frac{dy}{dx} = \frac{6xy - 2x - 4}{2y - 3x^2}$$

Example 14

Find the gradients of the curve at each of the points on the curve $x^3 + y^2 - xy + 2x = 5$ with x-coordinate 1.

Solution

$$\frac{d}{dx}(x^3 + y^2 - xy + 2x) = \frac{d}{dx}(5)$$

$$3x^2 + 2y\,\frac{dy}{dx} - \left(x\frac{dy}{dx} + y\right) + 2 = 0$$

$$(2y - x)\,\frac{dy}{dx} = y - 3x^2 - 2$$

$$\frac{dy}{dx} = \frac{y - 3x^2 - 2}{2y - x}$$

$$x = 1, \quad 1 + y^2 - y + 2 = 5$$

$$y^2 - y - 2 = 0$$

$$(y - 2)(y + 1) = 0$$

$$y = 2 \text{ or } -1$$

At $(1, -1)$, $\frac{dy}{dx} = \frac{-1 - 3 - 2}{-2 - 1}$

$$= \frac{-6}{-3}$$

$$= 2$$

At $(1, 2)$, $\frac{dy}{dx} = \frac{2 - 3 - 2}{4 - 1} = -1$

Exercise 13 C

1. Find the equation of each of the following curves with the given parametric equations:
   - (a) $(t + 3, 3t - 2)$
   - (b) $(t + 1, 2t^2 + 3)$
   - (c) $(2\cos\theta, 3\sin\theta)$
   - (d) $(a\cos\theta, a\sin\theta)$
   - (e) $(a\cos\theta, b\sin\theta)$
   - (f) $\left(t + 1, \frac{1}{2t - 3}\right)$
   - (g) $(2\tan\theta, 3\tan 2\theta)$
   - (h) $(\sin\theta, \sin 2\theta)$

2. Find the equations of the tangent and the normal to each of the following curves at the given point on the curve
   - (a) $x = 2t^2, y = 6t$ at $(8, 12)$
   - (b) $x = \frac{2}{t}, y = 3t$ at $(-2, -3)$
   - (c) $x = 2 + t^2, y = 7 - 3t$ at $(3, 10)$
   - (d) $x = 3t + \frac{1}{t}, y = 2t - \frac{3}{t}$ at $(4, -1)$
   - (e) $x = 3t^3 + 2t, y = t^2 + 3t + 2$ at $(5, 6)$
   - (f) $x = at^2, y = 2at$ at $(ap^2, 2ap)$
   - (g) $x = ct, y = \frac{c}{t}$ at $\left(cp, \frac{c}{p}\right)$
   - (h) $x = 3t^3, y = 2t^2$ at $(3p^3, 2p^2)$
   - (i) $x = 2 - t^2, y = 3 + 2t^3$ at $(2 - p^2, 3 + 2p^3)$
   - (j) $x = 2 + 3t, y = 5 - 4t^2$ at $(2 + 3p, 5 - 4p^2)$

3. Find the equation of the normal to the curve $x = 2t^2, y = 4t$ at the point $(18, 12)$. Find also the coordinates of the point at which this normal cuts the curve again.

4. Find the equation of the normal to the curve $x = 3t, y = \frac{2}{t}$ at the point $(3, 2)$. Find also the coordinates of the point at which this normal meets the curve again.

5. The parametric equations of a curve are $x = t + \frac{1}{t}, y = t - \frac{1}{t}$, where $t \neq 0$. At a point P on the curve, $t = 3$ and the tangent to the curve at P meets the x-axis at Q. The normal to the curve at P meets the x-axis at R. Calculate the area of the triangle PQR.

6. Find $\frac{dy}{dx}$ in terms of x and y, given:

(a) $x^2 - y^2 = 9$
(b) $x^2 + xy^2 - x^2y + y^2 = 12$
(c) $3x^3 - 2x^2y - 9xy^2 = 4$
(d) $(2x + y)^3 = 12$
(e) $\frac{1}{x} + \frac{1}{y} = \frac{1}{9}$
(f) $\frac{1}{x^2} - \frac{1}{y^2} = \frac{1}{4}$
(g) $\sqrt{x} - \sqrt{y} = 4$
(h) $y\sin x + x\sin y = 3$
(i) $xe^y - ye^x = 3$
(j) $y\ln x + x\ln y = 1$

7. Find the gradients of the curve $\frac{x^2}{9} - \frac{y^2}{18} = \frac{7}{9}$ at each of the points on the curve with y-coordinate 2.

8. Find the gradients of the curve $9x^2 - 6xy + 2y^2 = 5$ at each of the points with x-coordinate 1.

9. Find the x-coordinates of the stationary points of the curve $2x^3 - 2y^3 - 2x^2 - 8y - 2x = 11$.

10. Find the x-coordinates of the points on the curve $2y^3 - 4x^2 - 54y - 2x = 3$ at which the tangents are parallel to the y-axis.

11. Find the equation of the tangent to the curve $3x^2 + 2y^2 - 6xy = 26$ at the point $(2, -1)$ on the curve.

12. The variables x and y are connected by the equation $2x^2 - y^2 + 3x - 5y = 9$. Find the value of $\frac{dy}{dx}$ at the point $(1, -1)$.
Given that x is increasing at the rate of 0.3 unit per second at the point $(1, -1)$, find the corresponding rate of change of y.

Miscellaneous Exercise 13

1. (a) Differentiate with respect to x:
   (i) $x\sqrt{9 + 3x}$
   (ii) $\cos^3 2x$

   (b) Find the gradient of the curve $y = \frac{2x + 3}{x - 1}$ at the point where it crosses the x-axis. State whether or not this curve has a turning point and justify your answer. [C]

2. Differentiate with respect to x:
   (a) $x\sin 3x$ (b) $\tan^2 x$ [C]

3. (a) Find the stationary value of $(x + 5)\sqrt{7 - x}$

   (b) Differentiate $\frac{x^2}{2x - 1}$ with respect to x, and hence evaluate $\int_1^2 \frac{x(x - 1)}{(2x - 1)^2}\,dx$. [C]

4. The equations of a curve, in terms of a parameter θ, are given by $x = \sec\theta - \tan\theta$, $y = \sec\theta + 2\tan\theta$. Express $\sec\theta$ and $\tan\theta$ in terms of x and y. Hence, obtain, in simplified form, the cartesian equation of the curve. [C]

5. A curve is represented parametrically by the equations
$$x = \frac{4}{(2+t)^2},\quad y = \frac{12}{2+t}$$
(a) Find the equation of the chord joining the points P and Q with parameters –1 and 0 respectively.

(b) Show that $\frac{dy}{dx} = \frac{3}{2}(2+t)$. Hence, find the equation of the normal to the curve at the point on the curve where the normal is perpendicular to PQ. [C]

6. (a) Differentiate with respect to x:
(i) $\ln(3x+1)$ (ii) $x\cos 2x$

(b) Find the equation of the tangent to the curve $y = \frac{3+x}{1-2x}$ at the point where it crosses the line $y = -1$.

(c) Given that $x^2y + y^2 = 10$, find $\frac{dy}{dx}$ in terms of x and y. [C]

7. (a) The parametric equations of a curve are $x = 3t + 1$, $y = 2t^2$. A point P on the curve has parameter p. Given that the tangent at P passes through the point (1, –8), calculate the possible values of p.

(b) The cartesian equation of a curve is $y(y-2) = x$. Given that x is defined parametrically by $x = t^2 -$ and that $y = 4$ when $t = 3$, express y in terms of t.

(c) The parametric equations of a curve are $x = t^3 - t$, $y = t^2 + t$. Express $\frac{x}{y}$ in terms of t in the simplest possible form. Hence, or otherwise, find the cartesian equation of the curve. [C]

8. (a) Differentiate with respect to x:
(i) $\sqrt{1+x^2}$ (ii) $x(1+3x)^5$

(b) Show that the tangents to the curve $y^2 = 2y + 8x - 17$ at the points where $x = 4$ are perpendicular. [C]

(c) Determine the value of x, where $0 \leqslant x \leqslant \pi$, for which the curve $y = 2\cos x - 3\sin x$ has a stationary point and determine whether this is a maximum or a minimum.

9. (a) Differentiate with respect to x:
(i) $(4x+1)^3$ (ii) $x\tan 3x$

(b) Given that $xy^3 + y = x^3$, obtain $\frac{dy}{dx}$ in terms of x and y.

(c) Given that $y = \frac{\sqrt{x}}{x-2}$, prove that $\frac{dy}{dx} = -\frac{(x+2)}{2\sqrt{x}(x-2)^2}$. Hence, obtain the equation of the normal to the curve $y = \frac{\sqrt{x}}{x-2}$ at the point on the curve where $x = 4$. [C]

10. (a) Find the value of x between 0 and π for which the curve $y = e^x \sin x$ has a stationary point.

(b) Differentiate $\frac{2x+1}{x-4}$ with respect to x, simplifying your answer.

(c) Differentiate $x \ln x - x$ with respect to x and hence evaluate $\int_1^2 \ln x \, dx$.

(d) Calculate the value of $\frac{dy}{dx}$ at the point (1, 2) on the curve $x^3 + y^3 - 8y + 7 = 0$. [C]

11. (a) Differentiate with respect to x:

(i) $\frac{1}{\sqrt{x-1}}$ (ii) $\ln (3x-1)^2$

(b) The gradient of the curve $2x^2 + y^2 + 3x - 6y + 9 = 0$ at the point (a, b) is $\frac{1}{2}$. Show that $4a + b = 0$.

(c) Given that $y = e^x \cos 2x$, find $\frac{dy}{dx}$ and hence determine, for $0 \leqslant x \leqslant \pi$, the value of x for which y is stationary. [C]

12. (a) Differentiate with respect to x:

(i) $\ln (5x+2)$ (ii) $\frac{x}{2x+1}$

(b) The curve $y^2 + xy = 1 + \sin x$ crosses the positive y-axis at P. Show that the tangent at P is parallel to the x-axis.

(c) Find the coordinates of the stationary points of the curve $y = x(4-x)^3$. [C]

13. (a) Given that $y = \frac{\ln x}{x}$, calculate the value of $\frac{dy}{dx}$ when x = 1.

(b) Given that $y = x^2 e^{3x}$, write down an expression for $\frac{dy}{dx}$ and hence determine the values of x for which y is stationary.

(c) Given that $y = \frac{1}{3}\cos^3 x - \cos x$, prove that $\frac{dy}{dx} = \sin^3 x$.

(d) Find the gradient of the tangent at point (2, –1) on the curve $xy^3 + y^2 + 1 = 0$.

14. (a) Differentiate the following expressions with respect to x:

(i) $x^2 \ln x$ (ii) $\left(5x^2 - 2\right)^{\frac{3}{2}}$

(b) Find the gradient of the curve $x^2 + 9y^2 = 4x - 6y + 20$ at the point (5, 1).

(c) Given that $y = \frac{\sin x}{2 - \cos x}$, find the values of x between 0 and 2 for which y is stationary. [C]

15. (a) The curve whose parametric equations are $x = p^2 + 1$, $y = p - 2$ intersects the line $x - 4y = 6$ at the points A and B.

    (i) Obtain the equation in p which gives the value of the parameter at A and at B.

    (ii) Find the coordinates of A and of B.

    (b) A curve has parametric equations $x = t^2 - 2t$, $y = t^2 + 2t$. Find:

    (i) an expression for $\frac{dy}{dx}$ in terms of t.

    (ii) the equation of the tangent to the curve at the point where the gradient is $\frac{1}{2}$.

    (iii) the coordinates of the point on the curve at which the tangent is parallel to the y-axis.

    (iv) the cartesian equation of the curve. [C]

16. (a) Differentiate with respect to x:

    (i) $\frac{2x + 5}{1 + 3x}$ (ii) $x^2\cos 2x$

    (b) The variables x and y are related by the equation $2x^2 + xy - 3y^2 = 7$.

    Find the value of $\frac{dy}{dx}$ at the point (2, 1).

    Given that x is increasing at the rate of 0.08 units per second when x = 2,

    (i) find the corresponding rate of change of y.

    (ii) state whether y is increasing or decreasing.

    (c) The equation of a curve is $y = \tan(x + 1)$. Find, to two decimal places, the gradient of the normal to the curve at the point where x = 0. [C]

17. (a) Differentiate the following expressions with respect to x:

    (i) $\frac{2x - 1}{x + 5}$ (ii) $\sqrt{25 - 5x^3}$

    (b) Find the gradient of the curve $y^2 + 2xy = (x - 3)^3 + 16$ at the point (1, 2).

    (c) Determine the x-coordinate of the stationary point of the curve $y = x\ln x - 2x$. [C]

18. The parametric equations of a curve are $x = t - \frac{1}{t}$, $y = t + \frac{1}{t}$, where $t \neq 0$.

    (a) P is the point on the curve where t = 2. The tangent to the curve at P meets the x-axis at Q. The normal to the curve at P meets the x-axis at R. Calculate the area of the triangle PQR.

    (b) A is the point on the curve where t = k and B is the point on the curve where $t = -\frac{1}{k}$. Show that the line AB is parallel to the y-axis. [C]

19. (a) Differentiate with respect to x:

    (i) tan x (ii) $\cos^2 x$

    (b) Find the coordinates of the point on the curve $y = \ln(5 - 2x)$ at which the normal to the curve is parallel to $2y = x + 3$.

(c) Given the curve $x^2 + y^2 - 8x + 4y + 2 = 0$,

(i) obtain an expression for $\frac{dy}{dx}$.

(ii) show that the tangents to the curve at the points where $x = 7$ are perpendicular. [C]

20. (a) Differentiate with respect to x:

(i) $\frac{3x + 2}{2 - x}$ (ii) $x\tan^2 x$

(b) The variables x and y are related by the equation $x^2 + y^2 + 4x + 2y = 24$. Find the value of $\frac{dy}{dx}$ at the point (3, 1).

Given that x is increasing at the rate of 0.2 units per second at the point (3, 1), find the corresponding rate of decrease of y.

(c) The equation of a curve is $y = \ln (x^2 + 2x)$, where $x > 0$. Find the x-coordinate of the point on the curve at which the tangent to the curve is parallel to the line $5y = 12x$. [C]

CHAPTER 14

> INTEGRATION 2

14.1 Integration involving trigonometric, exponential and logarithmic functions

14.1.1 Trigonometric functions

Since $\frac{d}{dx}\sin x = \cos x$, $\frac{d}{dx}\cos x = -\sin x$, $\frac{d}{dx}\tan x = \sec^2 x$,

$$\int \cos x\, dx = \sin x + c$$

$$\int \sin x\, dx = -\cos x + c$$

$$\int \sec^2 x\, dx = \tan x + c$$

Since $\frac{d}{dx}\sin(ax + b)\, dx = a\cos(ax + b)$

$$\int a\cos(ax + b)\, dx = \sin(ax + b) + c$$

$$\int \cos(ax + b)\, dx = \frac{1}{a}\sin(ax + b) + k$$

Similarly $\int \sin(ax + b)\, dx = -\frac{1}{a}\cos(ax + b) + k$

$$\int \sec^2(ax + b)\, dx = \frac{1}{a}\tan(ax + b) + k$$

Example 1

Find: (a) $\int_0^{\frac{\pi}{6}} \sin 2x\, dx$ (b) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos\left(3x + \frac{\pi}{4}\right) dx$ (c) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sec^2\left(2x - \frac{\pi}{6}\right) dx$

Solution

(a) $\int_0^{\frac{\pi}{6}} \sin 2x\, dx = \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{6}}$

$$= -\frac{1}{2}\cos\frac{\pi}{3} + \frac{1}{2}\cos 0$$

$$= -\frac{1}{4} + \frac{1}{2}$$

$$= \frac{1}{4}$$

(b) $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos\left(3x + \frac{\pi}{4}\right) dx = \left[\frac{1}{3}\sin\left(3x + \frac{\pi}{4}\right)\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{1}{3}\sin\pi - \frac{1}{3}\sin\left(-\frac{\pi}{2}\right)$$

$$= 0 + \frac{1}{3}$$

$$= \frac{1}{3}$$

(c) $$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sec^2\left(2x - \frac{\pi}{6}\right) dx = \left[\frac{1}{2}\tan\left(2x - \frac{\pi}{6}\right)\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\tan\frac{5\pi}{6} - \frac{1}{2}\tan\frac{\pi}{3}$$

$$= \frac{-1}{2\sqrt{3}} - \frac{1\sqrt{3}}{2}$$

$$= \frac{-1-3}{2\sqrt{3}}$$

$$= \frac{-2}{\sqrt{3}}$$

$$= \frac{-2\sqrt{3}}{3}$$

14.1.2 Exponential and logarithmic functions

Since $$\frac{d}{dx}\left(e^{ax+b}\right) = ae^{ax+b}$$

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$$

Also $$\frac{d}{dx}\ln(ax+b) = \frac{a}{ax+b}$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$$

Note: When working with definite integrals involving ln functions, it is advisable to write $\int_{x_1}^{x_2} \frac{1}{ax+b}\,dx$ *as* $\left[\frac{1}{a}\ln|ax+b|\right]_{x_1}^{x_2}$.

Example 2

Find: (a) $\int e^{2x+3}\,dx$ (b) $\int \frac{1}{3x-4}\,dx$ (c) $\int \frac{2}{5x}\,dx$ (d) $\int_{-3}^{-2} \frac{2}{5x-1}\,dx$

Solution

(a) $\int e^{2x+3}\,dx = \frac{1}{2}e^{2x+3} + c$

(b) $\int \frac{1}{3x-4}\,dx = \frac{1}{3}\ln|3x-4| + c$

(c)
$$\int \frac{2}{5x}\,dx = \frac{2}{5}\int \frac{1}{x}\,dx$$
$$= \frac{2}{5}\ln|x| + c$$

(d)
$$\int_{-3}^{-2} \frac{2}{5x-1}\,dx = \left[\frac{2}{5}\ln|5x-1|\right]_{-3}^{-2}$$
$$= \frac{2}{5}\ln 11 - \frac{2}{5}\ln 16$$
$$= \frac{2}{5}\ln\frac{11}{16}$$

14.1.3 Integral of cos² x and sin² x

We recall that $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ and $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. We use these two formulae to find $\int \cos^2 x\,dx$ and $\int \sin^2 x\,dx$.

$$\int \cos^2 x\,dx = \int \frac{1}{2}(1 + \cos 2x)\,dx$$
$$= \frac{1}{2}\int (1 + \cos 2x)\,dx$$
$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + c$$
$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + c$$

$$\int \sin^2 x\,dx = \frac{1}{2}\int (1 - \cos 2x)\,dx$$
$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

Example 3

Find: (a) $\int \cos^2 3x\,dx$ (b) $\int \sin^2 \frac{1}{4}x\,dx$ (c) $\int_0^{\frac{\pi}{4}} 3\cos^2 \frac{1}{2}x\,dx$

Solution

(a) $\int \cos^2 3x\,dx = \frac{1}{2}\int (1 + \cos 6x)\,dx$

$$= \frac{1}{2}x + \frac{1}{12}\sin 6x + c$$

(b) $\int \sin^2 \frac{1}{4}x\,dx = \frac{1}{2}\int \left(1 - \cos\frac{1}{2}x\right) dx$

$$= \frac{1}{2}\left[x - 2\sin\frac{1}{2}x\right] + c$$

$$= \frac{1}{2}x - \sin\frac{1}{2}x + c$$

(c) $\int_0^{\frac{\pi}{4}} 3\cos^2 \frac{1}{2}x\,dx = 3 \times \frac{1}{2}\int_0^{\frac{\pi}{4}} (1 + \cos x)\,dx$

$$= \frac{3}{2}\left[x + \sin x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{3\pi}{8} + \frac{3}{2\sqrt{2}}$$

Exercise 14 A

1. Find the integrals with respect to x of:

(a) $\sin 2x$ (b) $\cos 4x$ (c) $\sec^2 5x$ (d) $\sin \frac{1}{2}x$

(e) $\cos \frac{2}{3}x$ (f) $\sec^2 \frac{3}{4}x$ (g) $\sin(3x + \pi)$ (h) $\cos\left(5x - \frac{\pi}{2}\right)$

(i) $\sec^2\left(3x + \frac{\pi}{6}\right)$ (j) $\sin\left(\frac{3}{4}x - \frac{\pi}{2}\right)$ (k) $\cos\left(\frac{2}{3}x + \frac{\pi}{6}\right)$ (l) $\sec^2\left(\frac{4}{3}x - \frac{\pi}{2}\right)$

2. Find the integrals with respect to x of:

(a) e^{4x} (b) e^{-5x} (c) $e^{\frac{1}{3}x}$ (d) $e^{-\frac{1}{5}x}$

(e) $e^{\frac{2}{3}x}$ (f) $e^{\frac{3}{4}x}$ (g) e^{3x+2} (h) e^{-4x+5}

(i) $e^{\frac{2}{3}x+\frac{1}{4}}$ (j) $e^{\frac{3}{5}-\frac{4}{7}x}$ (k) $(e^x + 1)(e^x + 1)$ (l) $(e^{2x} + 3)(e^{-x} - 4)$

3. Find the integrals with respect to x of:

(a) $\frac{1}{4x}$ (b) $\frac{2}{5x}$ (c) $\frac{2}{5x-3}$ (d) $\frac{4}{7-8x}$

(e) $4+\frac{2}{3x+1}$ (f) $\frac{2}{x-1}+\frac{3}{x-2}$ (g) $\frac{4}{2x+1}-\frac{6}{3x+1}$

4. Evaluate:

(a) $\int_{0.3}^{0.4} e^{-2x}\,dx$ (b) $\int_{0.1}^{0.2} e^{3x}\,dx$ (c) $\int_{1}^{2} \frac{1}{x}\,dx$

(d) $\int_{-1}^{0} \frac{2}{3x-1}\,dx$ (e) $\int_{2}^{3} \frac{5}{2-3x}\,dx$ (f) $\int_{-2}^{-1} \frac{2}{4t-1}\,dt$

(g) $\int_{0}^{\frac{\pi}{6}} \sin\left(2x+\frac{\pi}{6}\right)$ (h) $\int_{-\frac{\pi}{4}}^{0} \cos\left(3x-\frac{\pi}{4}\right)dx$ (i) $\int_{\frac{\pi}{12}}^{0} \sec^2\left(\frac{1}{2}x+\frac{\pi}{6}\right)dx$

5. Find the integrals with respect to x of:

(a) $\cos^2 2x$ (b) $\sin^2 3x$ (c) $\cos^2 \frac{1}{6}x$ (d) $\sin^2 \frac{3}{4}x$

(e) $\cos^2 4x$ (f) $\sin^2 \frac{1}{3}x$ (g) $\cos^2 \frac{2}{3}x$ (h) $\sin^2 6x$

6. Evaluate:

(a) $\int_{0}^{\frac{\pi}{8}} \sin^2 2x\,dx$ (b) $\int_{-\frac{\pi}{6}}^{0} \cos^2 3x\,dx$ (c) $\int_{\frac{\pi}{2}}^{\pi} \sin^2 \frac{1}{3}x\,dx$ (d) $\int_{0}^{2\pi} \cos^2 \frac{1}{2}x\,dx$

7. Use the identity $\sec^2 x = 1 + \tan^2 x$ to find $\int_{0}^{\frac{\pi}{4}} \tan^2 x\,dx$.

8. Evaluate $\int_{0}^{\frac{\pi}{9}} \tan^2 3x\,dx$.

9. Show that $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$. Hence, find $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \text{cosec}^2\,x\,dx$.

14.2 The trapezium rule

14.2.1 Formula for the trapezium rule

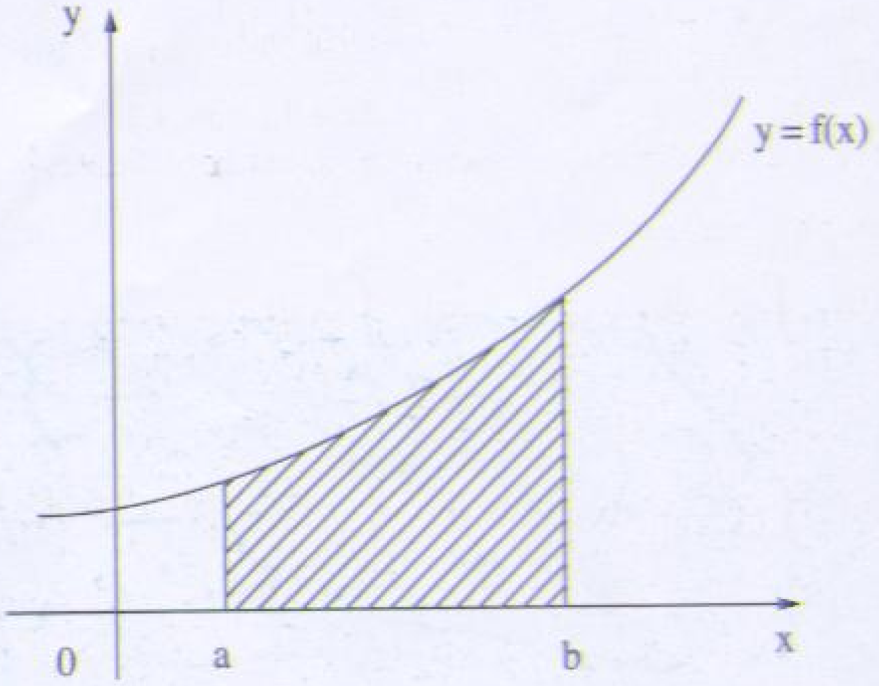


Figure 14.1

Consider the shaded region in Figure 14.1. Its area is $\int_a^b f(x)$ dx. Sometimes, it is not possible to find $\int_a^b f(x)$ dx an
we use an approximation.

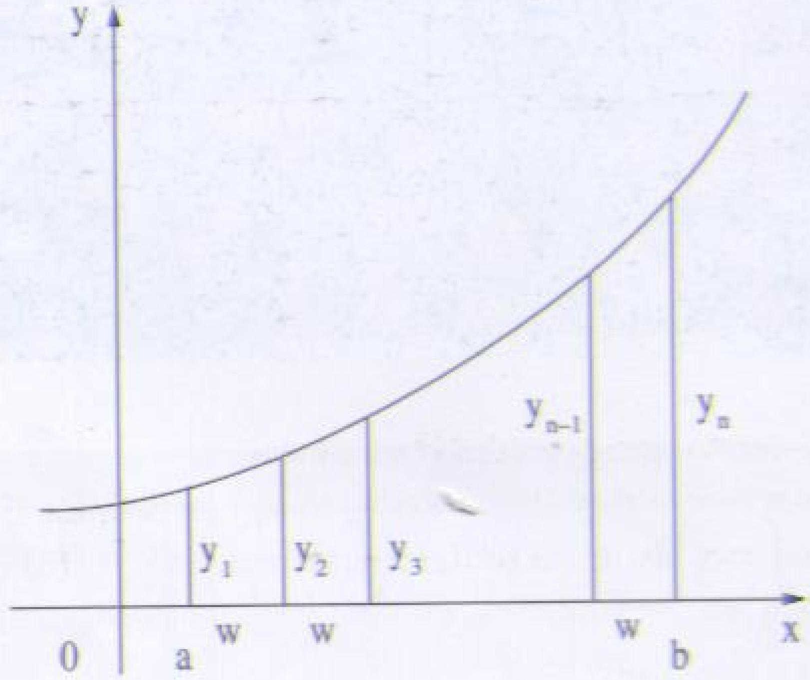


Figure 14.2

We divide the region into approximate trapezia of equal widths w. Let the ordinates be $y_1, y_2, y_3, \ldots\ldots y_n$ as sho
in Figure 14.2.

$$\int_a^b f(x) \approx \text{sum of approximate trapezia}$$

$$= \frac{1}{2}(y_1 + y_2)\,w + \frac{1}{2}(y_2 + y_3)\,w + \frac{1}{2}(y_3 + y_4)\,w + \ldots\ldots\ldots + \frac{1}{2}(y_{n-2} + y_{n-1})\,w + \frac{1}{2}(y_{n-1} + y_n)\,w$$

$$= \frac{1}{2}w[y_1 + y_2 + y_2 + y_3 + y_3 + y_4 + \ldots\ldots\ldots + y_{n-2} + y_{n-1} + y_{n-1} + y_n]$$

$$= \frac{1}{2}w[y_1 + 2y_2 + 2y_3 + 2y_4 + \ldots\ldots\ldots + 2y_{n-1} + y_n]$$

$$= \frac{1}{2}w[(y_1 + y_n) + 2y_2 + 2y_3 + 2y_4 + \text{..........} + 2y_{n-1}]$$

$$= w\left[\frac{(y_1 + y_n)}{2} + y_2 + y_3 + y_4 + \text{..........} + y_{n-1}\right]$$

or

$$\text{Area} = \text{Width} \times \left[\frac{1}{2}\text{ (sum of first and last ordinates)} + \text{sum of all other ordinates}\right]$$

This result is known as the *trapezium rule*.

Example 4

Find the area of the region enclosed by the curve $y = \frac{1}{x^3 + 2}$, the x-axis, the lines $x = 1$ and $x = 3$ by dividing it into 4 approximate trapezia of equal widths giving your result correct to 3 decimal places.

Solution

The width is $\frac{3-1}{4} = \frac{1}{2}$

So, we take x = 1, 1.5, 2, 2.5 and 3.

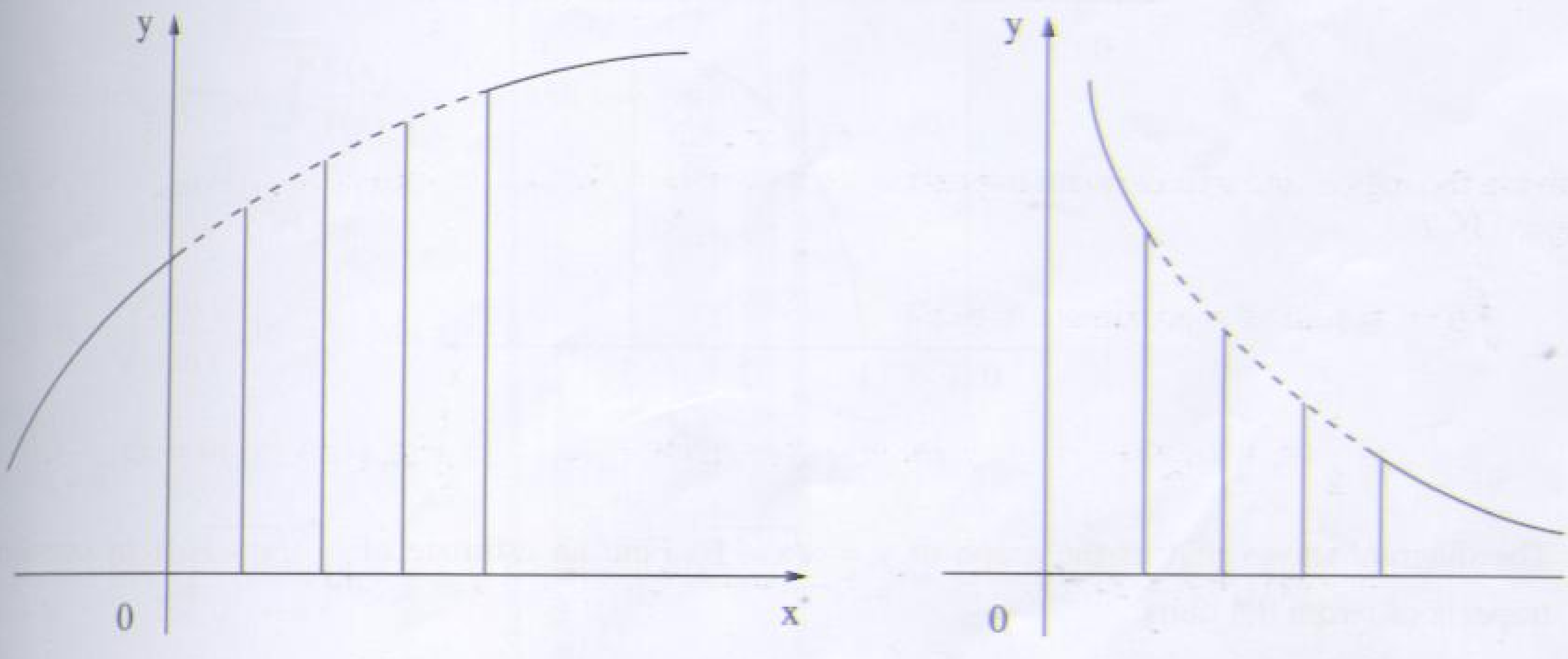
x	1	1.5	2	2.5	3
y	0.3333	0.1860	0.1	0.0567	0.0345

$$\text{Approximate area} = 0.5\left[\frac{0.3333 + 0.0345}{2} + 0.186 + 0.1 + 0.0567\right]$$

$$= 0.263 \text{ (3 d.p)}$$

14.2.2 Overestimates and underestimates

Depending upon the shape of the curve y = f(x), it is possible to say sometimes if the area obtained by using the trapezium rule is an overestimate or underestimate of the actual area.



(a) (b)

Figure 14.3

In Figure 14.3 (a), the sum of the trapezia is less than the actual area and the value obtained is an underestimate of the actual value.

In Figure 14.3 (b), the sum of the trapezia is greater than the actual area and the value obtained is an overestimate of the actual area.

Exercise 14 B

1. Find to 2 d.p the area enclosed by the curve $y = \sqrt{1 + x^3}$, the lines $x = 0$, $x = 4$ and the x-axis by considering trapezia of width 1 unit.

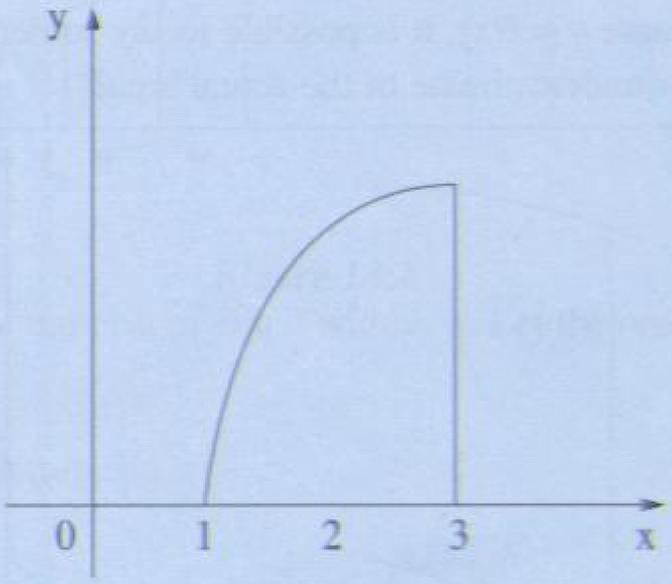
2. Find to 2 d.p the area enclosed by the curve $y = \ln x$, the lines $x = 1$ and $x = 3$ and the x-axis by considering trapezia of width 0.5 units.
   Sketch the graph of $y = \ln x$ for $1 \leqslant x \leqslant 3$. Hence, state whether the approximation obtained is an overestimate or an underestimate of the true area.

3. Find to 3 d.p the area enclosed by the curve $y = \dfrac{1}{\sqrt{x + 1}}$, the lines $x = 1$, $x = 3$ and the x-axis by considering trapezia of width 0.5 units.

4. (a) Find the exact value of $\displaystyle\int_1^3 \frac{1}{1 + x}\, dx$

   (b) Obtain the value to 2 d.p of $\displaystyle\int_1^3 \frac{1}{1 + x}\, dx$ by considering trapezia of width 0.5 units. Hence, obtain the % error done in using the trapezium rule with trapezia of width 0.5 units to evaluate $\displaystyle\int_1^3 \frac{1}{1 + x}\, dx$. Give your answer to 1 d.p.

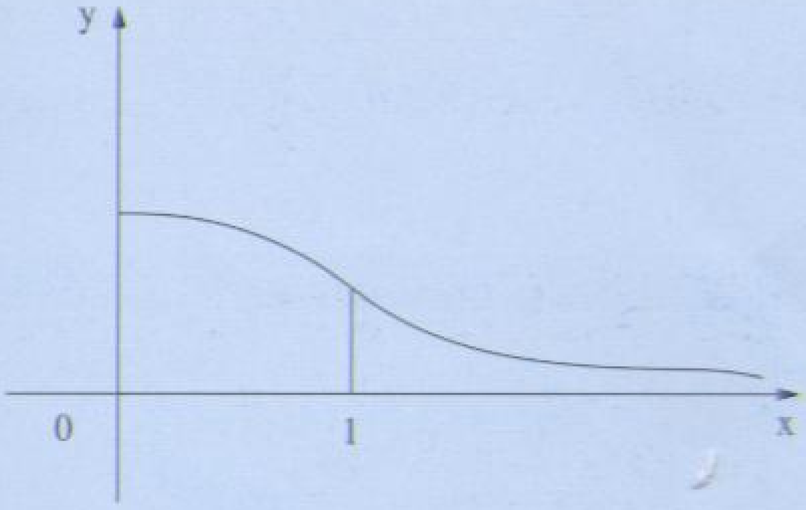
5.



The diagram shows part of the graph of $y = \sqrt{x - 1}$. Find an estimate of $\displaystyle\int_1^3 \sqrt{x - 1}\, dx$ by considering trapezia of width 0.4 units.

Is this estimate an underestimate or an overestimate of the true value?

6.



The diagram shows the region bounded by the axes, the curve $y = \frac{1}{\sqrt{x^2+1}}$ and the line $x = 1$. Find the area of this region by considering trapezia of width 0.2 units.
Is this area an overestimate or an underestimate of the true value?

7. The region bounded by the axes, the curve $y = \frac{1}{\sqrt{x^2+1}}$ and the line $x = 1$ is rotated though 4 right angles about the x-axis. Use the formula $\pi\int y^2\,dx$ and the trapezium rule with trapezia of width 0.2 units to obtain an appropriate value of the solid of revolution obtained.

14.3 Integration of rational functions

14.3.1 $\int \frac{k\,f'(x)}{f(x)}\,dx$

As $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

For the definite integral $\int_a^b \frac{f'(x)}{f(x)}\,dx$, we will use $\left[\ln|f(x)|\right]_a^b + c$

$$\int \frac{2x}{x^2-9}\,dx = \ln(x^2-9) + c$$

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + c$$

$$\int \frac{2x+1}{x^2+x+8}\,dx = \ln(x^2+x+8) + c$$

$$\int \frac{x}{x^2+8}\,dx = \frac{1}{2}\int \frac{2x}{x^2+8}\,dx$$

$$= \frac{1}{2}\ln(x^2+8) + c$$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

$$= -\int \frac{-\sin x}{\cos x} \, dx$$

$$= -\ln|\cos x| + c$$

$$= \ln(\cos x)^{-1} + c$$

$$= \ln|\sec x| + c$$

14.3.2 Integrations involving partial fractions

To integrate $\frac{f(x)}{g(x)}$ with respect to x where f(x) and g(x) are polynomials in x, we find the partial fractions of $\frac{f(x}{g(x}$

as illustrated in the following examples:

Example 5

Find: (a) $\int \frac{x-1}{(x-2)(x+3)} \, dx$ (b) $\int \frac{x-1}{(x+1)(x-2)^2} \, dx$ (c) $\int \frac{4x^2-2x+16}{(x-1)(x^2+8)} \, dx$

Solution

(a) $\frac{x-1}{(x-2)(x+3)} \, dx \equiv \frac{A}{x-2} + \frac{B}{x+3}$

$$\equiv \frac{A(x+3)+B(x-2)}{(x-2)(x+3)}$$

$(x-1) \equiv A(x+3) + B(x-2)$

$x = 2, \quad 1 = 5A$

$A = \frac{1}{5}$

$x = -3, \; -4 = -5B$

$B = \frac{4}{5}$

So, $\frac{x-1}{(x-2)(x+3)} \equiv \frac{1}{5(x-2)} + \frac{4}{5(x+3)}$

$$\int \frac{x-1}{(x-2)(x+3)} \, dx = \int \frac{1}{5(x-2)} \, dx + \int \frac{4}{5(x+3)} \, dx$$

$$= \frac{1}{5}\int \frac{1}{x-2} \, dx + \frac{4}{5}\int \frac{1}{x+3} \, dx$$

$$= \frac{1}{5}\ln|x-2| + \frac{4}{5}\ln|x+3| + c$$

$$= \ln|x-2|^{\frac{1}{5}} + \ln|x+3|^{\frac{4}{5}} + c$$

(b) $\int \frac{x-1}{(x+1)(x-2)^2} \, dx \equiv \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$

$$\equiv \frac{A(x-2)^2 + B(x+1)(x-2) + C(x+1)}{(x+1)(x-2)^2}$$

$x - 1 \equiv A(x-2)^2 + B(x+1)(x-2) + C(x+1)$

$x = -1,\ -2 = 9A$

$A = -\frac{2}{9}$

$x = 2,\quad 1 = 3C$

$C = \frac{1}{3}$

Coefficients of x^2, $A + B = 0$

$B = \frac{2}{9}$

$$\frac{x-1}{(x+1)(x-2)^2} = \frac{-2}{9(x+1)} + \frac{2}{9(x-2)} + \frac{1}{3(x-2)^2}$$

$$\int \frac{x-1}{(x+1)(x-2)^2} = -\frac{2}{9}\int \frac{1}{x+1}\,dx + \frac{2}{9}\int \frac{1}{x-2}\,dx + \frac{1}{3}\int (x-2)^{-2}\,dx$$

$$= -\frac{2}{9}\ln|x+1| + \frac{2}{9}\ln|x-2| - \frac{1}{3}(x-2)^{-1} + c$$

$$= \frac{2}{9}\left[\ln(x-2) - \ln(x+1)\right] - \frac{1}{3(x-2)} + c$$

$$= \frac{2}{9}\ln\left|\frac{x-2}{x+1}\right| - \frac{1}{3(x-2)} + c$$

(c) $\displaystyle\int \frac{4x^2 - 2x + 16}{(x-1)(x^2+8)}\,dx$

$$\frac{4x^2 - 2x + 16}{(x-1)(x^2+8)} \equiv \frac{A}{x-1} + \frac{Bx + C}{x^2+8}$$

$$\equiv \frac{A\left(x^2+8\right) + Bx\left(x-1\right) + C(x-1)}{\left(x-1\right)\left(x^2+8\right)}$$

$4x^2 - 2x + 16 \equiv A(x^2+8) + Bx(x-1) + C(x-1)$

Put $x = 1$, $4 - 2 + 16 = 9A$

$A = 2$

$x = 0,\quad 16 = 8A - C$

$C = 0$

Coefficients of x^2 $\quad 4 = A + B$

$B = 2$

So, $\displaystyle\frac{4x^2 - 2x + 16}{(x-1)(x^2+8)} \equiv \frac{2}{x-1} + \frac{2x}{x^2+8}$

$$\int \frac{4x^2 - 2x + 16}{(x-1)(x^2+8)}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{2x}{x^2+8}\,dx$$

$$= 2\ln|x-1| + \ln(x^2+8) + c$$

$$= \ln(x-1)^2 + \ln(x^2+8) + c$$

$$= \ln\left[(x-1)^2(x^2+8)\right] + c$$

Exercise 14 C

1. Find the integrals with respect to x of:

(a) $\frac{6x^2}{2x^3 + 4}$ (b) $\frac{2x + 1}{x^2 + x + 8}$ (c) $\frac{x + 3}{x^2 + 6x - 9}$

(d) $\frac{e^x}{e^x + 1}$ (e) $\cot 3x$ (f) $\tan \frac{1}{2}x$

(g) $\frac{4x^2 + 4}{2x^3 + 6x + 1}$ (h) $\cot \frac{1}{4}x$ (i) $\tan 5x$

(j) $\frac{1}{x \ln x}$ (k) $\frac{\sin x - \cos x}{\cos x + \sin x}$ (l) $\frac{\tan x + 1}{\tan x - 1}$

2. Find in each case the partial fractions of f(x) and obtain $\int f(x)\,dx$:

(a) $\frac{5x - 1}{x^2 + x - 2}$ (b) $\frac{x + 1}{(2x - 1)(3x - 1)}$ (c) $\frac{6x^2 - 22x + 18}{(x + 1)(x - 2)(x - 3)}$

(d) $\frac{6x^2 - x - 16}{(2x - 1)(2x + 1)(x - 3)}$ (e) $\frac{3x^2 - 5x + 8}{(x + 1)(x - 3)^2}$ (f) $\frac{7x^2 - 12x + 5}{(x - 1)(2x - 1)^2}$

(g) $\frac{12x^2 + 10x - 5}{4x^2(2x - 1)}$ (h) $\frac{12x^2 - 14x + 2}{(2x + 1)(2x - 1)^2}$ (i) $\frac{-2x^2 - 4x + 2}{(x + 1)(x^2 + 1)}$

(j) $\frac{13x^2 - 5x + 24}{(2x - 1)(x^2 + 8)}$

3. Find the exact value of:

(a) $\int_4^5 \frac{5x - 9}{(x - 1)(x - 3)}\,dx$ (b) $\int_{-2}^{-1} \frac{1 - x}{(2x + 1)(3x + 2)}\,dx$ (c) $\int_4^5 \frac{x^2 - 4x + 11}{(x + 1)(x - 3)^2}\,dx$

(d) $\int_2^3 \frac{10x^2 + 4x + 1}{(x + 1)(2x - 1)^2}\,dx$ (e) $\int_0^1 \frac{5x^2 + 2x + 12}{(x + 1)(x^2 + 4)}\,dx$

4. Find the partial fractions of $\frac{x^3 - x^2 + x + 1}{(x - 1)(x - 2)}$. Hence, obtain $\int_3^4 \frac{x^3 - x^2 + x + 1}{(x - 1)(x - 2)}\,dx$ to 2 d.p.

14.4 Further integration

14.4.1 Integration by parts

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Ignoring the constant of integration, $\int \left(u\frac{dv}{dx} + v\frac{du}{dx}\right) dx = uv$

$$\int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx = uv$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx\,uv$$

This formula is more easily remembered as

$$\int udv = uv - \int vdu$$

It is used to find integrals of the type $\int x\sin kx\,dx$, $\int x\cos kx\,dx$, $\int xe^{ax+b}\,dx$, $\int x^2\sin kx\,dx$, etc. as illustrated in the following examples:

Example 6

Find: (a) $\int xe^{2x}\,dx$ (b) $\int x\sin 3x\,dx$ (c) $\int x\cos\frac{1}{2}x\,dx$

Solution

(a) Put $u = x$, $\frac{dv}{dx} = e^{2x}$

$$\frac{du}{dx} = 1,\quad v = \frac{1}{2}e^{2x}$$

$$\int xe^{2x}\,dx = \int u\frac{dv}{dx}\,dx$$

$$= uv - \int v\frac{du}{dx}\,dx$$

$$= \frac{1}{2}x\,e^{2x} - \int \frac{1}{2}e^{2x}\,dx$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$$

(b) $\int x\sin 3x\,dx = \int u\frac{dv}{dx}\,dx$ where $u = x$, $\frac{dv}{dx} = \sin 3x$, i.e. $\frac{du}{dx} = 1$, $v = -\frac{1}{3}\cos 3x$

$$= uv - \int v\frac{du}{dx}\,dx$$

$$= -\frac{1}{3}x\cos 3x - \int -\frac{1}{3}\cos 3x\,dx$$

$$= -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x\,dx + c$$

$$= -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + c$$

(c) $\int x\cos\frac{1}{2}x\,dx = \int u\frac{dv}{dx}\,dx$ where $u = x$, $\frac{dv}{dx} = \cos\frac{1}{2}x$, i.e. $\frac{du}{dx} = 1$, $v = 2\sin\frac{1}{2}x$.

$$\int x\cos\frac{1}{2}x\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$= 2x\sin\frac{1}{2}x - \int 2\sin\frac{1}{2}x\,dx$$

$$= 2x\sin\frac{1}{2}x + 4\cos\frac{1}{2}x + c.$$

Example 7

Find: (a) $\int x^2e^{-x}\,dx$ *(b)* $\int x^2\sin x\,dx$

Solution

(a) $\int x^2e^{-x}\,dx = \int u\frac{dv}{dx}\,dx$ where $u = x^2$, $\frac{dv}{dx} = e^{-x}$, i.e. $\frac{du}{dx} = 2x$, $v = -e^{-x}$.

So, $\int x^2e^{-x}\,dx = uv - \int v\frac{du}{dx}\,dx$

$$= -x^2e^{-x} - \int -2xe^{-x}\,dx$$

$$= -x^2e^{-x} + \int 2xe^{-x}\,dx \qquad (1)$$

$\int 2xe^{-x}\,dx = \int u\frac{dv}{dx}\,dx$ where $u = 2x$, $\frac{dv}{dx} = e^{-x}$, i.e. $\frac{du}{dx} = 2$, $v = e^{-x}$

$$\int 2xe^{-x}\,dx = 2xe^{-x} - \int 2e^{-x}\,dx$$
$$= 2xe^{-x} + 2e^{-x} + c \qquad (2)$$

From (1) & (2) $\int x^2e^{-x}\,dx = -x^2e^{-x} + 2xe^{-x} + 2e^{-x} + c$

$$= -e^{-x}(x^2 - 2x - 2) + c$$

(b) $\int x^2\sin x\,dx = \int u\frac{dv}{dx}\,dx$ where $u = x^2$, $\frac{dv}{dx} = \sin x$, i.e. $\frac{du}{dx} = 2x$, $v = -\cos x$

So, $\int x^2\sin x\,dx = -x^2\cos x - \int -2x\cos x\,dx$

$$= -x^2\cos x + \int 2x\cos x\,dx \qquad (1)$$

$\int 2x\cos x\,dx = \int u\frac{dv}{dx}\,dx$ where $u = 2x$, $\frac{dv}{dx} = \cos x$, i.e. $\frac{du}{dx} = 2$, $v = \sin x$

So, $\int 2x\cos x\,dx = 2x\sin x - \int 2\cos x\,dx$

$$= 2x\sin x - 2\sin x + c \qquad (2)$$

From (1) & (2) $\int x^2\sin x\,dx = -x^2\cos x + 2x\sin x - 2\sin x + c$

Example 8

Find $\int e^x\sin x\,dx$.

Solution

$\int e^x\sin x = \int u\frac{dv}{dx}\,dx$ where $u = e^x$, $\frac{dv}{dx} = \sin x$, i.e. $\frac{du}{dx} = e^x$, $v = -\cos x$

So, $\int e^x\sin x\,dx = -e^x\cos x - \int -e^x\cos x\,dx$

$$= -e^x\cos x + \int e^x\cos x\,dx \qquad (1)$$

$\int e^x\cos x\,dx = \int u\frac{dv}{dx}\,dx$ where $u = e^x$, $\frac{dv}{dx} = \cos x$, i.e. $\frac{du}{dx} = e^x$, $v = \sin x$

$$\int e^x\cos x = e^x\sin x - \int e^x\sin x\,dx \qquad (2)$$

From (1) & (2) $\int e^x\sin x\,dx = -e^x\cos x + e^x\sin x - \int e^x\sin x\,dx$

$$2\int e^x\sin x\,dx = e^x(\sin x - \cos x) + k$$

$$\int e^x\sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + c$$

including the constant of integration.

14.4.2 Integration by substitution

Consider $\frac{dy}{dx} = f(x)$

$$y = \int f(x)\,dx \qquad (1)$$

Writing $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{dy}{du} \times \frac{du}{dx} = f(x)$$

$$\frac{dy}{du} = f(x)\frac{dx}{du}$$

$$y = \int f(x)\frac{dx}{du}\,du \quad (2)$$

From (1) & (2)

$$\int f(x)\,dx = \int f(x)\frac{dx}{du}\,du$$

This is a very helpful formula to integrate certain types of functions.

Consider $\int P(x)\left[Q(x)\right]^n\,dx$ where $P(x) = kQ'(x)$

e.g. $\int 2x\left(x^2+1\right)^5\,dx$, $\int 3e^x\left(e^x-4\right)^{-2}\,dx$.

Any integral of this type can be obtained by putting $u = Q(x)$.

Thus, in $\int 2x\left(x^2+1\right)^5\,dx$, putting $u = x^2+1$, $\frac{du}{dx} = 2x$, $\frac{dx}{du} = \frac{1}{2x}$

$$\text{So, } \int 2x\left(x^2+1\right)^5\,dx = \int 2x\left(x^2+1\right)^5\frac{dx}{du}\,du$$

$$= \int 2xu^5 \times \frac{1}{2x}\,du$$

$$= \int u^5\,du$$

$$= \frac{u^6}{6} + c$$

$$= \frac{\left(x^2+1\right)^6}{6} + c$$

Also, in $\int 3e^x\left(e^x-4\right)^{-2}\,dx$, putting $u = e^x - 4$, $\frac{du}{dx} = e^x$.

$$\text{So, } \int 3e^x\left(e^x-4\right)^{-2}\,dx = \int 3e^x\left(e^x-4\right)^{-2}\frac{dx}{du}\,du$$

$$= \int 3e^xu^{-2} \times \frac{1}{e^x}\,du$$

$$= \int 3u^{-2}\,du$$

$$= \frac{-3}{u} + c$$

$$= \frac{-3}{\left(e^x-4\right)} + c$$

Example 9

Find: (a) $\int \frac{3\sin x}{\left(4+2\cos x\right)^2}\,dx$ *(b)* $\int \frac{1}{x}\left(3+2\ln x\right)^3\,dx$

Solution

(a) Put $u = 4 + 2\cos x$

$$\frac{du}{dx} = -2\sin x$$

$$\sin x = -\frac{1}{2}\frac{du}{dx}$$

$$\int \frac{3\sin x}{\left(4+2\cos x\right)^2}\,dx = \int \frac{-3}{2u^2}\frac{du}{dx}\,dx$$

$$= \int -\frac{3}{2}u^{-2}\ du$$

$$= \frac{3}{2u} + c$$

$$= \frac{3}{2(4 + 2\cos x)} + c$$

(b) Put $u = 3 + 2\ln x$

$$\frac{du}{dx} = \frac{2}{x}$$

$$\frac{dx}{du} = \frac{x}{2}$$

$$\int \frac{1}{x}(3 + 2\ln x)^3\ dx = \int \frac{1}{x}(3 + 2\ln x)^3\ \frac{dx}{du}\ du$$

$$= \int \frac{1}{x} \times u^3 \times \frac{x}{2}\ du$$

$$= \int \frac{u^3}{2}\ du$$

$$= \frac{1}{8}u^4 + c$$

$$= \frac{1}{8}(3 + 2\ln x)^4 + c$$

Example 10

Use the substitution $\cos 2x = 2\cos^2 x - 1$ *to find* $\int_0^{\frac{\pi}{4}} \frac{4}{\cos 2x + 1}$.

Solution

$$\int_0^{\frac{\pi}{4}} \frac{4}{\cos 2x + 1}\ dx = \int_0^{\frac{\pi}{4}} \frac{4}{2\cos^2 x - 1 + 1}\ dx$$

$$= \int_0^{\frac{\pi}{4}} 2\sec^2 x\ dx$$

$$= [2\tan x]_0^{\frac{\pi}{4}}$$

$$= 2\tan \frac{\pi}{4} - 2\tan 0$$

$$= 2$$

Example 11

Use the substitution $s = \sqrt{t} - 9$ *to find* $\int_1^4 \frac{25}{9-\sqrt{t}}\,dt$.

Solution

$$s = \sqrt{t} - 9$$

$$\frac{dt}{ds} = \frac{1}{2\sqrt{t}}$$

$$= \frac{1}{2(s+9)}$$

$$\frac{dt}{ds} = 2(s+9)$$

Also t = 1, s = –8, t = 4, s = –7

$$\text{So,}\quad \int_1^4 \frac{25}{9-\sqrt{t}}\,dt = \int_{-8}^{-7} \frac{25}{-s} \times 2(s+9)$$

$$= \int_{-8}^{-7} \left(-50 - \frac{450}{s}\right) ds$$

$$= \left[-50s - 450\ln|s|\right]_{-8}^{-7}$$

$$= \left[350 - 450\ln 7\right] - \left[400 - 450\ln 8\right]$$

$$= 450\ln\frac{8}{7} - 50$$

Exercise 14 D

1. Find the integrals with respect to x of:

(a) xe^{-x}
(b) $2xe^{3x}$
(c) $4xe^{-\frac{1}{2}x}$
(d) $6xe^{-\frac{2}{3}x}$
(e) $x\sin\left(2x - \frac{\pi}{4}\right)$
(f) $2x\cos\left(3x + \frac{\pi}{4}\right)$
(g) $x\sec^2 x$
(h) $x\sin\frac{1}{4}x$
(i) $3x\cos\frac{2}{3}x$
(j) $\ln x$
(k) $x\ln x$
(l) x^2e^x
(m) $x^2\sin 3x$
(n) $x^2\cos 3x$
(o) $x^2\ln x$
(p) $e^x\cos x$
(q) $e^x\sin 2x$
(r) $x^2\sin\left(2x + \frac{\pi}{3}\right)$
(s) $x^2\cos\left(3x - \frac{\pi}{2}\right)$
(t) x^2e^{2x+3}

2. Evaluate:

(a) $\int_0^1 xe^x\,dx$
(b) $\int_0^{\frac{\pi}{4}} x\sin 2x\,dx$
(c) $\int_0^{\frac{\pi}{3}} x\cos\left(x + \frac{\pi}{3}\right)dx$

(d) $\int_{0}^{\frac{\pi}{4}} x\sec^2 x \, dx$ (e) $\int_{0}^{1} te^{-2t} \, dt$ (f) $\int_{-\frac{\pi}{8}}^{\frac{\pi}{4}} t\sin\left(2t + \frac{\pi}{4}\right) dt$

(g) $\int_{0}^{\frac{\pi}{4}} e^x\sin x \, dx$ (h) $\int_{-\frac{\pi}{2}}^{0} e^x\cos 2x \, dx$ (i) $\int_{1}^{2} t\ln t \, dt$

3. Find the integrals with respect to x of:

(a) $x(x^2 + 1)^{-3}$ (b) $\frac{3x^3}{\left(2x^4 + 1\right)^4}$ (c) $\sin x\,(2 + \cos x)^3$

(d) $3e^{2x}(2e^{2x} - 1)^{-4}$ (e) $\sec^2 x\left(3 + \tan x\right)^{\frac{1}{2}}$ (f) $\frac{\left(2 + \ln x\right)^3}{3x}$

(g) $(\sin x + \cos x)(\sin x - \cos x)^3$ (h) $\frac{3\sin x}{\left(5 - 2\cos x\right)^{\frac{1}{2}}}$

4. Find the following integrals, using the given substitutions:

(a) $\int 2x\sqrt{(5x + 3)} \, dx$ using $u = \sqrt{5x+3}$ (b) $\int \frac{3x}{\sqrt{x + 4}} \, dx$ using $u = \sqrt{x+4}$

(c) $\int (x - 3)^5(x + 2)^2 \, dx$ using $u = x - 3$ (d) $\int \frac{x + 1}{\sqrt{2x - 3}} \, dx$ using $u = \sqrt{2x - 3}$

(e) $\int \frac{1}{1 + x^2} \, dx$ using $x = \tan u$ (f) $\int \frac{1}{9 + 4x^2} \, dx$ using $x = \frac{3}{2}\tan u$

5. Evaluate the following integrals, using the given substitutions:

(a) $\int_{0}^{4} x\sqrt{3x+4} \, dx$ using $\sqrt{3x+4} = u$ (b) $\int_{0}^{\frac{\pi}{2}} \cos x(1 + \sin x)^2 \, dx$ using $u = 1 + \sin x$

(c) $\int_{0}^{\frac{\pi}{4}} \frac{1}{1 + x^2} \, dx$ using $x = \tan u$ (d) $\int_{0}^{\frac{5}{3}} \frac{1}{25 + 9x^2} \, dx$ using $x = \frac{5}{3}\tan u$

(e) $\int_{2}^{6} \frac{x - 1}{\sqrt{2x - 3}} \, dx$ using $\sqrt{2x - 3} = u$.

14.5 Improper integrals

Consider the integral $\int_1^\infty \frac{1}{x^2}$ known as an improper integral.

Proceeding as usual, we have $\int_1^\infty \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^\infty$

As infinity is not a real number, we cannot continue any further.

To evaluate the integral, we consider $\int_1^t \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^t$

$$= -\frac{1}{t} + 1$$

$\int_1^\infty \frac{1}{x^2}\,dx$ is then the value of $-\frac{1}{t} + 1$, when $t \to \infty$, i.e. when t becomes very large.

As $t \to \infty$ $-\frac{1}{t}$ approaches 0.

So, $\int_1^\infty \frac{1}{x^2}\,dx = 1$

We will therefore write $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$

$$= \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t$$

$$= \lim_{t\to\infty}\left[-\frac{1}{t}+1\right]$$

$$= 1$$

Example 12

Find $\int_1^\infty e^{-x}dx$

Solution

$$\int_1^\infty e^{-x}\,dx \quad = \lim_{t\to\infty} \int_1^t e^{-x}\,dx$$

$$= \lim_{t\to\infty}\left[\frac{e^{-x}}{-1}\right]_1^t$$

$$= \lim_{t\to\infty}\left[\frac{e^{-t}}{-1}+e^{-1}\right]$$

$$= e^{-1} \text{ as } \lim_{t\to\infty} e^{-t} = 0$$

Sometimes the integral does not have a finite value as shown in the next example.

Example 13

Find $\int_1^{\infty} \frac{1}{x}\,dx$

Solution

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty}\left[\ln|t|\right]_1^t$$

$$= \infty, \text{ as } \left[\ln|t|\right] \to \infty \text{ as } t \to \infty$$

Exercise 14 E

Evaluate each of the following integrals:

1. $\int_{-\infty}^{2} e^{2x}\,dx$
2. $\int_{1}^{\infty} e^{-3x}\,dx$
3. $\int_{2}^{\infty} \frac{1}{x^3}\,dx$
4. $\int_{-\infty}^{-3} -\frac{1}{x^2}\,dx$
5. $\int_{1}^{\infty} \ln x\,dx$
6. $\int_{1}^{\infty} 5x^{-3}\,dx$
7. $\int_{-\infty}^{0} \frac{1}{\sqrt{4-x}}\,dx$
8. $\int_{-\infty}^{0} xe^{-x^2}\,dx$

Miscellaneous Exercise 14

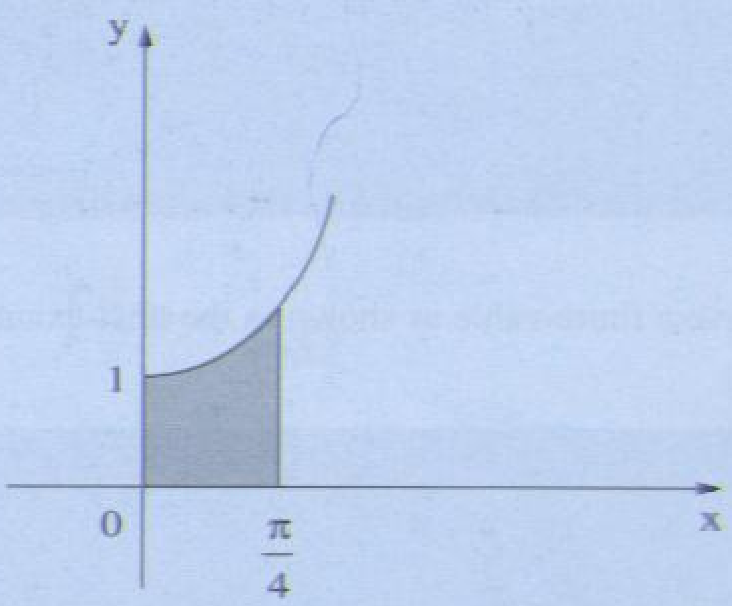
1. Find the exact value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\cos^2 x}\,dx$ [C]

2. Evaluate $\int_0^2 \sqrt{4-x^2}$ by means of the substitution $x = 2\sin\theta$ or otherwise. [C]

3. (a) Evaluate to two decimal places where necessary:

(i) $\int_{1}^{2} \frac{2}{x+1}\, dx$ (ii) $\int_{0}^{\frac{\pi}{6}} 3\sin 2x\, dx$

(b)



The figure shows part of the curve y = sec x.
Find the volume obtained when the shaded region is rotated through 360° about the x-axis. [C]

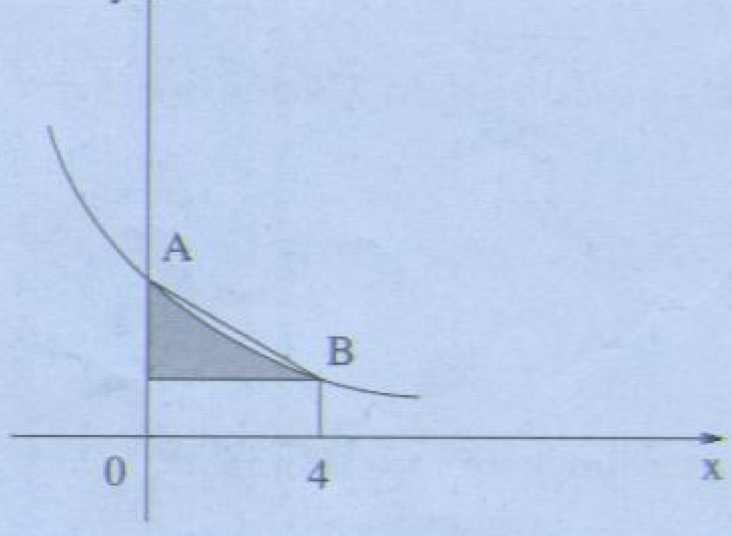
4. C is the curve given by y = sin x for values of x between 0 and π. Find:
   (a) the area of the region enclosed between C and the x-axis
   (b) the volume of the solid formed by rotating this region through four right angles about the x-axis. [C]

5. Use the trapezium rule with subdivisions at x = 3 and x = 5 to obtain an approximation to $\int_{1}^{7} \frac{x^3}{1+x^4}\, dx$, giving your answer correct to three places of decimals.
By evaluating the integral exactly, show that the error in the approximation is about 4.1%. [C]

6. Use the trapezium rule with ordinates at x = 1, x = 2 and x = 3 to estimate the value of $\int_{1}^{3} \sqrt{40-x^2}\, dx$. [C]

7. (a) Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3\cos x + 2\sin 2x)\, dx$.

(b)



The figure shows part of the curve $y = \frac{12}{x+2}$ and the chord AB. Find correct to two decimal places, the area of:

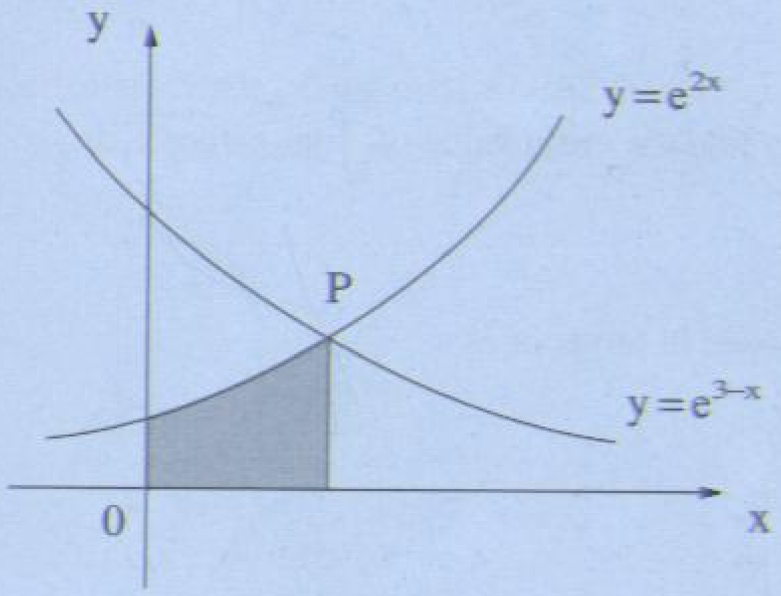
(i) the shaded region

(ii) the region enclosed by the curve and the chord AB. [C]

8. In the diagram, P is the point of intersection of the curves $y = e^{2x}$ and $y = e^{3-x}$. Find the x-coordinate of P and hence evaluate, to two decimal places:

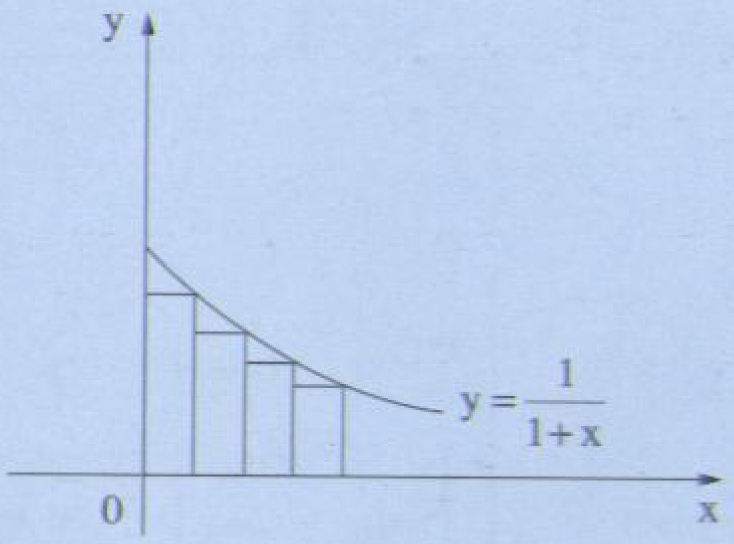
(i) the area of the shaded region

(ii) the area of the region bounded by the two curves and the y-axis.



[C]

9.



The diagram shows the part of the graph of $y = \frac{1}{1+x}$ between $x = 0$ and $x = 1$. The four rectangles drawn under the curve are of equal width and their total area is an approximation to the area under the curve from $x = 0$ to $x = 1$. Calculate this approximation to the area under the curve, giving 2 significant figures in your answer. When there are n rectangles of equal width under the curve between $x = 0$ and $x = 1$ (instead of just 4), find an expression for their total area. [C]

10. Express $\frac{6x+4}{(1-2x)(1+3x^2)}$ in partial fractions.

Hence, show that $\int_1^2 \frac{6x+4}{(1-2x)(1+3x^2)}\,dx = \ln\left(\frac{13}{36}\right)$. [C]

11. Find: (a) $\int x\sin 2x\,dx$ (b) $\int \frac{x-1}{x+1}\,dx$ [C

12. Find: (a) $\int xe^{4x}\,dx$ (b) $\int x^2\ln x\,dx$ [C

13. Find $\int \frac{1}{x(x+1)}\,dx$. [C

14. Show that $\int_1^2 \frac{6x+7}{(2x-1)(x+2)}\,dx = \ln 12$. [C

15. Use the substitution u = cos x to find the exact value of $\int_0^{\frac{\pi}{2}} \sin x\sqrt{\cos x}\,dx$. [C

16. Find $\int_0^1 xe^{-2x}\,dx$ leaving your answer in terms of e. [C

17. (a) Evaluate $\int_1^e \ln x\,dx$. [C

(b) Show that $\int_7^{10} \frac{7}{x^2-5x-6}\,dx = \ln\left(\frac{32}{11}\right)$. [C

18. (a) Find $\int \frac{(x-1)}{(x+4)(x+1)}\,dx$.

(b) Evaluate $\int_2^4 \frac{\ln x}{x}\,dx$

(i) by using the substitution u = ln x

(ii) by using the trapezium rule with four strips, giving your answer correct to three significant figures. [C

19. (a) Find $\int \frac{1}{(1+x)(2-x)}\,dx$.

(b) Use the substitution $x = \sin^2\theta$ to show that

$\int_0^1 \sqrt{\frac{1-x}{x}} = \int_0^{\frac{\pi}{2}} 2\cos^2\theta$ and evaluate either integral. [C

20. (a) By use of partial fractions, or otherwise, find $\int \frac{x}{x^2-2x-3}\,dx$.

(b) Using the substitution z = 1 – x, or otherwise, evaluate $\int_0^1 x^2(1-x)^{\frac{1}{2}}\,dx$. [C

CHAPTER 15

> NUMERICAL SOLUTIONS OF EQUATIONS

15.1 Location of the approximate root of an equation

We have considered solutions of different types of equations, linear, quadratic and cubic such as $3(x + 1) = -2(x - 5)$, $x^2 - 7x + 6 = 0$, $x^3 + 3x^2 - 7x - 6 = 0$. All the roots of these equations. i.e. the value(s) of x satisfying them can be obtained.

We now consider equations other than quadratic having roots which cannot be obtained exactly, e.g. $x^3 + 2x^2 - 6x - 1 = 0$. Before trying to solve this equation, we consider the graph of $y = f(x)$ where f is a continuous function in the interval in which the root lies.

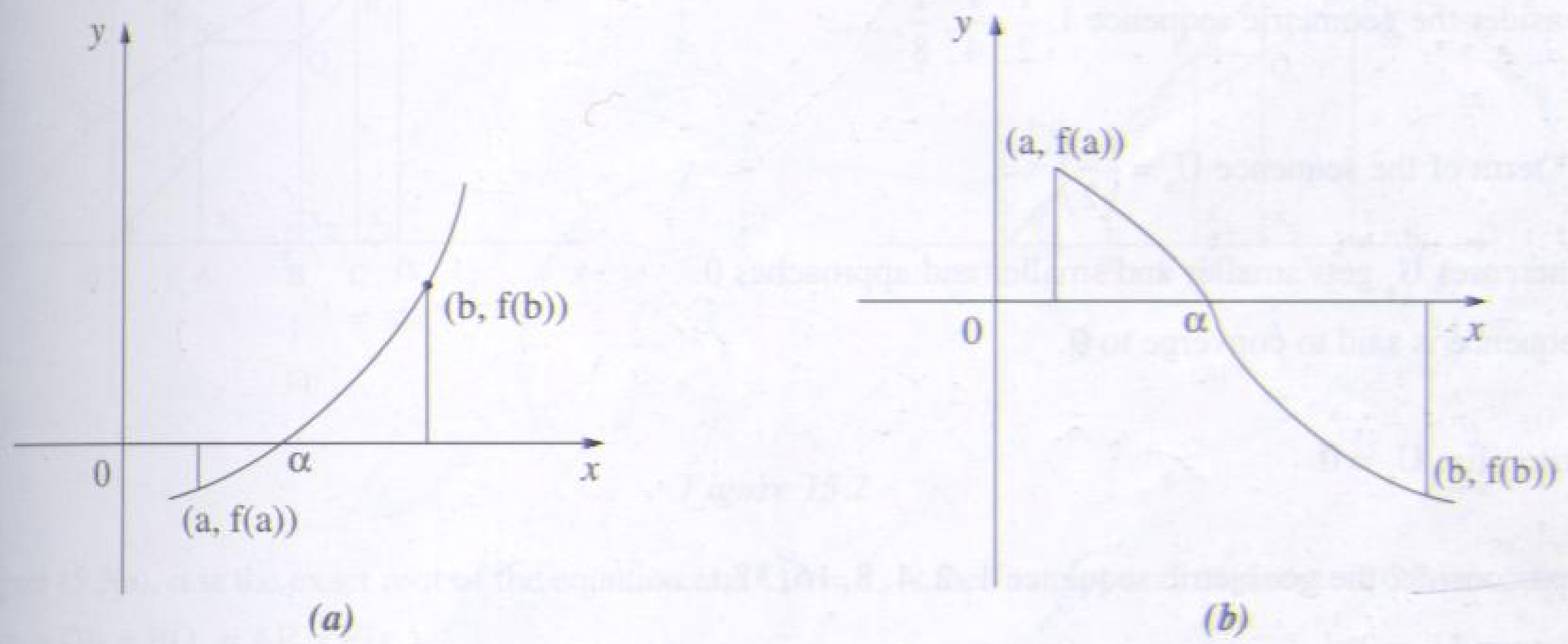


Figure 15.1

We note that if f(a) is of opposite sign to f(b), $f(\alpha) = 0$ for $a < \alpha < b$.

This result can be used to show that $f(x) = 0$ has a root in a given interval.

Example 1

Show that $x^3 + 2x^2 - 6x - 1 = 0$ has a root between 1 and 2.

Solution

Put $f(x) = x^3 + 2x^2 - 6x - 1$

$f(1) = -4$

$f(2) = 3$

Since f(1) is of opposite sign to f(2), f(x) has a root α for $1 < \alpha < 2$.

Example 2

Show that sin x = 2x – 1 has a root between 0 and 1.

Solution

Put $f(x) = \sin x - 2x + 1$
$f(0) = 1$
$f(1) = -0.159$

Since f(0) is of opposite sign to f(1), there is a root α for $0 < \alpha < 1$

Note: x is to be taken in radians.

15.1.2 Convergency and divergency

(a) Consider the geometric sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\ldots$

The nth term of the sequence $U_n = \left(\frac{1}{2}\right)^{n-1}$

As n increases U_n gets smaller and smaller and approaches 0.

The sequence is said to converge to 0.

We write : $\lim\limits_{n \to \infty} U_n = 0$

(b) Next consider the geometric sequence 1, 2, 4, 8, 16, 32,..........

The nth term $U_n = 2^{n-1}$

As n increases U_n gets bigger and bigger and does not approach any value.

The sequence is said to diverge.

In this case, $\lim\limits_{n \to \infty} U_n$ = infinity

15.1.3 Iterative method for solution of an equation

Whenever it is not possible to find an exact root of an equation f(x) = 0, we find a first approximation x_1. With this value of x_1, we find a better approximation x_2 and repeat this process until the approximate root x_{n+1} converges to a limit which is taken to be the root of the given equation. This method of finding the root of an equation is known as the iterative method.

15.1.4 Use of x = F(x) to solve an equation

Given the equation $f(x) = 0$, we write it in the form $x = F(x)$. There may be several ways of writing $f(x) = 0$ in the form $x = F(x)$.

Thus, we can write $2x^3 + x^2 - x - 3 = 0$ as either $x = 2x^3 + x^2 - 3$ or $x = \sqrt[3]{\dfrac{3+x-x^2}{2}}$ or $x = \pm\sqrt{3+x-2x^3}$

If x_1 is the first approximate root, then a better approximate root x_2 is given by $x_2 = F(x_1)$. A still better approximate root is x_3, where $x_3 = F(x_2)$. If x_{n+1} converges to a limit l then l is taken to be a root of the equation $f(x) = 0$.

An iteration procedure may fail to converge.

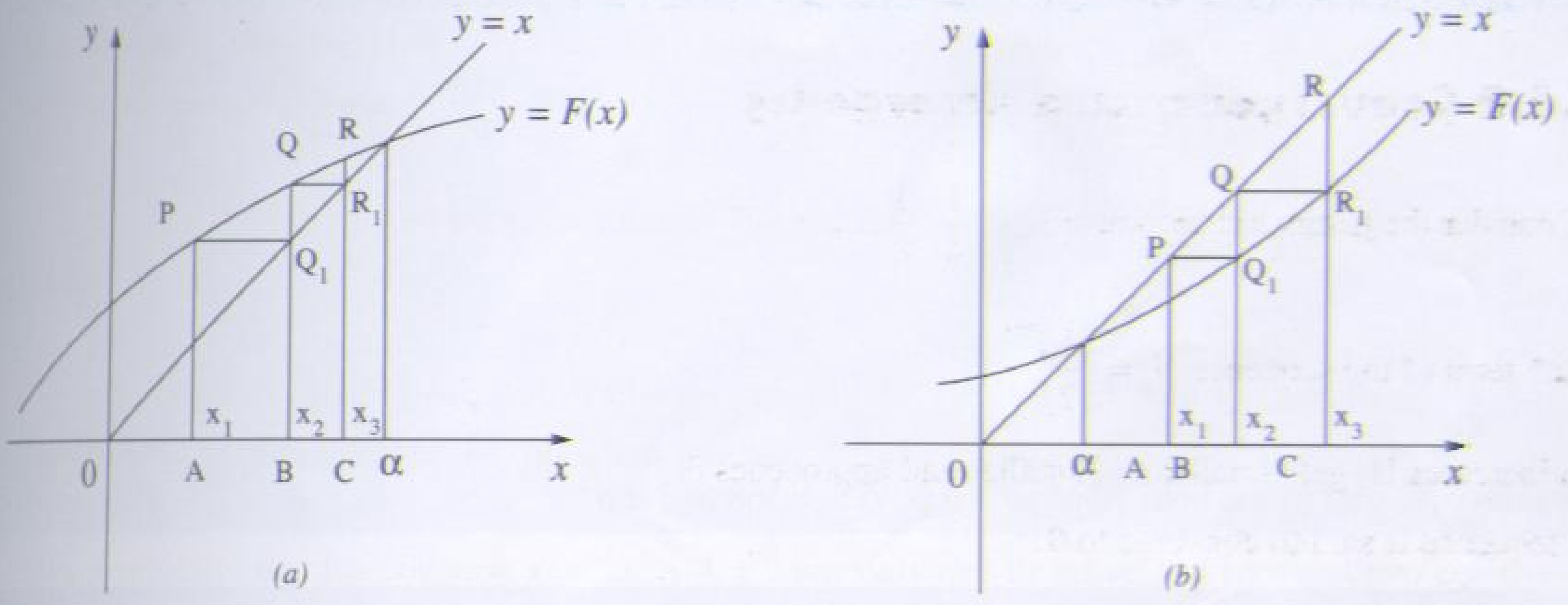


Figure 15.2

In Figure 15.2(a), α is the exact root of the equation and $OA = x_1$ is the first approximate root. If $OB = x_2$ is a better root, $x_2 = OB = BQ_1 = AP = F(x_1)$.

Similarly, if $OC = x_3$ is a better root, $x_3 = OC = CR_1 = BQ = F(x_2)$. Continuing the process $x_{n+1} = F(x_n)$. From the diagram, it can be seen that x_{n+1} converges to α.

In Figure 15.2(b), $OA = x_1$ is the first approximate root and OB the next approximate root, $x_2 = OB = BQ_1 = AP = F(x_1)$.

It can be seen that x_2 is further away from α than x_1. In this case, the sequence of roots diverges and the method fails.

Example 3

Show that the equation $3x^3 - x - 10 = 0$ has a root between 1 and 2. Use the iteration $x_{n+1} = f(x_n)$ to obtain the value of this root correct to 3 d.p.

Solution

$f(x) = 3x^3 - x - 10$

$f(1) = -8$

$f(2) = 12$

As f(1) is of opposite sign to f(2), there is a root α for $1 < \alpha < 2$.

We can write $x = 3x^3 - 10$ or $x = \sqrt[3]{\frac{x+10}{3}}$. So, $x_{n+1} = 3x_n^3 - 10$ or $x_{n+1} = \sqrt[3]{\frac{x_n+10}{3}}$

Using $x = 3x^3 - 10$

Take $x_1 = 1.5$

$x_2 = 3 \times 1.5^3 - 10 = 0.125$

$x_3 = 3 \times 0.125^3 - 10 = -9.99$

So, x_{n+1} diverges as n increases and the method fails.

Using $x_{n+1} = \sqrt[3]{\frac{x_n+10}{3}}$

$x_1 = 1.5$

$x_2 = 1.5650....$

$x_3 = \sqrt[3]{\frac{x_2+10}{3}}$

$= 1.5679$

$x_4 = \sqrt[3]{\frac{x_3+10}{3}}$

$= 1.5681....$

As values of x_3 and x_4 are both 1.568 to 3 d.p., $\alpha = 1.568$ to 3 d.p.

Example 4

Show that the equation $2e^{-x} - 5x + 2 = 0$ has a root between 0 and 1 and use the iteration $x_{n+1} = \frac{1}{5}(2x^{-x_n} +$ to find the value of this root correct to 3 places of decimals.

Solution

$f(x) = 2e^{-x} - 5x + 2$

$f(0) = 4$

$f(1) = -2.264$

So, there is a root α, for $0 < \alpha < 1$.

We reduce $2e^{-x} - 5x + 2$ to the form $x = F(x)$.

$x = \frac{1}{5}\left(2e^{-x} + 2\right)$

We use the iteration $x_{n+1} = \frac{1}{5}\left(2x^{-x_n} + 2\right)$

Put $x_1 = 0.5$

$x_2 = 0.6426....$

$x_3 = 0.6103....$

$x_4 = 0.6172....$

$x_5 = 0.6157....$

$x_6 = 0.6160....$

As x_5 and x_6 are both 0.616 to 3 d.p., $\alpha = 0.616$ to 3 d.p.

Exercise 15 A

1. Show that $x^3 - 2x^2 - 4 = 0$ has a root between 2 and 3. Use the iteration $x_{n+1} = \sqrt[3]{2x_n^2 + 4}$ to obtain the value of this root correct to 2 d.p.

2. Show that $e^x - 2x - 3 = 0$ has a root between 1 and 2. Use the iteration $x_{n+1} = \ln(2x_n + 3)$ to obtain the value of this root correct to 2 d.p.

3. Show that $\sin x + x - 2 = 0$ has a root between 1.1 and 1.2. Use the iteration $x_{n+1} = 2 - \sin x_n$ to obtain the value of this root correct to 3 d.p.

4. Show that $\ln x - x + 2 = 0$ has a root between 3 and 4. Use the iteration $x_{n+1} = \ln x_n + 2$ to obtain the value of this root correct to 2 d.p.

5. Show that $\ln x = e^{-x}$ has a root between 1 and 2. Use the iteration $x_{n+1} = e^{e^{-x_n}}$ with $x_1 = 1$ to obtain this root correct to 2 d.p.

6. Sketch the graph of $y = x^3$. Hence, show that the equation $x^3 = 2x + 3$ has only one real root. Show that this root lies between 1 and 2 and obtain its value correct to 2 d.p.

7. Show graphically that the equation $x^3 - 2x^2 + 4 = 0$ has only one negative real root and two positive real roots. Denoting the negative root by α, find the integer value of k such that $-k < \alpha < -k + 1$ and find α correct to 2 d.p, taking -1.12 to be the first approximation.

8. Show graphically or otherwise that $e^x + x^2 - 5 = 0$ has only two real roots, one positive and one negative. Denoting the positive root by α and the negative root by β, obtain two integers k and l such that $k < \alpha < k + 1$ and $l < \beta < l + 1$. Obtain the values of α and of β correct to 2 d.p.

9. Show that $2\sin x - x + 1 = 0$ has a root between $\frac{\pi}{2}$ and π. Starting with x_1 as 2.35, use the iteration $x_{n+1} = F(x_n)$ to find out whether x_n converges when:

   (a) $F(x_n) = 2\sin x_n + 1$ (b) $F(x_n) = \sin^{-1}\left(\frac{x_n - 1}{2}\right)$.

Miscellaneous Exercise 15

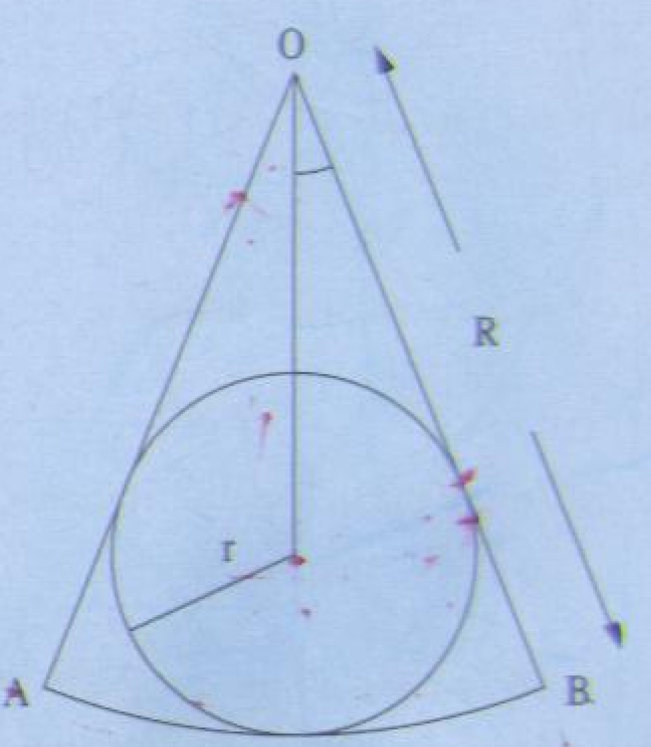
1. Show that the equation $x = \dfrac{1}{2 + \sqrt{x}}$ has a root between 0 and 1. By using an iterative formula of the form $x_{n+1} = F(X_n)$, find α correct to two decimal places.

   You should show clearly your sequence of approximations. [C]

2. Show, by means of a graphical argument, that the equation $x = 10^{-\frac{1}{4}x}$ has exactly one real root (denoted by α) and determine the pair of consecutive integers between which α lies. [C]

The iterative formula $x_{n+1} = 10e^{-\frac{1}{4}x_n}$ can be used to find α. Starting with the nearest integer above α, carry out four applications of this iteration and state the number of significant figures to which α has been determined as a result of these calculations.

3.



A new logo design is in the form of a sector OAB of a circle of radius R and angle 2θ radians where $0 < \theta < \frac{1}{2}\pi$. The logo includes a circle of radius r inscribed in the sector in such a way that it touches OA and OB and the arc AB (see diagram).

Show that $R = r\left(1 + \frac{1}{\sin\theta}\right)$

It is required that the area of the circle must be half the area of the sector. Show that $\theta = 2\pi\left(\frac{\sin\theta}{1+\sin\theta}\right)^2$

It is given that this equation has a solution θ = α, where $0.2 < \alpha < 0.3$.

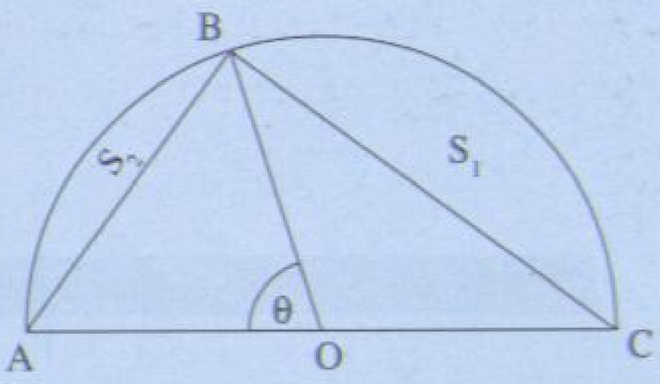
Starting with $\theta_1 = 0.3$ the iteration $\theta_{n+1} = 2\pi\left(\frac{\sin\theta_n}{1 + \sin\theta_n}\right)^2$, gives $\theta_2 = 0.327$, $\theta_3 = 0.371$, approximately.

Comment on the suitability of this iteration as a means of finding α.

Rearrange the equation $\theta = 2\pi\left(\frac{\sin\theta}{1 + \sin\theta}\right)^2$ to obtain $\theta = \sin^{-1}\left(\frac{1}{\sqrt{\frac{2\pi}{\theta}} - 1}\right)$

Use an iterative method based on this rearranged equation, starting with $\theta_1 = 0.25$, to find θ_2 and θ_3 giving your answers correct to 3 decimal places. [C]

4.



The diagram shows a semicircle ABC on AC as diameter. The mid-point of AC is O and angle AOB = θ radians, where $0 < \theta < \frac{1}{2}\pi$. The area of the segment S_1 bounded by the chord BC is twice the area of the segment S_2 bounded by the chord AB. Show that $3\theta = \pi + \sin\theta$.

Use the iterative formula $\theta_{n+1} = \frac{1}{3}(\pi + \sin\theta_n)$, together with a suitable starting value, to find correct to three-significant figures. You should show the value of each approximation you calculate. [C]

5. Find $\int xe^{-x}\,dx$.

The value of X is such that $\int_0^X xe^{-x}\,dx = 0.9$. Find an equation for X and show that it may be written in the form $X = \ln(10 + 10X)$. Use the iteration $X_{n+1} = \ln(10 + 10X_n)$, with initial approximation $X_1 = 4$, to find the value of X correct to three significant figures. You should show the value of each approximation that you calculate.

[C]

CHAPTER 16

> VECTORS 2

16.1 Vector equation of a straight line

16.1.1 Equation of line through a given point A with position vector *a* parallel to a vector *b*

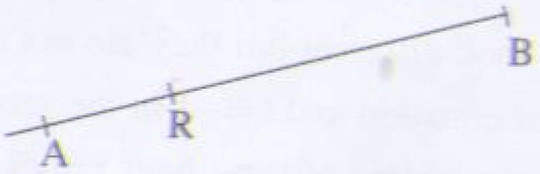
The cartesian equation of a line gives the relation between the coordinates of any point on the line. A vector equation of a line gives the position vector of any point on the line.

Taking a point R with position vector **r** on the line through A, we have

AR = tb where t is a real parameter

$\mathbf{r} - \mathbf{a} = t\mathbf{b}$

$\mathbf{r} = \mathbf{a} + t\mathbf{b}$



Example 1

Find a vector equation of the line through (2, 3, –4) parallel to the vector $2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$.

Solution

Equation of line is $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + t(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$

$= (2 + 2t)\mathbf{i} + (3 - t)\mathbf{j} + (-4 + 5t)\mathbf{k}$

Example 2

Find a vector equation of the line through (–2, 1, 5) parallel to the line passing through (1, –2, 3) and (–3, 0, 1).

Solution

A vector in the direction of the line is $-3\mathbf{i} + 0\mathbf{j} + \mathbf{k} - (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = -4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

Equation of line is $\mathbf{r} = -2\mathbf{i} + \mathbf{j} + 5\mathbf{k} + t(-4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$

$= (-2 - 4t)\mathbf{i} + (1 + 2t)\mathbf{j} + (5 - 2t)\mathbf{k}$

16.1.2 Equation of line through 2 points A and B

For any point R on the line

$$\begin{aligned} AR &= tAB \\ \mathbf{r} - \mathbf{a} &= t(\mathbf{b} - \mathbf{a}) \\ \mathbf{r} &= \mathbf{a} + t(\mathbf{b} - \mathbf{a}) \end{aligned}$$

Example 3

Find a vector equation of the line passing through the points (1, –2, 3) and (3, 0, –2).

Solution

Taking A to be (1, –2, 3) and B (3, 0, –2)

$\mathbf{b} - \mathbf{a} = 2\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$

So, equation of line is $\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$

$$\begin{aligned}\mathbf{r} &= \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + t(2\mathbf{i} + 2\mathbf{j} - 5\mathbf{k})\\ &= (1 + 2t)\mathbf{i} + (-2 + 2t)\mathbf{j} + (3 - 5t)\mathbf{k}\end{aligned}$$

Note: It is immaterial whether A is taken to be (1, –2, 3) or (3, 0, –2).

If A is taken to be (3, 0, –2), B(1, –2, 3)

Equation of line is $\mathbf{r} = 3\mathbf{i} + 0\mathbf{j} - 2\mathbf{k} + t(-2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k})$

$$= (3 - 2t)\mathbf{i} - 2t\mathbf{j} + (-2 + 5t)\mathbf{k}$$

The equations look different but they are not as can be checked by giving different values to t in each case, e.g. $t = 2$ in the first equation and $t = -1$ in the second.

For these different values of t, we have $\mathbf{r} = 5\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}$ in both cases.

16.1.3 Intersection of lines

In two dimensions, two lines always intersect unless they are parallel. In three dimensions, two lines may not intersect even if they are not parallel.

Two non–parallel lines which do not intersect are called skew lines.

Consider the lines l and m with equations

$\mathbf{r} = (1 - 2t)\mathbf{i} + (3 + 2t)\mathbf{j} + (4 - 3t)\mathbf{k}$

$\mathbf{r} = (1 + 2s)\mathbf{i} + (2 - 3s)\mathbf{j} + (3 + 2s)\mathbf{k}$

To find whether or not they intersect, we solve their equations.

If they intersect $\quad 1 - 2t = 1 + 2s \qquad (1)$

$3 + 2t = 2 - 3s \qquad (2)$

$4 - 3t = 3 + 2s \qquad (3)$

We note that we have 3 equations in 2 unknowns. These have solutions only if they are consistent, i.e. they are all satisfied by the same value of t and of s.

(1) + (2) give $4 = 3 - s$

$s = -1$

Substituting in (2), $3 + 2t = 2 + 3$

$2t = 2$

$t = 1$

We check whether $s = -1$ and $t = 1$ obtained by solving equations (1) & (2) satisfy equation (3).

$t = 1, 4 - 3t = 1$

$s = -1, 3 + 2s = 3 - 2 = 1$

So, the equations are consistent and the lines intersect at the point with position vector $\mathbf{r} = -\mathbf{i} + 5\mathbf{j} + \mathbf{k}$, i.e. at $(-1, 5, 1)$.

Example 4

Find a vector equation of the line through (0, 0, 0) and (1, 1, 1) and a vector equation of the line through (0, 1, 1) and (1, 1, 0). Do these two lines intersect? Find the coordinates of the point of intersection, if any.

Solution

Equation of line through (0, 0, 0) and (1, 1, 1) is

$\mathbf{r} = t(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$= t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$

Equation of line through (0, 1, 1) and (1, 1, 0) is

$\mathbf{r} = 0\mathbf{i} + \mathbf{j} + \mathbf{k} + s\left[(\mathbf{i} + \mathbf{j} + 0\mathbf{k} - (0\mathbf{i} + \mathbf{j} + \mathbf{k})\right]$

$= \mathbf{j} + \mathbf{k} + s(\mathbf{i} - \mathbf{k})$

$= s\mathbf{i} + \mathbf{j} + (1 - s)\mathbf{k}$

Note: We use different letters for the parameters for the two lines.

At point of intersection, if any,

$t\mathbf{i} + t\mathbf{j} + t\mathbf{k} = s\mathbf{i} + \mathbf{j} + (1 - s)\mathbf{k}$

$t = s$ (1)

$t = 1$ (2)

$t = 1 - s$ (3)

From (1) & (2), $t = s = 1$

Equation (3) is not satisfied by $t = 1, s = 1$

The equations (1), (2) and (3) are therefore not consistent. So, the lines do not intersect.

Exercise 16 A

1. Find a vector equation for the straight line passing through:
   - (a) O and the point A(1, –2, –3)
   - (b) O and the point B(2, –1, –5)
   - (c) A and B
   - (d) C(–1, 2, –4) and D(0, –2, 1)
   - (e) E(3, –5, 1) and F(–5, 1, 2)
   - (f) A, parallel to the vector $-\mathbf{i} + \mathbf{j} + \mathbf{k}$
   - (g) B, parallel to the vector $2\mathbf{i} - \mathbf{j}$
   - (h) C, parallel to the vector $3\mathbf{i} - 5\mathbf{k}$
   - (i) A, parallel to **BC**
   - (j) C, parallel to **DE**
   - (k) E, parallel to **AF**
   - (l) F, parallel to **AB**

2. Find the coordinates of the point of intersection, if any, of the following pairs of lines:
   - (a) $\mathbf{r} = (1 - 2t)\mathbf{i} + (3 + t)\mathbf{j} + (1 - 3t)\mathbf{k}$ and $\mathbf{r} = s\mathbf{i} + (1 - s)\mathbf{j} + (s - 3)\mathbf{k}$
   - (b) $\mathbf{r} = (2 + 3t)\mathbf{i} + (-3 + 2t)\mathbf{j} + (5 - t)\mathbf{k}$ and $\mathbf{r} = (1 + 4s)\mathbf{i} + (s - 2)\mathbf{j} + 4s\mathbf{k}$
   - (c) $\mathbf{r} = (1 + 2t)\mathbf{i} + (2 - 3t)\mathbf{j} + (5 - 2t)\mathbf{k}$ and $\mathbf{r} = (6 - s)\mathbf{i} - 4s\mathbf{j} + (2 - s)\mathbf{k}$
   - (d) $\mathbf{r} = (5 - 3t)\mathbf{i} + (2 + 3t)\mathbf{j} + (5 - t)\mathbf{k}$ and $\mathbf{r} = (-1 + 3s)\mathbf{i} + (12 - 2s)\mathbf{j} + (3 + s)\mathbf{k}$
   - (e) $\mathbf{r} = (1 + t)\mathbf{i} + (2 - 5t)\mathbf{j} + (1 - 3t)\mathbf{k}$ and $\mathbf{r} = (2 - s)\mathbf{i} + (1 + 3s)\mathbf{j} + (3 - 2s)\mathbf{k}$

3.

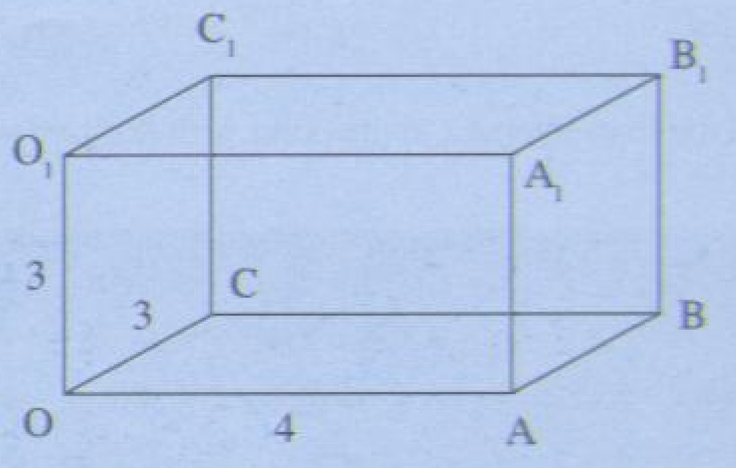


Figure 16.1

The diagram shows a cuboid with OA = 4 units, OC = 3 units and OO_1 = 3 units. OA, OC and OO_1 are mutually perpendicular and OABC, $O_1A_1B_1C_1$ are rectangles.

Taking OA, OC and OO_1 to represent the positive directions of the x–axis, y–axis and z–axis respectively, find the coordinates of all the vertices of the cuboid.

Find the vector equations of the lines OB_1 and AC_1. Do these lines intersect? If so, find the coordinates of the point of intersection.

16.2.1 Parallel lines

In the last section, we obtained vector equations of lines through 2 points or through a point parallel to a given vector. Given the vector equation of a line, we may find the coordinates of points on the line and the direction of the line.

Thus, the equation $\mathbf{r} = (1 + 2t)\mathbf{i} + (3 - 4t)\mathbf{j} + (2 + 3t)\mathbf{k}$ can be rewritten as $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k} + t(2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k})$.

It therefore represents a straight line through (1, 3, 2) parallel to the vector $2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$. By giving different values to the parameter t, we can obtain different points on the line. Thus, if t = 1, we have the point (3, –1, 5) on the line.

By considering the vector equations of two lines, we may find out whether or not they are parallel.

Example 5

Find out whether the lines with equations $\mathbf{r} = (-1 + 2t)\,\mathbf{i} + (3 - t)\,\mathbf{j} + (-5 + 3t)\,\mathbf{k}$ *and* $\mathbf{r} = (2 - 4s)\,\mathbf{i} + (1 + 2s)\,\mathbf{j} + (-3 - 6s)\,\mathbf{k}$ *are parallel or not.*

Solution

Rewriting the equations as

$\mathbf{r} = (-\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}) + t(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) + s(-4\mathbf{i} + 2\mathbf{j} - 6\mathbf{k})$, the first equation represents a line through (–1, 3, –5) parallel to the vector $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and the second equation represents a line through (2, 1, –3) parallel to the vector $-4\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$.

As $-4\mathbf{i} + 2\mathbf{j} - 6\mathbf{k} = -2(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$, the two vectors are parallel.

Hence, the lines being parallel to these two vectors are themselves parallel.

16.2.2 Angle between two lines

We recall that the angle θ between two vectors a and b is given by $\cos\theta = \dfrac{\mathbf{a}\,.\,\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$. To find the angle between two lines, we therefore find the angle between the vectors in the directions of the lines.

Example 6

Find the angle between the lines with equations $\mathbf{r} = (1 - t)\mathbf{i} + (2 + 3t)\mathbf{j} + (1 - t)\mathbf{k}$ *and* $r = (2 + 3s)\mathbf{i} + (4 - s)\mathbf{j} + (5 + s)\mathbf{k}$.

Solution

We rewrite the equations as $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(-\mathbf{i} + 3\mathbf{j} - \mathbf{k})$ and $\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} + s(3\mathbf{i} - \mathbf{j} + \mathbf{k})$.

A vector in the direction of the first line is $-\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and a vector in the direction of the second line is $3\mathbf{i} - \mathbf{j} + \mathbf{k}$.

If θ is the angle between the two lines,

$$\cos\theta = \frac{(-\mathbf{i} + 3\mathbf{j} - \mathbf{k})\,.\,(3\mathbf{i} - \mathbf{j} + \mathbf{k})}{|-\mathbf{i} + 3\mathbf{j} - \mathbf{k}||3\mathbf{i} - \mathbf{j} + \mathbf{k}|}$$

$$= \frac{-3 - 3 - 1}{\sqrt{1 + 9 + 1}\sqrt{9 + 1 + 1}}$$

$$= -\frac{7}{11}$$

$\theta = 129.5^\circ$ (to 1 d.p)

Exercise 16 B

1. Find out which of the following pairs of straight lines are parallel. If they are not parallel, find out if they are skew or whether they intersect. If they are neither parallel nor skew, find the coordinates of their point of intersection:

   (a) $\mathbf{r} = (1 - 2t)\mathbf{i} + (-1 + t)\mathbf{j} + (1 - 3t)\mathbf{k}$ and $\mathbf{r} = (-1 + 4s)\mathbf{i} + (1 - 2s)\mathbf{j} + (-1 - 6s)\mathbf{k}$

   (b) $\mathbf{r} = (1 - 3t)\mathbf{i} + (-1 + 5t)\mathbf{j} + (-1 - 3t)\mathbf{k}$ and $\mathbf{r} = (-2 + 6s)\mathbf{i} + (1 - 10s)\mathbf{j} + (1 + 6s)\mathbf{k}$

   (c) $\mathbf{r} = (-1 - 4t)\mathbf{i} + (-1 - 5t)\mathbf{j} + (-1 - 3t)\mathbf{k}$ and $\mathbf{r} = (2 - 10s)\mathbf{i} + (4 - 15s)\mathbf{j} + (-3 + 5s)\mathbf{k}$

   (d) $\mathbf{r} = (1 - 2t)\mathbf{i} + (3 + t)\mathbf{j} + (1 - 2t)\mathbf{k}$ and $\mathbf{r} = (2 + s)\mathbf{i} + (1 - 2s)\mathbf{j} + (3 - 2s)\mathbf{k}$

   (e) $\mathbf{r} = (2 + 3t)\mathbf{i} + (1 - 2t)\mathbf{j} + (3 - 4t)\mathbf{k}$ and $\mathbf{r} = (1 - 6s)\mathbf{i} + (3 + 6s)\mathbf{j} + (5 + 8s)\mathbf{k}$

   (f) $\mathbf{r} = (1 - 4t)\mathbf{i} + (2 + t)\mathbf{j} + (1 - t)\mathbf{k}$ and $\mathbf{r} = (2 + 3s)\mathbf{i} + s\mathbf{j} + 2s\mathbf{k}$

   (g) $\mathbf{r} = (5 + 3t)\mathbf{i} + (2 - 3t)\mathbf{j} + (5 - t)\mathbf{k}$ and $\mathbf{r} = (6 + 9s)\mathbf{i} + (4 - 9s)\mathbf{j} + (1 - 3s)\mathbf{k}$

   (h) $\mathbf{r} = (1 - t)\mathbf{i} + (2 - 5t)\mathbf{j} + (-2 + 3t)\mathbf{k}$ and $\mathbf{r} = -3s\mathbf{i} + 7s\mathbf{j} + (s - 3)\mathbf{k}$.

2. Find the angle between each of the following pairs of lines:

   (a) $\mathbf{r} = (1 + 2t)\mathbf{i} + (-1 - 3t)\mathbf{j} + (1 - t)\mathbf{k}$ and $\mathbf{r} = (-1 - s)\mathbf{i} + (-1 + 3s)\mathbf{j} + (-3 + s)\mathbf{k}$

   (b) $\mathbf{r} = (-1 - t)\mathbf{i} + (1 - 3t)\mathbf{j} + (-1 + 2t)\mathbf{k}$ and $\mathbf{r} = (1 + 2s)\mathbf{i} + (-1 - s)\mathbf{j} + (4 + 2s)\mathbf{k}$

   (c) $\mathbf{r} = (1 + 9t)\mathbf{i} + (1 + 4t)\mathbf{j} + (-1 - 8t)\mathbf{k}$ and $\mathbf{r} = (1 - 3s)\mathbf{i} + (1 - 2s)\mathbf{j} + (-1 - s)\mathbf{k}$

(d) $\mathbf{r} = (1 - t)\mathbf{i} + (2 + t)\mathbf{j} + (3 - 2t)\mathbf{k}$ and $\mathbf{r} = (3 + 2s)\mathbf{i} + (1 + s)\mathbf{j} + (4 - s)\mathbf{k}$

(e) $\mathbf{r} = (2 + 3t)\mathbf{i} + (1 - 2t)\mathbf{j} + (-5 + t)\mathbf{k}$ and $\mathbf{r} = (1 - 2s)\mathbf{i} + (3 + s)\mathbf{j} + (1 - 3s)\mathbf{k}$

(f) $\mathbf{r} = (2 - t)\mathbf{i} + (1 - t)\mathbf{j} + (3 + 4t)\mathbf{k}$ and $\mathbf{r} = (-1 - 2s)\mathbf{i} + (3 - 4s)\mathbf{j} + (-2 + s)\mathbf{k}$

(g) $\mathbf{r} = (3 + 2t)\mathbf{i} + (1 - 2t)\mathbf{j} + (4 - t)\mathbf{k}$ and $\mathbf{r} = (-2 + 2.5s)\mathbf{i} + (-1 - s)\mathbf{j} + (-1 + 2.5s)\mathbf{k}$

16.3.1 Normal vectors

A line is perpendicular to a plane if it is perpendicular to all lines in the plane. A vector is normal to a plane if it is perpendicular to any vector in the plane.

Theorem:
If a vector **n** is perpendicular to any two non–parallel vectors **a** and **b** in a plane, it is perpendicular to the plane.

Proof:

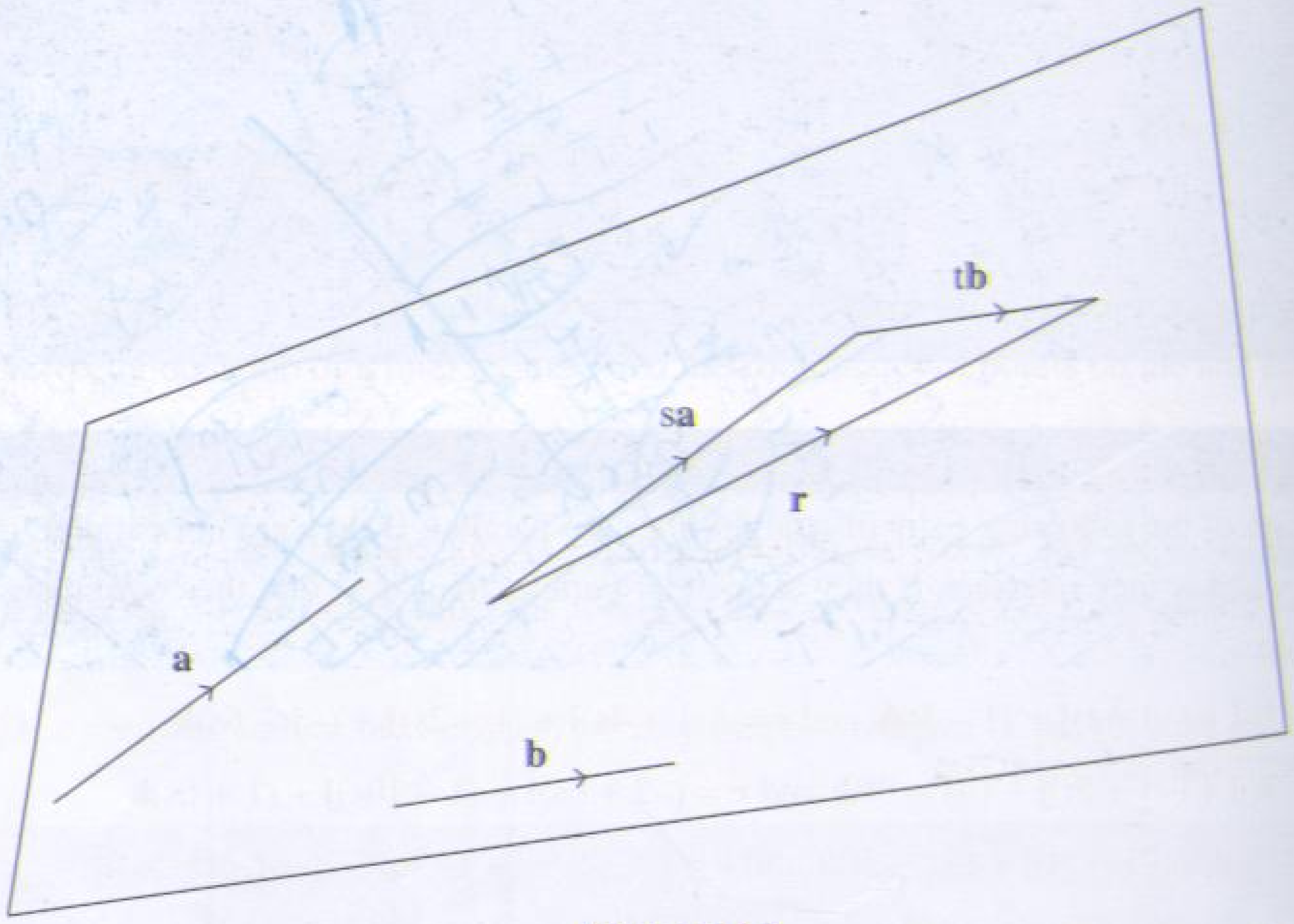


Figure 16.2

Consider two non–parallel vectors **a** and **b** in the plane. Any vector **r** in the plane can be written as sa + tb where s and t are parameters.

$\mathbf{r} = s\mathbf{a} + t\mathbf{b}$

To show **n** is perpendicular to the plane, we show **n** is perpendicular to **r**.

$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot (s\mathbf{a} + t\mathbf{b})$

$\quad = \mathbf{n} \cdot s\mathbf{a} + \mathbf{n} \cdot t\mathbf{b}$

$\quad = s(\mathbf{n} \cdot \mathbf{a}) + t(\mathbf{n} \cdot \mathbf{b})$

$\quad = s \times 0 + t \times 0$ (**n** is perpendicular to **a** and **b**)

$\quad = 0$

So, **n** is perpendicular to **r**.

Hence, **n** is normal to the plane.

To find a vector normal to a plane, it is therefore sufficient to consider only two non–parallel vectors in the plane.

Example 7

Find a normal vector to the plane containing the points (1, –1, 2), (2, 1, –1) and (–1, 2, 1).

Solution

Put A = (1, –1, 2), B = (2, 1, –1) and C = (–1, 2, 1).

AB = (**i** + 2**j** – 3**k**)

AC = (–2**i** + 3**j** – **k**)

Let normal vector be p**i** + q**j** + r**k**

n . AB = 0 and **n** . **AC** = 0

$p + 2q - 3r = 0 \quad (1)$

$-2p + 3q - r = 0 \quad (2)$

These two equations can be solved for p and q in terms of r.

$2p + 4q - 6r = 0 \quad (3), (1) \times 2$

$7q - 7r = 0 \quad (2) + (3)$

$q = r$

From (1) $p + 2r - 3r = 0$

$p = r$

So, a vector normal to plane is r**i** + r**j** + r**k**.

We can choose any value for r say 1. So, a vector normal to plane is **i** + **j** + **k**

Note: Any multiple of ***i*** *+* ***j*** *+* ***k*** *is a vector normal to plane.*

16.3.2 Equation of plane

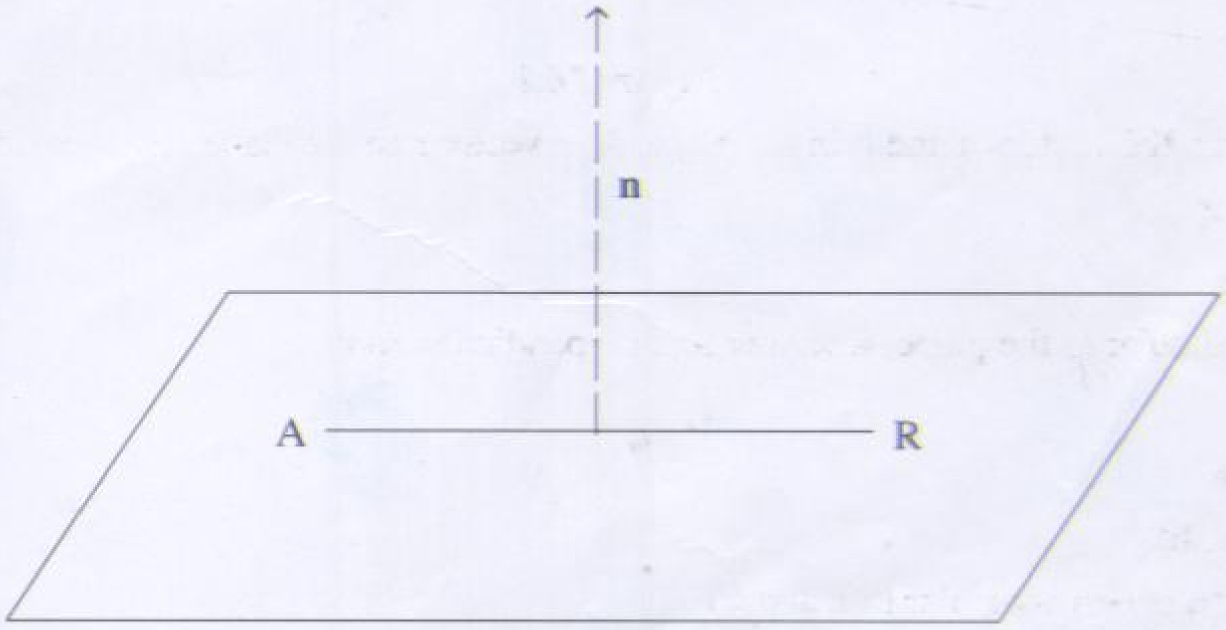


Figure 16.3

Consider a fixed point A and a general point R in a plane. If **n** is a normal vector to the plane, **n** is perpendicular to **AR**.

Hence, $\mathbf{n} \cdot \mathbf{AR} = 0$

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{a}) = 0$$

This equation is the equation of the plane. This can be rewritten as $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{a}$.

If $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is a vector normal to plane, R is the general point (x, y, z) and A is a fixed point (x_1, y_1, z_1) we have:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$ax + by + cz = ax_1 + by_1 + cz_1$$

$$= k \text{ (say)}$$

$ax + by + cz - k = 0$ or $ax + by + cz + d = 0$

This is the cartesian equation of the plane.

It also follows that if the cartesian equation of a plane is $ax + by + cz + d = 0$, a vector normal to the plane is $ai + bj + ck$ or $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Example 8

$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ *is a vector normal to a plane which contains the point (3, 4, −1). Find the cartesian equation of the plane.*

Solution

The cartesian equation of a plane is of the form $ax + by + cz + d = 0$.

$a = 2, b = -1$ and $c = 3$

So, equation is $2x - y + 3z + d = 0$

As the plane contains the point (3, 4, −1)

$$6 - 4 - 3 + d = 0$$

$$d = 1$$

Equation of plane is $2x - y + 3z + 1 = 0$

16.3.3 Use of r . n = a . n to find the cartesian equation of a plane containing three non-collinear points.

The method is illustrated in the following example:

Example 9

Find the cartesian equation of the plane containing (1, 1, 2), (3, –1, 4) and (–2, 3 ,–2).

Solution

We find a vector normal to the plane.

Taking A as (1, 1, 2), B as (3, –1, 4) and C as (–2, 3, –2)

$$\mathbf{AB} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} \text{ and } \mathbf{AC} = \begin{pmatrix} -3 \\ 2 \\ -4 \end{pmatrix}$$

Let a vector normal to plane be $\mathbf{n} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

$\mathbf{n} . \mathbf{AB} = 0$ and $\mathbf{n} . \mathbf{AC} = 0$

$2p - 2q + 2r = 0$

$p - q + r = 0$ (1)

$-3p + 2q - 4r = 0$ (2)

$2p - 2q + 2r = 0$ (3), multiplying (1) by 2

$-p - 2r = 0$ (2) + (3)

$p = -2r$

In (1) $p - q + r = 0$

$-2r - q + r = 0$

$q = -r$

Taking $r = -1$, $p = 2$ and $q = 1$

So, a vector normal to plane is $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

The cartesian equation is $2x + y - z + d = 0$

Choosing any point in the plane, e.g. (1, 1, 2)

$2 + 1 - 2 + d = 0$

$d = -1$

Equation of plane is $2x + y - z - 1 = 0$

Exercise 16 C

1. Find a normal vector to each of the planes containing the three given points and hence obtain the cartesian equation of each plane:

   (a) (1, 1, 1), (3, 2, 4) and (–2, 3, 0)
   (b) (0, 1, 0), (3, 1, 3) and (3, –1, 2)
   (c) (1, 0, –1), (3, –1, 3) and (–1, 2, –7)
   (d) (1, –2, 2), (3, 0, 6) and (2, –3, –4)
   (e) (–1, 3, 0), (2, –6, 1) and (0, 0, –2)

2. Show that **i** – **j** + 2**k** is perpendicular to the plane containing the points (1, 0, 1), (–1, 2, 3) and (4, 3, 1). Show also that (2, 7, 4) lies in this plane.

3. A plane is normal to the vector $\begin{pmatrix} -1 \\ 2 \\ -5 \end{pmatrix}$ and contains the point (2, 1, 0). Find its cartesian equation.

4. Write down the cartesian equation of the plane through the origin having normal vector 2**i** – **j** + **k**. Find the equation of the plane through the point (1, –2, 3) parallel to this plane.

5. Write down the cartesian equation of the plane through (–1, 1, 2) parallel to the plane $2x + y + z = 5$.

6. Find the cartesian equation of the plane through (–1, 1, 2) and (2, 0, –1) perpendicular to the plane $x + y - 2z = 1$.

16.4 Line and Plane

16.4.1 Condition for a line to lie in a plane

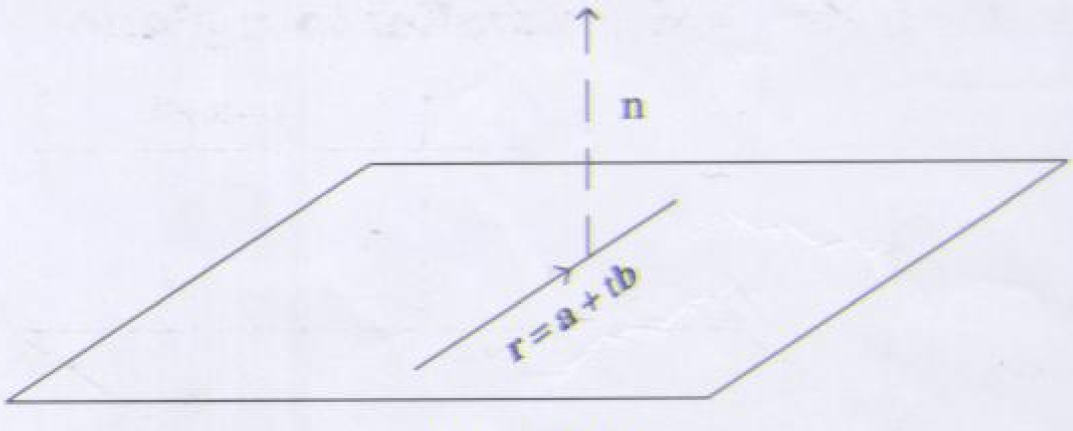


Figure 16.4

We know from previous work that the equation of a plane can be written as **n** . **r** = **d**. For the line **r** = **a** + t**b** to lie entirely in the plane, **n** . (**a** + t**b**) = d for all values of t.

Note that **n** . **b** = 0

Example 10

Show that the line $\mathbf{r} = (2\mathbf{i} - \mathbf{j} + \mathbf{k}) + t(2\mathbf{i} - \mathbf{j} - \mathbf{k})$ *lies in the plane* $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2$.

Solution

A vector normal to plane is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$$\begin{pmatrix} 2+2t \\ -1-t \\ 1-t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2 + 2t - 1 - t + 1 - t$$

$$= 2 \text{ for all values of t}$$

So, line lies in the plane.

Example 11

Show that the line with equation $\mathbf{r} = (1 + t)\mathbf{i} + (2 + t)\mathbf{j} + (1 + t)\mathbf{k}$ *lies entirely in the plane* $2x + y - 3z = 1$.

Solution

A vector normal to the plane is $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$.

$$\begin{pmatrix} t+1 \\ t+2 \\ t+1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = 1$$

Hence, the line lies in the plane.

16.4.2 Condition for a line to be parallel to a plane

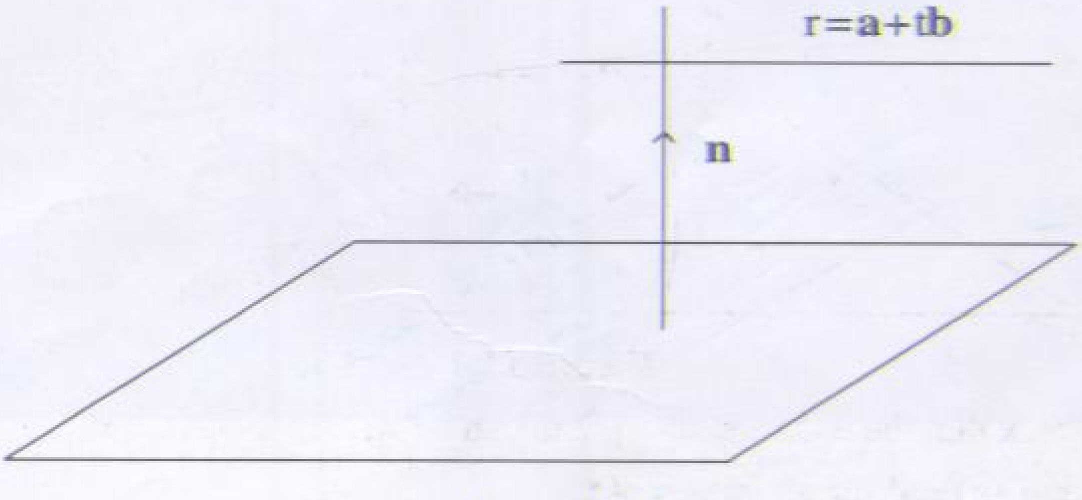


Figure 16.5

If a line is parallel to a plane, then the normal vector to the plane must be perpendicular to the direction vector of the line. If the normal vector to the plane is **n** and the line has equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, $\mathbf{n} \cdot \mathbf{b} = 0$.

Example 12

Show that the line $\mathbf{r} = (1 - t)\mathbf{i} + (-1 + 2t)\mathbf{j} + (-1 + t)\mathbf{k}$ *is parallel to the plane* $4x + 3y - 2z = 1$.

Solution

The plane has normal vector $\mathbf{n} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$

Line does not lie in plane as $\mathbf{n} \cdot \mathbf{r} \neq 1$ for all t.

The line has direction vector $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

$$\begin{aligned} \mathbf{n} \cdot \mathbf{b} &= -4 + 6 - 2 \\ &= 0 \end{aligned}$$

So, the line is parallel to the plane.

16.4.3 Intersection of a line and a plane

If a line is not parallel to a plane or it does not lie in the plane it intersects the plane at a point. The coordinates of the point are obtained by solving their equations.

Example 13

Find the point of intersection of the line with equation $\mathbf{r} = (-1 + 2t)\mathbf{i} + (1 + t)\mathbf{j} + (2 + 3t)\mathbf{k}$ *and the plane* $2x + y + z = 5$.

Solution

Any point on the line has coordinates (2t – 1, t + 1, 3t + 2).

If this point lies in the plane

$$\begin{aligned} 2(2t - 1) + (t + 1) + (3t + 2) &= 5 \\ 8t &= 4 \\ t &= \frac{1}{2} \end{aligned}$$

$x = 0, y = \frac{3}{2}, z = \frac{7}{2}$

Point of intersection = $\left(0, \frac{3}{2}, \frac{7}{2}\right)$

16.4.4 Intersection of 2 planes

Unless they are parallel, two planes intersect in a line. To find the equation of this line, we choose two points on the line.

Consider the planes $x + y + z = 5$ and $2x - y + 2z = 4$. Any point on the line satisfies both equations.

We solve the 2 equations

$$x + y + z = 5 \qquad (1)$$

$$2x - y + 2z = 4 \qquad (2)$$

Adding (1) & (2) $3x + 3z = 9$

$$x = 3 - z$$

Substituting in (1) $3 - z + y + z = 5$

$$y = 2$$

So, any point in both planes has coordinates $(3 - t, 2, t)$ where t is a parameter.

If we put $t = 0$ and $t = 1$, say we have two points $(3, 2, 0)$ and $(2, 2, 1)$ which lie in both planes, hence on the line of intersection of the planes.

The line has direction vector $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and its equation is:

$$\mathbf{r} = (3\mathbf{i} + 2\mathbf{j} + 0\mathbf{k}) + t(-\mathbf{i} + 0\mathbf{j} + \mathbf{k})$$

$$= (3 - t)\mathbf{i} + 2\mathbf{j} + t\mathbf{k}.$$

Exercise 16 D

1. Find in each case whether the given line is parallel to the given plane, lies entirely in the plane or intersects the plane at a point. In the latter case, find the coordinates of the point of intersection.

(a) The line $\mathbf{r} = (1 + 2t)\mathbf{i} + (2 - t)\mathbf{j} + (3 - t)\mathbf{k}$ and the plane $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} . \mathbf{r} = 7$

(b) The line $\mathbf{r} = (1 - 3t)\mathbf{i} + (3 - 2t)\mathbf{j} + (2 - 4t)\mathbf{k}$ and the plane $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} . \mathbf{r} = 5$

(c) The line $\mathbf{r} = (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) + t(-2\mathbf{i} + \mathbf{j} - 7\mathbf{k})$ and the plane $\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} . \mathbf{r} = -3$

(d) The line $\mathbf{r} = (2 + t)\mathbf{i} + (-1 + 2t)\mathbf{j} + 2t\mathbf{k}$ and the plane $3x - y - z = 6$

(e) The line $\mathbf{r} = (1 + t)\mathbf{i} + (-1 - 2t)\mathbf{j} + (1 + t)\mathbf{k}$ and the plane $2x + y - z = 1$

(f) The line $\mathbf{r} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) + t(-\mathbf{i} + \mathbf{j} + \mathbf{k})$ and the plane $x + 2y - z = 5$

(g) The line $\mathbf{r} = (2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) + t(-\mathbf{i} + \mathbf{j} + \mathbf{k})$ and the plane $3x + 2y + z = 0$

(h) The line $\mathbf{r} = (\mathbf{i} - \mathbf{j} - \mathbf{k}) + t(-2\mathbf{i} + \mathbf{j} - \mathbf{k})$ and the plane $2x + y - z = 1$.

2. Find in each case, the equation of the line of intersection of the two given planes:
   - (a) $x + y + z - 1 = 0$ and $2x + y - z - 3 = 0$
   - (b) $x - 2y - z + 1 = 0$ and $2x - y + z - 1 = 0$
   - (c) $2x + y - z - 1 = 0$ and $x - y - z + 2 = 0$
   - (d) $x - 2y - z - 3 = 0$ and $3x - y - 2z - 1 = 0$

16.5.1 Length of perpendicular from a point to a line

To find the length of the perpendicular from a point to a line, we find the coordinates of the foot of the perpendicular from the point to the line.

Consider the line with equation $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + t(\mathbf{i} - \mathbf{j} - \mathbf{k})$. Any point on the line has coordinates $(2 + t, -1 - t, 1 - t)$

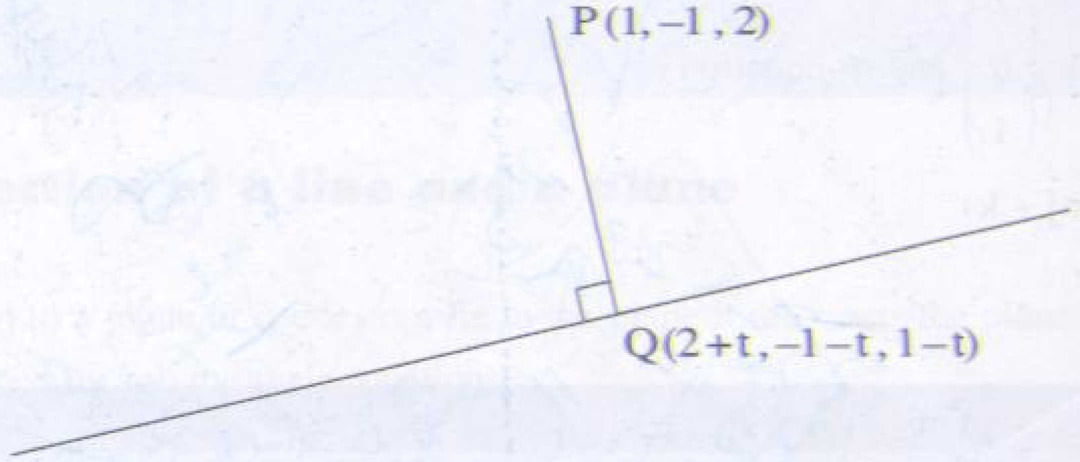


Figure 16.6

To find the length of the perpendicular from the point P(1, –1, 2) to the line, we find **PQ**.

$\mathbf{PQ} = \mathbf{q} - \mathbf{p}$

$= (1 + t)\mathbf{i} + -t\mathbf{j} + (-1 - t)\mathbf{k}$.

As **PQ** is perpendicular to the line,

$\mathbf{PQ} \cdot (\mathbf{i} - \mathbf{j} - \mathbf{k}) = 0$

$1 + t + t + 1 + t = 0$

$3t = -2$

$t = -\frac{2}{3}$

$$\mathbf{PQ} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$$

$$|\mathbf{PQ}| = \frac{1}{3}\sqrt{1^2 + 2^2 + (-1)^2} = \frac{\sqrt{6}}{3}$$

Example 14

Find the length of the perpendicular from the point (–1, 2, 3) to AB where A and B have coordinates A(2, 1, –
and B(3, 0, 1) respectively.

Solution

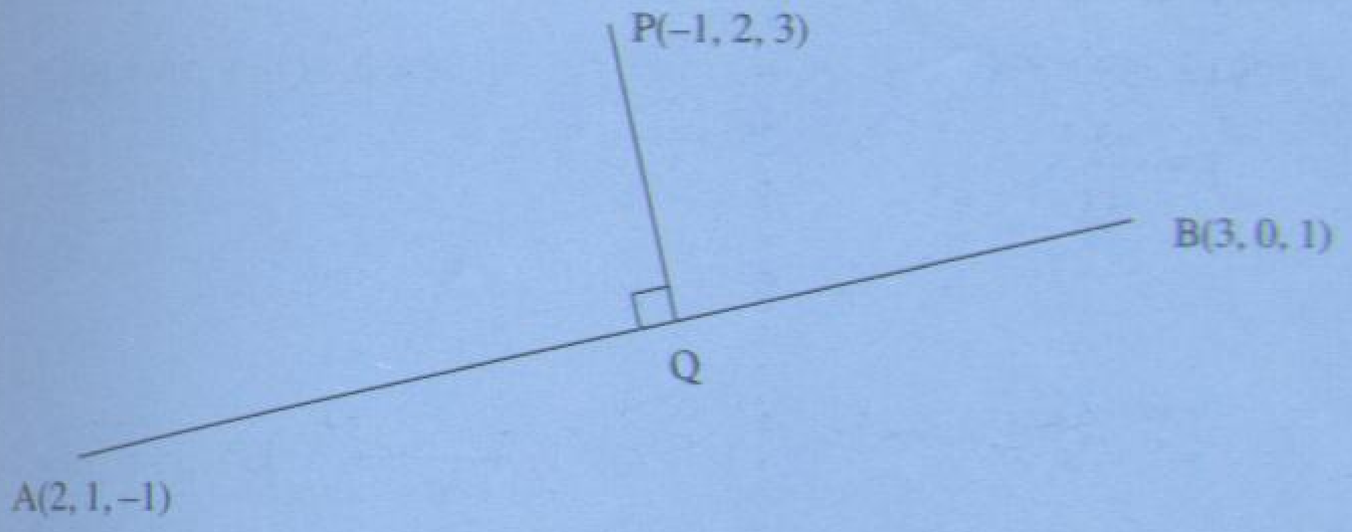


Figure 16.7

Direction vector of $\mathbf{AB} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$

Its equation is $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - \mathbf{k} + t(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

$$= (2 + t)\mathbf{i} + (1 - t)\mathbf{j} + (-1 + 2t)\mathbf{k}$$

Any point on line has coordinates $(2 + t, 1 - t, -1 + 2t)$

$$\mathbf{PQ} = (3 + t)\mathbf{i} + (-1 - t)\mathbf{j} + (-4 + 2t)\mathbf{k}$$

$$\mathbf{PQ} \,.\, \mathbf{AB} = 0$$

$$3 + t + 1 + t - 8 + 4t = 0$$

$$6t = 4$$

$$t = \frac{2}{3}$$

$$\mathbf{PQ} = \frac{11}{3}\mathbf{i} - \frac{5}{3}\mathbf{j} - \frac{8}{3}\mathbf{k}$$

$$|\mathbf{PQ}| = \sqrt{\frac{121}{9} + \frac{25}{9} + \frac{64}{9}}$$

$$= \frac{\sqrt{210}}{3}$$

16.5.2 Coordinates of the foot of the perpendicular from a point to a plane

Consider the plane with equation $2x + 3y + 4z - 5 = 0$. A normal vector to the plane is $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$.

If P is the point $(-1, 2, 1)$ and Q is the foot of the perpendicular from P to the plane, $\mathbf{PQ} = t\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$.

$$\mathbf{q} - \mathbf{p} = \begin{pmatrix} 2t \\ 3t \\ 4t \end{pmatrix}$$

$$\mathbf{q} = \begin{pmatrix} -1 + 2t \\ 2 + 3t \\ 1 + 4t \end{pmatrix}$$

As Q lies in the plane, $2(-1 + 2t) + 3(2 + 3t) + 4(1 + 4t) - 5 = 0$

$$29\,t = -3$$

$$t = -\frac{3}{29}$$

$$\mathbf{q} = \begin{pmatrix} -1-\frac{6}{29} \\ 2-\frac{9}{29} \\ 1-\frac{12}{29} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{35}{29} \\ \frac{49}{29} \\ \frac{17}{29} \end{pmatrix}$$

$$Q = \left(-\frac{35}{29}, \frac{49}{29}, \frac{17}{29}\right)$$

16.5.3 Length of perpendicular from a point to a plane

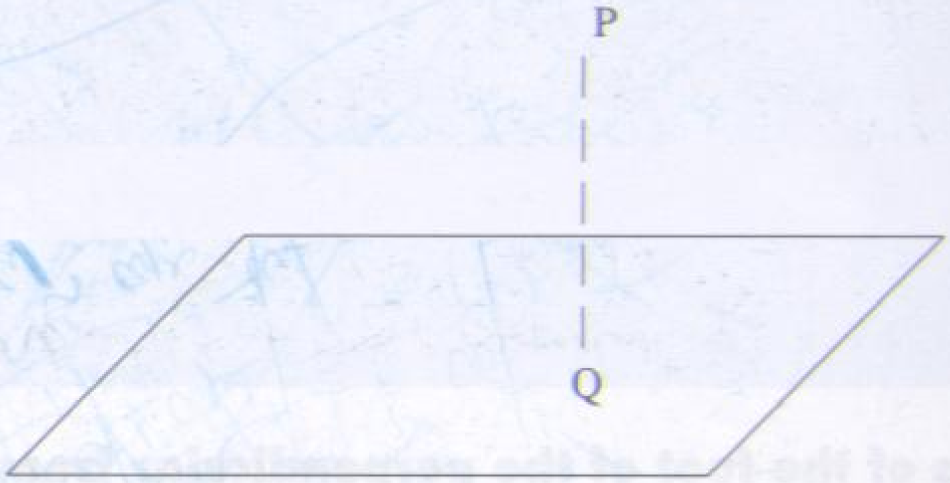


Figure 16.8

If the foot of the perpendicular is Q, then **PQ** = k**n** where **n** is the normal vector.

So, $\mathbf{q} - \mathbf{p} = k\mathbf{n}$

$$\mathbf{q} = \mathbf{p} + k\mathbf{n}$$

$$= \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + k \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= \begin{pmatrix} x_1 + ka \\ y_1 + kb \\ z_1 + kc \end{pmatrix}$$

As Q lies in the plane ax + by + cz + d = 0

$$a(x_1 + ka) + b(y_1 + kb) + c(z_1 + kc) + d = 0$$

$$k = -\frac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

$$\begin{aligned}|PQ| &= |k|\,|\mathbf{n}| \\ &= \frac{|ax_1 + by_1 + cz_1 + d|}{a^2 + b^2 + c^2} \times \sqrt{a^2 + b^2 + c^2} \\ &= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}\end{aligned}$$

Example 15

Find the length of the perpendicular from (–1, 2, 0) to the plane 3x + 4y – 12z – 8 = 0.

Solution

Using the formula, length of perpendicular $= \frac{|3 \times -1 + 4 \times 2 - 12 \times 0 - 8|}{\sqrt{3^2 + 4^2 + 12^2}}$

$$= \frac{|-3|}{13}$$

$$= \frac{3}{13}$$

Exercise 16 E

1. Find the coordinates of the foot of the perpendicular from the point (1, –1, 2) to the line with equation $\mathbf{r} = \mathbf{i} + \mathbf{j} - \mathbf{k} + t(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$.
   Deduce the length of the perpendicular.

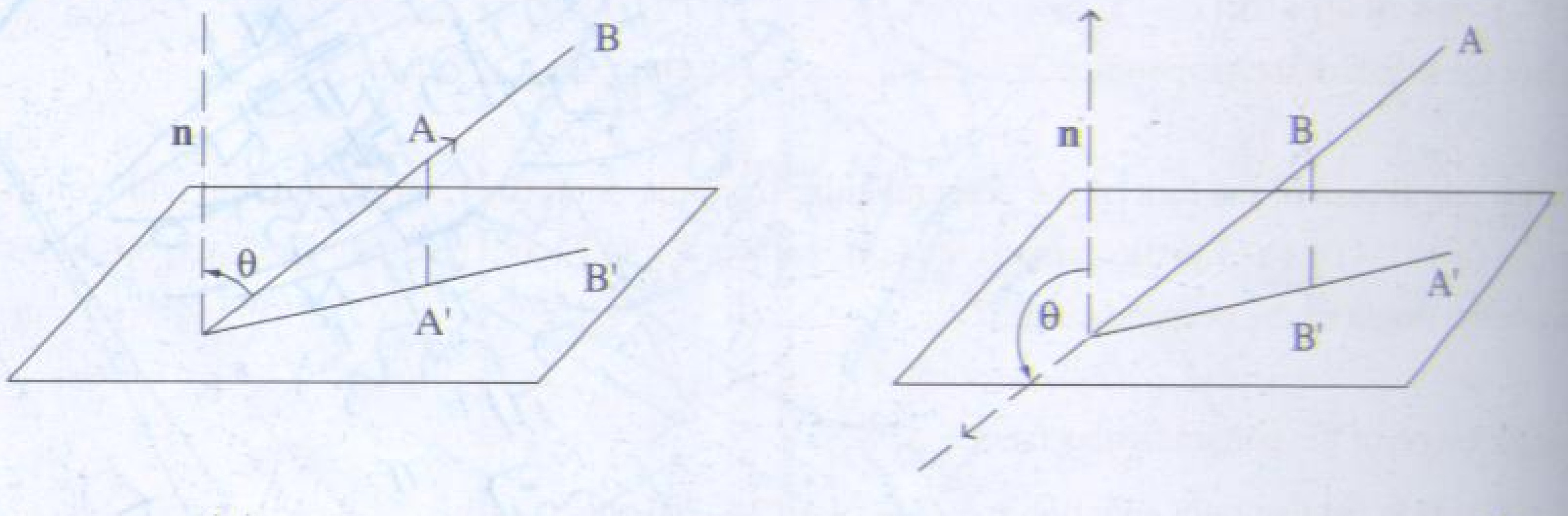
2. Find the coordinates of the foot of the perpendicular from the point (0, 1, –1) to the line with equation $\mathbf{r} = (2 + t)\mathbf{i} + (1 - t)\mathbf{j} + (-1 + t)\mathbf{k}$.
   Deduce the length of the perpendicular.

3. Find the length of the perpendicular from:
   (a) (1, –1, 1) to the line with equation $\mathbf{r} = (1 + t)\mathbf{i} + (2 + 2t)\mathbf{j} + (-1 - t)\mathbf{k}$
   (b) (0, –1, 1) to the line through the points A(1, 2, 0) and B(2, –1, 1)
   (c) (–1, 1, 2) to the line with equation $\mathbf{r} = -\mathbf{i} + \mathbf{j} + \mathbf{k} + t(\mathbf{i} - \mathbf{j} - 2\mathbf{k})$
   (d) (3, –1, 1) to the line with equation $\mathbf{r} = (1 - t)\mathbf{i} + (2 + t)\mathbf{j} + (1 - 2t)\mathbf{k}$
   (e) (1, 2, 0) to the line with equation $\mathbf{r} = (-1 - t)\mathbf{i} + \mathbf{j} + (1 + t)\mathbf{k}$.

4. Find the coordinates of the foot of the perpendicular from:
   - (a) $(1, -1, 1)$ to the plane $2x + y - z - 2 = 0$
   - (b) $(-1, 2, 0)$ to the plane $3x - 4y - 12z = 158$
   - (c) $(-1, 0, 1)$ to the plane $x + 5y - 7z = 7$
   - (d) $(2, -1, 1)$ to the plane $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} . \mathbf{r} = 21$
   - (e) $(-1, 2, 0)$ to the plane $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} . \mathbf{r} = 5.$

5. Find the length of the perpendicular from the given point to the given plane leaving your answer in $\sqrt{}$ form where necessary:
   - (a) $(1, 2, 1)$ to the plane $3x - y + z - 13 = 0$
   - (b) $(2, -1, 1)$ to the plane $x + 2y - z = 11$
   - (c) $(-1, 1, 2)$ to the plane $3x + 4y - 2z = 84$
   - (d) $2, -1, 1)$ to the plane $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} . \mathbf{r} = 3$
   - (e) $(-1, 2, 3)$ to the plane $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} . \mathbf{r} = 7$

16.6.1 Angle between a line and a plane



(a) (b)

Figure 16.9

The angle between a line AB and a plane is the angle between the line and its projection A'B' on the plane. The angle between line through A and B and the plane shown is either $(90 - \theta)$ for acute angle θ, where θ is the angle between **AB** and the normal vector; or $(\theta - 90^\circ)$ for obtuse angle θ.

Example 16

Find the angle between the line with equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + t(2\mathbf{i} - \mathbf{j} - \mathbf{k})$ *and the plane* $x - 2y - z = 5$.

Solution

Line has direction vector $2\mathbf{i} - \mathbf{j} - \mathbf{k}$

A normal to the plane is $\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

Angle θ between the two vectors is given by:

$$\cos\theta = \frac{2+2+1}{\sqrt{4+1+1}\sqrt{1+4+1}}$$

$$= \frac{5}{6}$$

$$\theta \approx 33.6^\circ$$

Required angle $= 90^\circ - 33.6^\circ$

$= 56.4^\circ$

Example 17

Find the angle between the line $\mathbf{r} = (1 - 2t)\mathbf{i} + (-1 + 3t)\mathbf{j} - \mathbf{k}$ *and the plane* $3x - y - z = 2$.

Solution

Line has direction vector $-2\mathbf{i} + 3\mathbf{j} + 0\mathbf{k}$

A normal to the plane is $3\mathbf{i} - \mathbf{j} - \mathbf{k}$.

Angle θ between the two vectors is given by:

$$\cos\theta = \frac{-6-3+0}{\sqrt{4+9+0}\sqrt{9+1+1}}$$

$$= -\frac{9}{\sqrt{13}\sqrt{11}}$$

$\theta = 138.8^\circ$ (to 1 d.p)

Required angle $= 138.8^\circ - 90^\circ$

$= 48.8^\circ$ (to 1 d.p)

16.6.2 Angle between 2 planes

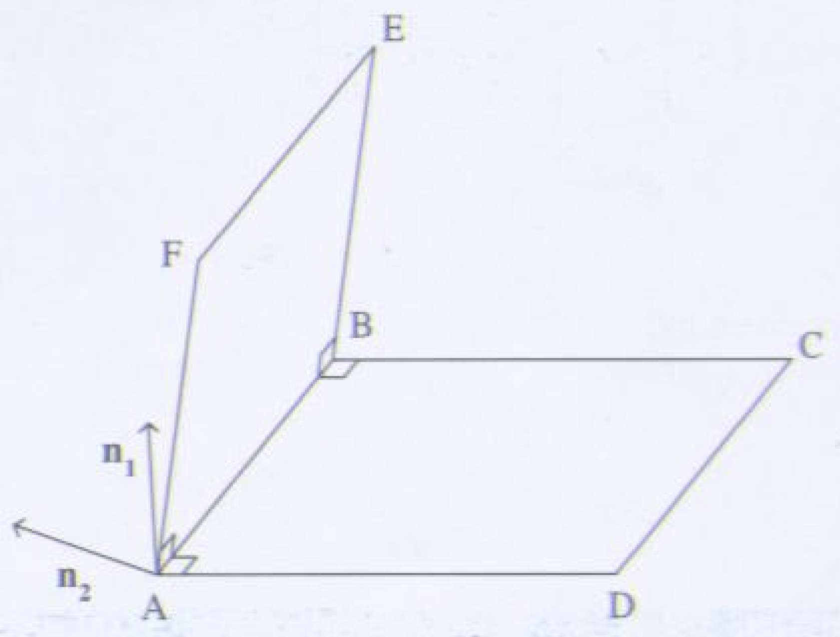


Figure 16.10

Unless they are parallel, two planes meet in a line. Consider two planes meeting in a line AB. The angle between the two planes is then the angle between 2 perpendiculars to AB, one in each plane.
In the given diagram, the angle is either ∠EBC or ∠FAD. This is the same as the angle between the normals to the two given planes.
To find the angle between two planes, we therefore find the angle between the normals to the two planes.

Example 18

Find the angle between the planes $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 3$ *and* $\mathbf{r} \cdot \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = 1.$

Solution

The required angle is the angle θ between $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ given by:

$$\cos\theta = \frac{-1-2+2}{\sqrt{1+4+1}\sqrt{1+1+4}}$$
$$= -\frac{1}{6}$$
$$\theta = 99.6^\circ \text{ (to 1 d.p)}$$

Example 19

Find the angle between the planes $x + y - 2z = 3$ *and* $3x - y - z = 5$.

Solution

Required angle is the angle θ between $\begin{pmatrix}1\\1\\-2\end{pmatrix}$ and $\begin{pmatrix}3\\-1\\-1\end{pmatrix}$ given by:

$$\cos\theta = \frac{3-1+2}{\sqrt{1+1+4}\sqrt{9+1+1}}$$

$$= \frac{4}{\sqrt{6}\sqrt{11}}$$

$\theta \approx 60.5°$ (to 1 d.p)

Exercise 16 F

1. Find the angle between:
   (a) the vector **i** – **j** – **k** and the plane $2x - y - z + 3 = 0$.
   (b) the vector 2**i** + **j** + **k** and the plane $\begin{pmatrix}1\\-2\\1\end{pmatrix}.\mathbf{r} = 3$.
   (c) the line through the points (1, –1, 2), (2, 0, 1) and the plane $x + 2y + z = 3$.
   (d) the line through the points (–1, 2, 1), (–3, –1, 0) and the plane $2x + y + z = 5$.
   (e) the line with equation $\mathbf{r} = (1 - 2t)\,\mathbf{i} + (2 - 3t)\,\mathbf{j} + (-1 - t)\,\mathbf{k}$ and the plane $x + 2y - z = 3$.
   (f) the line with equation $\mathbf{r} = (1 + 3t)\,\mathbf{i} + (-1 - 2t)\,\mathbf{j} + (-1 - 3t)\,\mathbf{k}$ and the plane $2x - y - z = 1$.
   (g) the line with equation $\mathbf{r} = \begin{pmatrix}-1\\1\\-1\end{pmatrix} + \mathbf{t}\begin{pmatrix}9\\8\\15\end{pmatrix}$ and the plane $x - y - 2z = 1$.
   (h) the line with equation $\mathbf{r} = \begin{pmatrix}1\\-1\\-1\end{pmatrix} + \mathbf{t}\begin{pmatrix}4\\15\\12\end{pmatrix}$ and the plane $x + y + z = 2$.

2. Find the angles between the following pairs of planes:
   (a) $2x - y - z = 1$ and $x + y + z = 3$
   (b) $3x + 2y - z = -3$ and $x - y - 2z = 5$
   (c) $x - 2y - 3z + 1 = 0$ and $2x + y - z - 4 = 0$
   (d) $\begin{pmatrix}1\\-1\\2\end{pmatrix}.\mathbf{r} = 3$ and $\begin{pmatrix}2\\1\\-3\end{pmatrix}.\mathbf{r} = 1$
   (e) $\begin{pmatrix}-1\\2\\0\end{pmatrix}.\mathbf{r} = 1$ and $\begin{pmatrix}2\\-1\\3\end{pmatrix}.\mathbf{r} = 5$.

Miscellaneous Exercise 16

1. The points A, B, C, D have position vectors given by
   $\mathbf{a} = \mathbf{i} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{c} = -2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{d} = 2\mathbf{i} + \mathbf{k}$
   respectively. Find:
   (i) a unit vector perpendicular to the plane ABC
   (ii) the length of the perpendicular from D to the plane ABC
   (iii) the cosine of the angle between the line AD and the perpendicular to the plane ABC. [C]

2. A pyramid has a square base OABC and vertex V. The position vectors of A, B, C, V referred to O as origin are given by $\mathbf{OA} = 2\mathbf{i}$, $\mathbf{OB} = 2\mathbf{i} + 2\mathbf{j}$, $\mathbf{OC} = 2\mathbf{j}$, $\mathbf{OV} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$.
   (i) Express **AV** in terms of **i**, **j** and **k**.
   (ii) Using scalar products, or otherwise, find a vector **x** which is perpendicular to both **OV** and **AV**.
   (iii) Calculate the angle between the vector **x**, found in (ii), and **VB**, giving your answer to the nearest degree.
   (iv) Write down the acute angle between **VB** and the plane OVA. [C]

3. The point O is the origin and points A, B, C and D have position vectors $\begin{pmatrix}4\\3\\4\end{pmatrix}$, $\begin{pmatrix}6\\1\\2\end{pmatrix}$, $\begin{pmatrix}0\\9\\-6\end{pmatrix}$ and $\begin{pmatrix}-1\\1\\1\end{pmatrix}$ respectively.
   Prove that:
   (a) the triangle OAB is isosceles
   (b) D lies in the plane OAB
   (c) **CD** is perpendicular to the plane OAB
   (d) **AC** is inclined at angle of 60° to the plane OAB. [C]

4. The planes π_1 and π_2 have equations $3x - y - z = 2$ and $x + 5y + z = 14$ respectively. Show that the point $(1, 3, -2)$ lies in both planes. By finding the coordinates of another point lying in both planes, or otherwise, show that the line of intersection, l_1, of π_1 and π_2 has equation $\mathbf{r} = \begin{pmatrix}1\\3\\-2\end{pmatrix} + t\begin{pmatrix}1\\-1\\4\end{pmatrix}$.
   The line l_2 has equation $\mathbf{r} = \begin{pmatrix}1\\0\\8\end{pmatrix} + s\begin{pmatrix}1\\-3\\-1\end{pmatrix}$.
   (a) Show that l_1 and l_2 are perpendicular.
   (b) Find the coordinates of the point of intersection, P, of l_2 and π_1.
   (c) Show that the perpendicular distance of P from π_2 is $\dfrac{20}{3\sqrt{3}}$. [C]

5. The position vectors of A and B relative to the origin O, are $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ respectively. The point C is given by $\mathbf{OC} = 2\mathbf{OA}$. Find:
   (a) the length OC
   (b) $\cos \angle AOB$
   (c) a vector equation of the line through A and B. [C]

6. The planes π_1 and π_2 have equations $3x - 4y + 12z = 13$ and $\mathbf{r} \cdot \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = 1$ respectively.

The line l has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 8 \\ 2 \end{pmatrix}$, and the point A has position vector $\begin{pmatrix} 3 \\ -6 \\ 4 \end{pmatrix}$.

(a) Calculate, to the nearest one tenth of a degree, the acute angle between the planes π_1 and π_2.

(b) Calculate the shortest distance from A to the plane π_1.

(c) Find the position vectors of the two points in which l meets the planes π_1 and π_2.

(d) Find an equation of the plane containing the line l and the point A. [C]

7. The point A has coordinates (3, −1, 5) and the line l has equation $\mathbf{r} = \begin{pmatrix} 8 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} -6 \\ 1 \\ 4 \end{pmatrix}$. Find the coordinates of the point B on l such that AB is perpendicular to l.

The plane π has equation $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = 15$

Find the coordinates of the point C where l intersects π. Find a vector perpendicular to the plane ABC. Hence, show that the angle between π and the plane ABC is 68°, correct to the nearest degree. [C]

8. The points P and Q have position vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix}$ respectively, referred to the origin O.

(a) Find the position vector of the point where the line PQ meets the plane $z = 0$.

(b) Find the equation of the plane through P normal to **PQ**.

(c) Find the angle OPQ, giving your answer to the nearest 0.1°.

(d) Find the values of a, b, c such that the equation of the plane OPQ is $\mathbf{r} \cdot \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = c$. [C]

9. Two planes P_1 and P_2 have equations $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = 4$ and $\mathbf{r} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 1$, respectively.

(a) Find the acute angle between P_1 and P_2, giving your answer to the nearest 0.1°.

(b) Find the position vector of the foot of the perpendicular from the origin O to P_1.

(c) The line l passes through the point P with position vector $\begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix}$ and is parallel to $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$. Verify that l lies in P_1 and find the position vector of the point where l meets P_2.

(d) Find the length of the projection of OP to l. [C]

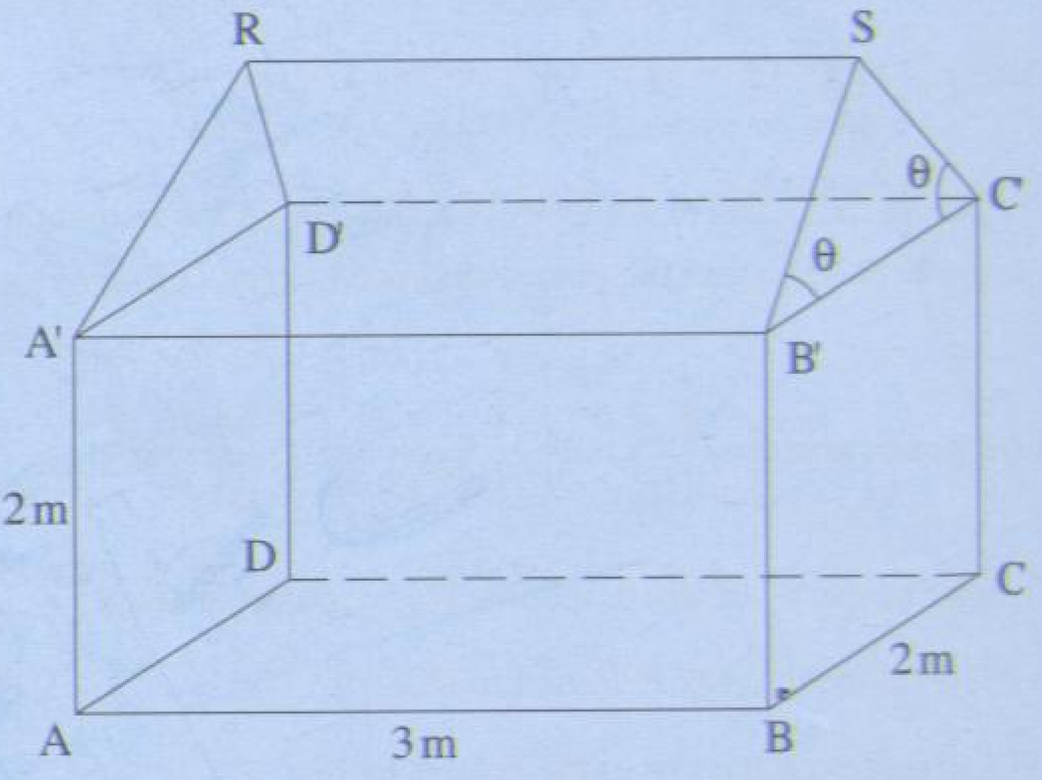
10. The points P and Q have coordinates (1, 6, 1) and (4, 0, –8) respectively. Find an equation, in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, for the straight line through P and Q.

The line *l*, which passes through the origin, has equation $\mathbf{r} = s\begin{pmatrix}1\\2\\-1\end{pmatrix}$.

Write down three equations which must be satisfied by the real parameters s and t if the lines *l* and PQ intersect. Find the values of s and t satisfying the three equations, and hence find the coordinates of the point of intersection of the two lines.

The line *m* has equation $\mathbf{r} = \begin{pmatrix}0\\1\\1\end{pmatrix} + u\begin{pmatrix}5\\a\\5\end{pmatrix}$ where u is a real parameter and a is a constant. Find the positive value of a for which the angle between *l* and *m* is 60°. [C]

11.



The diagram shows a garden shed with horizontal rectangular base ABCD where AB = 3 m, BC = 2 m and vertical walls. There are two rectangular walls, ABB'A' and DCC'D', where AA' = BB' = CC' = DD' = 2 m. The roof consists of the planes A'B' SR and C'D' RS, where RS is horizontal. Each section of the roof is inclined at an angle θ to the horizontal, where $\tan\theta = \frac{3}{4}$. The point A is taken as origin and vectors **i**, **j**, **k**, each of length 1 m, are taken along AB, AD, AA' respectively.

(a) Verify that the plane with equation $\mathbf{r} \cdot (22\mathbf{i} + 33\mathbf{j} - 12\mathbf{k}) = 66$ passes through B, D and S.

(b) Find the perpendicular distance, in metres, from A to the plane BDS. Give three significant figures in your answer.

(c) Find a vector equation of the straight line A'S.

(d) Show that the perpendicular distance from C to the straight line A'S is 2.91 m, correct to three significant figures. [C]

12. The plane p has equation $3x + 2y - z + 1 = 0$ and the line l has equation $\mathbf{r} = \begin{pmatrix} 4 \\ -3 \\ 7 \end{pmatrix} + s\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$. Show that l lies in p.

The line m has vector equation $\mathbf{r} = \begin{pmatrix} 0 \\ 10 \\ 7 \end{pmatrix} + \mathbf{t}\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ and m intersects p at the point A. Find the coordinates of A.

Write down the cartesian equation of the plane passing through A and perpendicular to l.
By finding where the plane intersects l, or otherwise, find the vector equation for the line through A which lies in p and is perpendicular to 1.

13. The points A and B have position vectors $2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{i} + \mathbf{j}$ respectively, relative to the origin O. The point P lies on OA produced and is such that OP = 2OA; Q lies on OB produced and is such that OQ = 3OB. Find the vector equations of the lines AQ and BP and find also the position vector of the point of intersection of these two lines.

The point Z has position vector $\mathbf{k}$ relative to O. Show that the cosine of the acute angle between ZA and ZB is $\sqrt{\frac{6}{7}}$ and find the area of the triangle ZAB, giving your answer in surd form. [C]

14. Planes π_1 and π_2 have equations $x - 2y + 3z = 0$ and $3x + y + 2z = 0$ respectively.
(a) Show that the acute angle between π_1 and π_2 is 60°.
(b) Show that the point P_1 with coordinates $(7, 2, -1)$ lies in π_1. Find the perpendicular distance of P from π_2.
(c) Deduce, or find otherwise, the perpendicular distance from P to the line of intersection of π_1 and π_2.
[C]

CHAPTER 17

> DIFFERENTIAL EQUATION

17.1 Examples of differential equations

Equations of the type $x\frac{dy}{dx} + y = 0, 2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$ are known as differential equations. All differential equations contain a differential coefficient such as $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, etc.

In $x\frac{dy}{dx} + y = 0$*, the highest derivative present is the 1st derivative. It is called a first order differential equation.*

In $2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$*, the highest derivative present is the second derivative. It is therefore called a second order differential equation.*

17.1.2 Solution of a first order differential equation using the method of separation of variables

If $\frac{dy}{dx}$ is a function of x, i.e. f(x), $y = \int f(x)\,dx$ and so y can be obtained by integrating f(x) with respect to x.

Thus, if $\frac{dy}{dx} = e^{2x}$, $y = \int e^{2x}\,dx = \frac{1}{2}e^{2x} + c$. If $\frac{dy}{dx}$ is a function of both x and y, i.e. $\frac{dy}{dx} = f(x, y)$, y cannot be obtained as $\int f(x, y)dx$.

Thus, if $\frac{dy}{dx} = \frac{x}{y}$, y cannot be obtained directly. In this particular case, we may separate x and y completely.

Thus, $\frac{dy}{dx} = \frac{x}{y}$ can be written as

$y\,dy = x\,dx$

Hence, $\int y\,dy = \int x\,dx$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + c$$

$$y^2 = x^2 + 2c$$

$$y^2 = x^2 + A$$

This method is known as the method of separation of variables.

Example 1

Solve the equation $\frac{dy}{dx} = \frac{\sin x}{\cos y}$

Solution

$$\frac{dy}{dx} = \frac{\sin x}{\cos y}$$

$$\cos y \, dy = \sin x \, dx$$

$$\int \cos y \, dy = \int \sin x \, dx$$

$$-\sin y = \cos x + c$$

$$\sin y = -(\cos x + c)$$

$$y = \sin^{-1}(-\cos x + c)$$

Example 2

Solve the equation $y\frac{dy}{dx} = \frac{y^2 + 1}{x - 2}$

Solution

$$y\frac{dy}{dx} = \frac{y^2 + 1}{x - 2}$$

$$\frac{y}{y^2 + 1} dy = \frac{dx}{x - 2}$$

$$\int \frac{y}{y^2 + 1} dy = \int \frac{dx}{x - 2}$$

$$\frac{1}{2}\ln(y^2 + 1) = \ln(x - 2) + c$$

$$\ln(y^2 + 1) = 2\ln(x - 2) + 2c$$

$$= \ln(x - 2)^2 + \ln A$$

$$= \ln\left[A(x - 2)^2\right]$$

$$y^2 + 1 = A(x - 2)^2$$

$$y^2 = A(x - 2)^2 - 1$$

17.1.3 Particular solutions of differential equations

The constant can be obtained if we are given a pair of values of the variables. Thus, if $\frac{dy}{dx} = \frac{y^2}{x^2}$ and $y = 2$ when $x = 1$, we may obtain y in terms of x.

$$\frac{dy}{dx} = \frac{y^2}{x^2}$$

$$\frac{1}{y^2}\,dy = \frac{1}{x^2}\,dx$$

$$\int \frac{1}{y^2}\,dy = \int \frac{1}{x^2}\,dx$$

$$-\frac{1}{y} = -\frac{1}{x} + c$$

As $y = 2$ when $x = 1$, $-\frac{1}{2} = -1 + c$

$$c = \frac{1}{2}$$

So, $-\frac{1}{y} = -\frac{1}{x} + \frac{1}{2}$

$$\frac{1}{y} = \frac{1}{x} - \frac{1}{2}$$

$$\frac{1}{y} = \frac{2 - x}{2x}$$

$$y = \frac{2x}{2 - x}$$

Example 3

Solve the differential equation $\frac{dy}{dx} = \frac{\sin x}{\tan y}$, *given* $y = \frac{\pi}{4}$ *when* $x = \frac{\pi}{6}$.

Solution

$$\frac{dy}{dx} = \frac{\sin x}{\tan y}$$

$$\tan y\,dy = \sin x\,dx$$

$$\int \tan y\,dy = \int \sin x\,dx$$

$$\int \frac{\sin \cancel{x}}{\cos y}\,dy = \int \sin x\,dx$$

$$-\ln \cos y = -\cos x + c$$

$$-\ln \cos \frac{\pi}{4} = -\cos \frac{\pi}{6} + c$$

$$-\ln \frac{1}{\sqrt{2}} = -\frac{\sqrt{3}}{2} + c$$

$$\ln\sqrt{2} + \frac{\sqrt{3}}{2} = c$$

$$-\ln\cos y = -\cos x + \ln\sqrt{2} + \frac{\sqrt{3}}{2}$$

$$\ln\cos y + \ln\sqrt{2} = \cos x - \frac{\sqrt{3}}{2}$$

$$\ln(\sqrt{2}\cos y) = \cos x - \frac{\sqrt{3}}{2}$$

$$\sqrt{2}\cos y = e^{\cos x - \frac{\sqrt{3}}{2}}$$

$$\cos y = \frac{1}{\sqrt{2}}e^{\cos x - \frac{\sqrt{3}}{2}}$$

Exercise 17 A

1. Solve the following differential equations:

(a) $\frac{dy}{dx} = y$

(b) $\frac{dy}{dx} = \frac{y}{x}$

(c) $\frac{dy}{dx} = \frac{1}{y}$

(d) $\frac{dy}{dx} = e^y$

(e) $\frac{dy}{dx} = e^{x+y}$

(f) $\frac{dy}{dx} = y(y+1)$

(g) $\frac{dy}{dx} = \frac{\tan y}{\tan x}$

(h) $\frac{dy}{dx} = \frac{x+2}{y-2}$

(i) $\frac{dy}{dx} = \frac{y-2}{x+2}$

(j) $\sin y \frac{dy}{dx} = \cos\left(x + \frac{\pi}{3}\right)$

(k) $\frac{dy}{dx} = \frac{\cos x}{\tan y}$

(l) $x\frac{dy}{dx} = \cot y$

(m) $(x^2 - 1)\frac{dy}{dx} = \frac{x}{y}$

(n) $\frac{1}{x}\frac{dy}{dx} = \sqrt{x+1}$

(o) $\frac{ds}{dt} = s(s-1)$

(p) $e^y \frac{dy}{dx} = xe^x$

2. Find the particular solutions of the differential equations which satisfy the given conditions:

(a) $\frac{dy}{dx} = e^{-x}$ given $y = -1$ when $x = 0$

(b) $\frac{dy}{dx} = \frac{y}{2x+1}$ given $y = 1$ when $x = 0$

(c) $\frac{dy}{dx} = \frac{2x}{y}$ given $y = 1$ when $x = 0$

(d) $\frac{dy}{dx} = \frac{x-1}{y+1}$ given $y = 0$ when $x = 1$

(e) $\frac{dy}{dx} = \frac{y+1}{x-1}$ given $y = 1$ when $x = 2$

(f) $\frac{dy}{dx} = e^{x-y}$ given $y = 1$ when $x = 1$

(g) $\sin y \frac{dy}{dx} = \cos\left(x + \frac{\pi}{4}\right)$ and $y = -\frac{\pi}{2}$ when $x = -\frac{\pi}{4}$

(h) $\frac{dy}{dx} = \frac{\cos x}{\tan y}$ given $y = -\frac{\pi}{4}$ when $x = -\frac{\pi}{2}$

(i) $\frac{d^2y}{dx^2} = \cos x$ given $y = 0$, $\frac{dy}{dx} = 0$ when $x = 0$

(j) $\frac{d^2y}{dx} = \cos\left(2x - \frac{\pi}{3}\right)$ given $y = 1$, $\frac{dy}{dx} = \frac{1}{2}$ when $x = \frac{\pi}{3}$.

17.2 Formulation of statements involving differential equations

(a) Consider the statement 'The distance s is increasing at a rate proportional to the distance'. The rate of increase s is $\frac{ds}{dt}$. As it is proportional to the distance s, $\frac{ds}{dt} = ks\ (k > 0)$.

(b) Consider next the statement 'The number of bacteria present in a solution is decreasing at a rate proportional to the number present'.

We assume that the initial number of bacteria is A i.e. when time $t = 0$, the number of bacteria is A. A is therefore a constant.

After time t, if the number of bacteria present is x, then x is a variable. In time t, decrease in the number of bacteria is $A - x$.

$$\begin{aligned} \text{Rate of decrease } &= \frac{d}{dt}(A - x) \\ &= \frac{dA}{dt} - \frac{dx}{dt} \\ &= -\frac{dx}{dt} \left(\frac{dA}{dt} = 0, \text{ as A is a constant}\right) \\ -\frac{dx}{dt} &= kx \\ \frac{dx}{dt} &= -kx\ (k > 0) \end{aligned}$$

(c) Consider the statement 'The rate of cooling of a liquid is proportional to its excess temperature over the surroundings, assumed constant'.

Let the initial temperature be A (constant), temperature at time t be x (variable).

The rate of change is $\frac{d}{dt}(A - x) = -\frac{dx}{dt}$.

If the temperature of surroundings is θ_o (constant), excess temperature over surroundings is $x - \theta_o$.

$$\text{So, } -\frac{dx}{dt} = k(x - \theta_o) \quad (k > 0)$$

$$\frac{dx}{dt} = -k(x - \theta_o)$$

Example 4

The birth rate in Outerland is one hundredth of the population x present. Every year, 2000 inhabitants of Outerland emigrate. Modeling x as a continuous variable, form a differential equation involving x and t where t is the time in years.

Solution

Let initial number of inhabitants be A.

Number of inhabitants at time t = x

Increase in time t is x – A.

$$\text{Rate of increase} = \frac{d}{dt}(x - A) = \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{1}{100}x - 2000$$

$$= \frac{x - 200000}{100}$$

$$100\frac{dx}{dt} = x - 200000$$

17.2.2 Problems leading to differential equations

We consider solutions of problems leading to differential equations.

Example 5

The rate of decay of a radioactive substance is proportional to the amount present. It takes twenty five years for one quarter of the amount to decompose, how long will it take for half of the amount to decompose?

Solution

Let initial amount of radioactive substance be A.

Let amount at time t be x.

Amount decomposed in time t = A – x.

$$\text{Rate of decomposition} = \frac{d}{dt}(-x + A) = -\frac{dx}{dt}$$

Note: $\frac{A - x}{t}$ *is the average rate of decomposition, not the actual rate of decomposition at time t.*

$$-\frac{dx}{dt} = kx \qquad (k > 0)$$

$$\int \frac{dx}{x} = \int -k\, dt$$

$$\ln x = -kt + c$$

$$x = e^{-kt + c}$$

$$= e^{-kt} \times e^{c}$$

$$= ae^{-kt}$$

At $t = 0$, $x = A$, $a = A$

$$x = Ae^{-kt}$$

$t = 25$, $x = \frac{3}{4}A \left(\text{as } \frac{1}{4}A \text{ has decomposed}\right)$

$$\frac{3}{4}A = Ae^{-25k}$$

$$e^{-25k} = \frac{3}{4}$$

$$-25k = \ln\frac{3}{4}$$

$$k = \frac{\ln\frac{3}{4}}{-25}$$

$$= 0.01150\text{......} \text{ (calculator)}$$

When $\frac{1}{2}A$ decomposes, $x = \frac{1}{2}A$

$$\frac{1}{2}A = Ae^{-0.01150\text{.....}t}$$

$$-0.01150t = \ln\frac{1}{2}$$

$$t = \frac{\ln\frac{1}{2}}{-0.01150}$$

$$\approx 60$$

So, it takes approximately 60 years for half of the amount to decompose.

Example 6

Newton's law of cooling states that the rate of cooling is proportional to its excess temperature over the temperature of the surroundings. It takes 2 minutes for the body to cool from 100° C to 80° C, find how long it takes to cool to 50° C, given that the temperature of the surroundings is 25° C.

Solution

Let initial temperature be A and temperature at time t be x.

Rate of cooling $= \frac{d}{dt}(A - x) = -\frac{dx}{dt}$

$-\frac{dx}{dt} = k(x - 25)(k > 0)$

$$\int \frac{dx}{x - 25} = \int -k\,dt$$

$$\begin{aligned}
\ln(x - 25) &= -kt + c \\
x - 25 &= e^{-kt + c} \\
&= e^{-kt} \times e^{c} \\
&= ae^{-kt} \\
x &= ae^{-kt} + 25
\end{aligned}$$

$$\begin{aligned}
t &= 0, x = 100 \\
100 &= a + 25 \\
a &= 75 \\
x &= 75e^{-kt} + 25 \\
t &= 2, x = 80 \\
80 &= 75e^{-2k} + 25 \\
55 &= 75e^{-2k} \\
e^{-2k} &= \frac{55}{75} \\
-2k &= \ln\left(\frac{55}{75}\right) \\
k &= -\frac{1}{2}\ln\frac{55}{75} \\
&\approx -0.1550\text{...... (calculator)}
\end{aligned}$$

$$\begin{aligned}
x = 50,\ 50 &= 75e^{-0.1550\,t} + 25 \\
25 &= 75e^{-0.1550\,t} \\
e^{-0.1550\,t} &= \frac{1}{3} \\
t &= \frac{\ln\frac{1}{3}}{-0.1550} \\
t &= 7.08 \text{ (to 3 s.f)}
\end{aligned}$$

Exercise 17 B

1. For each of the following examples, write a differential equation stating clearly what the symbols used represent:
   (a) A colony of insects grows at a rate proportional to the number present.
   (b) The displacement of a particle increases at a rate proportional to the displacement.
   (c) The rate of decay of a radioactive substance is proportional to the amount present.
   (d) The water level in a container decreases at a rate proportional to the square of the water level.
   (e) In a chemical reaction the mass x of a substance is being decomposed at a rate proportional to the mass left.
   (f) The number of births per unit time on an island is proportional to the population at any time and the number of deaths per unit time is proportional to the square of the population. The population at time t is p.
   (g) A vessel contains liquid, which is flowing out from a small hole at a point O in the base of the vessel. The height of the liquid above O is decreasing at a rate proportional to the square root of the height.

2. Sand is poured on level ground to form a large conical mound. In a simple mathematical model the rate at which the height h of the cone increases is inversely proportional to h^2.

   Express this model as a differential equation relating h with the time t, and find its general solution, expressing h in terms of t.

   If after 5 minutes its height is 10 m, find at what time the height is 18 m.

   It is assumed that the volume v of the cone increases at a constant rate and that v varies directly as h^3. Show that this assumption leads to the model described above.

3. A disease is spreading in a country at a rate which is proportional to the product of the fraction of the population infected by the disease and of the fraction of the population not infected by the disease.

   Taking x to be the fraction of the population infected by the disease at time t hours, form a differential equation connecting x and t and solve the equation.

   If initially, $\frac{1}{100}$ of the population was infected by the disease, show that $x = \frac{e^{kt}}{99 + e^{kt}}$ where k is a constant.

4. In a given model, the value of a car at any instant is decreasing at a rate proportional to its value v at that instant. Write down this statement as a differential equation involving v and the time t in years. Initially, the value of the car was R 600 000 and, a year later its value was R 510 000. Express v in terms of t.

   In another model, the value of the car, t years after sale as new, satisfies the differential equation $\frac{dv}{dt} = -k(v + 100\,000)$, where k is a positive constant. Given that $t = 0$, $v = 600\,000$ and that when $t = 3$, $v = 420\,000$, calculate the value of the car one year after sale.

5. A race called the Matrices lives on an isolated island called Vector. The number of births per unit time is proportional to the population at any time and the number of deaths per unit is proportional to the square of the population. If the population at time t is p, show that $\frac{dp}{dt} = ap - bp^2$ where a and b are positive constants.

Solve the equation for p in terms of t, given that $p = \frac{2a}{3b}$ when $t = 0$. Show that there is a limit to the size of the population. [C]

6. A population of insects is allowed to grow in an experimental environment. The rate of increase of the population is proportional to the number, n, of insects at any time t days after the start of the experiment. Regarding n and t as continuous variables, form a differential equation relating n and t and solve it to show that $n = Ae^{kt}$ where A and k are constants.

   The net increases during the fourth and fifth days are 350 and 500 insects respectively. Determine the population at the beginning of the fourth day. Hence or otherwise determine the population at the beginning of the first day. [C]

7. Newton's law of cooling states that the rate of decrease of temperature of a hot body is proportional to the excess of the temperature of the body over that of its surroundings. Using t for the time in minutes, the temperature in °C and θ_o for the temperature of the surroundings (assumed constant), express the law in the form of a differential equation.
   In a particular case $\theta_o = 20$, $\theta = 80$ and $t = 0$ and $\theta = 70$ when $t = 5$. Prove that, at time t, $\theta = 20 + 60\left(\frac{5}{6}\right)^{\frac{t}{5}}$. [C]

Miscellaneous Exercise 17

1. (a) Find the general solution of the differential equation $\frac{dx}{dt} = kx$, where k is a positive constant, expressing x in terms of k and t in your answer.

   (b) Solve the differential equation $\frac{dx}{dt} = kx(a - x)$ $(0 < x < a)$ where k and a are positive constants, given that $x = \frac{1}{2}a$ where $t = 0$. Express x in terms of k, a and t in your answer. [C]

2. Find y in terms of x, given that $\frac{dy}{dx} = \frac{y(1 - x)}{x}$ and that $y = 2$ when $x = 1$. [C]

3. Find the general solution of the differential equation $\frac{dy}{dx} - \frac{2y}{x} = 0$.
   Sketch the two solution curves passing respectively through the points (2, 2) and (–1, –1). [C]

4. Given that $\frac{dy}{dx} = (\tan x)\sqrt{y}$ $0 \leqslant x < \frac{1}{2}\pi$ and that $y = 1$ when $x = 0$, find an expression for y in terms of x. [C]

5. The variables x and y are connected by the differential equation $\frac{dy}{dx} = \frac{1 + x + y}{1 - x - y}$.

   Show that the substitution $u = x + y$ reduces the equation to $\frac{du}{dx} = \frac{2}{1 - u}$ and solve this differential equation. Deduce that $(x + y)^2 + 2(x - y) = A$, where A is an arbitrary constant. [C]

6. Two variables x and t are connected by the differential equation $\frac{dy}{dx} = \frac{kx}{10 - x}$, where $0 < x < 10$ and where k is a constant. It is given that $x = 1$ when $t = 0$ and that $x = 2$ when $t = 1$. Find the value of t when $x = 5$, giving three significant figures in your answer.

7. At time $t = 0$, there are 8 000 fish in a lake. At time t days, the birthrate of fish is equal to one fiftieth of the number N of fish present. Fish are taken from the lake at the rate of 100 per day. Modeling N as a continuous variable show that $50\frac{dN}{dt} = N - 5000$. [C]

   Solve the differential equation to find N in terms of t. Find the time taken for the population of fish in the lake to increase to 11 000.

   When the population of fish has reached 11 000, it is decided to increase the number of fish taken from the lake from 100 per day to F per day. Write down, in terms of F, the new differential equation satisfied by N.

   Show that if $F > 220$, then $\frac{dN}{dt} < 0$ when $N = 11\,000$.

   For this range of values of F, give a reason why the population of fish in the lake continues to decrease. [C]

8. The organiser of a sale, which lasted for 3 hours and raised a total of £ 1 000, attempted to create a model to represent the relationship between s and t, where £ s is the amount which has been raised at time t hours after the start of the sale. In the model, s and t were taken to be continuous variables. The organiser assumed that the rate of raising money varied directly as the time remaining and inversely as the amount already raised.

   Show that, for this model, $\frac{ds}{dt} = k\left(\frac{3 - t}{s}\right)$, where k is a constant.

   Solve the differential equation and show that the solution can be written in the form $\frac{s^2}{1000^2} + \frac{(3 - t)^2}{3^2} = 1$.

   Hence,

   (a) find the amount raised during the first hour of the sale;

   (b) find the rate of raising money one hour after the start of the sale. [C]

9. The rate of destruction of a drug by the kidneys is proportional to the amount of drug in the body. The constant of proportionality is denoted by k. At time t, the quantity of drug in the body is x. Write down a differential equation relating x and t and show that the general solution is $x = Ae^{-kt}$, where A is an arbitrary constant.

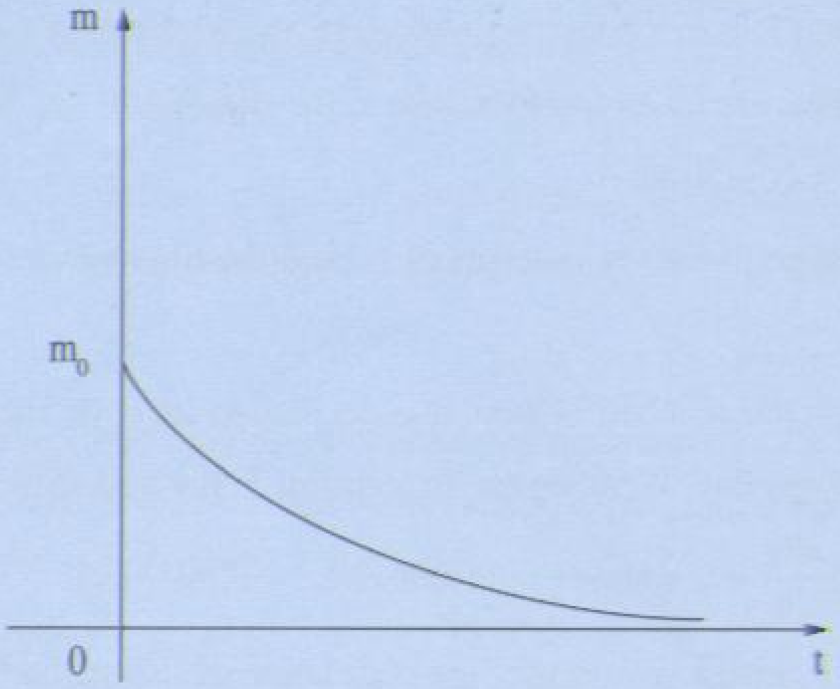
   Before $t = 0$, there is no drug in the body, but at $t = 0$, a quantity Q of the drug is administered. When $t = 1$, the amount of drug in the body is $Q\alpha$, where is a constant such that $0 < \alpha < 1$. Show that $x = Q\alpha^t$.

   Sketch the graph of x against t for $0 < t < 1$.

   When $t = 1$ and again when $t = 2$, another dose Q is administered. Show that the amount of drug in the body immediately after $t = 2$ is $Q(1 + \alpha + \alpha^2)$. [C]

10. During a chemical reaction, the mass m of one of the chemicals involved decreases at a rate which is proportional to m. Express this information as a differential equation involving m and the time t.

When $t = 0$, the mass of the chemicals is m_0. Show by integration that the solution of the differential equation is $m = m_0e^{-kt}$, where k is a positive constant.



A sketch of the graph of $m = m_0e^{-kt}$ is shown in the diagram. The mass M of another chemical involved in the reaction varies in such a way that $m + M = A$, where A is a constant.

(a) Sketch the graph of M against t.

(b) Describe what happens to the value of M as t becomes large.

(c) Show that $\frac{dM}{dt} = k(A - M)$. [C]

CHAPTER 18

> COMPLEX NUMBERS

18.1 The number denoted by i

Consider the equation $x^2 + 2x + 5 = 0$

Using the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm 4\sqrt{-1}}{2}$$

$$= -1 \pm 2\sqrt{-1}$$

This equation does not have real roots as there is no real number whose square is –1. A new number is invented such that $\sqrt{-1} = i$, i.e. $i^2 = -1$. With this new invention, all quadratic equations can be solved.

Example 1

Solve the equations: (a) $x^2 + 9 = 0$ (b) $2x^2 + 2x + 5 = 0$

Solution

(a) $x^2 + 9 = 0$

$$x^2 = -9$$

$$x = \pm\sqrt{-9}$$

$$= \pm 3i$$

(b) $2x^2 + 2x + 5 = 0$

$$x = \frac{-2 \pm \sqrt{4-40}}{4}$$

$$= \frac{-2 \pm \sqrt{-36}}{4}$$

$$= \frac{-2 \pm 6i}{4} \text{ (where } i^2 = -1\text{)}$$

$$= \frac{-1+3i}{2} \text{ or } \frac{-1-3i}{2}$$

Any number of the form $a + ib$ is called a complex number. a is called the real part and b the imaginary part.

If $z = a + ib$, we write $\text{Re}(z) = a$ and $\text{Im}(z) = b$

A complex number is usually denoted by the letter z.

Thus, we may write $z_1 = 2 + 3i$, $z_2 = 3 - 4i$, $z_3 = -4 + i$, etc.

18.1.2 Addition of complex numbers

To add 2 complex numbers, we add the real parts and the imaginary parts separately.

Thus, $(2 + 3i) + (3 + 5i) = (2 + 3) + (3i + 5i)$

$= 5 + 8i$

$(3 - 4i) + (-7 + 6i) = (3 + -7) + (-4i + 6i)$

$= -4 + 2i$

Generally, $(a + bi) + (c + di) = (a + c) + (b + d)i$

18.1.3 Subtraction of complex numbers

$(3 + 4i) - (2 + 5i) = (3 - 2) + (4i - 5i)$

$= 1 - i$

$(2 - 3i) - (7 - 6i) = (2 - 7) + (-3i + 6i)$

$= -5 + 3i$

Generally, $(a + bi) - (c + di) = (a - c) + (b - d)i$

18.1.4 Multiplication of complex numbers

$$\begin{aligned}(2 + 3i)(3 - 2i) &= 2(3 - 2i) + 3i(3 - 2i)\\ &= 6 - 4i + 9i - 6i^2\\ &= 6 + 5i + 6\\ &= 12 + 5i\end{aligned}$$

$$\begin{aligned}(3 + 4i)(2 + i) &= 3(2 + i) + 4i(2 + i)\\ &= 6 + 3i + 8i + 4i^2\\ &= 2 + 11i\end{aligned}$$

Generally, $(a + bi)(c + di) = a(c + di) + bi(c + di)$

$= ac + adi + bci - bd$

$= (ac - bd) + (ad + bc)i$

Note that $(2 + 3i)(2 - 3i) = 4 + 9 = 13$

$(3 + 5i)(3 - 5i) = 9 + 25 = 34$

Generally, $(a + bi)(a - bi) = a^2 + b^2$

Two complex numbers of the form $a + bi$ and $a - bi$ are said to be conjugate.

If z is a complex number, its conjugate is written as z^* or $\bar{z}$

18.1.5 Division of complex numbers

To divide z_1 by z_2, i.e. to find $\frac{z_1}{z_2}$, we multiply the numerator and the denominator by the conjugate of z_2.

$$\text{Thus, } \frac{1}{3-4i} = \frac{1(3+4i)}{(3-4i)(3+4i)}$$
$$= \frac{3+4i}{9+16}$$
$$= \frac{3}{25} + \frac{4}{25}i$$
$$\frac{3+4i}{2+3i} = \frac{(3+4i)(2-3i)}{(2+3i)(2-3i)}$$
$$= \frac{6-9i+8i-12i^2}{4+9}$$
$$= \frac{18-i}{13}$$
$$= \frac{18}{13} - \frac{i}{13}$$

$$\text{Generally, } \frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)}$$
$$= \frac{ac+bd+(bc-ad)i}{c^2+d^2}$$
$$= \frac{ac+bd}{c^2+d^2} + \frac{(bc-ad)i}{c^2+d^2}$$

Example 2

Simplify:

(a) i^5 *(b)* $(2-3i)+(1-4i)$ *(c)* $(3-2i)-(2-5i)$

(d) $(3+i)(2-5i)$ *(e)* $\frac{2+3i}{4i}$ *(f)* $\frac{3-2i}{2-i}$

Solution

(a) $i^5 = (i^4) \times i$
$= (i^2)^2 \times i$
$= i$

(b) $(2-3i)+(1-4i) = (2+1)+(-3i-4i)$
$= 3-7i$

(c) $(3-2i)-(2-5i) = (3-2)+(-2i+5i)$
$= 1+3i$

(d) $(3 + i)(2 - 5i) = 6 - 15i + 2i - 5i^2$

$$= 11 - 13i$$

(e) $\dfrac{2 + 3i}{4i} = \dfrac{2i + 3i^2}{4i^2}$

$$= \frac{2i - 3}{-4}$$

$$= \frac{3}{4} - \frac{1}{2}i$$

(f) $\dfrac{3 - 2i}{2 - i} = \dfrac{(3 - 2i)(2 + i)}{(2 - i)(2 + i)}$

$$= \frac{6 + 3i - 4i - 2i^2}{4 + 1}$$

$$= \frac{8 - i}{5}$$

$$= \frac{8}{5} - \frac{i}{5}$$

Exercise 18 A

1. Simplify:

   (a) i^3 (b) $3i^5$ (c) $-2i^4$ (d) $5i^6$ (e) $\dfrac{3}{i}$ (f) $\dfrac{2}{i^3}$

2. Simplify:

   (a) $(4 + 3i) + (2 - 6i)$
   (b) $\left(4 - \sqrt{3}i\right) + \left(5 + 2\sqrt{3}i\right)$
   (c) $(5 - 2i) - (3 - 6i)$
   (d) $(3 - 5i) - (2 - 5i)$
   (e) $\left(2\sqrt{3} + \sqrt{3}i\right) + \left(5\sqrt{3} - 3\sqrt{3}i\right)$
   (f) $(6 - 8i) - (3 - 5i)$

3. Simplify:

   (a) $(2 + 3i)(4 + 3i)$
   (b) $(5 - 2i)(3 + 7i)$
   (c) $\left(\dfrac{1}{\sqrt{2}} + i\right)\left(\dfrac{3}{\sqrt{2}} - i\right)$
   (d) $\left(2\sqrt{3} + 5i\right)\left(3\sqrt{3} - 2i\right)$
   (e) $(2x + 3yi)(4x - 5yi)$
   (f) $2i(3 - 5i)$

4. Simplify, giving your answer in the form a + bi:

   (a) $\dfrac{2 - i}{2 + i}$ (b) $\dfrac{3}{4 - 5i}$ (c) $\dfrac{2i - 5}{3i + 4}$ (d) $\dfrac{5 + \sqrt{3}i}{5 - \sqrt{3}i}$

   (e) $\dfrac{(2 + i)(3 - i)}{1 - 2i}$ (f) $\dfrac{2}{3 + 5i} + \dfrac{1}{2 - 3i}$ (g) $\dfrac{2 + 3i}{1 + 4i} + \dfrac{3 - 2i}{1 - 4i}$ (h) $\dfrac{2}{1 - 3i} + \dfrac{5}{1 + 2i}$

5. Simplify the following expressions, giving your answers in the form a + bi:

(a) $(2 - 5i)^2$ (b) $(3 + 7i)^2$ (c) $(2 - i)^3$

(d) $(4 - i)^3$ (e) $\dfrac{2}{(3+i)^3}$ (f) $\dfrac{1}{\cos\theta + i\sin\theta}$

(g) $\dfrac{1}{1+\cos\theta - i\sin\theta}$ (h) $\dfrac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta}$ (i) $\dfrac{1}{1+\cos\theta + i\sin\theta}$

6. Find the roots of the following equations:

(a) $x^2 + 4x + 5 = 0$ (b) $x^2 + 3x + 3 = 0$

(c) $3x^2 + x + 1 = 0$ (d) $2x^2 + 5x + 8 = 0$

18.2 Geometrical representation of complex numbers

18.2.1 The Argand diagram

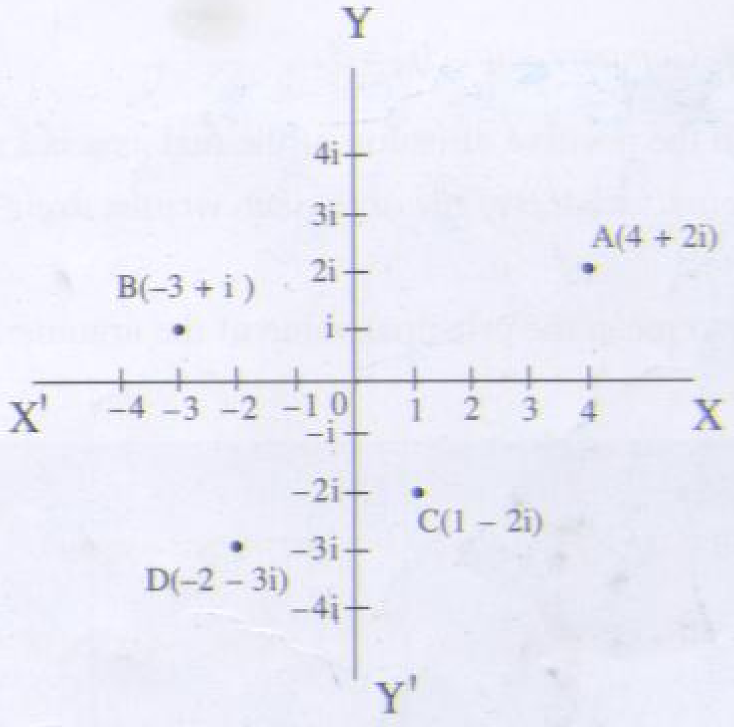


Figure 18.1

In the cartesian plane, any point can be represented by an ordered pair (a, b) and any ordered pair represents a point.

In a similar way, a complex number can be represented by a point on a diagram known as an *Argand diagram* (Figure 18.1). Conversely, any point on the Argand diagram represents a complex number.

The complex number 4 + 2i is represented by A and the point B represents the complex number –3 + i, C the number 1 – 2i, D the number –2 – 3i, etc. X'OX is known as the real axis and Y'OY as the imaginary axis.

18.2.2 Modulus and argument of a complex number

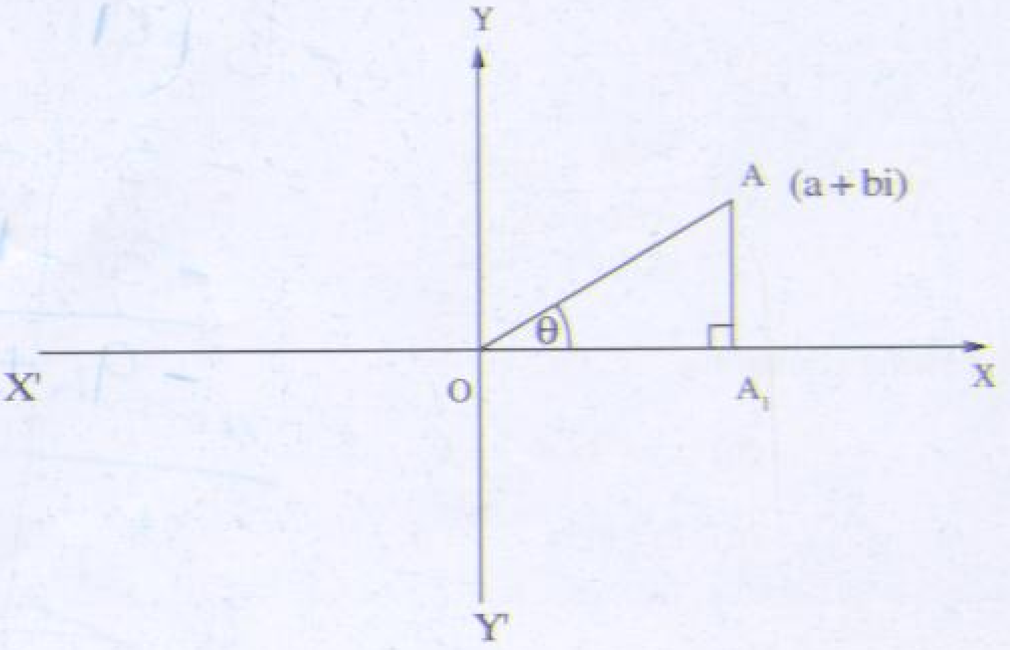


Figure 18.2

In Figure 18.2, A represents the complex number a + bi. The length of **OA** is called the modulus of a + bi.

$|\mathbf{OA}| = \sqrt{a^2 + b^2}$

Note: The length of AA$_1$ is b not bi. Compare with $|3j| = 3$.

The angle θ which **OA** makes with the positive direction of the real axis is known as the argument of a + bi. *If z is a complex number, the principal value θ of the argument, written arg z is the value of θ:*

$-\pi < \theta \leq \pi$

Henceforth, we will use argument to mean the principal value of the argument.

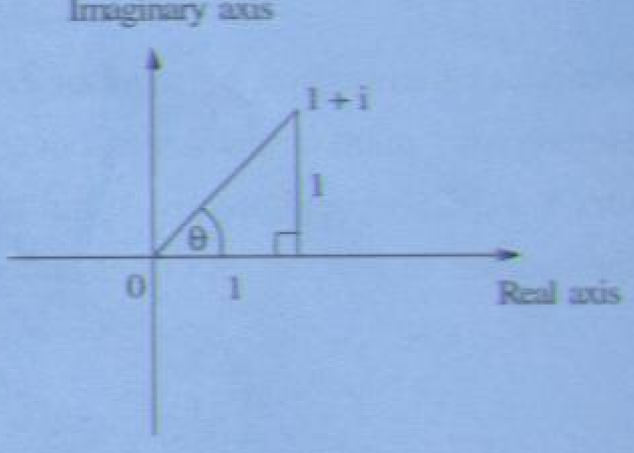
Example 3

Find the modulus and argument of each of the following complex numbers:

(a) $1 + i$ *(b)* $-1 + i\sqrt{3}$ *(c)* $-3 - 4i$ *(d)* $2 - 3i$

Solution

We represent the numbers on Argand diagram.



(a) $|1 + i| = \sqrt{1^2 + 1^2}$

$= \sqrt{2}$

$\tan\theta = 1$

$\theta = \frac{\pi}{4}$

$\arg(1 + i) = \frac{\pi}{4}$

(b) $\left|-1+i\sqrt{3}\right| = \sqrt{1^2+\left(\sqrt{3}\right)^2}$

$= 2$

$\theta = \pi - \alpha$

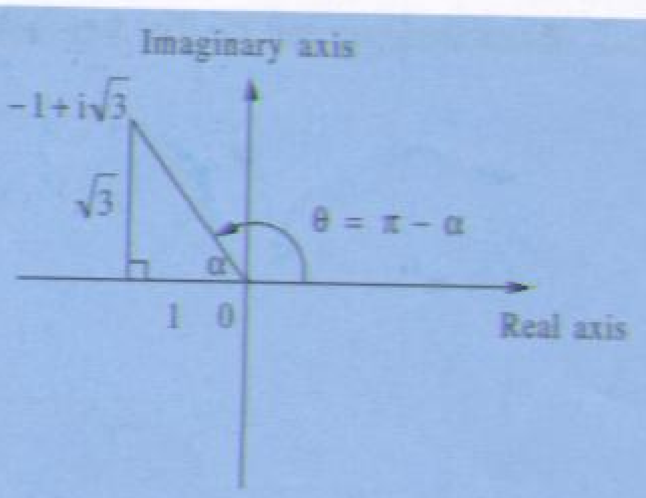
$\tan\alpha = \sqrt{3}$

$\alpha = \dfrac{\pi}{3}$

$\theta = \pi - \dfrac{\pi}{3}$

$= \dfrac{2\pi}{3}$

$\arg\left(-1+i\sqrt{3}\right) = \dfrac{2\pi}{3}$



(c) $\left|-3-4i\right| = \sqrt{3^2+4^2}$

$= 5$

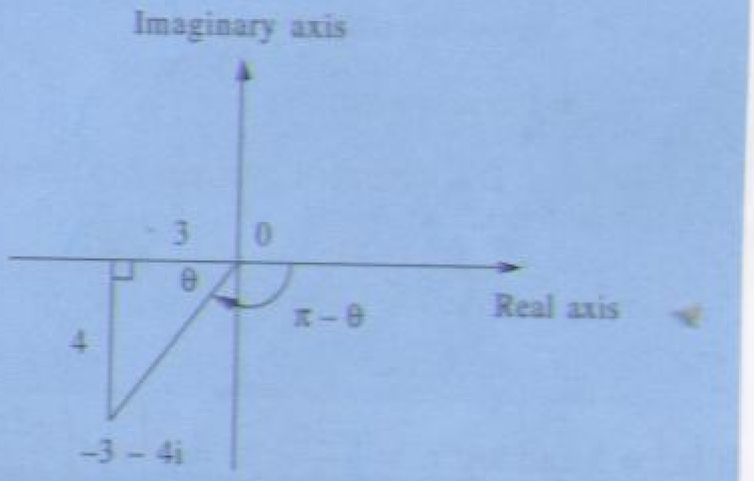
Since the angle measured anticlockwise is greater than 180°, we find the angle $\pi - \theta$ measured clockwise.

$\tan\theta = \dfrac{4}{3}$

$\theta = 0.927$ rad (calculator)

$\pi - \theta = 2.21$ rad

$\text{Arg}(-3-4i) = -2.21$ rad



Note: The angle is measured clockwise as the angle measured anticlockwise is greater than π radians.

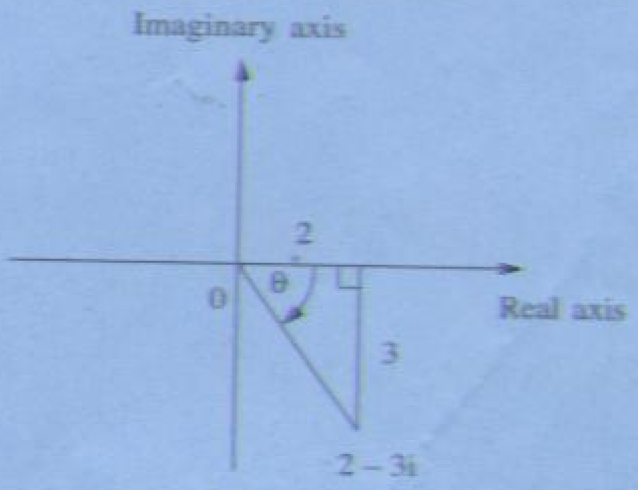
(d) $\left|2-3i\right| = \sqrt{2^2+3^2}$

$= \sqrt{13}$

$\tan\theta = \dfrac{3}{2}$

$\theta = 0.983$ rad

$\text{Arg}(2-3i) = -0.983$ rad



18.2.3 Reducing a + ib to the form r(cos θ + isin θ)

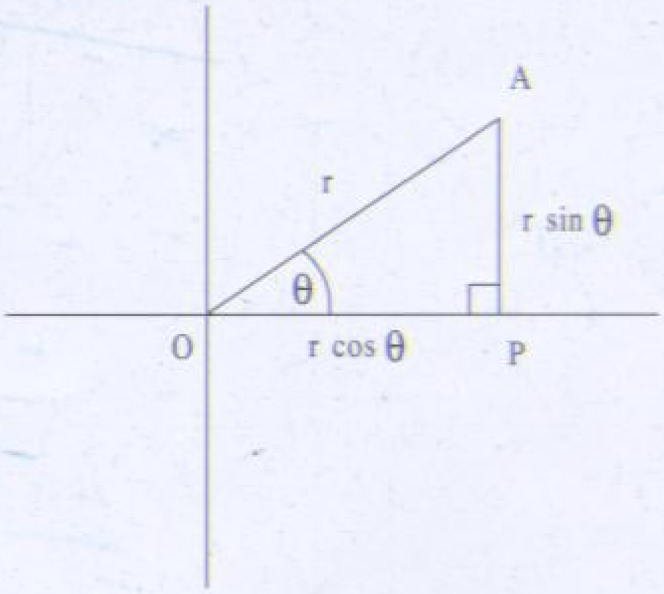


Figure 18.3

Consider the complex number a + ib which has modulus r and argument θ.

In Figure 18.3, OP = rcos θ and PA = rsin θ.

So, $a + ib \equiv r\cos\theta + ir\sin\theta$

$\equiv r(\cos\theta + i\sin\theta)$

It follows that if a complex number has modulus r and argument θ, it can be written as r(cos θ + isin θ).

If $|z_1| = 4$ and $\arg z_1 = \frac{\pi}{6}$, $z_1 = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.

If $|z_2| = 5$ and $\arg z_2 = \frac{-2\pi}{3}$, $z_2 = 5\left[\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right]$

$$= 5\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)$$

Example 4

Reduce 4 + 3i to the form r(cos θ + isin θ).

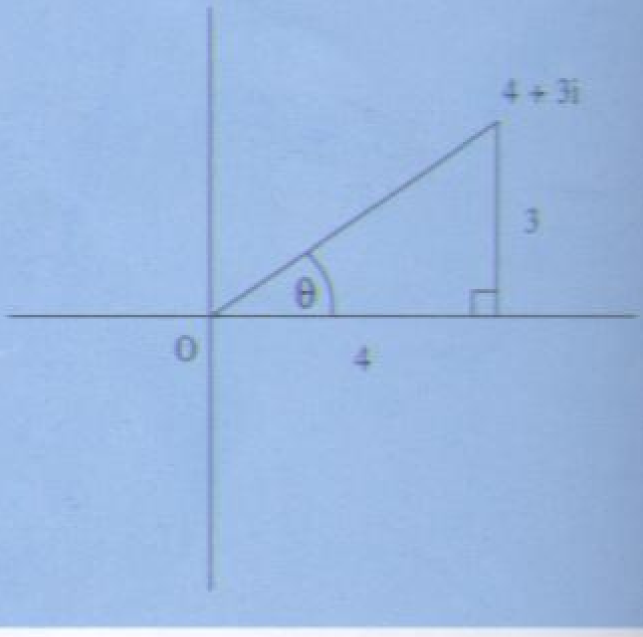
Solution

$$|4 + 3i| = \sqrt{4^2 + 3^2}$$
$$= 5$$

From the diagram, $\tan\theta = \frac{3}{4}$

$$\theta = 0.644 \text{ rad}$$

So, 4 + 3i = 5(cos 0.644 + isin 0.644).

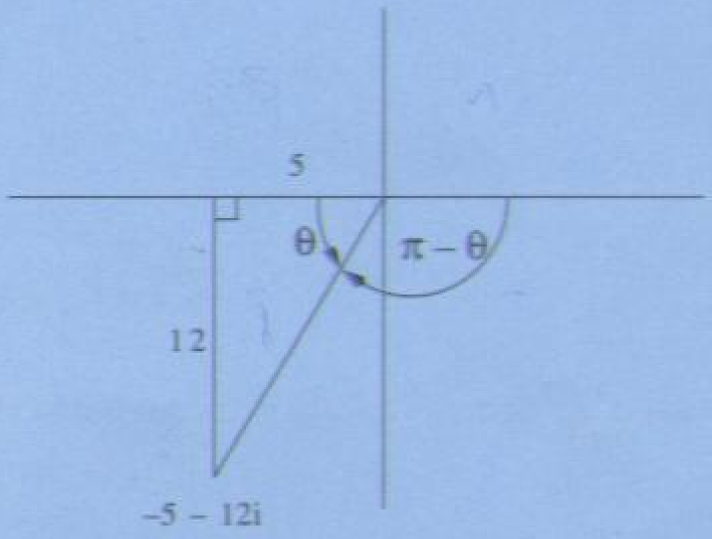


Example 5

Reduce $-5 - 12i$ to the form $r(\cos \theta + i\sin \theta)$.

Solution

$$|-5 - 12i| = \sqrt{5^2 + 12^2}$$
$$= 13$$



From the diagram, $\tan \theta = \dfrac{12}{5}$

$\theta = 1.176$ rad

$\pi - \theta = 1.966$ rad

$\arg(-5 - 12i) = -1.97$ rad

So, $-5 - 12i = 13(\cos(-1.97) + i\sin(-1.97))$

$= 13(\cos 1.97 - i\sin 1.97)$

Example 6

Convert $5(\cos 150° + i\sin 150°)$ to the form $a + bi$.

Solution

$$5(\cos 150° + i\sin 150°) = 5\left(-\cos 30° + \frac{i}{2}\right)$$

$$= \frac{-5\sqrt{3}}{2} + \frac{5i}{2}$$

Exercise 18 B

1. Represent each of the following complex numbers on an Argand diagram:

(a) $2 + i$
(b) $-1 + 2i$
(c) $-2 - i$
(d) $1 - 2i$
(e) $3i$
(f) -4
(g) $-5i$
(h) $3 + \frac{1}{2}i$
(i) $-4 + \frac{3}{2}i$
(j) $-5 - \frac{7}{2}i$
(k) $\sqrt{3} - i$
(l) $1 - i\sqrt{3}$

2. Find the modulus and argument of each of the following complex numbers, giving the argument in terms of π where possible:

(a) $\sqrt{3} + i$
(b) $-1 + i$
(c) $-1 - i\sqrt{3}$
(d) $4 - 3i$
(e) $12 + 5i$
(f) $-8 + 6i$
(g) $-5 - 12i$
(h) $3 - 2i$
(i) $3i$
(j) $4i$
(k) $-5i$
(l) $-\sqrt{3}$
(m) $3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$
(n) $2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$
(o) $-4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$
(p) $-2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

3. Write each of the following complex numbers in the form r(cos + isin):

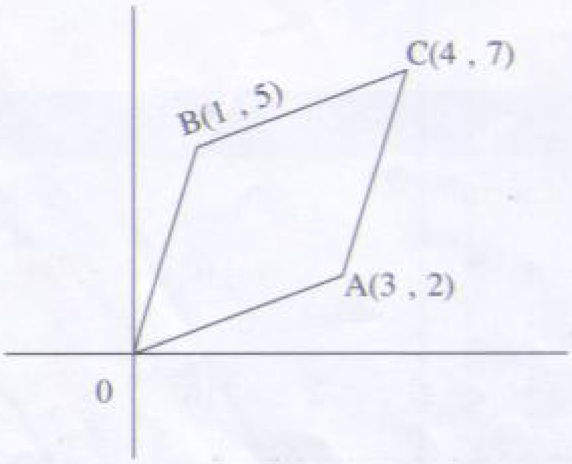
(a) $1 + i\sqrt{3}$
(b) $-2 + 2i$
(c) $-3 - 3i\sqrt{3}$
(d) $5 - 12i$
(e) $6 + 8i$
(f) $-4 + 5i$
(g) $-3 - 7i$
(h) $3 - 8i$
(i) $2i$
(j) -5
(k) $-6i$
(l) $8i$

4. Reduce each of the following to the form a + ib:

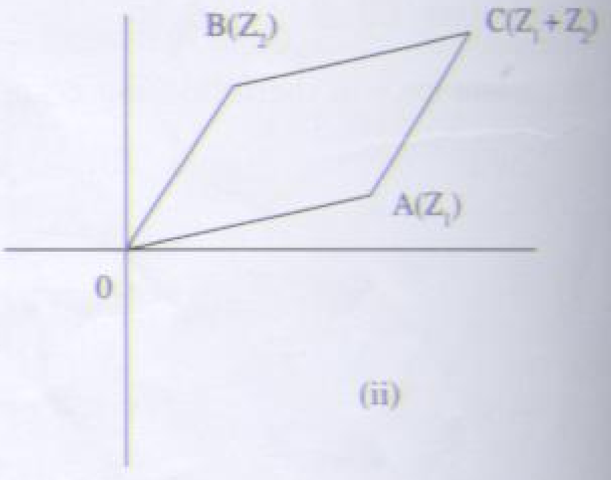
(a) $3\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$
(b) $4\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$
(c) $-4\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$
(d) $3\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

18.3 Geometrical interpretation of operations on complex numbers

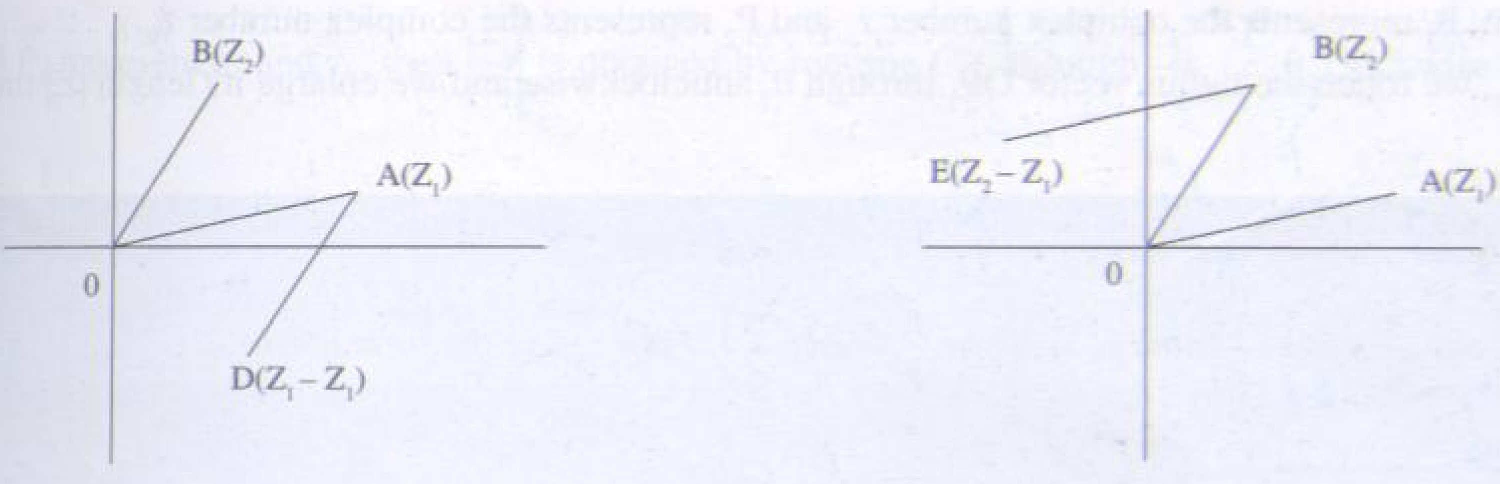
18.3.1 Addition and subtraction



(i)



(ii)



(iii) (iv)

Figure 18.4

Consider the points with coordinates (3, 2) and (1, 5) in the cartesian plane. Their position vectors are **OA** and **OB**. **OA** + **OB** is obtained by completing the parallelogram OACB.

We note that C has coordinates (4, 7) = (1, 5) + (3, 2).

Similarly, if A and B represent the complex numbers z_1 and z_2, the sum is obtained by finding **OA** + **OB** which is **OC**.

So, C represents $z_1 + z_2$.

Also, D represents $z_1 - z_2$ where **AD** = – **AC** and E represents $z_2 - z_1$ where **BE** = –**BC**.

18.3.2 Product

Consider complex numbers z_1 and z_2 with $|z_1| = r_1$, arg $z_1 = \theta_1$, $|z_2| = r_2$ and arg $z_2 = \theta_2$.

$z_1 = r_1(\cos \theta_1 + i\sin \theta_1)$

$z_2 = r_2(\cos \theta_2 + i\sin \theta_2)$

$z_1z_2 = r_1r_2(\cos \theta_1 + i\sin \theta_1)(\cos \theta_2 + i\sin \theta_2)$

$= r_1r_2[\cos\theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i(\sin\theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)]$

$= r_1r_2[\cos (\theta_1 + \theta_2) + i\sin (\theta_1 + \theta_2)]$

So, $|z_1z_2| = r_1r_2 = |r_1| \times |r_2|$

arg z_1z_2 = arg z_1 + arg z_2

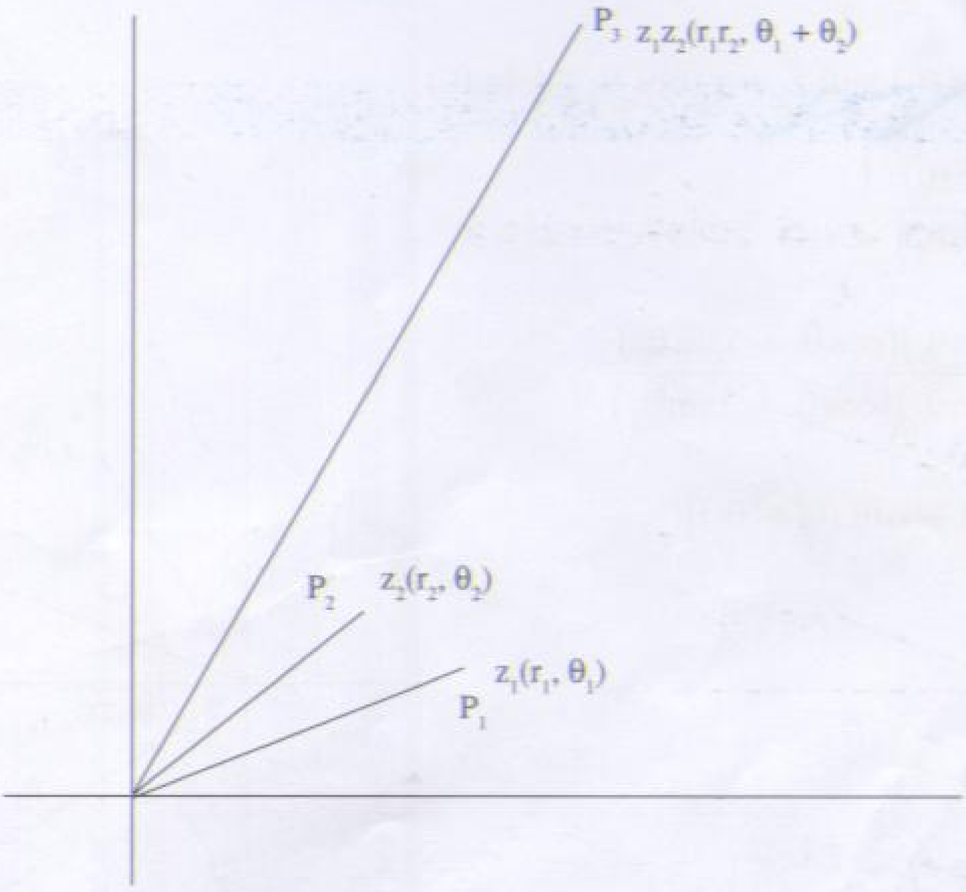


Figure 18.5

In the diagram, P_1 represents the complex number z_1 and P_2 represents the complex number z_2.
To obtain z_1z_2, we rotate the radius vector OP_1 through θ_2 anticlockwise and we enlarge its length $|z_2|$ times.

Example 7

Given $z_1 = 3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ *and* $z_2 = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$. *Find* z_1z_2 *in the form* $a + bi$.

Solution

$$|z_1| = 3, \arg z_1 = \frac{\pi}{4}$$

$$|z_1| = 2, \arg z_2 = \frac{-\pi}{6}$$

$$\begin{aligned} |z_1z_2| &= |z_1| \times |z_2| \\ &= 3 \times 2 \\ &= 6 \end{aligned}$$

$$\begin{aligned} \arg z_1z_2 &= \arg z_1 + \arg z_2 \\ &= \frac{\pi}{4} + \frac{-\pi}{6} \\ &= \frac{\pi}{12} \end{aligned}$$

$$\begin{aligned} z_1z_2 &= 6\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \\ &= 5.80 + 1.55i \end{aligned}$$

18.3.3 Division

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \\ &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)} \\ &= \frac{r_1}{r_2}\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \end{aligned}$$

So, $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

$$\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$$

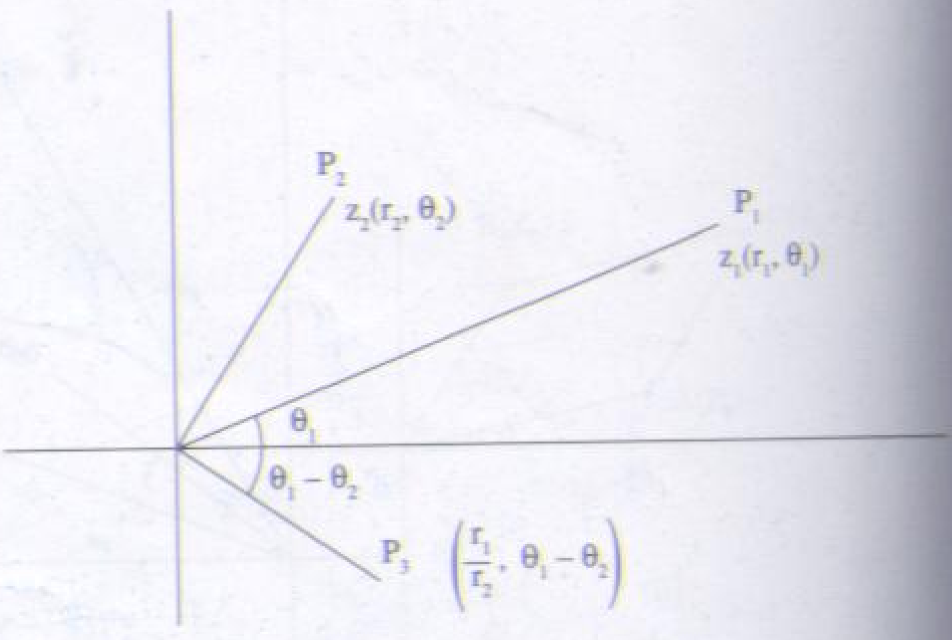


Figure 18.3

If P_1 and P_2 represent z_1 and z_2, then $\left|\frac{z_1}{z_2}\right|$ is obtained by rotating OP_1 through $-\theta_2$ i.e. θ_2 clockwise and enlarging its length $\frac{1}{|z_2|}$ time.

Example 8

Given $z_1 = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, $z_2 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$,

(a) *find in the form* $a + bi$ (i) $\frac{z_2}{z_1}$ (ii) $\frac{z_1}{z_2}$

(b) *Represent* z_1, z_2, $\frac{z_1}{z_2}$ *and* $\frac{z_1}{z_2}$ *on the same Argand diagram*

Solution

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$= \frac{3}{2}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

$$\frac{z_1}{z_2} = \frac{3}{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

$$= 1.45 + 0.388i$$

$$\left|\frac{z_2}{z_1}\right| = \frac{|z_2|}{|z_1|}$$

$$= \frac{2}{3}$$

$$\arg\left(\frac{z_2}{z_1}\right) = \arg z_2 - \arg z_1$$

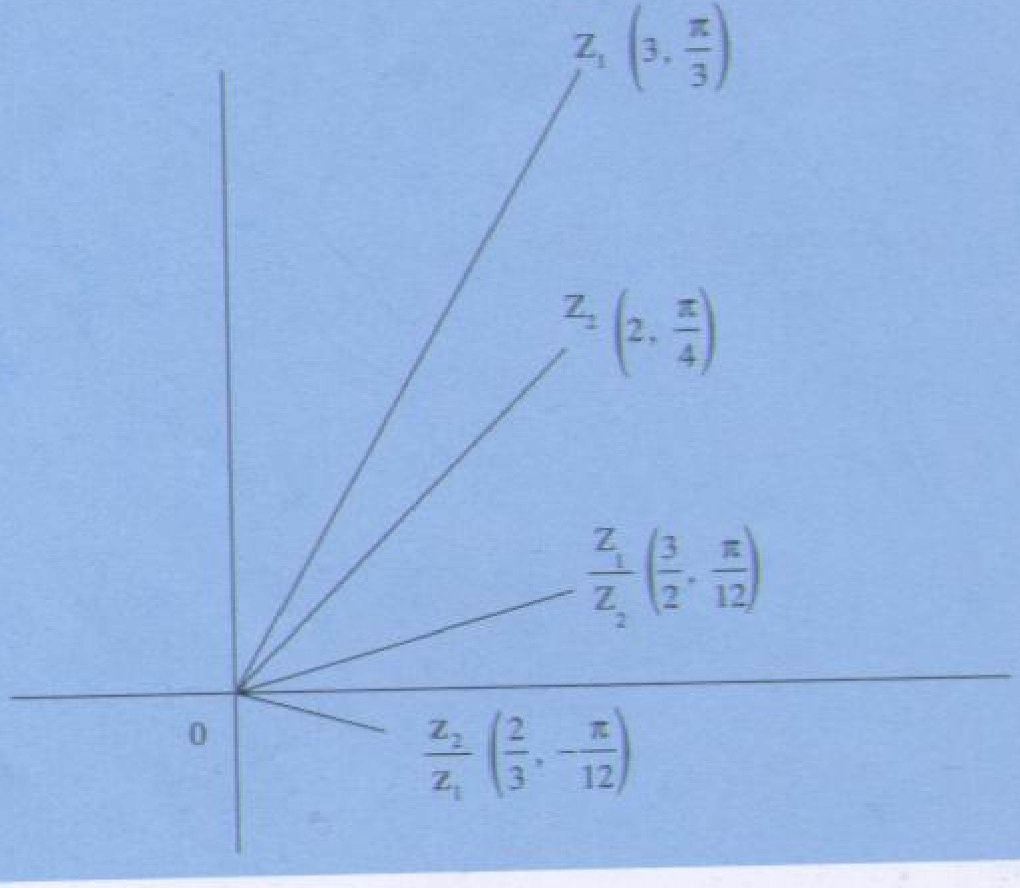
$$= \frac{\pi}{4} - \frac{\pi}{3}$$

$$= -\frac{\pi}{12}$$

$$\frac{z_2}{z_1} = \frac{2}{3}\left(\cos\left(\frac{-\pi}{12}\right) + i\sin\left(\frac{-\pi}{12}\right)\right)$$

$$= \frac{2}{3}\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right)$$

$$= 0.644 - 0.173i$$



18.3.4 The form $re^{i\theta}$

It can be shown that cos θ + isin θ can be written as $e^{i\theta}$. The complex number r(cos θ + isin θ) can therefore be written as $re^{i\theta}$.

This method of writing a complex number is very helpful for multiplying or dividing complex numbers expressed in the form r(cos θ + isin θ).

Example 9

Find: (a) $\left(2\cos\frac{3\pi}{4} + 2i\sin\frac{3\pi}{4}\right)\left(3\cos\frac{\pi}{6} - 3i\sin\frac{\pi}{6}\right)$ *(b)* $\dfrac{\left(3\cos\frac{\pi}{12} - 3i\sin\frac{\pi}{12}\right)}{\left(2\cos\frac{\pi}{6} - 2i\sin\frac{\pi}{6}\right)}$

Solution

(a) $\left(2\cos\frac{3\pi}{4} + 2i\sin\frac{3\pi}{4}\right) = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$$= 2e^{\frac{i3\pi}{4}}$$

$$\left(3\cos\frac{\pi}{6} + 3i\sin\frac{\pi}{6}\right) = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$= 3e^{\frac{-i\pi}{6}}$$

$$\left(2\cos\frac{3\pi}{4} + 2i\sin\frac{3\pi}{4}\right)\left(3\cos\frac{\pi}{6} - 3i\sin\frac{\pi}{6}\right) = 2e^{\frac{i3\pi}{4}} \times 3e^{\frac{-i\pi}{6}}$$

$$= 6e^{\frac{i7\pi}{12}}$$

$$= 6\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$$

$$= -1.55 + 5.80i$$

(b) $$\frac{\left(3\cos\frac{\pi}{12} - 3i\sin\frac{\pi}{12}\right)}{\left(2\cos\frac{\pi}{6} - 2i\sin\frac{\pi}{6}\right)} = \frac{3e^{\frac{-i\pi}{12}}}{2e^{\frac{-i\pi}{6}}}$$

$$= \frac{3}{2}e^{\frac{i\pi}{12}}$$

$$= \frac{3}{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

$$= 1.45 + 0.388i$$

18.3.5 Equality of complex numbers

Consider two complex numbers $a + bi$ and $c + di$. If they are equal, i.e. $a + bi = c + di$, it can be shown that $a = c$ and $b = d$, i.e. their real parts are equal and their imaginary parts are equal.
So, if $a + bi = 3 - 2i$, $a = 3$ and $b = -2$.

18.3.6 Square roots of a complex number

To find the square roots of the complex number $5 - 12i$, we assume that the square roots are of the form $a + bi$ where a and b are real numbers.

Let $\sqrt{5 - 12i} = a + bi,\ a, b \in \mathbb{R}$

So, $(a + bi)^2 = 5 - 12i$

$a^2 - b^2 + 2abi = 5 - 12i$

Comparing real and imaginary points,

$$a^2 - b^2 = 5 \qquad (1)$$

$$2ab = -12 \qquad (2)$$

From (2) $b = \frac{-6}{a}$ (A)

Solving (1) & (A), we have $a^2 - \left(\frac{-6}{a}\right)^2 = 5$

$$a^2 - \frac{36}{a^2} = 5$$

$$a^4 - 5a^2 - 36 = 0$$

$$(a^2 - 9)(a^2 + 4) = 0$$

$$a^2 = 9 \text{ or } -4$$

As $a \in \mathbb{R}$ $a^2 = 9$, $a = \pm 3$

From (A) $b = -2$ or $+2$

So, the square roots of $5 - 12i$ are $3 - 2i$ or $-3 + 2i$.

Exercise 18 C

1. In each of the following, find z_1z_2, $\frac{z_1}{z_2}$ and $\frac{z_2}{z_1}$ giving each result in

   (i) the form $re^{i\theta}$ and

   (ii) the form $a + ib$, where θ is the principal value of arg z:

   (a) $z_1 = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$, $z_2 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

   (b) $z_1 = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, $z_2 = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

   (c) $z_1 = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$, $z_2 = \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

   (d) $z_1 = 4\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$, $z_2 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

   (e) $z_1 = 3\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$, $z_2 = \left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$

2. Reduce $\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$ to the form $re^{i\theta}$. Hence, obtain $\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)^2$ in the form $a + ib$.

3. Write z_1 and z_2 in the form $re^{i\theta}$ where $z_1 = \frac{1}{2} + \frac{1}{2}\sqrt{3}i$ and $z_2 = \frac{1}{2}\sqrt{3} + \frac{1}{2}i$. Hence, obtain z_1z_2, $\frac{z_1}{z_2}$ and $\frac{z_2}{z_1}$ in the form a + ib.

4. Write z_1 and z_2 in the form $re^{i\theta}$ where $z_1 = 1 - \sqrt{3}i$ and $z_2 = \sqrt{3} + i$. Hence, obtain z_1z_2, $\frac{z_1}{z_2}$ and $\frac{z_2}{z_1}$ in the form a + ib.

5. Write z_1 and z_2 in the form $re^{i\theta}$ where $z_1 = 1 + i$ and $z_2 = 1 - \sqrt{3}i$. Hence, obtain z_1^3, z_2^2, $z_1^3z_2^2$, $\frac{z_1^3}{z_2^2}$ and $\frac{z_2^2}{z_1^3}$ in the form a + ib.

6. Find the two square roots of each of the following:

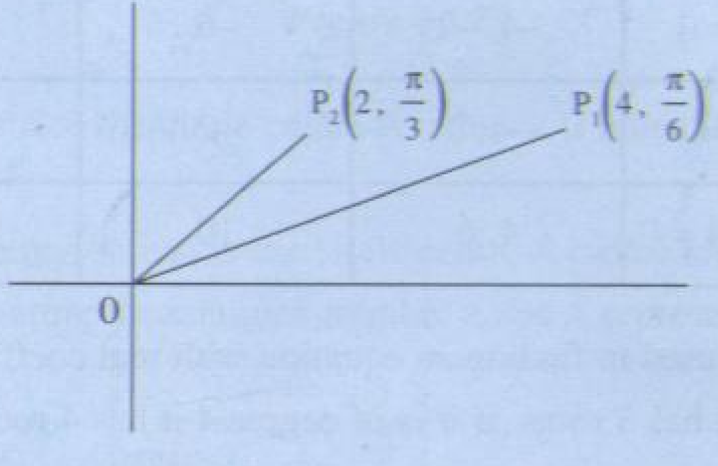
(a) $3 - 4i$ (b) $-18 + 24i$ (c) $21 - 20i$

(d) $7 - 24i$ (e) $-5 - 12i$ (f) $16 + 30i$

7. On an argand diagram, mark the points P_1, P_2 and P representing the complex numbers z_1, z_2 and $z_1 + z_2$ respectively. Deduce that $|z_1 + z_2| \leq |z_1| + |z_2|$.

Illustrate the case $|z_1 + z_2| = |z_1| + |z_2|$ on a separate diagram.

8.



In the given diagram, P_1 represents $z_1 = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$, P_2 represents $z_2 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$. Mark on the same diagram the points P_3, P_4, P_5 and P_6 representing $z_1 + z_2$, $z_1 - z_2$, z_1z_2 and $\frac{z_1}{z_2}$ respectively.

9. Use the fact that the conjugate of a + ib is a − ib to show that the conjugate of $r(\cos\theta + i\sin\theta)$ is $r(\cos\theta - i\sin\theta)$.

Represent on a single diagram z, z* and zz*. What geometrical interpretation can be given to the multiplication of a complex number by its conjugate?

10. Given $|z| = r$ and $\arg z = \theta$, what are the values of $|zz^*|$, $\left|\frac{z}{z^*}\right|$, $\arg(zz^*)$ and $\arg\frac{z}{z^*}$?

18.4 Complex roots of an equation

Consider the general quadratic equation $ax^2 + bx + c = 0$. We have seen that the roots are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. If $b^2 - 4ac < 0$, put $b^2 - 4ac = -k^2$ where k is a real number.

$$\text{So, } x = \frac{-b \pm \sqrt{-k^2}}{2a}$$

$$= -\frac{b}{2a} + \frac{ik}{2a} \text{ or } -\frac{b}{2a} - \frac{ik}{2a}$$

We note that these two roots are conjugate.
So, if one root of a quadratic is complex, the other root is its conjugate. **This result applies to any polynomial equation with real coefficients. Non–real roots occur always in conjugate pairs.** Thus, for example, if $2 + 3i$ satisfies the equation $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$ where a, b, c, d, e and f are real, another root is $2 - 3i$.

We note that the sum of the roots of the quadratic equation $x^2 + dx + e = 0$ (i.e. where the coefficient of x^2 is 1) is always equal to –d (minus the coefficient of x) and the product of the roots is always equal to the constant term e, as illustrated in the table below.

Equation	Roots	Sum	Product
$x^2 + 6x + 8 = 0$	–4, –2	–6	8
$x^2 + 2x - 8 = 0$	–4, 2	–2	–8
$x^2 - 10x + 24 = 0$	4, 6	10	24

This is an important result which is used in finding an equation with real coefficients and having complex roots. Also, if an equation is of degree 3 it has 3 roots, if it is of degree 4 it has 4 roots. Generally if it is of degree n it has n roots.

Example 10

Find the equation with degree 4 having (3 – 2i) and (5 + 2i) as two of its roots.

Solution

As two of its roots are $3 - 2i$ and $5 + 2i$ the two other roots are $3 + 2i$ and $5 - 2i$.
Consider the quadratic with conjugate roots $3 - 2i$ and $3 + 2i$.
The sum of its roots is 6 and the product is 13.
So the quadratic is $z^2 - 6z + 13 = 0$
Next consider the quadratic with conjugate roots $5 + 2i$ and $5 - 2i$.
The sum of its roots is 10 and the product is 29.

So the quadratic is $z^2 - 10z + 29 = 0$.

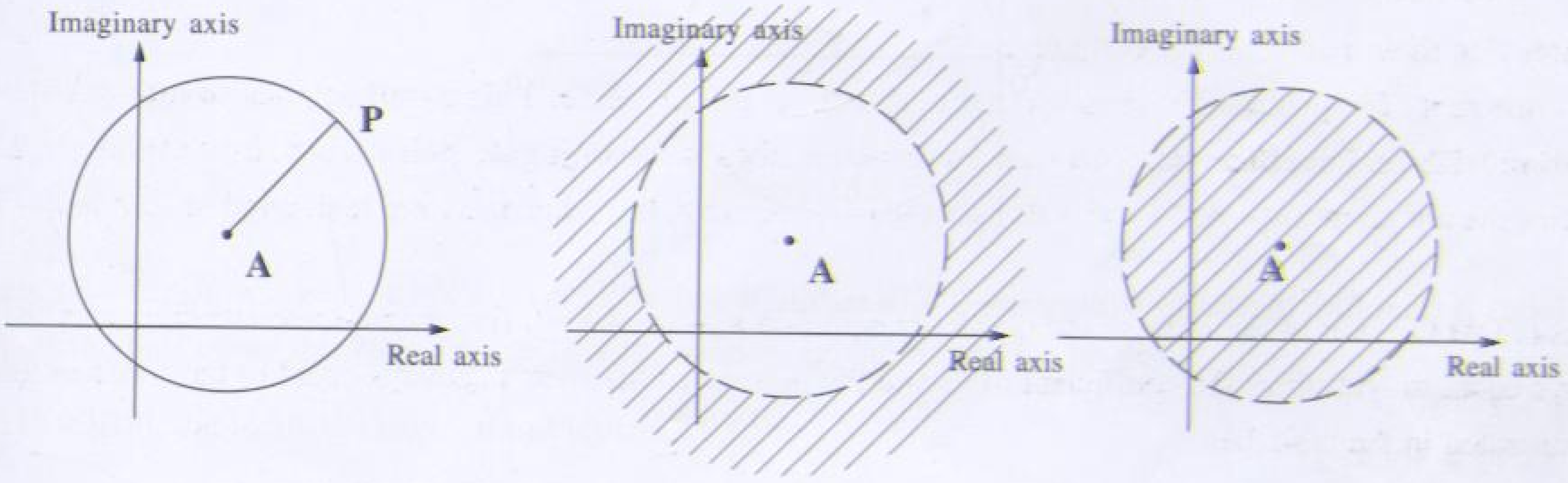
Hence the required equation is $(z^2 - 6z + 13)(z^2 - 10z + 29) = 0$

i.e. $z^4 - 16z^3 + 102z^2 - 304z + 377 = 0$

18.4.2 Loci in an Argand diagram

Type 1 $|z - a| = k$, $|z - a| > k$ *and* $|z - a| < k$

If A is a fixed point in a plane and P is a point in the plane, then if $|\mathbf{AP}| = k$, P lies on a circle with centre A and radius k (Figure 18.7 (a)).



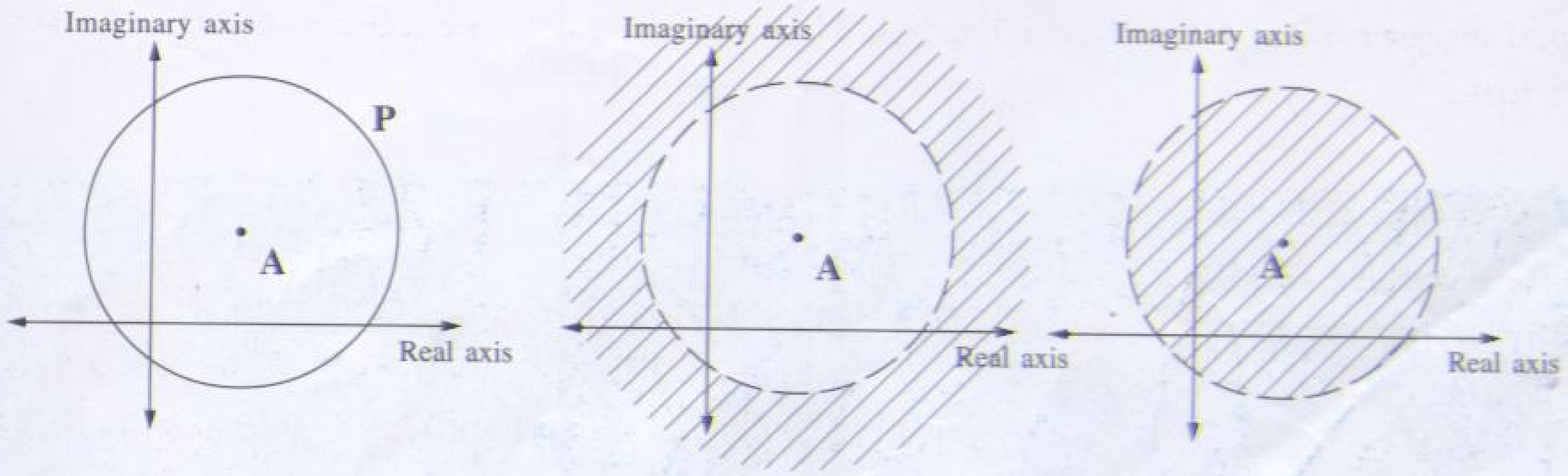
(a) *(b)* *(c)*

Figure 18.7

If $|\mathbf{AP}| > k$, P lies in the region outside the circle centre A radius k (Figure 18.7 (b)). The circle is not included in the region and is shown broken.

Similarly, if $|\mathbf{AP}| < k$, P lies in the region inside the circle centre A radius k (Figure 18.7 (c)).

Next, consider the point P representing the complex number z and A representing a fixed complex number a.



(a) *(b)* *(c)*

Figure 18.8

The length of radius vector **AP** represents $|z - a|$ and so $|z - a| = k$, $|z - a| > k$ and $|z - a| < k$ are equivalent to $|\mathbf{AP}| = k$, $|\mathbf{AP}| > k$ and $|\mathbf{AP}| < k$ giving the loci in Figure 18.7 (a), Figure 18.7 (b) and Figure 18.7 (c) respectively.

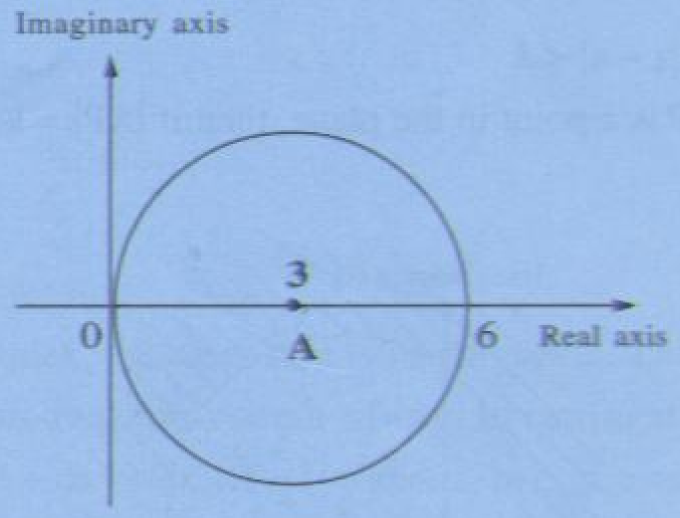
Example 11

Illustrate on separate Argand diagrams:

(a) $|z-3| = 3$ *(b)* $|z-2-3i| \geqslant 2$ *(c)* $|z+1i| < \sqrt{2}$

Solution

(a) $|z-3| = 3$ is equivalent to $|\mathbf{AP}| = 3$ where P represents z and A represents 3.

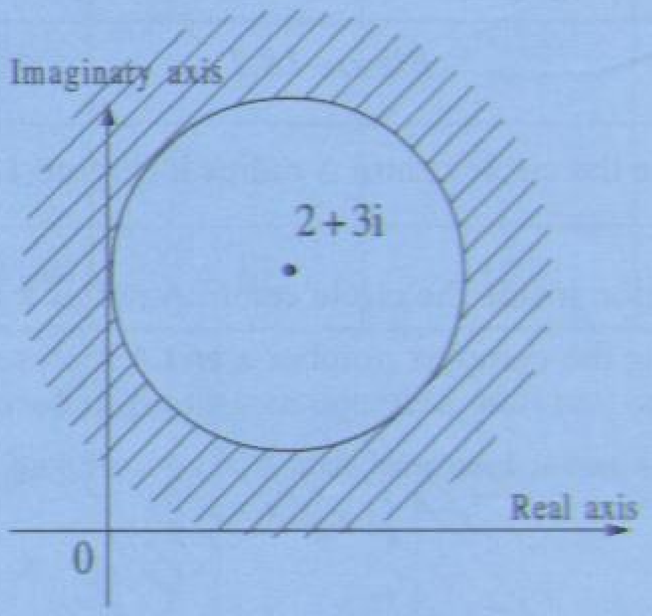


The locus is therefore the circle with centre 3 + 0i radius 3.

(b) $|z-2-3i| \geqslant 2$

$|z-(2+3i)| \geqslant 2$

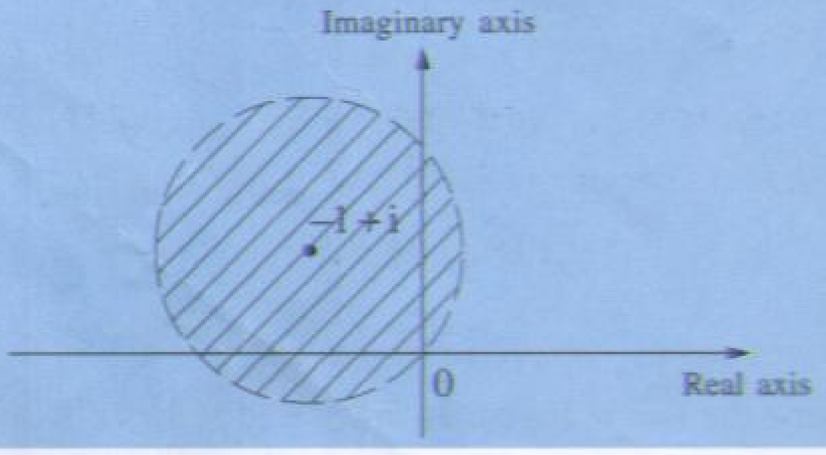
This is equivalent to $|\mathbf{AP}| \geqslant 2$where P represents z and A (2 + 3i).



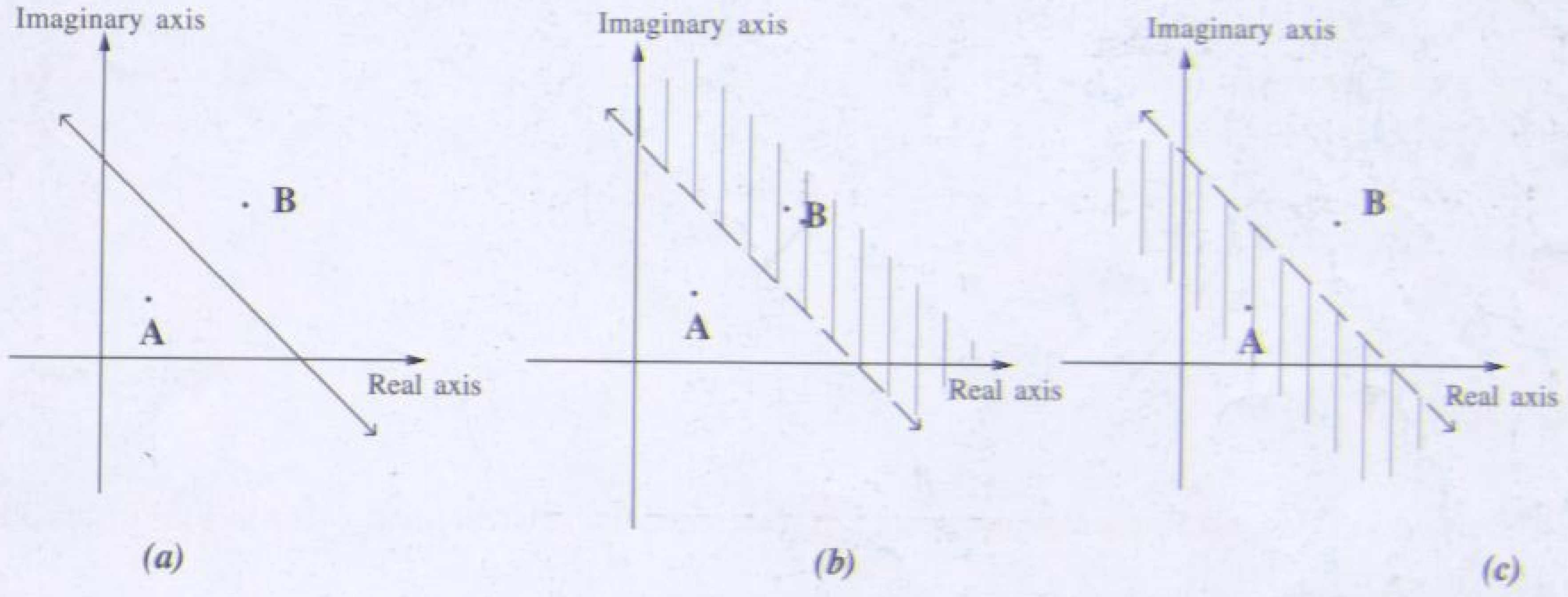
(c) $|z+1-i| < \sqrt{2}$

$|z-(-1+i)| < \sqrt{2}$

This is equivalent to $|\mathbf{AP}| < \sqrt{2}$ where A = (−1 + i). As $|(-1+i)| = \sqrt{2}$, circle passes through 0.



Type 2 $|z - a| = |z - b|$, $|z - a| > |z - b|$ *and* $|z - a| < |z - b|$

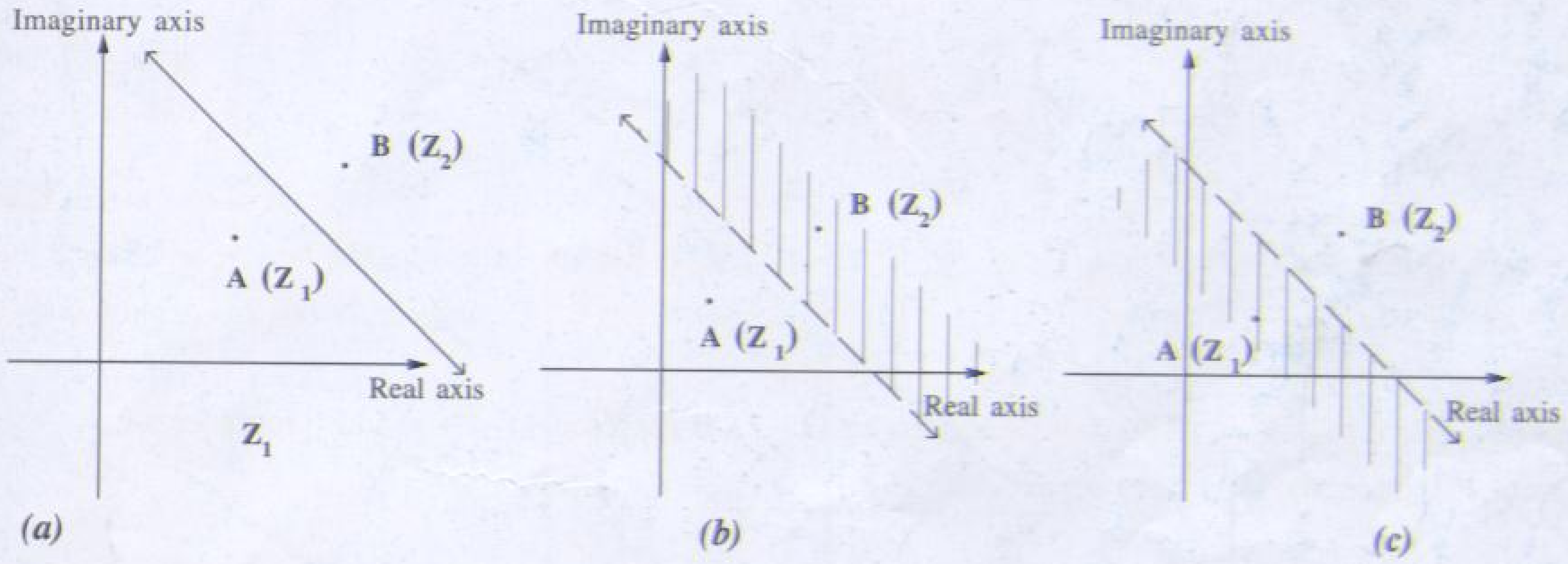


(a) *(b)* *(c)*

Figure 18.9

Figure 18.9 (a) shows the locus of a points P such that **|AP|** = **|BP|**. This is the mediator of AB. If **|AP|** > **|BP|**, the locus is the half plane shown in Figure 18.9 (b) shaded, excluding the mediator.

If **|AP|** < **|BP|**, the locus is as shown in Figure 18.9 (c).



(a) *(b)* *(c)*

Figure 18.10

If A represents the complex number z_1, B the complex number z_2 and P a complex number z, then $|z - z_1| = |z - z_2|$ is equivalent to **|AP|** = **|BP|**.

The locus of P is then the mediator of AB.

Similarly, $|z - z_1| > |z - z_2|$ is equivalent to **|AP|** > **|BP|** and $|z - z_1| = |z - z_2|$ is equivalent to **|AP|** < **|BP|** and the loci are as shown in Figure 18.10 (b) & (c) respectively.

Example 12

Illustrate on separate Argand diagrams the locus of a point P representing z given:

(a) $|z - 1| = |z + 3|$

(b) $|z - 1 + i| \leq |z + 1 - i|$

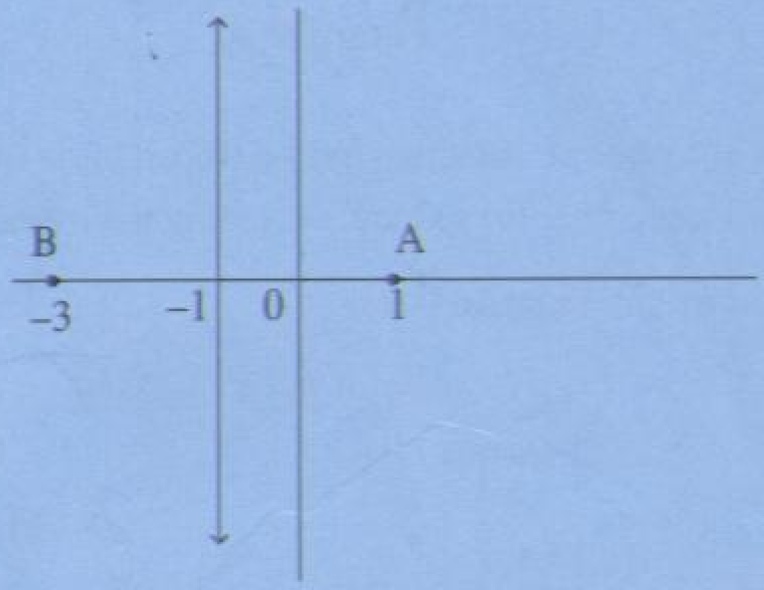
(c) $|z + 3 - 4i| > |z - 4 + 3i|$.

Solution

(a) $|z - 1| = |z + 3|$

$|z - 1| = |z - (-3)|$

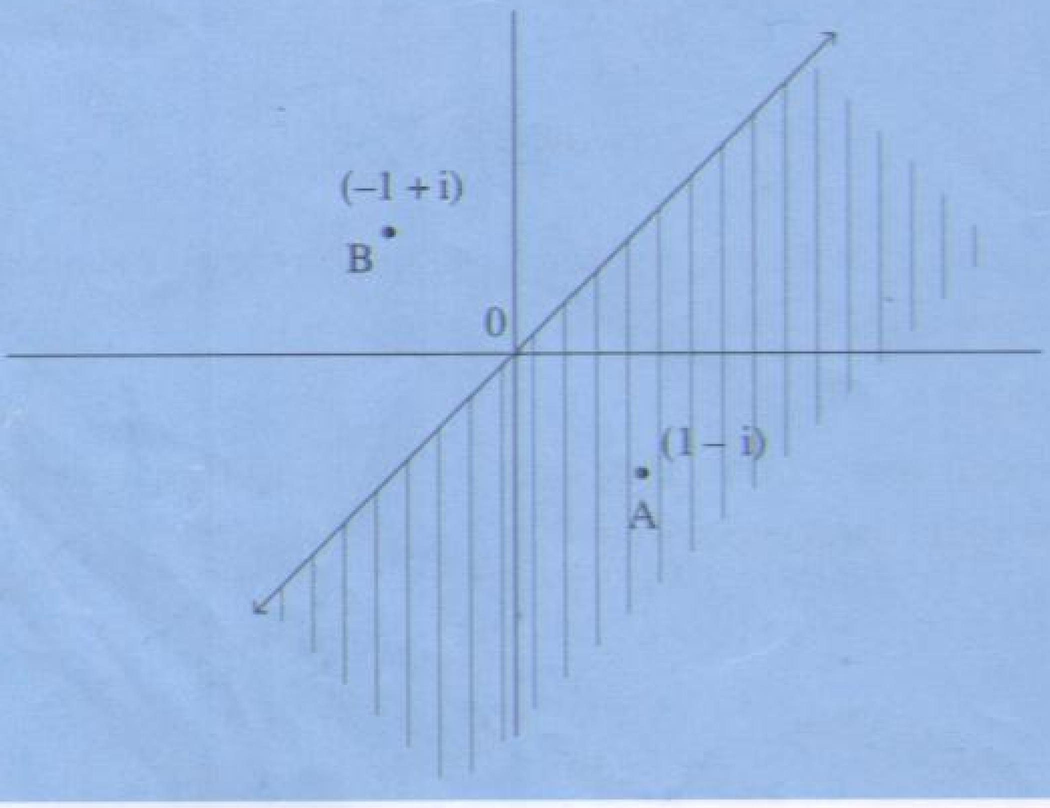
This is equivalent to $|\mathbf{AP}| = |\mathbf{BP}|$ where A represents 1, B represents –3 and P represents z. The locus is the mediator of AB as shown.



(b) $|z - 1 + i| \leq |z + 1 - i|$

$|z - (1 - i)| \leq |z - (-1 + i)|$

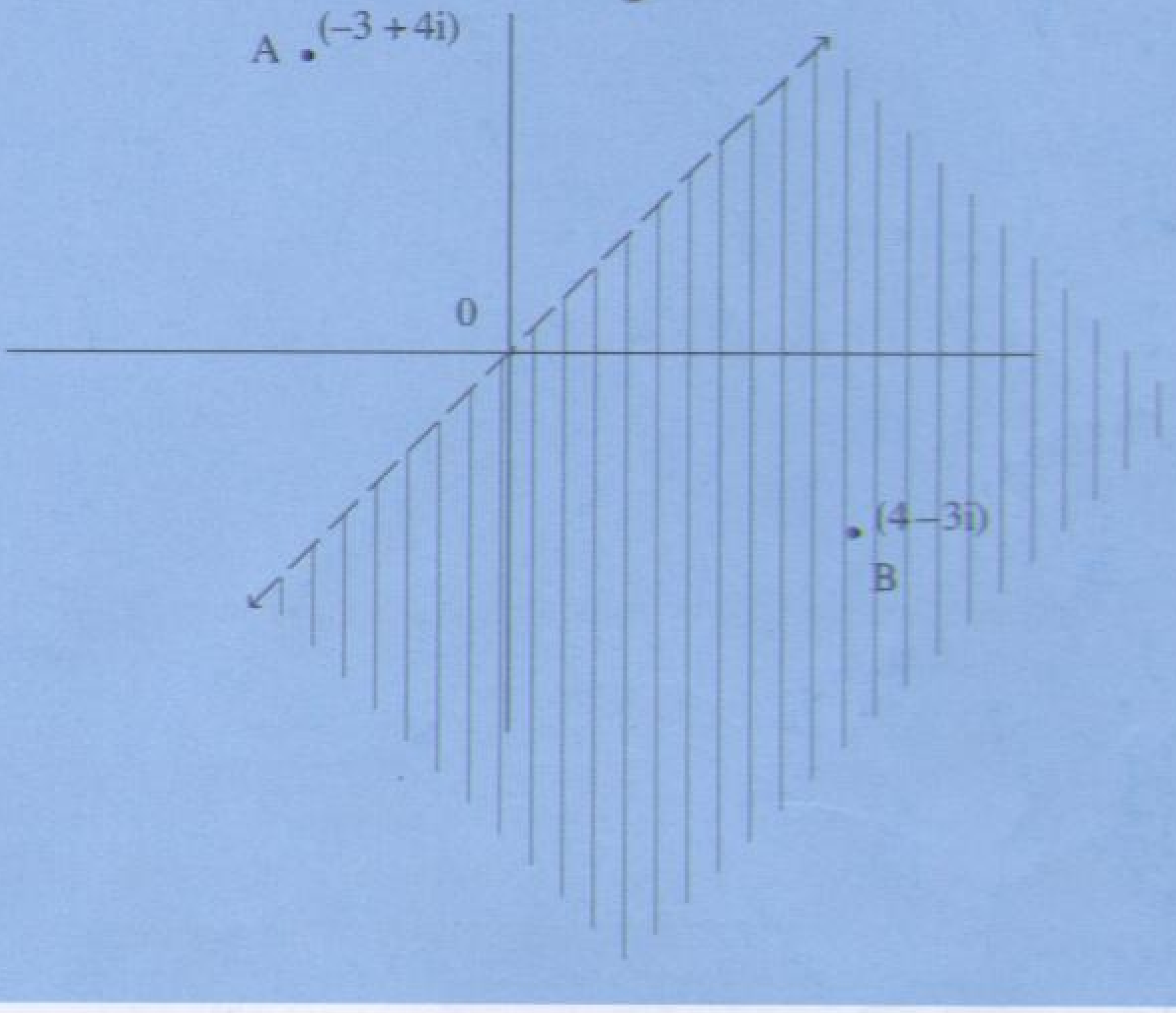
This is equivalent to $|\mathbf{AP}| \leq |\mathbf{BP}|$ where A represents (1 – i), B represents (–1 + i) and P represents z. The locus is then as shown.



(c) $|z + 3 - 4i| > |z - 4 + 3i|$

$|z - (-3 + 4i)| > |z - (4 - 3i)|$

This is equivalent to $|\mathbf{AP}| > |\mathbf{BP}|$ where A represents $(-3 + 4i)$, B represents $(4 - 3i)$ and P represents z.
The locus is as shown.



Type 3 arg (z – a) = α

If A is a fixed point and **AP** makes a constant angle α with OX, the locus of P is a half line starting at A (but excluding A) as shown in Figure 18.11.

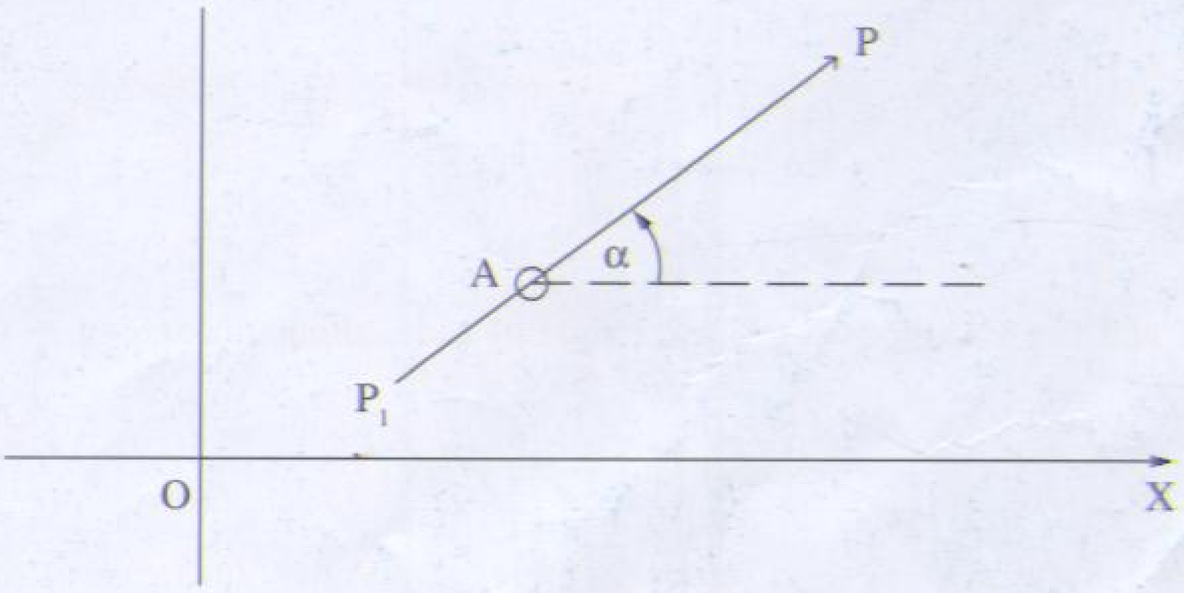


Figure 18.11

Note that the locus is not the full line for $\mathbf{AP}_1$ does not make an angle α with OX.

Note also that A is excluded and this is indicated by an unshaded circle at A. arg (z – a) is equivalent to arg **AP** where A represents a, P represents z.

The locus is then the half line starting from A, excluding A.

Example 13

Sketch on separate Argand diagrams the locus of the point P representing the complex number z such that:

(a) $\arg(z+1) = -\frac{5\pi}{6}$

(b) $\arg(z-1-2i) = \frac{\pi}{3}$

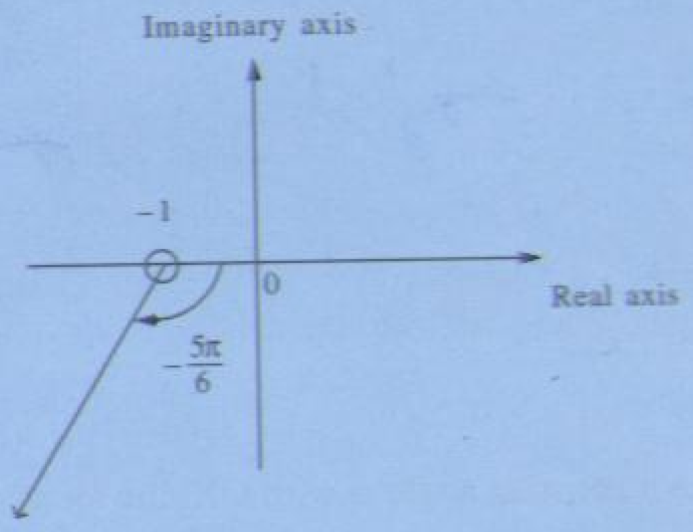
(c) $\arg(z+2+i) = -\frac{\pi}{4}$

Solution

(a) $\arg(z+1) = -\frac{5\pi}{6}$

$\arg(z-(-1)) = -\frac{5\pi}{6}$

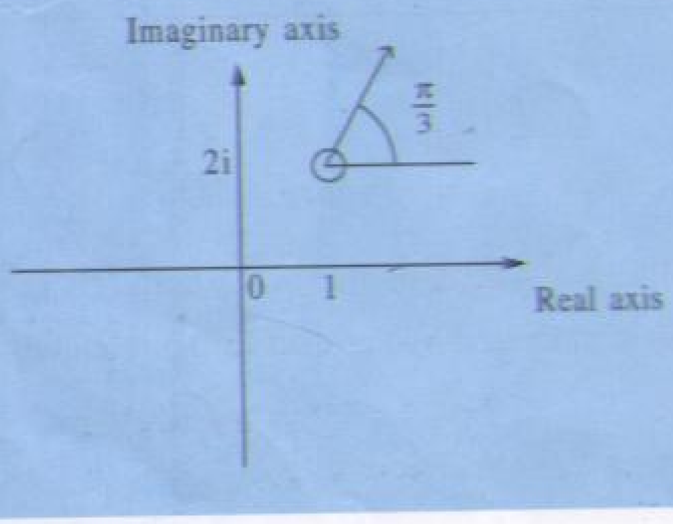
Locus of P is then the half line starting at $-1+0i$ (excluding it) making an angle of $\frac{5\pi}{6}$ clockwise with the positive real axis.



(b) $\arg(z-1-2i) = \frac{\pi}{3}$

$\arg(z-(1+2i)) = \frac{\pi}{3}$

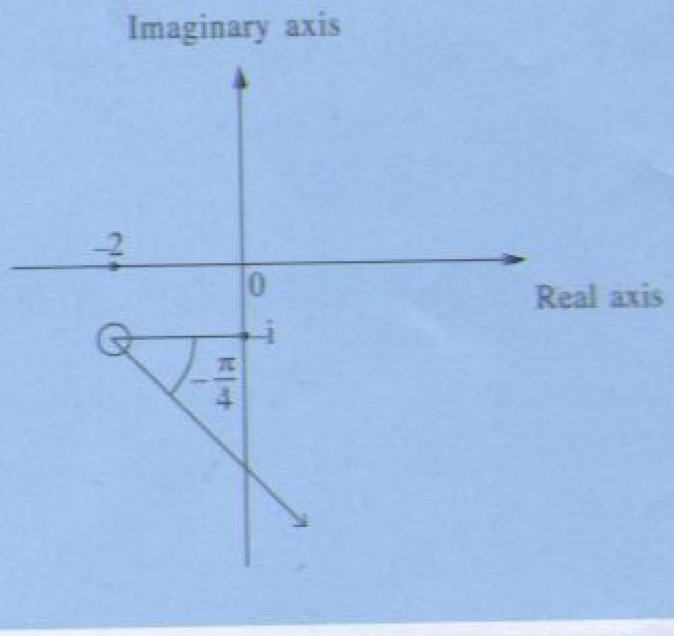
The locus of P is then the half line starting at $1+2i$ (excluding it) and making an angle of $\frac{\pi}{3}$ with the positive real axis.



(c) $\arg(z + 2 + i) = -\frac{\pi}{4}$

$\arg[z - (-2 - i)] = -\frac{\pi}{4}$

The locus of P is the half line starting at –2 – i (excluding it) making an angle of $\frac{\pi}{4}$ clockwise with the positive real axis.



Exercise 18 D

1. In each of the following, a few roots of the polynomial of given degree are given. State the remaining roots in each case and write the equation of each polynomial:
   - (a) Degree 3 – Two roots are 3 and $1 - 2i$
   - (b) Degree 4 – Two roots are $3 - 2i$, $2 + 5i$
   - (c) Degree 5 – Three roots are 1, i, $\sqrt{3} + i$
   - (d) Degree 6 – Three roots are $2 + 3i$, $3 - 2i$, $4 - i$.

2. Sketch on separate free hand argand diagrams the locus of the point P representing the complex number Z such that:
   - (a) $|z - 1| = 1$
   - (b) $|z - 2i| \leqslant 1$
   - (c) $|z + 2| \geqslant 2$
   - (d) $|z + 3i| < 2$
   - (e) $|z - 2 + i| \geqslant 2$
   - (f) $|z - 3 - i| \leqslant 1$
   - (g) $|z + 3–2i| = 2$
   - (h) $|z + 2 + 2i| \geqslant 2$
   - (i) $|z - 1 - i| < \sqrt{2}$
   - (j) $|z + 2 - i| \leqslant \sqrt{5}$
   - (k) $|z + 3 + 4i| \leqslant 5$
   - (l) $|z - 3 - 2i| \geqslant \sqrt{13}$

3. Sketch on separate free hand Argand diagrams the locus of the point P representing the complex number Z such that:
   - (a) $|z - 1| = |z + 3|$
   - (b) $|z -1| \geqslant |z + i|$
   - (c) $|z + 2i| < |z + 2|$
   - (d) $|z + 1 - i| = | z - 1 + i|$
   - (e) $|z + 2 - 3i| \leqslant |z - 3 + 2i|$
   - (f) $|z + 4 - 3i| \leqslant |z - 3 + 4i|$
   - (g) $|z - 1 - 4i| = |z + 4 - i|$
   - (h) $|z - 3 - 2i| \geqslant |z + 2 + 3i|$
   - (i) $|z - 2 + i| = |z + 1 - 2i|$

4. Sketch on separate free hand Argand diagrams the locus of the point P representing the complex number z such that:

(a) $\arg(z-2)=0$
(b) $\arg(z-i)=\frac{\pi}{2}$
(c) $\arg(z+3)=-\frac{\pi}{4}$
(d) $\arg(z+2i)=-\frac{3\pi}{4}$
(e) $\arg(z-1-i)=\pi$
(f) $\arg(z+2-i)=\frac{5\pi}{6}$
(g) $\arg(z-3-i)=-\frac{2\pi}{3}$
(h) $\arg(z+2+2i)=-\frac{\pi}{4}$
(i) $\arg(z+2+3i)=-\frac{\pi}{2}$
(j) $\arg(z+1-3i)=-\frac{\pi}{4}$
(k) $\arg(z+3+i)=\frac{\pi}{2}$
(l) $\arg(z+2-3i)=-\frac{5\pi}{6}$

5. Indicate clearly on separate diagrams the locus of the point P representing the complex number z such that:

(a) $2 \leqslant |z-2| \leqslant 3$

(b) $|z-2| \leqslant 2$ and $-\frac{\pi}{6} \leqslant \arg(z-2) \leqslant \frac{\pi}{6}$

(c) $1 \leqslant |z-1-i| \leqslant 2$ and $0 \leqslant \arg(z-1-i) \leqslant \frac{\pi}{3}$

(d) $|z+1+2i| > \sqrt{5}$ and $-\frac{\pi}{6} \leqslant \arg(z+1+2i) \leqslant \frac{\pi}{6}$.

Miscellaneous Exercise 18

1. (a) The complex numbers z and w are such that $w = 1 + ia$, $z = -b - i$ where a and b are real and positive. Given that $wz = 3 - 4i$, find the exact values of a and b.

(b) The complex number z is such that $|z| = 2$, $\arg z = -\frac{2}{3}\pi$ Find the exact values of the real part of z and the imaginary part of z. [C]

2. (a) Obtain a quadratic equation, with integer coefficients, having roots $2 + i\sqrt{5}$ and $2 - i\sqrt{5}$.

(b) Find the roots of the equation $z^2 + 2z = -4$. Show that the roots satisfy the equations:

(i) $|z| = 2$
(ii) $|z-1| = \sqrt{7}$
(iii) $z + z^* = -2$.

On a single, clearly labelled diagram, sketch the loci defined by the equations in (i), (ii) and (iii).
[z* is the complex conjugate of z]. [C]

3. (a) Find the modulus of the complex number $\frac{2i}{3-4i}$ and show that the argument, in radians, is 2.5, correct to one decimal place.

Hence find, correct to one decimal place, the value of x and a value of y such that $e^{x+iy} = \frac{2i}{3-4i}$.

(b) The point A in an Argand diagram represents the complex number z. Show by means of separate, clearly labelled sketches the set of all possible positions of P in each of the following cases:

(i) $|z + 2| = 2$

(ii) $\frac{1}{4}\pi < \arg\left(\frac{1}{z}\right) < \frac{1}{2}\pi$

[C]

4. Sketch the following loci, in separate Argand diagrams:

(a) $|z - 1 - i|^2 = 2$ (b) $|z - 1| = |z - i|$ (c) $z + z^* = 2$ [C]

5. (a) Given $(2 + 3i) z = 4 - i$, find the complex number z, giving your answer in the form a + bi.

(b) Find the modulus and argument of the complex number $5 - 3i$.

The complex number w is represented in an Argand diagram by the point w. Describe geometrically the locus of w such that $|w| = |5 - 3i|$. [C]

6. (a) The roots of the quadratic equation $z^2 + z + 1 = 0$ are denoted by z_1 and z_2.

(i) Obtain z_1 and z_2 in the form a + bi where $a, b \in \mathbb{R}$.

(ii) Verify that $z_1 = z_2^2$ and $z_2 = z_1^2$.

(b) In an Argand diagram, the points P_1 and P_2 represent the complex numbers z and z^2 respectively, where $z = \sqrt{2} + \left(\sqrt{2}\right)i$ and P_0 represents the complex number $1 + 0i$.

(i) Write down the modulus and argument of z.

(ii) Find the area of triangle OP_0P_1, where O is the origin and find the area of the triangle OP_1P_2. [C]

7. Two complex numbers w and z are such that

$w + z = 6 + 2i$

$w - 3z = \frac{20}{2 - i}$

Find w and z, giving each answer in the form x + iy. [C]

8. The complex number z is given by $z = \frac{i}{1 - i}$.

(a) Express z in the form a + ib, where a and b are real.

(b) Hence, or otherwise, find the modulus and argument of z.

(c) Draw an Argand diagram showing the points representing the complex numbers i, $1 - i$ and z. Show that these three points form a right–angled triangle. [C]

9. The complex numbers z and w are such that

$|z| = 2$, $\arg(z) = -\frac{2}{3}\pi$, $|w| = 5$, $\arg(w) = \frac{3}{4}\pi$.

Find the exact values of:

(a) the real part of z and the imaginary part of z.

(b) the modulus and argument of $\frac{w}{z^2}$. [C]

10. The point P in an Argand diagram represents the complex number z and the point Q represents the complex number w, where $w = \dfrac{1}{z+1}$

    (a) Find w when $z = -i$ and when $z = i$, expressing your answer in the form $u + iv$.

    (b) Find z in terms of w.

    (c) Given that P lies on the circle with centre the origin and radius 1, prove that $|w| = |w - 1|$.

    Hence, sketch the locus of Q in this case. [C]

11. Two of the roots of a cubic equation, in which all the coefficients are real, are 2 and $1 + 3i$.

    (a) State the third root of the equation.

    (b) Find the equation, giving it in the form $z^3 + az^2 + bz + c = 0$. [C]

12. Given that z is a complex number such that $z + 3z^* = 12 + 18i$, find z, giving your answer in the form $x + iy$. [C]

13. Given that $(5 + 12i)\,z = 63 + 16i$, find $|z|$ and arg z, giving your answer in radians correct to 3 significant figures.

    Given also that $w = 3\left(\cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi\right)$, find:

    (a) $\left|\dfrac{z}{w}\right|$

    (b) arg (zw) [C]

14. In an Argand diagram, the point P represents the complex number z. On a single diagram, illustrate the set of possible positions for each of the following cases. Any relevant points or directions should be clearly indicated.

    (a) $|z - 3i| \leq 3$

    (b) $\arg(z + 3 - 3i) = \frac{1}{4}\pi$.

    Given that z satisfies both (a) and (b), find the greatest possible value of $|z|$. [C]

15. The complex number z satisfies the equation $|z| = |z + 2|$. Show that the real part of z is -1.

    The complex number z also satisfies the equation $|z| = 2$. By sketching two loci in an Argand diagram, find the two possible values of the imaginary part of z and state two possible values of arg z.

    The two possible values of z are denoted by z_1 and z_2, where $\text{Im}z_1 > \text{Im}z_2$. Determine the square roots of z_1, giving your answers in the form $x + iy$.

    (Imz is the imaginary part of z)

ANSWERS

Exercise 1 A

1. $(a+4)^2+1$
2. $(x-9)^2-81$
3. $2\left(x-\frac{11}{4}\right)^2-\frac{41}{8}$
4. $3\left(x-\frac{5}{3}\right)^2-\frac{25}{3}$
5. $(x+2)^2-4$
6. $(x-2)^2-4$
7. $\left(x+\frac{1}{8}\right)^2-\frac{1}{64}$
8. $2(x-1)^2-7$
9. $3\left(x+\frac{2}{3}\right)^2-\frac{7}{3}$
10. $-10\left(x+\frac{9}{20}\right)^2+\frac{401}{40}$
11. $\left(x+\frac{5}{2}\right)^2-\frac{49}{4}$
12. $2\left(x+\frac{3}{4}\right)^2-\frac{25}{8}$
13. $-3\left(x+\frac{1}{3}\right)^2+\frac{37}{3}$
14. $-4\left(x+\frac{1}{2}\right)^2+26$
15. $3\left(x+\frac{11}{6}\right)^2+\frac{119}{12}$
16. $5\left(x-\frac{6}{5}\right)^2-\frac{1}{5}$

Exercise 1 B

1. $(x+4)^2+3$; minimum value 3 at $x=-4$.
2. $(x-3)^2+1$; minimum value 1 at $x=3$.
3. $-(x-3)^2-6$; maximum value -6 at $x=3$.
4. $36-(x+5)^2$; maximum value 36 at $x=-5$.
5. $2\left(x-\frac{3}{2}\right)^2+\frac{21}{2}$; minimum value $10\frac{1}{2}$ at $x=1\frac{1}{2}$.
6. $3\left(x+\frac{11}{6}\right)^2+\frac{119}{12}$; minimum value $9\frac{11}{12}$ at $x=-1\frac{5}{6}$.
7. $-3\left(x+\frac{1}{3}\right)^2+\frac{37}{3}$; maximum value $12\frac{1}{3}$ at $x=-\frac{1}{3}$.
8. $26-4\left(x+\frac{1}{2}\right)^2$; maximum value 26 at $x=\frac{-1}{2}$.
9. $3\left(x-\frac{4}{3}\right)^2-\frac{16}{3}$; minimum value $-5\frac{1}{3}$ at $x=1\frac{1}{3}$.
10. $\frac{36}{5}-5\left(x-\frac{6}{5}\right)^2$; maximum value $7\frac{1}{5}$ at $x=1\frac{1}{5}$.

Exercise 1 D

1. **(a)** imaginary **(b)** real, distinct **(c)** real, distinct **(d)** real, equal **(e)** real, equal
(f) real, distinct **(g)** imaginary **(h)** real, equal **(i)** real, distinct **(j)** real, distinct.

2. $1\frac{1}{4}$
3. 0 or -12
5. $\frac{1}{4}$

Exercise 1 E

1. $2 \leqslant x \leqslant 6$
2. $x \leqslant \frac{-5}{2}$ or $x \geqslant 2$
3. $\frac{1}{3} < x < 3$
4. $x < -1$ or $x > \frac{3}{4}$
5. $x \leqslant -5$ or $x \geqslant 2$
6. $-\frac{2}{3} \leqslant x \leqslant 1$
7. $x < -5$ or $x > 5$
8. $x \leqslant -12$ or $x \geqslant 12$
9. $\frac{-1-\sqrt{31}}{3} < x < \frac{-1+\sqrt{31}}{3}$
10. $x < \frac{3-\sqrt{109}}{10}$ or $x > \frac{3+\sqrt{109}}{10}$
11. $k \leqslant -10$ or $k \geqslant 10$
12. $k \leqslant 1$
13. $-6 < k < 6$
14. $-10 < k < 14$.

Exercise 1 F

(i) $x = -\frac{1}{3}, y = 4\frac{2}{3}; x = 3, y = -2$

(ii) $x = -2, y = 1; x = 1\frac{10}{27}, y = -\frac{4}{9}$

(iii) $x = 1, y = -3; x = 1\frac{12}{13}, y = -2\frac{5}{13}$

(iv) $x = -1, y = -1; x = -2\frac{13}{21}, y = 1\frac{3}{7}$

(v) $x = 1, y = -1; x = 2\frac{11}{13}, y = \frac{3}{13}$

(vi) $x = 37$ or 2; $y = -20\frac{1}{3}$ or 3

(vii) $x = 7, y = 1$

(viii) $x = 3, y = 2; x = 3\frac{2}{3}, y = 3\frac{1}{3}$

Exercise 1 G

1. $x = -1$ or 0

2. $x = \frac{1}{4}$ or 4

3. $x = 2^{\frac{1}{4}}$ or $2^{\frac{1}{2}}$

4. $x = \ln 3$ or $\ln 5$

5. $x = 0, -1, \pm \frac{\sqrt{5} - 1}{2}$

6. $x = 0$ or -2

7. $x = 0.585$ or -2.32

8. $x = \pm\sqrt{\frac{2}{3}}$

9. $x = -0.636$ or -0.225

10. $x = 0.382$ or 0.624

Miscellaneous Exercise 1

1. $x = 3, y = -1; x = 6, y = 3$

2. $x \leqslant -\frac{1}{2}$ or $x \geqslant 2$

3. $\left(\frac{1}{2}, 3\right), \left(4, \frac{2}{3}\right)$

4. $\frac{3}{2} \leqslant x \leqslant \frac{5}{2}$

5. $2\left(x - \frac{3}{2}\right)^2 + \frac{41}{2}; 20\frac{1}{2}, x = 1\frac{1}{2}$

6. $x = -1, y = \frac{1}{2}; x = 2, y = 2$

7. $x \leqslant -3$ or $x \geqslant -2$

8. $27, x = -1$

9. $p = 3$ or $-\frac{3}{4}$

10. $\pm\frac{1}{\sqrt{2}}, \pm 1$

11. $\left(\frac{5}{3}, 2\right)$

12. $x = \frac{1}{2}$

14. **(a)** $p = -4, q = -12$; **(b)** $r < -16$

15. **(a)** $x < \frac{1}{3}$ or $x > 3$

16. **(a)** $x = 3, y = 2$

(b) line is a tangent to curve

17. $x < -1$ or $x > -\frac{1}{3}$

18. $x = -\sqrt[3]{15}, y = \sqrt[3]{22}; x = 2, y = -1$

19. **(a)** $x < -\sqrt{3}$ or $x > \sqrt{3}$ **(b)** $-1 < x < 1$

20. $(x - 4)^2 + 3$; 3 at $x = 4$

21. $\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}; k = -10$

Exercise 2 A

1. $\{f(x) \in \mathbb{R} : f(x) \geqslant 6\}$; not one-one; e.g. 0 and –4

2. $\{f(x) \in \mathbb{R} : f(x) \neq 0\}$; one-one.

3. $\{f(x) \in \mathbb{R} : f(x) > 0\}$; not one-one; e.g. 1 and –1

4. $\{f(x) \in \mathbb{R} : 0 \leqslant f(x) \leqslant 1\}$; not one-one; e.g. $\frac{\pi}{3}$ and $-\frac{\pi}{3}$

5. $\{f(x) \in \mathbb{R} : f(x) \geqslant 0\}$; one-one

6. $\{f(x) \in \mathbb{R} : f(x) \leqslant 16\}$; one-one.

7. $\{f(x) \in \mathbb{R}\}$; one-one

8. $\{f(x) \in \mathbb{R} : f(x) > 0\}$; one-one

9. $\{f(x) \in \mathbb{R} : f(x) \geqslant 0\}$; one-one

10. $\{f(x) \in \mathbb{R} : f(x) \geqslant -18\}$; one-one.

Exercise 2 B

1. (a) $f^{-1}:x \mapsto \dfrac{3+2x}{x-2}, x \neq 2.$ (b) $f^{-1}:x \mapsto \dfrac{3(1-x)}{x}, x \neq 0.$ (c) $f^{-1}:x \mapsto \dfrac{(4-x)^2}{9} + 2, x \leqslant 4.$

   (d) $f^{-1}:x \mapsto \dfrac{5x+4}{6x-2}, x \neq \dfrac{1}{3}.$ (e) $f^{-1}:x \mapsto \sqrt[3]{\dfrac{x-3}{2}}$ (f) $f^{-1}:x \mapsto \dfrac{3-x}{2}.$

   (g) $f^{-1}:x \mapsto \dfrac{2}{x-3}, x \neq 3.$ (h) $f^{-1}:x \mapsto -3 + \sqrt{x+13}, x \geqslant -13.$ (i) $f^{-1}:x \mapsto \dfrac{1-\sqrt{x-3}}{2}, x \geqslant 3$

   (j) $f^{-1}:x \mapsto 3 + \sqrt{9-x}, x \leqslant 9.$

2. (a) $\{x \in \mathbb{R} : x > 0\}; \{f^{-1}(x) \in \mathbb{R} : f^{-1}(x) > 0\}; f^{-1}(x) = \dfrac{1}{x}.$

   (b) $\{x \in \mathbb{R} : x \neq 0\}; \{f^{-1}(x) \in \mathbb{R} : f^{-1}(x) \neq 0\}; f^{-1}(x) = \dfrac{1}{x}.$

   (c) $\{x \in \mathbb{R} : x > 0\}; \{f^{-1}(x) \in \mathbb{R} : f^{-1}(x) > 0\}; f^{-1}(x) = \dfrac{1}{\sqrt{x}}.$

   (d) $\{x \in \mathbb{R} : x > 0\}; \{f^{-1}(x) \in \mathbb{R} : f^{-1}(x) < 0\}; f^{-1}(x) = -\dfrac{1}{\sqrt{x}}.$

   (e) $\{x \in \mathbb{R} : x \geqslant 0\}; \{f^{-1}(x) \in \mathbb{R} : f^{-1}(x) \geqslant 0\}; f^{-1}(x) = \sqrt{x}.$

   (f) $\{x \in \mathbb{R} : x \leqslant 0\}; \{f^{-1}(x) \in \mathbb{R} : f^{-1}(x) \geqslant 0\}; f^{-1}(x) = \sqrt{-x}.$

   (g) $\{x \in \mathbb{R} : x \geqslant 1\}; \{f^{-1}(x) \in \mathbb{R} : f^{-1}(x) \geqslant 0\}; f^{-1}(x) = \sqrt[3]{\dfrac{x-1}{3}}.$

   (h) $\{x \in \mathbb{R} : x \leqslant 4\}; \{f^{-1}(x) \in \mathbb{R} : f^{-1}(x) \leqslant 2\}; f^{-1}:(x) = 2 - \sqrt{4-x}.$

   (i) $\{x \in \mathbb{R} : x \geqslant -4\}; \{f^{-1}(x) \in \mathbb{R} : f^{-1}(x) \leqslant -1\}; f^{-1}:(x) = -1 - \sqrt{x+4}.$

   (j) $\{x \in \mathbb{R} : x \geqslant -7\}; \{f^{-1}(x) \in \mathbb{R} : f^{-1}(x) \geqslant 4\}; f^{-1}(x) = 4 + \sqrt{x+7}$

Exercise 2 D

1. (a) $fg:x \mapsto 13 - 9x$; $gf:x \mapsto 11 - 9x$

   (b) $fg:x \mapsto \dfrac{2}{x} + 1, x \neq 0$; gf does not exist as range of f is not a subset of domain of g.

   (c) fg does not exist as range of g is not a subset of domain of f ; $gf:x \mapsto 4 - \dfrac{5}{x-3}, x \neq 3$

   (d) $fg:x \mapsto 3|x| - 4$; $gf:x \mapsto |3x - 4|$

   (e) $fg:x \mapsto 1 - 2\ln x, x > 0$; gf does not exist as range of f is not a subset of domain of g.

   (f) fg does not exist as range of g is not a subset of domain of f ; $gf:x \mapsto 2\sqrt{x} + 3$

   (g) $fg:x \mapsto \sqrt[3]{3x-1}$; $gf:x \mapsto 3\sqrt[3]{x} - 1$

   (h) $fg:x \mapsto 2^{x^2}$; $gf:x \mapsto 2^{2x}$

2. fg:x $\mapsto 3^{3-2x}$; Range = $\{fg(x) \in \mathbb{R} : fg(x) > 0\}$.

gf:x $\mapsto 3 - 2 \times 3^x$; Range = $\{gf(x) \in \mathbb{R} : gf(x) < 3\}$.

3. fg:x $\mapsto 5^{x^2}$; Range = $\{fg(x) \in \mathbb{R} : fg(x) \geqslant 5\}$.

gf:x $\mapsto 5^{2x}$; Range = $\{gf(x) \in \mathbb{R} : gf(x) > 0\}$.

4. fg does not exist as range of g is not a subset of domain of f.

gf is a function as range of f $\subset$ domain of g. Range of gf = $\{gf(x) \in \mathbb{R} ; \cos 1 \leqslant gf(x) \leqslant 1\}$.

■ *Miscellaneous Exercise 2*

1. **(a)** $(x + 2)^2 - 4$ **(b)** $\{x \in \mathbb{R} : x \geqslant -4\}$, $f^{-1}(x) = -2 + \sqrt{x+4}$

2. $fg(x) = \dfrac{1}{1-x}$; $gf(x) = 1 - \dfrac{1}{x}$.

3. $gf(x) = 3(x - 1)^2 + 8$; $a = 2, b = 5; a = -2, b = -3$.

4. $f^{-1}:x \mapsto \sqrt[3]{\dfrac{x-3}{4}}$, $x \in \mathbb{R}$. Graph of f^{-1} obtained from graph of f by a reflection in the line y = x.

5. $(gh)^{-1}(y) = y^2 - 2$.

6. Not one-one; e.g. 0 and 1 have same image −2; $\{f(x) \in \mathbb{R} : -2\frac{1}{4} \leqslant f(x) \leqslant 4\}$.

7. $\mathbb{R}$; $\{g(x) \in \mathbb{R} : g(x) \geqslant 1\}$; f one-one; g not one-one, as y-line cuts curve in two points; $fg(x) = 3x^2 + 5$; $gf(x) = (3x + 2)^2 + 1$; 0, −2; $(fg)^{-1}:x \mapsto \pm\sqrt{\dfrac{x-5}{3}}$; Two values of $(fg)^{-1}$ for $x > 5$.

8. $f^{-1}:x \mapsto -2 + \sqrt{x-3}$, $x \geqslant 3$. **9.** $f^{-1}:x \mapsto 3 - \sqrt{x-1}$, $x \geqslant 1$.

10. $\left\{f(x) \in \mathbb{R} : f(x) \geqslant -\dfrac{25}{4}\right\}$; 4, −1.

11. $D = \{x \in \mathbb{R} : x \geqslant 2\}$; $g^{-1}:x \mapsto 2 + \sqrt{x+1}$, domain = $\{x \in \mathbb{R} : x \geqslant -1\}$, range = $\{g^{-1}(x) \in \mathbb{R} : g^{-1}(x) \geqslant 2\}$

12. **(a)** $\dfrac{1}{1+x}$, $x \neq -1$ **(b)** $\dfrac{x}{1-x}$, $x \neq 1$ **(c)** $\dfrac{1-x}{x}$, $x \neq 0$, $x = -\dfrac{1}{2}$ or 1

■ Exercise 3 A

1. **(a)** 5 **(b)** $\sqrt{34}$ **(c)** 5 **(d)** $\sqrt{65}$

(e) $\sqrt{26}$ **(f)** $5\sqrt{2}$ **(g)** 6 **(h)** 9

(i) $(b - a)\sqrt{2}$ or $(a - b)\sqrt{2}$ **(j)** $\sqrt{(a+2)^2+9}$

2. **(a)** $\left(\dfrac{1}{2}, -3\right)$ **(b)** $\left(\dfrac{3}{2}, -\dfrac{5}{2}\right)$ **(c)** $\left(1, \dfrac{5}{2}\right)$ **(d)** $\left(-\dfrac{3}{2}, -1\right)$

(e) $\left(\dfrac{1}{2}, \dfrac{3}{2}\right)$ **(f)** $\left(\dfrac{3}{2}, \dfrac{3}{2}\right)$ **(g)** $(1, 0)$ **(h)** $\left(0, \dfrac{5}{2}\right)$

(i) $\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$ **(j)** $\left(\frac{a-2}{2}, \frac{-1}{2}\right)$

3. (1, 7); 5 or –3 **5.** $\frac{1}{2}\sqrt{197}$; $\frac{1}{2}\sqrt{53}$; $4\sqrt{5}$ **6.** $\left(0, -\frac{3}{2}\right)$; (–4, –11)

8. **(a)** yes **(b)** yes **(c)** no **9.** $\sqrt{58}$, $\sqrt{58}$; square

11. (3, 1)

12. **(a)** $y = 4x - 8$ **(b)** $y + 4x = 7$ **(c)** $y + 6x + 21 = 0$ **(d)** $4y = 3x + 7$

(e) $12y + 60x = 17$ **(f)** $20y + 15x + 19 = 0$ **(g)** $60y + 15x + 11 = 0$ **(h)** $y = mx + b - ma$

(i) $ey = x + e^3$ **(j)** $y + tx = at^3 + 2at$ **(k)** $t^2y + x = 2ct$ **(l)** $\sqrt{2}y = x - 2$

13. **(a)** $y = 2x - 5$ **(b)** $2y = x + 11$ **(c)** $4y = 3x + 11$ **(d)** $2y + 2x = 3$

(e) $2y = 3x - 2$ **(f)** $y + x = a + b$ **(g)** $(q + p)y = 2x + 2apq$

(h) $y = -1$ **(i)** $x = -4$ **(j)** $pqy = (p + q)x - 1$

14. $3y + 4x = 20$; $3y + 14x = 50$; $x = 3$ **15.** $2y = 6x + 1$; $2y + 3x = 10$; $4y = 3x + 20$

■Exercise 3 B

1. **(a)** $\frac{5}{2}, \frac{7}{2}$ **(b)** $-\frac{2}{3}, 3$ **(c)** $-\frac{4}{9}, \frac{10}{3}$ **(d)** $-\frac{b}{a}, b$ **(e)** $-\frac{a}{b}, \frac{c}{b}$

(f) $-\cot\theta$, $p \operatorname{cosec}\theta$ **(g)** 2, 5 **(h)** $-\frac{5}{2}, \frac{19}{2}$ **(i)** $-\frac{5}{2}, -\frac{29}{2}$ **(j)** $-\frac{5}{2}, -5$

2. **(a)** $\frac{4}{3}, -\frac{3}{4}$ **(b)** $-\frac{4}{3}, \frac{3}{4}$ **(c)** $3, -\frac{1}{3}$ **(d)** $p, -\frac{1}{p}$ **(e)** $-\frac{9}{10}, \frac{10}{9}$

(f) $-\cot\theta$, $\tan\theta$ **(g)** $\frac{5}{3}, -\frac{3}{5}$ **(h)** infinite, 0 **(j)** 0, infinite **(k)** $\frac{3}{7}, -\frac{7}{3}$

3. **(a)** $-\frac{1}{3}$ **(b)** $4\frac{1}{2}$ **(c)** $1\frac{4}{7}$ **(d)** $-5\frac{2}{3}$ **(e)** $4\frac{1}{2}$

6. $2y = 3x - 8$

7. **(a)** $2y + x = 3$ **(b)** $y + 3x = 5$ **(c)** $3y + 4x = 6$ **(d)** $4y = x - 13$

(e) $y = 3$ **(f)** $x = 1$ **(g)** $x = -1$ **(h)** $y = 5$

(i) $y\sqrt{3} = x + 3\sqrt{3} - 2$ **(j)** $y = tx + at^3 - 2at$ **(k)** $t^2y + x = 2ct$ **(l)** $py = x$

8. $y = \frac{1}{2}x$ **9.** $2y = x + 4$; $2y + x = 10$; $x = 3$ **10.** $y\sqrt{3} = x - 3$

■Exercise 3 C

1. (3, 1), $\sqrt{13}$ **2.** $\frac{1}{6}\sqrt{13}$ **3.** (6, 1) **4.** 2 **5.** $\left(\frac{23}{5}, \frac{19}{5}\right)$, $\frac{3\sqrt{5}}{5}$; $6\sqrt{2}$

6. $5y + 8x = 29$, $7y + x = 44$; $\left(-\frac{1}{3}, \frac{19}{3}\right)$; $2y = 7x + 15$.

7. mediator of AC, y = 5; of AB, x = 2; of BC, y = x + 3. **8.** (0, 0), (2, 6), (6, 2); $\left(2\frac{1}{2}, 2\frac{1}{2}\right)$.

■ Exercise 3 D

1. **(a)** (−3, 0), (2, 0) **(b)** $\left(1\frac{1}{2}, 0\right)$, (−1, 0) **(c)** (3, 0), (6, 0)
(d) (−1, 0), (2, 0) **(e)** (0, 0), (8, 0)

2. $7\sqrt{2}$ **3.** $80\sqrt{2}$ **4.** m < −8 or m > 8

5. 5, 9 **6.** **(a)** (−2, 4), (1, 4) **(b)** (0, 0), $\left(\sqrt[3]{4}, \sqrt[3]{16}\right)$
(c) (2, 3) **(d)** $\left(\frac{1}{3}, 8\frac{1}{9}\right)$, (3, −7) **(e)** (0, 0), (9, 9)

■ *Miscellaneous Exercise 3*

1. (5, −2) **2.** **(a)** 2y + x = 11 **(b)** (1, 5)
(c) (2, 7) **3.** **(a)** 2, $-\frac{1}{2}$ **(b)** $2\sqrt{5}$

4. **(a)** $\frac{1}{2}$ **(b)** (5, 7) **5.** **(a)** 5 **(b)** q ⩾ 8 or q ⩽ −8

6. 2

7. **(a)** 3y = x + 22 **(b)** (14, 12) **(c)** (16, 6)
(d) 80 square units **8.** $\left(-\frac{1}{2}, \frac{2}{3}\right)$

9. (2, 3); y + 5x = 13; 2y = 3x − 13; **(a)** (3, −2) **(b)** (1, 8)

10. **(a)** −6 < c < 6 **(b)** $\frac{11}{4}$

■ Exercise 4 A

1. **(a)** 30° **(b)** 120° **(c)** 225° **(d)** 67.5° **(e)** 123.75°
(f) 22.9° **(g)** 143.2° **(h)** 223.5° **(i)** 74.5° **(j)** 47.0°

2. **(a)** $\frac{\pi}{2}$ rad **(b)** $\frac{5\pi}{6}$ rad **(c)** $\frac{7\pi}{6}$ rad **(d)** $\frac{5\pi}{12}$ rad **(e)** $\frac{3\pi}{8}$ rad
(f) $\frac{7\pi}{9}$ rad **(g)** $\frac{7\pi}{45}$ rad **(h)** 0.648 rad **(i)** 2.42 rad **(j)** 5.07 rad

3. **(a)** 6.28 cm **(b)** 23.6 cm **(c)** 12.16 cm **(d)** 6.25 cm **(e)** 6.708 cm

4. 12.8 cm **5.** 1.28 : 1 **6.** **(a)** 14.6 cm **(b)** 31.8 cm **(c)** 30.9 cm
(e) 17.0 cm **(f)** 29.7 cm

Exercise 4 B

1. **(a)** 14.4 cm^2 **(b)** 48.6 cm^2 **(c)** 198.45 cm^2 **(d)** 79.524 cm^2

2. 27 cm^2 3. 0.072 cm^2 4. **(a)** 16 cm^2 **(b)** 41.5 cm^2

5. **(a)** 46.8 cm^2 **(b)** 0.673 cm^2 **(c)** $\frac{1}{2}r^2(\theta + \sin\theta)$ **(d)** $\frac{1}{2}r^2\left(\theta + 2\sin\frac{\theta}{2}\right)$
(e) 2.58cm^2 **(f)** 2.40 cm^2

Miscellaneous Exercise 4

1. 0.25 rad 2. 120°; 0.614 unit^2 3. **(a)** 16.5 m **(b)** 2.28 m^2

4. 12 cm. **(a)** 10.6 cm **(b)** 7.86 cm **(c)** 14.6 cm^2

5. **(a)** 51.5 cm **(b)** 4760 cm^2 **(c)** 94.8%

6. **(a)** 7.79 cm **(b)** 1.79 rad **(c)** 0.928 cm

7. **(a)** 42.1 cm^2 **(b)** 0.571 rad

8. **(a)** 10.5 cm **(b)** 9.06 cm^2 **(c)** 17.1 cm^2 **(d)** 13.3 cm

9. **(b)** $\frac{25\pi}{3}$ cm^2 **(c)** 17 cm^2

10. **(a)** $2r\cos\theta$ **(b)** $4r^2\theta\cos^2\theta$ **(c)** $\frac{1}{2}r^2\sin 2\theta$ **(d)** $\frac{1}{2}r^2(\pi - 2\theta - \sin 2\theta)$

Exercise 5 A

1. **(a)** sin 12° **(b)** cos 18° **(c)** tan 44° **(d)** –sin 39°
(e) –cos 29° **(f)** –tan 44° **(g)** –sin 59° **(h)** –cos 67°
(i) –cos 72° **(j)** –tan 61° **(k)** –sin 8° **(l)** cos 88°
(m) tan 27° **(n)** sin 44° **(o)** cos 57° **(p)** –tan 39°

2. I(cos 0°, sin 0°) or (1, 0); J(cos 90°, sin 90°) or (0, 1);
I^1(cos 180°, sin 180°) or (–1, 0); J^1(cos 270°, sin 270°) or (0, –1).
0, 1, 0, 1, 0, infinite and positive, –1, 0, 0, –1, infinite and negative.

3. **(a)** $\frac{1}{\sqrt{2}}$ **(b)** $-\frac{1}{\sqrt{2}}$ **(c)** -1 **(d)** $-\frac{1}{2}$
(e) $\frac{\sqrt{3}}{2}$ **(f)** $-\sqrt{3}$ **(g)** $-\frac{\sqrt{3}}{2}$ **(h)** $-\frac{\sqrt{3}}{2}$
(i) $-\frac{1}{\sqrt{3}}$ **(j)** $\frac{1}{2}$ **(k)** $-\frac{1}{2}$ **(l)** $\sqrt{3}$

■ Exercise 5 B

(a) sin 30°; $\frac{1}{2}$ **(b)** –cos 45°; $-\frac{1}{\sqrt{2}}$ **(c)** tan 15° **(d)** sin 60°; $\frac{\sqrt{3}}{2}$

(e) cos 45°; $\frac{1}{\sqrt{2}}$ **(f)** –tan 60°; $-\sqrt{3}$ **(g)** sin 45°; $\frac{1}{\sqrt{2}}$ **(h)** –cos 60°, $-\frac{1}{2}$

(i) –tan 65° **(j)** –sin 30°; $-\frac{1}{2}$ **(k)** cos 25° **(l)** tan 45°; 1.

■ Exercise 5 C

1. **(a)** 240°, 300° **(b)** 45°, 315° **(c)** 150°, 330° **(d)** 30°, 150°

   (e) 90°, 270° **(f)** 0°, 180°, 360° **(g)** 225°, 315° **(h)** 0°, 360°

   (i) 135°, 315° **(j)** 270° **(k)** 180° **(l)** 120°, 300°

2. **(a)** 36.9°, 143.1° **(b)** 130.0°, 230.0° **(c)** 99.0°, 279.0° **(d)** 218.0°, 322.0°

   (e) 47.0°, 313.0° **(f)** 29.0°, 209.0° **(g)** 25.6°, 154.4° **(h)** 105.4°, 254.6°

   (i) 112.5°, 292.5° **(j)** 226.7°, 313.3° **(k)** 40.1°, 319.9° **(l)** 57.7°, 237.7°

   (m) 16.8°, 163.2° **(n)** 99.0°, 261.0° **(o)** 114.8°, 294.8°

3. **(a)** –140°, –40° **(b)** ±35.7° **(c)** –74.2°, 105.8° **(d)** –125.1°, –54.9°

   (e) ±130° **(f)** –119.9°, 60.1° **(g)** –122.2°, –57.8° **(h)** ±62.4°

   (i) –140.4°, 39.6° **(j)** –120°, –60° **(k)** ±150° **(l)** –45°, 135°

4. **(a)** 20°, 40°, 140°, 160°, 260°, 280° **(b)** 67.5°, 112.5°, 247.5°, 292.5°

   (c) 45°, 105°, 165°, 225°, 285°, 345° **(d)** 97.5°, 157.5°, 277.5°, 337.5°

   (e) $26\frac{2}{3}^{\circ}, 226\frac{2}{3}^{\circ}, 266\frac{2}{3}^{\circ}$ **(f)** 30°, 120°, 210°, 300°

   (g) 110°, 290° **(h)** $21\frac{2}{3}^{\circ}, 111\frac{2}{3}^{\circ}, 141\frac{2}{3}^{\circ}, 231\frac{2}{3}^{\circ}, 261\frac{2}{3}^{\circ}, 351\frac{2}{3}^{\circ}$

   (i) 45° **(j)** 6.9°

   (k) 93.7° **(l)** 61.7°, 151.7°, 241.7°, 331.7°

5. **(a)** –120° **(b)** no solution

   (c) –127.5°, –37.5°, 52.5°, 142.5° **(d)** –70°, –10°, 110°, 170°

   (e) $-133\frac{1}{3}^{\circ}, -113\frac{1}{3}^{\circ}, -13\frac{1}{3}^{\circ}, 6\frac{2}{3}^{\circ}, 106\frac{2}{3}^{\circ}, 126\frac{2}{3}^{\circ}$ **(f)** 130°

(g) −163.4°, −148.6°, −43.4°, −28.6°, 76.6°, 91.4° **(h)** −52°, 148°

(i) −139.4°, −79.4°, −19.4°, 40.6°, 100.6°, 160.6° **(j)** −162.6°

(k) −137.8°, −82.2°, 42.2°, 97.8° **(l)** −146.5°, −56.5°, 33.5°, 123.5°

6. **(a)** 30°, 150°, 270° **(b)** 0°, 30°, 150°, 180°, 360°

(c) 75.5°, 180°, 284.5° **(d)** 26.6°, 135°, 206.6°, 315°

(e) 45°, 135°, 193.6°, 346.3° **(f)** 49.4°, 310.6°

(g) 33.7°, 213.7° **(h)** 41.8°, 138.2°, 270°

(i) 0°, 48.2°, 180°, 311.8°, 360° **(j)** 46.1°, 313.9°

■ Exercise 5 D

1. 1.13 rad 2. 0.13 rad, 1.44 rad 3. 0.86 rad

4. 2.41 rad 5. 2.34 rad or 134.1° 6. 2.83

■ Exercise 5 E

1. **(a)** $\frac{\pi}{4}$ **(b)** $\frac{\pi}{4}$ **(c)** $\frac{\pi}{6}$ **(d)** $-\frac{\pi}{2}$ **(e)** $\frac{\pi}{2}$

(f) $-\frac{\pi}{4}$ **(g)** $\frac{\pi}{3}$ **(h)** $\frac{2\pi}{3}$ **(i)** $-\frac{\pi}{3}$ **(j)** 0

(k) 0 **(l)** 0 **(m)** $-\frac{\pi}{4}$ **(n)** $\frac{\pi}{6}$ **(o)** $-\frac{\pi}{6}$

2. **(a)** 0.698 rad **(b)** 2.78 rad **(c)** 1.07 rad **(d)** −1.16 rad **(e)** 1.42 rad

(f) −1.43 rad **(g)** 0.491 rad **(h)** 2.01 rad **(i)** 1.08 rad **(j)** −0.464 rad

(k) 0.924 rad **(l)** −1.16 rad **(m)** 0.601 rad **(n)** 2.22 rad **(o)** 1.37 rad

■ *Miscellaneous Exercise 5*

1. $-\sqrt{1-p^2}$; $\frac{-p}{\sqrt{1-p^2}}$ 2. $\frac{-p}{\sqrt{1+p^2}}$; $\frac{-1}{\sqrt{1+p^2}}$ 3. $-\sqrt{1-p^2}$; $\frac{-\sqrt{1-p^2}}{p}$ 4. $\pi+\alpha, 2\pi-\alpha$

5. **(a)** 108.4°, 288.4° **(b)** 75°, 135°, 255°, 315°

6. **(a)** 30°, 150°, 210°, 330° **(b)** 41.8°, 138.2°, 270°

7. **(a)** 90°, 210°, 330° **(b)** 0°, 48.2°, 180°, 311.8°, 360°

8. **(a)** 80°, 260° **(b)** 0°, 109.5°, 180°, 250.5°, 360° **(c)** 45°, 135°, 225°, 315°

9. **(a)** $\frac{5\sqrt{2}}{2}$ m **(b)** $\frac{\pi}{3}$ s **10.** **(a)** 6 m **(b)** 0.659 s

11. $15\sin^2\theta + \sin\theta - 2 = 0$; 19.5°, 160.5°, 203.6°, 336.4° **13.** 3

14. **(a)** **(i)** 45°, 225° **(ii)** 105°, 165°, 285°, 345° **(b)** $p; -\sqrt{1-p^2}$

15. **(a)** **(i)** 240 **(ii)** $\frac{\sqrt{3}}{2}$

(b) **(i)** $-p$ **(ii)** $-p$ **(iii)** $\frac{1}{p}$

16. **(a)** **(i)** 240°, 300° **(ii)** 76.7°, 166.7°, 256.7°, 346.7° **(b)** $\frac{p}{\sqrt{1+p^2}}$

17. **(a)** 100°, 160°, 280°, 340° **(b)** 57.9°, 122.1° **18.** **(a)** 90°, 210°, 330°

(b) 60°, 300° **19.** **(a)** 15°, 75°, 195°, 255° **(b)** 63.4°, 243.4°

20. **(a)** 121.3°, 301.3° **(b)** 210°, 330° **(c)** 0°, 60°, 180°, 300°, 360°

■ Exercise 6 A

1. **(a)** $-4\mathbf{p} + 2\mathbf{q}$ **(b)** $8\mathbf{p} + 2\mathbf{q}$

2. $-\mathbf{p} + \mathbf{q}$; $-3\mathbf{p} + 3\mathbf{q}$; $|\mathbf{BC}| = \frac{1}{3}|\mathbf{DE}|$; Parallel, same direction

3. Parallelogram

4. **(a)** $\begin{pmatrix}2\\4\end{pmatrix}$ **(b)** $\begin{pmatrix}3\\-3\end{pmatrix}$ **(c)** $\begin{pmatrix}-7\\-1\end{pmatrix}$ **(d)** $\begin{pmatrix}0\\2\end{pmatrix}$ **(e)** $\begin{pmatrix}3\\0\end{pmatrix}$ **5.** Q(5, 2), R(3, 6), S(–1, 7)

7. **(a)** $\begin{pmatrix}9\\15\end{pmatrix}$ **(b)** $\begin{pmatrix}8\\12\end{pmatrix}$ **(c)** $\begin{pmatrix}-4\\-6\end{pmatrix}$ **(d)** $\begin{pmatrix}6\\9\end{pmatrix}$ **(e)** $\begin{pmatrix}11\\17\end{pmatrix}$ **(f)** $\begin{pmatrix}17\\27\end{pmatrix}$

8. **(a)** $2\mathbf{a}$ **(b)** $\mathbf{b}$ **(c)** $\mathbf{a}$ **(d)** $\mathbf{a}$ **(e)** $\mathbf{a} + \mathbf{b}$

(f) $2\mathbf{b} - \mathbf{a}$ **(g)** $3\mathbf{b}$ **(h)** $\mathbf{b} - 5\mathbf{a}$ **(i)** $-\mathbf{b}$ **(j)** $\mathbf{b} - \mathbf{a}$

■ Exercise 6 B

1. **(a)** $\frac{3\mathbf{i} - 4\mathbf{j}}{5}$ **(b)** $\frac{5\mathbf{i} + 12\mathbf{j}}{13}$ **(c)** $\frac{4\mathbf{i} - 3\mathbf{j}}{5}$ **(d)** $\frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}$ **(e)** $\frac{\mathbf{i} + \sqrt{3}\mathbf{j}}{2}$

(f) $\frac{6\mathbf{i} + \mathbf{j}}{\sqrt{37}}$ **(g)** $\frac{3\mathbf{i} + 4\mathbf{j}}{5}$ **(h)** $\frac{3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}}{13}$ **(i)** $\frac{4\mathbf{i} - \mathbf{j} - \mathbf{k}}{3\sqrt{2}}$ **(j)** $\frac{\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}}{\sqrt{14}}$

(k) $\frac{2\mathbf{i} - \mathbf{j} - \mathbf{k}}{\sqrt{6}}$ **(l)** $\frac{1}{13}\begin{pmatrix}5\\-12\end{pmatrix}$ **(m)** $\frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix}$ **(n)** $\frac{1}{\sqrt{6}}\begin{pmatrix}1\\-1\\2\end{pmatrix}$ **(o)** $\frac{1}{5\sqrt{2}}\begin{pmatrix}4\\-3\\-5\end{pmatrix}$

2. (a) $\frac{3\mathbf{i}+4\mathbf{j}}{5}$ (b) $\frac{-5\mathbf{i}+12\mathbf{j}}{13}$ (c) $\frac{-3\mathbf{i}+4\mathbf{j}}{5}$ (d) $\frac{-\mathbf{i}+\mathbf{j}}{\sqrt{2}}$ (e) $\frac{-9\mathbf{i}+4\mathbf{j}}{\sqrt{97}}$

(f) $\frac{-\mathbf{i}-4\mathbf{j}}{\sqrt{17}}$ 3. (a) $\frac{2\mathbf{i}-2\mathbf{j}-\mathbf{k}}{3}$ (b) $\frac{-\mathbf{i}+\mathbf{j}+2\mathbf{k}}{\sqrt{6}}$ (c) $\frac{-\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$

4. (a) $\pm\frac{4}{5}$ (b) $\frac{1}{13}, \frac{25}{13}$ (c) $\pm\frac{3}{13}$ (d) $\pm\frac{1}{\sqrt{5}}$ (e) $\pm\frac{1}{\sqrt{26}}$

(f) $\frac{1}{13}$ or $\frac{25}{13}$ 5. (a) $\left(\frac{3}{5}, \frac{-4}{5}\right)$ (b) $\left(\frac{3}{5\sqrt{2}}, \frac{-4}{5\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ 6. $-\frac{1}{5}, -\frac{9}{5}$

7. (a) $\frac{1}{5\sqrt{5}}\begin{pmatrix}2\\11\end{pmatrix}$ (b) $\begin{pmatrix}0\\1\end{pmatrix}$ 8. (a) $\frac{1}{\sqrt{105}}\begin{pmatrix}8\\-5\\4\end{pmatrix}$ (b) $\frac{1}{\sqrt{358}}\begin{pmatrix}3\\5\\18\end{pmatrix}$

9. $\frac{5\mathbf{i}-12\mathbf{j}}{13}$; $10\mathbf{i}-24\mathbf{j}$ 10. (a) $15\mathbf{i}-20\mathbf{j}$ (b) $3\mathbf{i}+3\mathbf{j}$ (c) $9\mathbf{i}+12\mathbf{j}-36\mathbf{k}$ (d) $\sqrt{6}\begin{pmatrix}1\\2\\-1\end{pmatrix}$

(e) $\begin{pmatrix}\sqrt{6}\\\sqrt{3}\\0\end{pmatrix}$ 11. (a) $-2\mathbf{i}-30\mathbf{j}$ (b) $\frac{-(\mathbf{i}+15\mathbf{j})}{\sqrt{226}}$ 12. $\frac{25\mathbf{i}-12\mathbf{j}-24\mathbf{k}}{\sqrt{1345}}$

13. $(-8, 6)$ 14. $(-48, -12, -16)$ 15. $\left(\frac{18}{13}, \frac{27}{13}\right)$ 16. $\left(\frac{4}{3}, -\frac{5}{3}, \frac{8}{3}\right)$ 17. $\frac{4}{5}\mathbf{i}-\frac{3}{5}\mathbf{j}$

18. $\frac{4\mathbf{i}+3\mathbf{j}-12\mathbf{k}}{13}$ 19. $\left(\frac{9}{5}, \frac{7}{5}\right)$ 20. $\left(2-\frac{1}{\sqrt{35}}, \frac{3}{\sqrt{35}}-1, 1-\frac{5}{\sqrt{35}}\right)$

■ Exercise 6 C

1. (a) 1 (b) 17 (c) 14 (d) 5 (e) 0

(f) 0 (g) –1 (h) –5 (i) 0 (j) 25

(k) 9 (l) 4 (m) 8

2. (a) 22.6° (b) 148.7° (c) 109.4° (d) 95.0° (e) 99.6°

(f) 78.5° (g) 94.1° (h) 107.6°

3. (a) $\angle A = 11.3^\circ, \angle B = 157.4^\circ, \angle C = 11.3^\circ$ (b) $\angle A = 36.9^\circ, \angle B = 34.7^\circ, \angle C = 108.4^\circ$

(c) $\angle A = 68.9^\circ, \angle B = 87.8^\circ, \angle C = 23.3^\circ$ (d) $\angle A = 90^\circ, \angle B = 35.3^\circ, \angle C = 54.7^\circ$

4. (a) 4 (b) $a = \pm 4$; $b = \pm 3$ (c) $-\frac{1}{3}$ (d) –1.2 (e) $\frac{7\sqrt{2}}{10}$

5. (a) 2 (b) $a = 2, b = 4$; $a = 4, b = 2$ (c) 10.5 (d) $-\frac{11}{10}$

6. $\begin{pmatrix}1+2\lambda\\-2+6\lambda\end{pmatrix}$; $\left(\frac{23}{10}, \frac{19}{10}\right)$; $\frac{\sqrt{10}}{10}$ 7. $\begin{pmatrix}1+3\lambda\\2+\lambda\\-1-3\lambda\end{pmatrix}$; $\left(-\frac{5}{19}, \frac{30}{19}, \frac{5}{19}\right)$; $\frac{5}{19}\sqrt{38}$

8. $\left(\frac{16}{23}, \frac{9}{23}, \frac{59}{23}\right); \frac{1}{23}\sqrt{2990}$ **9.** Perpendicular to the plane **10.** $\begin{pmatrix} -13 \\ -5 \\ 1 \end{pmatrix}$

■ *Miscellaneous Exercise 6*

1. **(a)** $2\sqrt{14}$ **(b)** 10.7° **2.** $\sqrt{10}$

3. 0; 90°; Angle in a semicircle is 90°

4. $\frac{12\mathbf{i} + 17\mathbf{j}}{5}$; $12\mathbf{i} + \mathbf{j}$ **5.** **(a)** $\frac{3\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}}{\sqrt{61}}$ **(b)** $8\mathbf{i} + (\lambda + 4)\mathbf{j} + 8\mathbf{k}$ **(c)** -9 **(d)** $\frac{12\mathbf{i} + \mathbf{j}}{13}$

6. **(a)** 1 **(b)** 82° **(c)** $\frac{9}{13}$ **(d)** 1.53 or –0.15

7. **(a)** $\mathbf{a}//\mathbf{b}$; $|\mathbf{b}| = 1$ **(b)** Circle with OA as diameter

8. **(a)** $\frac{10}{\sqrt{110}}, 12\sqrt{10}$; $7\mathbf{i} + \mathbf{j} + \mathbf{k}$; $6\mathbf{i} - 2\mathbf{j}$ **9.** $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$; $-2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$; 63.6°

10. $2\mathbf{i} - \frac{8}{5}\mathbf{j} + \frac{6}{5}\mathbf{k}$; $\mathbf{i} - \frac{21}{5}\mathbf{j} - \frac{3}{5}\mathbf{k}$ **11.** $\frac{5}{18}; \frac{3}{10}$

12. $\frac{27}{\sqrt{850}}; \frac{18}{11}$ **13.** **(a)** $\frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{k})$; $\frac{\sqrt{15}}{5}$; $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$; $\mathbf{i} - 4\mathbf{j}$

■ Exercise 7 A

1. **(a)** $16 + 32x + 24x^2 + 8x^3 + x^4$ **(b)** $243 - 810a + 1080a^2 - 720a^3 + 240a^4 - 32a^5$

(c) $81 + 54x + \frac{27}{2}x^2 + \frac{3}{2}x^3 + \frac{1}{16}x^4$ **(d)** $8a^3 - 36a^2b + 54ab^2 - 27b^3$

(e) $32 - \frac{80}{3}x + \frac{80}{9}x^2 - \frac{40}{27}x^3 + \frac{10}{81}x^4 - \frac{1}{243}x^5$ **(f)** $\frac{16}{81}a^4 - \frac{16}{9}a^3b + 6a^2b^2 - 9ab^3 + \frac{81}{16}b^4$

(g) $\frac{1}{8}a^3 - a^2b + \frac{8}{3}ab^2 - \frac{64}{27}b^3$ **(h)** $\frac{81}{16}a^4 - \frac{9}{2}a^3b + \frac{3}{2}a^2b^2 - \frac{2}{9}ab^3 + \frac{1}{81}b^4$

(i) $\frac{64}{27}x^3 - 8x^2y + 9xy^2 - \frac{27}{8}y^3$

2. **(a)** $1 + 12x + 60x^2 + 160x^3$ **(b)** $32 - 240x + 720x^2 - 1080x^3$

(c) $64 + 48x + 15x^2 + \frac{5}{2}x^3$ **(d)** $\frac{81}{16} - 36x + 6x^2 - \frac{8}{3}x^3$

(e) $\frac{729}{4096}y^6 - \frac{729}{256}y^5x + \frac{1215}{64}y^4x^2 - \frac{135}{2}y^3x^3$ **(f)** $\frac{32}{3125}y^5 - \frac{8}{25}y^4x + 4y^3x^2 - 25y^2x^3$

3. $1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$; 1. 010 040 **4.** $64 + 24x + \frac{15}{4}x^2 + \frac{5}{16}x^3$; 66.438

5. $32 - \frac{80}{3}x + \frac{80}{9}x^2 - \frac{40}{27}x^3$; 31.734 **6.** $1 + 14x + 84x^2 + 280x^3$; 1.316

Exercise 7 B

1. **(a)** $270x^3$ **(b)** $1120x^4$ **(c)** $6048x^5$ **(d)** $70x^4$ **(e)** $-\frac{880}{9x^3}$
2. **(a)** 40 **(b)** -160 **(c)** 90 **(d)** 40 **(e)** 672
   (f) 4320 **(g)** 160 **(h)** 5376
3. **(a)** **(i)** 4860 **(ii)** 4608 **(iii)** -1458 **(iv)** 960 **(b)** **(i)** 4320
   (ii) -8064 **(iii)** 90720 **(iv)** 135
4. **(a)** 48 **(b)** -450 **(c)** 9 **(d)** 120 **(e)** -228
5. $a = \frac{12}{7}, b = -\frac{4}{7}; a = -1, b = 2$
6. -160 **7.** 4; 10 **8.** -683

Exercise 7 C

1. **(a)** 81 **(b)** 6.5 **(c)** -103 **(d)** 6.3
   (e) $4n + 17$ **(f)** $\frac{1}{4}(79 - 3n)$ **(g)** $1.3r - 0.1$
2. **(a)** 26; 1729 **(b)** 33; 158.4 **(c)** 100; 5050 **(d)** 16; 200
   (e) 34; $259\frac{1}{4}$
3. 4 **4.** $-4; 43; -1; 100$ **5.** 775 **6.** $1, -3, -115$
7. **(a)** 1650 **(b)** 364.8 **(c)** -1500 **(d)** $581\frac{1}{4}$
   (e) $\frac{n(n+1)}{2}$ **(f)** n^2x **(g)** $n(n+1)x$ **(h)** $n(7.3 - 0.3n)$
8. 24; $\frac{75}{23}$ **9.** $2n^2$, 101 **10.** $2n(51-n)$, 52 **11.** $27, -4, -15$
12. 32 **13.** 4; 8 **14.** $1.5; -0.5; 10$ **15.** 30°, 70°, 110°, 150°
16. 5 cm, 7 cm, 9 cm, 11 cm, 13 cm **17.** 20 cm
18. **(a)** $\frac{n(n+1)}{2}$ **(b)** $\frac{3n^2 + 3n - 4}{2}$ **(c)** 21

Exercise 7 D

1. **(a)** 6×3^7 **(b)** $\frac{1}{2^{10}}$ **(c)** -3^{29} **(d)** $\frac{3^{14}}{2^{25}}$
   (e) 3 **(f)** $3 \times (-2)^r$ **(g)** $\left(\frac{3}{4}\right)^{2r}$

2. **(a)** 7 **(b)** 10 **(c)** 11 **(d)** 8

(e) 8 **3.** **(a)** $3; 10; 5(3^{10} - 1); |r| > 1$

4. $\pm\frac{1}{2}$; 64 or $21\frac{1}{3}$ **5.** $-\frac{1}{4}$; 32 **6.** 3 or $\frac{1}{5}$

7. **(a)** $72\left[1 - \left(\frac{1}{2}\right)^{10}\right]$ **(b)** $1 + 2^{15}$ **(c)** $12\left[1.5^8 - 1\right]$ **(d)** $64\left[1 - 0.25^{20}\right]$

(e) $\frac{1 + 3^{25}}{20}$ **(f)** $\frac{9}{2}\left[1 - \left(\frac{2}{3}\right)^9\right]$ **(g)** $\frac{27}{7}\left[1 + \left(\frac{4}{3}\right)^{15}\right]$ **(h)** $-\frac{3}{7}\left[1 - \left(\frac{3}{4}\right)^{10}\right]$

8. **(a)** $2[2^{10} - 1]$ **(b)** $\frac{1}{4}[1 + 3^7]$ **(c)** $0.1[4^6 - 1]$ **(d)** $\frac{2}{405}[1 - 1.5^8]$

(e) $-\frac{64}{3645}\left[1 - 1.5^8\right]$ **9.** 3; 4; 118 096

10. **(a)** 96 **(b)** 8 **(c)** 2.7 **(d)** no S_∞

(e) –0.9 **11.** $\frac{5}{7}$ **12.** 12; $\frac{2}{3}$ **13.** 32

14. Rs $(1.12 \times 500\,000 - P)$; Rs $[(1.12)^2 \times 500\,000 - 1.12P - P]$; Rs $\left[1.12^n \times 500\,000 - \frac{25}{3}p(1.12^n - 1)\right]$; Rs 73 400.

15. **(a)** Rs $(1.0125 \times 300\,000 - x)$; Rs $((1.0125)^2 \times 300\,000 - 2.0125x)$;
Rs $[(1.0125)^n \times 300\,000 - 80x(1.0125^n - 1)]$; Rs 7 140.

■ *Miscellaneous Exercise 7*

1. $1 + 6ax + 15a^2x^2 + \ldots$; $a = \frac{1}{2}, b = -3; a = -\frac{1}{2}, b = 3;$ **2.** **(a)** 4, **(b)** $\frac{2}{3}$

3. $1 - 15x + 90x^2; 16 + 32x + 24x^2; 984$ **4.** **(a)** $64 + 192x^2 + 240x^4 + 160x^6; 48$

5. **(a)** 15.2 cm **(b)** $\frac{2}{3}$, 54 **6.** $\frac{7}{9}; -\frac{35}{3}; -7$ **7.** **(a)** 1.5; 20; 30.5

(b) $\frac{1}{3}; \frac{1}{4}; \frac{64}{9}$ **8.** **(a)** 0.917, 40 **(b)** $\frac{10000 + 5n - 5n^2}{n}$; 73

9. **(a)** 105 **(b)** **(i)** £ 65 100 (ii) £ 22 700 **10.** 8; 2

■ Exercise 8 A

1. **(a)** $10x^4$ **(b)** $24x^3$ **(c)** $-15x^{-6}$ **(d)** $2x^{-\frac{2}{3}}$

(e) $-4x^{-\frac{5}{4}}$ **(f)** $-4x^{-\frac{3}{2}}$ **(g)** $-\frac{10}{3}x^{-\frac{4}{3}}$ **(h)** $-9x^{-\frac{7}{4}}$

2. **(a)** $24x^3 + 9x^2$ **(b)** $45x^4 + 6x^2 + 14x$ **(c)** $15x^2 - 4x - 3$ **(d)** $12x - 7$

(e) $30x^4 + 21x^2 - 3$ **(f)** $\frac{3}{2}(5x^2 - 1)$ **(g)** $12x - 15 - 5x^{-2}$ **(h)** $\frac{6\left(3x^2 + 4x + 4\right)}{7x^3}$

3. **(a)** $6 + 6r$ **(b)** $6t^2 - 6t + 1$ **(c)** $18x^2 - 20x + 3$ **(d)** $6 + \frac{3}{t^2}$

(e) $2\pi h + 4\pi r$

4. **(a)** $f':x \mapsto 9x^2 - 12x + 15$ **(b)** $f':t \mapsto 4t + 5$ **(c)** $f':r \mapsto 3r^2 + 4r - 1$ **(d)** $f':s \mapsto \frac{3 - 6s - 2s^2}{5s^4}$

(e) $f':t \mapsto t - \frac{1}{4} + \frac{3}{4t^2}$

5. **(a)** -9 **(b)** 19 **(c)** -3 **(d)** ± 7

(e) $\frac{1}{4}$

6. **(a)** $(1, 4)$ **(b)** $\left(\frac{1}{2}, \frac{5}{2}\right), \left(-\frac{1}{2}, -\frac{5}{2}\right)$ **(c)** $(5, -99), (-1, 9)$ **(d)** $(2, 12)$

■ Exercise 8 B

1. **(a)** $y = x - 3;\ y + x + 1 = 0$ **(b)** $y = 20x + 27;\ 20y + x + 262 = 0$

(c) $y + 8x = 8;\ 16y - 2x = 63$ **(d)** $y + 16x = 12;\ 32y = 2x + 127$

(e) $y = x;\ y + x + 2 = 0$ **(f)** $y = 1;\ x = 2$

2. **(a)** $\left(\frac{3}{2}, -\frac{3}{2}\right)$ **(b)** $(1, -1)$ **(c)** $\left(\frac{13}{8}, -\frac{47}{32}\right)$

3. $(-1, -5), (-3, -1);\ x = -1, x = -3$

4. $\left(-\frac{7}{3}, \frac{41}{27}\right); (1, 3)$ **5.** $(1, 0)$ **6.** $\left(-\frac{11}{12}, \frac{383}{48}\right)$

■ Exercise 8 C

1. **(a)** Minimum $(2, 1)$ **(b)** Maximum $\left(\frac{3}{2}, \frac{9}{2}\right)$

(c) Minimum $(1, 2)$; maximum $(-1, -2)$ **(d)** Minimum $(3, 27)$

(e) Minimum $(-1, -6)$; maximum $(-2, -5)$ **(f)** Minimum $(-4, -131), (1, -6)$; maximum $(0, -3)$

(g) Minimum $(-6, 108)$ **(h)** Maximum $\left(\frac{2}{3}, \frac{200}{27}\right)$; minimum $(3, -18)$

(i) Stationary point of inflexion $(0, 0)$; maximum $\left(\frac{3}{4}, \frac{27}{256}\right)$ **(j)** Maximum $\left(-\frac{2}{5}, \frac{16}{3125}\right)$; minimum $(0, 0)$

(k) Minimum $\left(4, 1\frac{1}{2}\right)$ **(l)** Minimum $(-1, 33), (2, -21)$;
maximum $(1, -19), (-2, 35)$

■ Exercise 8 D

1. **(a)** $x > 2$ **(b)** $x < 1$ or $x > 3$ **(c)** $x < -3$ or $x > \frac{1}{2}$ **(d)** $x < -1$ or $x > 1$

(e) $x > 2$ **3.** $b > 4$ **6.** $-1 < x < 2$ **7.** $k > 1$

8. **(a)** $12(4x + 2)^2$ **(b)** $2(4x + 1)^{-\frac{1}{2}}$ **(c)** $-\frac{4}{3}(5 - 4x)^{-\frac{2}{3}}$ **(d)** $-30(2x + 1)^{-6}$

(e) $3(x + 2)(x^2 + 4x + 1)^{\frac{1}{2}}$ **(f)** $-10(2x + 3)^{-2}$ **(g)** $24(1 - 3x)^{-3}$ **(h)** $6(5 - 2x)^{-2}$

(i) $-3(x + 2)(x^2 + 4x - 1)^{-\frac{1}{2}}$ **(j)** $-8x(x + 2)(2x^3 + 6x^2 + 1)^{-\frac{4}{3}}$

(k) $-(3x + 1)(3x^2 + 2x - 1)^{-\frac{5}{4}}$ **(l)** $-\frac{5}{2}(1 - 12x)(1 + x - 6x^2)^{-\frac{3}{2}}$

9. **(a)** 24 **(b)** 1 **(c)** $-\frac{3}{250}$ **(d)** $\frac{\pi}{3g}$

(e) $\frac{24}{5\pi}$

10. **(a)** $2\left(x - \frac{1}{x}\right)\left(1 + \frac{1}{x^2}\right)$ **(b)** $\frac{16\pi h}{\sqrt{h^2 + 16}}$ **(c)** $\left(1 + \frac{3}{t^3}\right)\left(2t - \frac{3}{t^2}\right)^{-\frac{1}{2}}$

(d) $-4\left(6x + \frac{2}{x^2}\right)\left(3x^2 - \frac{2}{x}\right)^{-5}$ **(e)** $-\frac{15}{4}\left(2 + \frac{1}{3r^2}\right)\left(2r - \frac{1}{3r}\right)^{-\frac{7}{4}}$

11. 0.52 cm s^{-1} **12.** **(a)** 3.2π cm^3 s^{-1} **(b)** 0.018π cm^3 s^{-1} **(c)** 7.2π cm^3 s^{-1}

13. **(a)** $\frac{5}{8\pi}$ cm s^{-1} **(b)** 0.214 cm s^{-1} **(c)** $\frac{1}{10\pi}$ cm s^{-1}

14. $\frac{1}{4\pi}$ cm s^{-1}; 25 cm^3 s^{-1} **15.** **(a)** 2.16 cm^3 s^{-1} **(b)** 0.96 cm^3 s^{-1} **(c)** 3.84×10^{-4} cm^3 s^{-1}

16. $-\frac{1}{16}$ cm s^{-1}; $-\frac{1}{9}$ cm s^{-1} **17.** **(a)** -4 cm^2 s^{-1} **(b)** -5 cm^2 s^{-1} **(c)** -8 cm^2 s^{-1}

18. -2.4π cm^3 s^{-1}

19. **(a)** $\frac{-2}{15\pi}$ cm s^{-1} **(b)** $-\frac{3}{80}$ cm^2 s^{-1} **20.** **(a)** $5x^3$ **(b)** -2.51 cm^2 s^{-1}

■ *Miscellaneous Exercise 8*

1. $y = -x$; $(0, 0)$ **2.** **(a)** $-12x(3 - x^2)^5$ **(b)** $24y + x = 26$

3. Minimum $y = 6\frac{2}{3}$ **4.** 4π cm^3 s^{-1}; $\frac{1}{2}$ cm s^{-1} **5.** $x = \frac{8}{3}$

6. $x = \frac{3}{4}$ **7.** max 3 ; $y = 2x$; $y + 2x = 8$ **8.** **(a)** 4 **(b)** 2 cm s^{-1}

9. min $(0, 0)$; max $(4, 32)$ $x < 0$ or $x > 4$; $y = 12x - 8$ **10.** $(-2, 4)$

11. $-\frac{1}{10}$ **12.** $p = \frac{5}{2}$ **13.** $(2, 0)$

14. Minimum at $\left(\frac{1}{2}, 8\right)$ **15.** $1 - \frac{4}{x^3}$; $k = 6\frac{1}{2}$

16. **(a)** 1 **(b)** -108 **17.** $6\frac{2}{3}$; minimum

18. **(a)** $3 + \frac{4}{x^2}$ **(b)** $4p$ **19.** 160π

20. **(a)** $\sqrt[3]{16}$ **(b)** $\frac{27}{2}x^{\frac{1}{2}}, 0.0140$

Exercise 9 A

1. **(a)** $y = x^2 - 3x + c$ **(b)** $y = x^3 - 2x^2 + 2x + c$ **(c)** $y = \frac{x^4}{2} + \frac{x^3}{3} - \frac{x^2}{2} - 5x + c$

(d) $y = \frac{x^4}{2} - \frac{x^3}{3} - 3x^2 + 3x + c$ **(e)** $y = \frac{3x^4}{4} - \frac{5x^3}{3} + \frac{13x^2}{2} + 5x + c$ **(f)** $y = \frac{-3}{x} - \frac{2}{x^2} + c$

(g) $y = \frac{-1}{x} + \frac{1}{x^2} - \frac{5}{3x^3} + c$ **(h)** $y = \frac{-2}{3x} - \frac{1}{2x^2} + \frac{7}{9x^3} - \frac{3}{4x^4} + c$ **(i)** $y = \frac{-3}{2x} - \frac{19}{8x^2} + \frac{4}{3x^3} - \frac{3}{16x^4} + c$

2. $y = \frac{2}{3}x^3 - \frac{1}{2}x^2 + 3x - 4\frac{1}{6}$ **3.** $A = \frac{1}{4}r^4 - \frac{2}{3}r^3 + \frac{r^2}{2} - 3r + 1$ **4.** $y = \frac{1}{2}(x^3 + x^2 - x + 1)$

5. $f(x) = \frac{2}{3}x^3 - \frac{1}{2}x^2 + 3x + 16\frac{1}{3}$ **6.** $A = \frac{-2}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + 2\frac{1}{6}$ **7.** **(a)** $\frac{x^4}{2} + c$

(b) $\frac{x^6}{2} + c$ **(c)** $-\frac{4}{5x^5} + c$ **(d)** $-\frac{9}{x} - \frac{7}{2x^2} + c$

(e) $6r - \frac{1}{2}r^2 - \frac{1}{3}r^3 + c$ **(f)** $t - \frac{1}{5}t^5 + c$ **(g)** $2s - \frac{1}{2}s^2 - 2s^3 + c$

(h) $-\frac{3}{2t} - \frac{1}{t^2} + \frac{1}{6t^3} + c$ **(i)** $-\frac{8}{3s} - \frac{5}{3s^2} + \frac{1}{3s^3} + c$ **8.** **(a)** $\frac{(3x + 2)^4}{12} + c$

(b) $\frac{(2x - 1)^6}{3} + c$ **(c)** $\frac{(4x + 1)^{\frac{3}{2}}}{6} + c$ **(d)** $\frac{-2}{3(3x - 1)} + c$

(e) $\frac{5}{4(1 - 2x)^2} + c$ **(f)** $4(3x + 1)^{\frac{1}{2}} + c$ **(g)** $-(5 - 3x)^{\frac{2}{3}} + c$

(h) $\frac{1}{(5 - 2r)^2} + c$ **(i)** $\frac{-1}{2(3t - 5)^4} + c$ **9.** $s = 2 - \frac{1}{(2t - 1)^2}$

10. $A = 2 + r - \frac{5}{2}r^2 - \frac{3}{2(2r + 1)}$ **11.** $y = \frac{2}{3}x^3 - \frac{1}{2}x^2 + 3x + 31\frac{1}{2}$ **12.** $y = 2x - \frac{3}{2}x^2 - \frac{1}{3}x^3 - 2\frac{1}{6}$

13. $y = \frac{31}{7} - \frac{3}{7}x^{\frac{3}{2}}$ **14.** $y = 1 + 3x - 2x^2$

Exercise 9 B

1. **(a)** -4 **(b)** $7\frac{1}{2}$ **(c)** 4 **(d)** $1\frac{2}{3}$ **(e)** $19\frac{1}{3}$

(f) $\frac{1}{6}$ **(g)** $\frac{1}{6}$ **(h)** 6 **(i)** $-8\frac{2}{3}$

2. **(a)** $3\frac{3}{4}$ **(b)** 0 **(c)** $1368\frac{1}{5}$ **(d)** $2\frac{8}{15}$ **(e)** 4
(f) $-\frac{1}{2}$ **(g)** 5 **(h)** 3

■ Exercise 9 C

1. **(a)** $24\frac{2}{3}$ **(b)** $10\frac{1}{2}$ **(c)** $18\frac{2}{3}$ **(d)** $20\frac{5}{6}$ **(e)** 21 **(f)** $4\frac{2}{3}$
2. **(a)** $\frac{2}{3}$ **(b)** $\frac{26}{27}$ **(c)** $22\frac{1}{2}$ **(d)** $4\frac{2}{3}$ **(e)** $8\frac{5}{6}$ or $3\frac{5}{6}$
3. **(a)** $1\frac{1}{3}$ **(b)** $10\frac{2}{3}$ **(c)** $21\frac{1}{2}$ **(d)** $11\frac{1}{4}$
4. **(a)** $10\frac{2}{3}$ **(b)** $\frac{1}{6}$ **(c)** $5\frac{1}{3}$ **5.** $49\frac{1}{2}$ **6.** $7^{\frac{1}{3}}$
7. **(a)** 6.2π **(b)** 69.6π **(c)** 7.5π **(d)** 5.5π **(e)** 48.4π **(f)** $\frac{129}{7}\pi$
(g) 24.4π
8. **(a)** 4π **(b)** 7.5π **(c)** 6.2π **(d)** $\frac{9}{2}\pi$ **(e)** 18.6π **(f)** 42.2π
9. 291.6π **10.** $10\frac{2}{3}\pi$ **11.** 53π **12. (a)** $y = 2(7 - x)$ **(b)** 36π **(c)** 18π
13. **(a)** 4.5π **(b)** 3.6π **14. (a)** 2 **(b)** $\frac{8}{15}\pi$

■ *Miscellaneous Exercise 9*

1. **(a)** $\frac{1}{2}$ **(b)** $3\frac{1}{2}$ **2.** $y = \frac{1}{9}[(3x - 2)^3 + 17]$
3. **(a)** $\frac{1}{2}(4x + 3)^{\frac{1}{2}} + c$ **(b)** 0 **4.** 10 sq. units **5. (a)** $\frac{(2x + 1)^{11}}{22} + c$
(b) $\frac{(2x + 1)^{\frac{3}{2}}}{3} + c$ **6. (a)** 1 **(b)** $\frac{10}{81}$ **7.** 2
8. $33\frac{1}{3}\pi$ **9.** 5 **10. (a)** $y = -x^2 - 5x + 6$ **(b)** $-\frac{5}{2}$
11. **(a)** $-\frac{1}{x} + \frac{1}{2x^2} + c$ **(b)** $\frac{\pi}{6}$ **12.** $2\frac{2}{3}$ **13. (a)** $y = x^2 + 3x + 5$
(b) 18π **14.** 60π **15.** -4; $y = x^2 - 4x + 9$ **17. (a)** $a = -1, b = 4$
(b) 10 square units **18.** 4.5 cubic units **20. (a)** 8 **(b)** $(0, 7)$

Exercise 10 A

1. **(a)** $-1, 5$ **(b)** $4, -1$ **(c)** $\frac{1}{5}, 1$

   (d) no solution **(e)** no solution **(f)** $\frac{1}{4}, \frac{3}{2}$

2. **(a)** $-3 \leqslant x \leqslant 2$ **(b)** $x \leqslant -\frac{1}{2}$ or $x \geqslant 1$ **(c)** $x < \frac{1}{7}$ or $x > 3$

   (d) $-\frac{3}{2} \leqslant x \leqslant -\frac{1}{10}$ **(e)** $\frac{2}{5} \leqslant x \leqslant 4$ **(f)** $0 < x < 4$

Exercise 10 B

1. Quotient $2x^2 + 9x + 22$, remainder 68
2. Quotient $6x^3 - 16x^2 + 42x - 118$, remainder 344
3. Quotient $4x^3 - x^2 + x - \frac{3}{2}$, remainder $-7\frac{1}{2}$
4. Quotient $6x^2 - 4x - 3$, remainder 11
5. Quotient $10x^3 + 16x^2 + 10x + 10$, remainder 12
6. Quotient $3x + 1$, remainder $-7x + 6$
7. Quotient $2x - 1$, remainder $3x - 6$
8. Quotient $10x^2 - 2x + 23$, remainder $-22x + 66$

Exercise 10 C

1. **(a)** -6 **(b)** 20 **(c)** 7 **(d)** 103 **(e)** 44
2. **(a)** $-3\frac{1}{4}$ **(b)** 0 **(c)** 8
3. **(a)** $(x - 1)(x + 2)(x + 3)$ **(b)** $(x + 1)(x + 2)(2x - 1)$ **(c)** $(x + 2)(x - 3)(2x + 1)$

   (d) $(x + 2)(6x^2 - 8x + 1)$ **(e)** $(2x - 1)(x^2 + 3x + 1)$

4. **(a)** $-1, -2, 3$ **(b)** $-2, \frac{1}{2}, 1$ **(c)** $-\frac{3}{2}, -2, \frac{1}{2}$ **(d)** $-1, -2 - \sqrt{6}, -2 + \sqrt{6}$

   (e) $-3, -\frac{1}{2}, 1$

5. $a = -1, b = 4$
6. $a = -6, b = 43$

Exercise 10 D

1. $\frac{2}{3(x+1)} + \frac{4}{3(x-2)}$
2. $\frac{18}{7(3x+2)} - \frac{5}{7(2x-1)}$
3. $\frac{2}{x-1} - \frac{9}{x-2} + \frac{7}{x-3}$
4. $\frac{-1}{6(x-1)} + \frac{1}{15(x+2)} + \frac{21}{10(x-3)}$
5. $\frac{5}{3(x+1)} + \frac{2}{3(2x-1)} - \frac{5}{3x+2}$
6. $\frac{6}{(x-1)} - \frac{5}{(x-2)} + \frac{9}{(x-2)^2}$
7. $\frac{7}{9x} - \frac{1}{3x^2} - \frac{7}{9(x+3)}$
8. $\frac{3}{x+1} - \frac{2}{x+2} - \frac{7}{(x+3)^2}$
9. $\frac{-10}{13(x+2)} + \frac{10x+19}{13(x^2+9)}$
10. $\frac{7}{11(x+2)} - \frac{3x+5}{11(2x^2+3)}$
11. $1 + \frac{1}{2(x-1)} - \frac{7}{(x-2)} + \frac{23}{2(x-3)}$
12. $1 - \frac{1}{(x-1)} + \frac{7}{(x-2)} + \frac{9}{(x-2)^2}$

13. $1 - \frac{5}{9x} + \frac{1}{3x^2} + \frac{32}{9(x-3)}$ **14.** $(x+5) - \frac{2}{x-1} + \frac{16}{x-3}$

15. $(x+2) - \frac{2}{3(x-1)} + \frac{34}{15(x+2)} + \frac{47}{5(x-3)}$

Exercise 10 E

1. **(a)** $1 + 6x + 24x^2 + 80x^3 \ldots\ldots, |x| < \frac{1}{2}$ **(b)** $1 + 2x - 2x^2 + 4x^3 \ldots., |x| < \frac{1}{4}$

(c) $1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 \ldots\ldots, |x| < \frac{1}{2}$ **(d)** $\frac{1}{4} - \frac{3}{4}x + \frac{27}{16}x^2 - \frac{27}{8}x^3 \ldots., |x| < \frac{2}{3}$

(e) $2 - \frac{1}{4}x - \frac{1}{64}x^2 - \frac{1}{512}x^3 \ldots\ldots, |x| < 4$ **(f)** $2 - \frac{1}{12}x - \frac{1}{288}x^2 - \frac{5}{20736}x^3 \ldots., |x| < 8$

(g) $\frac{1}{3} - \frac{1}{27}x + \frac{1}{162}x^2 - \frac{5}{4374}x^3 \ldots\ldots, |x| < \frac{9}{2}$ **(h)** $\frac{1}{2} - \frac{1}{48}x + \frac{1}{576}x^2 - \frac{7}{41472}x^3 \ldots\ldots, |x| < 8$

2. $1 + x - x^2 + \frac{5}{3}x^3, 2.23$ **3.** $1 - \frac{1}{3}x - \frac{1}{9}x^2, 3.3322$

4. $\frac{7}{6} + \frac{1}{36}x + \frac{43}{216}x^2 + \frac{49}{1296}x^3 \ldots\ldots, |x| < 2$ **5.** $\frac{2}{1-x} - \frac{x+2}{1+x^2}, x + 4x^2 + 3x^3 - x^5, |x| < 1$

Miscellaneous Exercise 10

1. -3 **2.** $x < -\frac{1}{3}$ or $x > \frac{1}{5}$ **3.** $a = -1, b = -10$

4. $\frac{82}{13(2x+3)} + \frac{3-28x}{13(x^2+1)}$ **5.** 1.000 200 00 **6.** $1 + \frac{3}{2}x + \frac{7}{4}x^2 + \frac{15}{8}x^3 \ldots, |x| < 1, 2 - \frac{1}{2^n}$

7. $1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3$ **8.** $-\frac{1}{2(x-1)} + \frac{5}{2(x+3)}, \frac{4}{3} + \frac{2}{9}x + \frac{16}{27}x^2 \ldots, \frac{2}{x} - \frac{8}{x^2} + \frac{11}{x^3}$

9. $c_0 = 0, c_1 = 5, c_2 = 6$ **10.** -6 **11.** $a = 6, b = -4,$

(c) $2x$ **(d)** $1, 1 \pm \sqrt{2}$ **12.** $\frac{8}{5(2+x)} + \frac{2x+1}{5(1+x^2)}; 0.2$

13. $1 + \frac{1}{4}x - \frac{3}{32}x^2 \ldots.$ **14.** $\frac{1}{2} - \frac{3}{8}x + \frac{7}{16}x^2$ **15.** $a = -3, b = -2, 8x + 12.$

16. **(b)** $x^2 + x + 2$ **(c)** $x < -2$ or $x > 1.$ **17.** $a = 3, b = 4; (x-1)(2x+1)(x-4)$

(a) $(x+1)(x-1)(x+2)(x-2)(2x^2+1)$ **(b)** $(x-1)(x+2)(4x-1)$

Exercise 11 A

1. **(a)** $5^2 = 25$ **(b)** $7^3 = 343$ **(c)** $16^{\frac{1}{2}} = 4$ **(d)** $27^{-\frac{1}{3}} = \frac{1}{3}$

(e) $3^{-2} = \frac{1}{9}$ **(f)** $e^y = x$ **(g)** $10^y = x$ **(h)** $y^z = x$

2. **(a)** 3 **(b)** 3 **(c)** 0 **(d)** $\frac{1}{3}$ **(e)** $\frac{3}{4}$
(f) $\frac{1}{2}$ **(g)** 2 **(h)** –2 **(i)** $-\frac{2}{3}$ **(j)** 3
(k) $\frac{1}{3}$ **(l)** $\frac{3}{2}$ **(m)** $-\frac{5}{3}$ **(n)** $\frac{5}{3}$ **(o)** $-\frac{n}{m}$

3. **(a)** $\lg a + 2\lg b + 3\lg c$ **(b)** $\lg c - \lg a - \lg b$ **(c)** $\lg a - 3\lg c$
(d) $\lg c - \lg a - 2\lg b$ **(e)** $3\lg a + 2\lg b - 5\lg c$ **(f)** $2 + \lg c - \lg a - 2\lg b$
(g) $\frac{1}{2}\lg a + 2\lg b$ **(h)** $1 + \lg b - \lg a - \frac{1}{2}\lg c$ **(i)** $3\lg a + 5\lg b - \frac{2}{3}\lg c$
(j) $\frac{3}{2}\lg a - \lg b - 2\lg c$ **(k)** $1 + \frac{2}{3}\lg a - \frac{1}{3}\lg b - \lg c$ **(l)** $\frac{1}{2} - \lg a - \frac{3}{2}g\ b - \frac{1}{2}\lg c$

4. **(a)** $\lg 14$ **(b)** $\lg \frac{4}{7}$ **(c)** $\lg 1.2$ **(d)** $\lg 128$ **(e)** $\lg 48$
(f) $\lg \frac{ab}{c}$ **(g)** $\lg a^2b^{\frac{1}{3}}c^{\frac{1}{2}}$ **(h)** $\lg 800$ **(i)** $\lg \frac{2}{5}$ **(j)** $\lg\frac{a^{\frac{1}{3}}b^{\frac{1}{2}}}{c^{\frac{4}{5}}}$
(k) $\lg\frac{a^2}{b^3c^{\frac{1}{2}}}$ **(l)** $\lg \frac{a^3b^2}{1000}$

5. **(a)** 3 **(b)** –2 **(c)** $\frac{3}{2}$ **(d)** 2 **(e)** $\frac{2}{3}$
(f) $\frac{3}{4}$ **(g)** $\frac{3}{5}$ **(h)** $\frac{2}{5}$

6. **(a)** 1.505 **(b)** 1.4313 **(c)** 2.0333 **(d)** 0.3522 **(e)** 0.6990
(f) 2.3522 **(g)** 0.8266 **(h)** 1.03955 **(i)** 0.3801 **(j)** –0.8293

■ Exercise 11 B

1. **(a)** 0.082 **(b)** 2.014 **(c)** 0.395 **(d)** 0.651 **(e)** 4.113
(f) 0.160 **(g)** 1.924 **(h)** 0.681

2. **(a)** 0.998 **(b)** 0.663 **(c)** –0.274 **(d)** 3.250 **(e)** 0.549
(f) 1.005 **(g)** 0.902 **(h)** 2.320

3. **(a)** 0.039 **(b)** 0.008 **(c)** 66.069 **(d)** 25.955 **(e)** 19.307
(f) 14.279 **(g)** 3.320 **(h)** 0.061 **(i)** 9.356 **(j)** 1.249
(k) 0.651 **(l)** 1.999

4. **(a)** 1.609 **(b)** 0.050 **(c)** 0.745 **(d)** 2.446 **(e)** –0.970
(f) –1.5865.

5. **(a)** –1.099, 0 **(b)** 1.609 **(c)** 1.686 **(d)** –0.147 **(e)** 0.693, –1.609

6. **(a)** 2.18 **(b)** 4.46 **(c)** –1.49 **(d)** 0.398 **(e)** 0.585
(f) –0.667 **(g)** 2.11 **(h)** –0.495 **(i)** 2.50

7. **(a)** $x > 6.84$ **(b)** $x < 3.21$ **(c)** $x \geqslant -0.111$ **(d)** $x \geqslant 0.957$ **(e)** $x \geqslant 5.91$
(f) $n > 6.35$ **(g)** $n \leqslant 5.419$ **(h)** $x > 0.877$ **(i)** $n \geqslant 6.35$

8. 8, 4 374 **9.** 22^{nd}, 0.076 **10.** 24 **11.** **(a)** 10 **(b)** 5

12. 71

■ Exercise 11 C

1. $y \approx 0.7e^{1.4x}$ **2.** $y \approx 3e^{-2x}$ **3.** $a \approx 2.3, b \approx -3.7$ **4.** $a \approx 1.7, b \approx 1.5$

5. $s = ab^t$ holds; $a \approx 1.8$, $b \approx 1.1$

■ *Miscellaneous Exercise 11*

1. **(a)** $x = 3, y = 2$ **(b) (i)** $x = \frac{2}{9}$ **(ii)** $y = 3.82$ **(c)** 2.5

2. **(a)** 1.43 **(b)** $p = 6.5$, $q = 1.5$

3. **(a)(i)** 171 **(ii)** 17.2 **(b)** 4, 12 **(c)** $\{f(x) \in \mathbb{R}: f(x) > 0\}, f^{-1}: x \mapsto \frac{1}{2}\ln\frac{x}{3}$

4. 10 000; 10, $\frac{1}{2}$

5. **(a)(i)** $\frac{3 \pm \sqrt{5}}{2}$ **(ii)** 10 **(b)** 0.347 **(c)(ii)** $\ln x = \frac{1}{2}x - 1, y = \frac{1}{2}x - 1$

6. **(b)** 1.65 **7.** $f^{-1}: x \mapsto e^x - 1 \ (x \in \mathbb{R})$ **8.** **(a)** $x = \frac{1}{\ln a - 2}$ **(b)** $x = \frac{\sqrt{ae}}{2}$

9. $a = 2.4, b = 1.5$ **10.** $72, 72\left[1 - \left(\frac{2}{3}\right)^n\right], 13$ **11.** **(a)** $p = \frac{1}{3}, q = -4$ **(b)** $x = 2\frac{1}{2}, y = 20$

12. $a = 0.25, b = 0.12$; $x = 1.83$, $y = 2.67$

13. **(a)** $p = 10, q = -4$ **(b)** $x = 1\frac{1}{3}$

14. **(a)** $x = 9$ **(b)** $p = 4, q = \frac{1}{2}$ **15.** $A = 90, k = 0.09$

Exercise 12 A

1. **(a)** -2 **(b)** $\sqrt{2}$ **(c)** $\sqrt{3}$ **(d)** $\frac{2}{\sqrt{3}}$ **(e)** $-\frac{2}{\sqrt{3}}$

(f) $-\sqrt{3}$ **(g)** $-\frac{2}{\sqrt{3}}$ **(h)** -2 **(i)** $-\frac{1}{\sqrt{3}}$ **(j)** $-\frac{2}{\sqrt{3}}$

(k) -2 **(l)** -1 **(m)** $-\sqrt{2}$ **(n)** $-\sqrt{2}$ **(o)** $\frac{1}{\sqrt{3}}$

(p) $\sqrt{2}$ **(q)** 2 **(r)** $-\sqrt{3}$

2. **(a)** -1.494 **(b)** 1.788 **(c)** 0.900 **(d)** 1.031 **(e)** -1.079

(f) -0.781 **(g)** -1.440 **(h)** -4.134 **(i)** -0.839 **(j)** -1.008

(k) -1.390 **(l)** -4.011 **(m)** -14.336 **(n)** -1.269 **(o)** 0.070

(p) 1.192 **(q)** 1.305 **(r)** -8.144

3. **(a)** 30°, 330° **(b)** 210°, 330° **(c)** 30°, 210° **(d)** 120°, 240° **(e)** 60°, 120°

(f) 120°, 300° **(g)** 45°, 315° **(h)** 225°, 315° **(i)** 45°, 225° **(j)** 150°, 210°

(k) no solution **(l)** 50°, 110°, 170°, 230°, 290°, 350° **(m)** 68.6°, 291.4° **(n)** 25.7°, 154.3°

(o) 20.7°, 200.7° **(p)** 124.8°, 235.2° **(q)** 205.7°, 334.3° **(r)** 114.3°, 294.3°

(s) 37.6°, 142.4°, 217.6°, 322.4° **(t)** no solution

(u) 43.6°, 103.6°, 163.6°, 223.6°, 283.6°, 343.6°

Exercise 12 B

1. **(a)** $\cos\theta = \pm\sqrt{\frac{2}{3}}$, $\tan\theta = \pm\frac{1}{\sqrt{2}}$, $\operatorname{cosec}\theta = \sqrt{3}$, $\sec\theta = \pm\sqrt{\frac{3}{2}}$, $\cot\theta = \pm\sqrt{2}$

(b) $\sin\theta = \pm\frac{1}{\sqrt{5}}$, $\tan\theta = \pm\frac{1}{2}$, $\operatorname{cosec}\theta = \pm\sqrt{5}$, $\sec\theta = \frac{\sqrt{5}}{2}$, $\cot\theta = \pm 2$

(c) $\sin\theta = \pm\frac{2}{\sqrt{13}}$, $\cos\theta = \pm\frac{3}{\sqrt{13}}$, $\operatorname{cosec}\theta = \pm\frac{\sqrt{13}}{2}$, $\sec\theta = \pm\frac{\sqrt{13}}{3}$, $\cot\theta = -\frac{3}{2}$

(d) $\sin\theta = -\frac{3}{5}$, $\cos\theta = \pm\frac{4}{5}$, $\tan\theta = \pm\frac{3}{4}$, $\sec\theta = \pm\frac{5}{4}$, $\cot\theta = \pm\frac{4}{3}$

(e) $\sin\theta = \pm\frac{\sqrt{7}}{4}$, $\cos\theta = -\frac{3}{4}$, $\tan\theta = \pm\frac{\sqrt{7}}{3}$, $\operatorname{cosec}\theta = \pm\frac{4}{\sqrt{7}}$, $\cot\theta = \pm\frac{3}{\sqrt{7}}$

(f) $\sin\theta = \pm\frac{3}{\sqrt{34}}$, $\cos\theta = \pm\frac{5}{\sqrt{34}}$, $\tan\theta = \frac{3}{5}$, $\operatorname{cosec}\theta = \pm\frac{\sqrt{34}}{3}$, $\sec\theta = \pm\frac{\sqrt{34}}{5}$

(g) $\cos\theta = -\frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\text{cosec}\,\theta = -\frac{5}{3}$, $\sec\theta = -\frac{5}{4}$, $\cot\theta = \frac{4}{3}$

(h) $\sin\theta = \frac{4}{5}$, $\cos\theta = -\frac{3}{5}$, $\text{cosec}\,\theta = \frac{5}{4}$, $\sec\theta = -\frac{5}{3}$, $\cot\theta = -\frac{3}{4}$

(i) $\sin\theta = -\frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = -\frac{4}{3}$, $\text{cosec}\,\theta = -\frac{5}{4}$, $\cot\theta = -\frac{3}{4}$

(j) $\sin\theta = -\frac{3}{5}$, $\cos\theta = -\frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\text{cosec}\,\theta = -\frac{5}{3}$, $\sec\theta = -\frac{5}{4}$

Exercise 12 C

1. **(a)** $\frac{\sqrt{3}+1}{2\sqrt{2}}$ **(b)** $\frac{\sqrt{3}-1}{2\sqrt{2}}$ **(c)** $2+\sqrt{3}$ **(d)** $\frac{2\sqrt{2}}{\sqrt{3}-1}$

(e) $\frac{2\sqrt{2}}{1-\sqrt{3}}$ **(f)** $-(2+\sqrt{3})$

2. **(a)** $-\frac{33}{65}; \frac{63}{65}$ **(b)** $\frac{65}{16}; \frac{65}{56}$ **(c)** $\frac{-16}{63}; \frac{-56}{33}$

3. **(a)** $-\frac{85}{13}$ **(b)** $\frac{36}{85}$ **(c)** $\frac{77}{36}$

4. **(a)** $\frac{140}{221}$ **(b)** $\frac{221}{21}$ **(c)** $-\frac{21}{220}$

5. **(a)** $-\frac{221}{220}$ **(b)** $\frac{221}{171}$ **(c)** $\frac{21}{220}$

7. $\frac{33}{47}$ **8.** $\frac{1}{4}$

9. **(a)** sin 53° **(b)** sin 18° **(c)** cos 98° **(d)** cos 64° **(e)** sin (60 + x)°

(f) sin (x – 45)° **(g)** sin (60 – x)° **(h)** tan (60 + x)° **(i)** cot (60 – x)° **(j)** tan (45 + x)°

10. $\sqrt{\frac{3}{2}}, \frac{1}{\sqrt{2}}$ **11.** $\sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}$ **12.** $A \approx 19.8°, B \approx 10.2°$ or $A = 79.8°, B = 70.2°$

Exercise 12 D

1. **(a)** $-\frac{24}{25}$ **(b)** $-\frac{527}{625}$ **(c)** $-\frac{24}{7}$ **(d)** $\frac{1}{\sqrt{10}}$ **(e)** 3

2. **(a)** $-\frac{169}{120}$ **(b)** $-\frac{28\,561}{239}$ **(c)** $-\frac{120}{119}$ **(d)** $\frac{1}{\sqrt{26}}$ **(e)** $-\frac{1}{5}$

3. **(a)** $\frac{169}{120}$ **(b)** $-\frac{28\,561}{239}$ **(c)** $-\frac{120}{119}$ **(d)** $-\frac{2}{\sqrt{13}}$ **(e)** $-\frac{3}{2}$

4. **(a)** $\sin\frac{A}{2}=\pm\frac{5}{\sqrt{26}}, \cos\frac{A}{2}=\pm\frac{1}{\sqrt{26}}$ **(b)** $\sin\frac{A}{2}=\frac{3}{\sqrt{34}}, \cos\frac{A}{2}=\frac{5}{\sqrt{34}}$

(c) $\sin\frac{A}{2}=\frac{1}{\sqrt{5}}, \cos\frac{A}{2}=-\frac{2}{\sqrt{5}}$ **(d)** $\sin\frac{A}{2}=\pm\frac{1}{\sqrt{26}}$ or $\pm\frac{5}{\sqrt{26}}, \cos\frac{A}{2}=\pm\frac{5}{\sqrt{26}}$ or $\pm\frac{1}{\sqrt{26}}$

5. **(a)** $2\sqrt{2}$ **(b)** $\frac{\sqrt{3}}{2}$ **(c)** 1 **(d)** $2\sqrt{3}$ **(e)** $-\frac{1}{\sqrt{3}}$

(f) $\frac{\sqrt{3}}{2}$ **6.** **(a)** $y = 2(1-x)^2$ **(b)** $y = 4-\frac{1}{2}(x+1)^2$ **(c)** $y = 2-\frac{4x}{4-x^2}$

(d) $\frac{1}{y-2}=\frac{2}{(x-1)^2}-1$ **(e)** $\frac{1}{y-1}=1-\frac{18}{(2-x)^2}$ **(f)** $(x+1)^2-9=6(x+1)(y-1)$

■ Exercise 12 E

1. **(a)** 120°, 360° **(b)** 37.4°, 277.4° **(c)** 0°, 112.6°, 360°

(d) 40.8°, 201.1° **(e)** 11.3°, 45°, 191.3°, 225° **(f)** 27.2°, 99.7°, 207.2°, 279.7°

2. **(a)** Maximum value $\sqrt{13}$ for $x = 56.3°$; minimum value $-\sqrt{13}$ for $x = 236.3°$.

(b) Maximum value 5 for $x = 323.1°$; minimum value -5 for $x = 143.1°$.

(c) Maximum value 2 for $x = 120°$; minimum value -2 for $x = 300°$.

(d) Maximum value $\sqrt{13}$ for $x = 56.3°$; minimum value $-\sqrt{13}$ for $x = 236.3°$.

(e) Maximum value $4+\sqrt{13}$ for $x = 326.3°$; minimum value $4-\sqrt{13}$ for $x = 146.3°$.

(f) Maximum value $\sqrt{5}-1$ for $x = 26.6°$; minimum value $-\sqrt{5}-1$ for $x = 206.6°$.

■ *Miscellaneous Exercise 12*

2. **(a)** 79.8°, 347.6° **(b)** **(i)** 45° **(ii)** $\frac{1}{7}$

3. **(a)** 41.6°, 244.7° **(b)** **(i)** $\frac{1}{\sqrt{1-c^2}}$ **(ii)** $\frac{c}{\sqrt{1-c^2}}$ **(iii)** $2c\sqrt{1-c^2}$

(iv) $\frac{c+\sqrt{1-c^2}}{c-\sqrt{1-c^2}}$ **5.** **(a)** 103.7°, 309.5° **(b)** $\frac{1}{7}; \frac{6}{7}$

6. **(a)** 72.3° **(b)** 90°, 48.6°, 131.4° **(c)** 63.4°, 146.3°

7. **(a)** **(i)** $-\frac{56}{65}$ **(ii)** $-\frac{7}{25}$ **(iii)** $\frac{120}{169}$ **(b)** 69.2°, 327.7°

9. **(a)** 43.7°, 283.7° **(b) (i)** $\frac{t}{\sqrt{1+t^2}}$ **(ii)** $\frac{2t}{1+t^2}$ **(iii)** $\frac{1-t}{1+t}$

11. **(a)** 30°, 90°, 210°, 270° **(b)** 48.2°, 90°, 270°, 311.8° **(c)** 41.8°, 138.2°, 210°, 330°

12. **(a)** 82.0°, 334.1° **(b) (i)** 7 **(ii)** $\frac{4}{5}, \frac{7\sqrt{2}}{10}$ **(iii)** 1

14. **(a)** 15cos (x – 36.87°) **(i)** 79.7° **(ii)** 15 **(b)** 60°, 90°, 120°

15. **(a) (i)** $\frac{7}{9}$ **(ii)** $\frac{17}{81}$ **(b) (ii)** $10\sqrt{5}\cos(\theta - 26.6^\circ)$; 70.9°, 25

■ Exercise 13 A

1. **(a)** $2e^{2x}$ **(b)** $-3e^{-3x}$ **(c)** $\frac{1}{2\sqrt{x+2}}e^{\sqrt{x+2}}$ **(d)** $\cos xe^{\sin x}$

(e) $-\frac{1}{x^2}e^{\frac{1}{x}}$ **(f)** $-\sin xe^{\cos x}$ **(g)** $15e^{5x}$ **(h)** $-2e^{-\frac{1}{2}x}$

(i) $2\sec^2 xe^{\tan x}$ **(j)** $4(2x+1)\,e^{(2x+1)^2}$ **(k)** $5(2x-1)\,e^{x(x-1)}$ **(l)** $\frac{1}{\sqrt{x}}\cos\sqrt{x}e^{\sin\sqrt{x}}$

(m) $\sin 2xe^{\sin^2 x}$ **(n)** $-6\cos^2 2x\sin 2xe^{\cos^3 2x}$

2. **(a)** $-\frac{1}{x}$ **(b)** $\frac{1}{x}$ **(c)** $\frac{1}{x}$ **(d)** $\frac{1}{2x}$

(e) $\frac{6}{2x+5}$ **(f)** –12tan 3x **(g)** $\frac{6e^{2x}}{e^{2x}+1}$ **(h)** 2cosec 2x

(i) $-\frac{7}{(2x+1)(x-3)}$ **(j)** $\frac{6-x}{(2x+1)(x^2+3)}$ **(k)** $-\frac{3}{x}$ **(l)** $\frac{20(x-1)}{(2x+1)(4x-3)}$

(m) $\frac{1-\tan x}{1+\tan x}$ **(n)** $\frac{2e^x}{e^{2x}-1}$ **(o)** 2sec 2x

3. **(a)** 6cos 2x **(b)** –12sin 3x **(c)** $\frac{5}{3}\sec^2\frac{1}{3}x$ **(d)** $2\cos\left(\frac{1}{4}x+\frac{\pi}{6}\right)$

(e) $-4\sin\left(\frac{1}{3}x-\frac{\pi}{4}\right)$ **(f)** $12\sec^2(2x-\pi)$ **(g)** $-12\sin^3 x\cos x$ **(h)** –20sin 10x

(i) $\frac{6\sec^2 4x}{\sqrt{\tan 4x}}$ **(j)** $-24\sin 3x\,(1+2\cos 3x)^3$

(k) $24\sec^2\left(2x-\frac{\pi}{6}\right)\tan^3\left(2x-\frac{\pi}{6}\right)$ **(l)** $\frac{15}{2}\cos\left(3x+\frac{\pi}{6}\right)\sin^{-\frac{1}{2}}\left(3x+\frac{\pi}{6}\right)$

4. $y = e^3(3x-2)$ **5.** 8

6. **(a)** $9.8\ \text{ms}^{-1}$ **(b)** 9.02 m **(c)** $-0.2\ \text{ms}^{-2}$

7. -3 **8.** $y = 5(x + 1)$ **9.** $(0.5, \ln 4.5)$ **10.** 2

Exercise 13 B

1. **(a)** $(2x + 1)(6x - 11)$ **(b)** $6(2x + 3)^2(3x - 1)^3(7x + 5)$ **(c)** $-3(3x - 1)(2 - 3x)^2(15x - 7)$

(d) $-2(3 - 2x)^4(4 - 3x)^3(38 - 27x)$ **(e)** $2(3x - 2)(12x^2 - 4x + 3)$ **(f)** $\dfrac{12x + 8}{\sqrt{4x + 1}}$

(g) $\dfrac{45x^2 + 6x - 12}{\sqrt{6x + 1}}$ **(h)** $\dfrac{63x + 16}{2\sqrt{5 + 7x}}$

2. **(a)** $(x\cos x + \sin x)\, e^{x\sin x}$ **(b)** $3(-4x\sin 4x + \cos 4x)\, e^{x\cos 4x}$ **(c)** $e^{x\cos x}(\cos x - x\sin x)$

(d) $2(\sin x + x\cos x)e^{2x\sin x}$ **(e)** $e^{x\tan x}(\tan x + x\sec^2 x)$ **(f)** $12e^{4x\sin 3x}(\sin 3x + 3x\cos 3x)$

(g) $-4e^{-3x}\left(\dfrac{1}{3}\sin\dfrac{1}{3}x + 3\cos\dfrac{1}{3}x\right)$ **(h)** $3e^{-4x}(3\sec^2 3x - 4\tan 3x)$ **(i)** $1 + \ln x$

(j) $xe^{3x}(3x + 2)$ **(k)** $\dfrac{1}{2}e^{\sqrt{x}}\left(\sqrt{x} + 2\right)$ **(l)** $e^{x\ln x}(1 + \ln x)$

(m) $3x\cos 3x + \sin 3x$ **(n)** $x(2\cos x - x\sin x)$ **(o)** $x\left(\dfrac{1}{2}x\sec^2\dfrac{1}{2}x + 2\tan\dfrac{1}{2}x\right)$

3. **(a)** $\dfrac{-5}{(3x - 4)^2}$ **(b)** $\dfrac{-23}{(5 + 2x)^2}$ **(c)** $\dfrac{-6x - 7}{2(3x + 1)^{\frac{1}{2}}(2x - 1)^2}$

(d) $\dfrac{(2x + 3)(6x - 29)}{(3x - 5)^2}$ **(e)** $\dfrac{49 + 12x}{(2 - 3x)^4}$ **(f)** $\dfrac{-(6x + 3)}{(2x - 1)^3}$

4. **(a)** $\dfrac{3\cos x + 1}{(3 + \cos x)^2}$ **(b)** $\dfrac{e^x(x - 1)}{x^2}$ **(c)** $\dfrac{\cos x - \sin x}{e^x}$

(d) $e^x\operatorname{cosec} x\,(1 - \cot x)$ **(e)** $\dfrac{\sin x - x\ln x\cos x}{x\sin^2 x}$ **(f)** $\dfrac{2}{\sin 2x - 1}$

(g) $\dfrac{-2e^x}{(1 + e^x)^2}$ **(h)** $\dfrac{e^x}{(e^x + 1)^2}$ **(i)** $-\operatorname{cosec}^2 x$

5. Minimum at $\left(\dfrac{1}{e}, -\dfrac{1}{e}\right)$ **6.** $2e^x\sin x$ **7.** Minimum at $x = 0$, maximum at $x = -\dfrac{2}{3}$

8. Maximum at $x = \dfrac{3\pi}{4}$ **9.** Maximum at $x = 1.11$ **10.** Maximum at $\left(e, \dfrac{1}{e}\right)$

Exercise 13 C

1. **(a)** $y = 3x - 11$ **(b)** $y = 2(x - 1)^2 + 3$ **(c)** $\frac{x^2}{4} + \frac{y^2}{9} = 1$ **(d)** $x^2 + y^2 = a^2$

(e) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **(f)** $y = \frac{1}{2x - 5}$ **(g)** $y = \frac{12x}{4 - x^2}$ **(h)** $y^2 = 4x^2(1 - x^2)$

2. **(a)** $4y = 3x + 24; 3y + 4x = 68$ **(b)** $2y + 3x + 12 = 0; 3y = 2x - 5$

(c) $2y = 3x + 11; 3y + 2x = 36$ **(d)** $2y = 5x - 22; 5y + 2x = 3$

(e) $11y = 5x + 41; 5y + 11x = 85$ **(f)** $py = x + ap^2; y + px = ap(p^2 + 2)$

(g) $p^2y + x = 2cp; py = p^3x + c - cp^4$ **(h)** $9py = 4x + 6p^3; 4y + 9px = 8p^2 + 27p^4$

(i) $y + 3px = 3 + 6p - p^3; 3py = x - 2 + 9p + p^2 + 6p^4$

(j) $3y + 8px = 12p^2 + 16p + 15; 8py = 3x - 6 + 31p - 32p^3$

3. $y + 3x = 66; \left(\frac{242}{9}, -\frac{44}{3}\right)$ **4.** $2y = 3x - 5; \left(-\frac{4}{3}, -\frac{9}{2}\right)$

5. $4y = 5x - 6; \frac{328}{45}$ square units

6. **(a)** $\frac{x}{y}$ **(b)** $\frac{2xy - 2x - y^2}{2xy + 2y - x^2}$ **(c)** $\frac{9x^2 - 4xy - 9y^2}{2x^2 + 18xy}$ **(d)** -2

(e) $\frac{-y^2}{x^2}$ **(f)** $\frac{y^3}{x^3}$ **(g)** $\frac{\sqrt{y}}{\sqrt{x}}$ **(h)** $\frac{-(\sin y + y\cos x)}{\sin x + x\cos y}$

(i) $\frac{ye^x - e^y}{xe^y - e^x}$ **(j)** $\frac{-y(x\ln y + y)}{x(y\ln x + x)}$

7. ± 3 **8.** $6, -3$ **9.** $-\frac{1}{3}, 1$

10. $(5.38, -3), (-4.89, -3)$ **11.** $8y = 9x - 26$ **12.** $\frac{7}{3}$, 0.7 units/s

Miscellaneous Exercise 13

1. **(a) (i)** $\frac{9x + 18}{2\sqrt{9 + 3x}}$ **(ii)** $-6\sin 2x \cos^2 2x$ **(b)** $-\frac{4}{5}$; No, $\frac{dy}{dx} < 0$ all $x \in \mathbb{R}$

2. **(a)** $3x\cos 3x + \sin 3x$ **(b)** $2\tan x \sec^2 x$

3. **(a)** 16 **(b)** $\frac{2x(x - 1)}{(2x - 1)^2}; \frac{1}{6}$

4. $\sec\theta = \frac{2x + y}{3}, \tan\theta = \frac{y - x}{3}; x^2 + 2xy = 3$ **5.** $y = 2x + 4; 8y + 4x = 81$

6. **(a) (i)** $\frac{3}{3x + 1}$ **(ii)** $\cos 2x - 2x\sin 2x$ **(b)** $7y = x - 11$ **(c)** $\frac{-2xy}{x^2 + 2y}$

7. **(a)** ± 2 **(b)** $y = 1 + t$ **(c)** $\frac{x}{y} = t - 1$; $y^3 = x^2 + 3xy + 2y^2$

8. **(a) (i)** $\frac{x}{\sqrt{1 + x^2}}$ **(ii)** $(1 + 3x)^4(18x + 1)$ **(c)** minimum at $x = 2.16$ radians

9. **(a) (i)** $12(4x + 1)^2$ **(ii)** $3x\sec^2 3x + \tan 3x$ **(b)** $\frac{3x^2 - y^3}{1 + 3x^2y}$

(c) $-\frac{(x + 2)}{2\sqrt{x}(x - 2)^2}$; $3y = 8x - 29$

10. **(a)** $\frac{3\pi}{4}$ **(b)** $-\frac{9}{(x - 4)^2}$ **(c)** $\ln x$; 0.386 **(d)** $-\frac{3}{4}$

11. **(a) (i)** $-\frac{1}{2(x-1)^{\frac{3}{2}}}$ **(ii)** $\frac{6}{3x - 1}$ **(c)** 0.232

12. **(a) (i)** $\frac{5}{5x + 2}$ **(ii)** $\frac{1}{(2x + 1)^2}$ **(c)** $(1, 27)$, $(4, 0)$

13. **(a)** 1 **(b)** $xe^{3x}(3x + 2)$; $0, -\frac{2}{3}$ **(d)** $\frac{1}{4}$

14. **(a) (i)** $x(1 + 2\ln x)$ **(ii)** $15x(5x^2 - 2)^{\frac{1}{2}}$ **(b)** $-\frac{1}{4}$ **(c)** $\frac{\pi}{3}, \frac{5\pi}{3}$

15. **(a) (i)** $p^2 - 4p + 3 = 0$ **(ii)** $(2, -1)$, $(10, 1)$ **(b) (i)** $\frac{t + 1}{t - 1}$

(ii) $2y = x - 9$ **(iii)** $(-3, 5)$ **(iv)** $x^2 + y^2 - 8x - 8y - 2xy = 0$

16. **(a) (i)** $\frac{-13}{\left(1 + 3x^2\right)}$ **(ii)** $2x\cos 2x - 2x^2\sin 2x$ **(b) (i)** $\frac{9}{4}$, 0.18 units/s

(ii) increasing **(c)** -0.29

17. **(a) (i)** $\frac{11}{(x + 5)^2}$ **(ii)** $\frac{-15x^2}{2\sqrt{25 - 5x^3}}$ **(b)** $\frac{4}{3}$ **(c)** e

18. **(a)** $\frac{85}{12}$ **19.** **(a) (i)** $\sec^2 x$ **(ii)** $-\sin 2x$

(b) $(2, 0)$ **(c) (i)** $\frac{4 - x}{y + 2}$ **20.** **(a) (i)** $\frac{8}{(2 - x)^2}$

(ii) $\tan x\ (2x\sec^2 x + \tan x)$ **(b)** $-\frac{5}{2}$; 0.5 units/s **(c)** $\frac{1}{2}$

Exercise 14 A

1. **(a)** $-\frac{1}{2}\cos 2x + c$ **(b)** $\frac{1}{4}\sin 4x + c$ **(c)** $\frac{1}{2}\tan 5x + c$

(d) $-2\cos \frac{1}{2}x + c$ **(e)** $\frac{3}{2}\sin \frac{2}{3}x + c$ **(f)** $\frac{4}{3}\tan \frac{3}{4}x + c$

(g) $-\frac{1}{3}\cos(3x + \pi) + c$ **(h)** $\frac{1}{5}\sin\left(5x - \frac{\pi}{2}\right) + c$ **(i)** $\frac{1}{3}\tan\left(3x + \frac{\pi}{6}\right) + c$

(j) $-\frac{4}{3}\cos\left(\frac{3}{4}x - \frac{\pi}{2}\right) + c$ **(k)** $\frac{3}{2}\sin\left(\frac{2}{3}x + \frac{\pi}{6}\right) + c$ **(l)** $\frac{3}{4}\tan\left(\frac{4}{3}x - \frac{\pi}{2}\right) + c$

2. **(a)** $\frac{1}{4}e^{4x} + c$ **(b)** $\frac{-1}{5}e^{-5x} + c$ **(c)** $3e^{\frac{1}{3}x} + c$

(d) $-5e^{-\frac{1}{5}x} + c$ **(e)** $\frac{3}{2}e^{\frac{2}{3}x} + c$ **(f)** $\frac{4}{3}e^{\frac{3}{4}x} + c$

(g) $\frac{1}{3}e^{3x+2} + c$ **(h)** $-\frac{1}{4}e^{-4x+5} + c$ **(i)** $\frac{3}{2}e^{\frac{2}{3}x+\frac{1}{4}} + c$

(j) $-\frac{7}{4}e^{\frac{3}{5}-\frac{4}{7}x} + c$ **(k)** $\frac{1}{2}e^{2x} + 2e^{x} + x + c$ **(l)** $e^{x} - 2e^{2x} - 3e^{-x} - 12x + c$

3. **(a)** $\frac{1}{4}\ln|x| + c$ **(b)** $\frac{2}{5}\ln|x| + c$ **(c)** $\frac{2}{5}\ln|5x - 3| + c$

(d) $-\frac{1}{2}\ln|7 - 8x| + c$ **(e)** $4x + \frac{2}{3}\ln|3x + 1| + c$ **(f)** $2\ln|x - 1| + 3\ln|x - 2| + c$

(g) $2\ln|2x + 1| - 2\ln|3x + 1| + c$

4. **(a)** 0.0497 **(b)** 0.157 **(c)** 0.693

(d) –0.924 **(e)** –0.933 **(f)** –0.294

(g) $\frac{\sqrt{3}}{4}$ **(h)** $\frac{-1}{3\sqrt{2}}$ **(i)** –0.38

5. **(a)** $\frac{1}{2}x + \frac{1}{8}\sin 4x + c$ **(b)** $\frac{1}{2}x - \frac{1}{12}\sin 6x + c$ **(c)** $\frac{1}{2}x + \frac{3}{2}\sin \frac{1}{3}x + c$

(d) $\frac{1}{2}x - \frac{1}{3}\sin \frac{3}{2}x + c$ **(e)** $\frac{1}{2}x + \frac{1}{16}\sin 8x + c$ **(f)** $\frac{1}{2}x - \frac{3}{4}\sin \frac{3}{2}x + c$

(g) $\frac{1}{2}x + \frac{3}{8}\sin \frac{4}{3}x + c$ **(h)** $\frac{1}{2}x - \frac{1}{24}\sin 12x + c$

6. **(a)** $\frac{\pi - 2}{16}$ **(b)** $\frac{\pi}{12}$ **(c)** $\frac{\pi}{4}$ **(d)** π

7. $1 - \frac{\pi}{4}$ **8.** $\frac{1}{3} - \frac{\pi}{4}$ **9.** $\frac{1}{\sqrt{3}}$

■ Exercise 14 B

1. 14.23
2. 1.28; underestimate
3. 1.174
4. **(a)** ln 2 **(b)** 0.70; 0.6%
5. 1.84; underestimate
6. 0.880; underestimate
7. 2.46 cubic units

■ Exercise 14 C

1. **(a)** $\ln|2x^3+4|+c$ **(b)** $\ln|x^2+x+8|+c$ **(c)** $\frac{1}{2}\ln|x^2+6x-9|+c$

   (d) $\ln(e^x+1)+c$ **(e)** $\frac{1}{3}\ln|\sin 3x|+c$ **(f)** $2\ln\left|\sec\frac{1}{2}x\right|+c$

   (g) $\frac{2}{3}\ln|2x^3+6x+1|+c$ **(h)** $4\ln\left|\sin\frac{1}{4}x\right|+c$ **(i)** $\frac{1}{5}\ln|\sec 5x|+c$

   (j) $\ln|\ln x|+c$ **(k)** $-\ln|\cos x+\sin x|+c$ **(l)** $\ln|\sin x-\cos x|+c$

2. **(a)** $\frac{11}{3}\ln|x+2|+\frac{4}{3}\ln|x-1|+c$ **(b)** $\frac{3}{2}\ln|2x-1|-\frac{4}{3}\ln|3x-1|+c$

   (c) $\ln|x-1|+2\ln|x-2|+3\ln|x-3|+c$ **(d)** $\frac{3}{2}\ln|2x-1|-\ln|2x+1|+\ln|x-3|+c$

   (e) $\ln|x+1|+2\ln|x-3|-\frac{5}{x-3}+c$ **(f)** $\frac{7}{4}\ln|2x-1|+\frac{3}{4(2x-1)}+c$

   (g) $-\frac{5}{4x}+\frac{3}{2}\ln|2x-1|+c$ **(h)** $\frac{3}{2}\ln|2x+1|+\frac{1}{2(2x-1)}+c$

   (i) $2\ln|x+1|-2\ln(x^2+1)+c$ **(j)** $\frac{3}{2}\ln|2x-1|+\frac{5}{2}\ln(x^2+8)+c$

3. **(a)** $\ln\frac{128}{9}$ **(b)** $\ln\frac{8\sqrt[3]{2}}{3\sqrt{3}}$

   (c) $\ln 1.2+1$ **(d)** $\frac{7}{9}\ln\frac{4}{3}+\frac{31}{18}\ln\frac{5}{3}+\frac{11}{45}$ **(e)** ln 10

4. $x+2-\frac{2}{x-1}+\frac{7}{x-2}$; 9.54

■ Exercise 14 D

1. **(a)** $-e^{-x}(x+1)+c$ **(b)** $\frac{2e^{3x}}{9}(3x-1)+c$

   (c) $-8e^{-\frac{1}{2}x}(x+2)+c$ **(d)** $-\frac{9}{2}e^{-\frac{2}{3}x}(2x+3)+c$

(e) $-\frac{1}{2}x\cos\left(2x-\frac{\pi}{4}\right)+\frac{1}{4}\sin\left(2x-\frac{\pi}{4}\right)+c$ **(f)** $\frac{2}{3}x\sin\left(3x+\frac{\pi}{4}\right)+\frac{2}{9}\cos\left(3x+\frac{\pi}{4}\right)+c$

(g) $x\tan x+\ln|\cos x|+c$ **(h)** $-4x\cos\frac{1}{4}x+16\sin\frac{1}{4}x+c$

(i) $\frac{9x}{2}\sin\frac{2x}{3}+\frac{27}{4}\cos\frac{2}{3}x+c$ **(j)** $x\ln|x|-x+c$

(k) $\frac{1}{4}x^2(2\ln|x|-1)+c$ **(l)** $e^x(x^2-2x+2)+c$

(m) $-\frac{1}{3}x^2\cos 3x+\frac{2}{9}x\sin 3x+\frac{2}{27}\cos 3x+c$ **(n)** $\frac{1}{3}x^2\sin 3x+\frac{2}{9}x\cos 3x-\frac{2}{27}\sin 3x+c$

(o) $\frac{1}{9}x^3(3\ln|x|-1)+c$ **(p)** $\frac{1}{2}e^x(\sin x+\cos x)+c$

(q) $\frac{1}{5}e^x(\sin 2x-2\cos 2x)+c$

(r) $-\frac{1}{2}x^2\cos\left(2x+\frac{\pi}{3}\right)+\frac{1}{2}x\sin\left(2x+\frac{\pi}{3}\right)+\frac{1}{4}\cos\left(2x+\frac{\pi}{3}\right)+c$

(s) $\frac{1}{3}x^2\sin\left(3x-\frac{\pi}{2}\right)+\frac{2x}{9}\cos\left(3x-\frac{\pi}{2}\right)-\frac{2}{27}\sin\left(3x-\frac{\pi}{2}\right)+c$

(t) $\frac{1}{4}e^{2x+3}(2x^2-2x+1)+c$

2. **(a)** 1 **(b)** $\frac{1}{4}$ **(c)** $\frac{\pi}{6}\sqrt{3}-1$ or -0.0931 **(d)** $\frac{\pi}{4}-\frac{1}{2}\ln 2$ or -0.439

(e) $\frac{1}{4}\left(1-\frac{3}{e^2}\right)$ or 0.148 **(f)** 0.258 **(g)** $\frac{1}{2}$ **(h)** $\frac{1}{5}\left(1+e^{-\frac{\pi}{2}}\right)$ or 0.242

(i) $2\ln 2-\frac{3}{4}$ or 0.636 **3.** **(a)** $-\frac{1}{4}(x^2+1)^{-2}+c$ **(b)** $-\frac{1}{8}(2x^4+1)^{-3}+c$ **(c)** $-\frac{(2+\cos x)^4}{4}+c$

(d) $-\frac{1}{4}(2e^{2x}-1)^{-3}+c$ **(e)** $\frac{2}{3}(3+\tan x)^{\frac{3}{2}}+c$ **(f)** $\frac{(2+\ln x)^4}{12}+c$ **(g)** $\frac{(\sin x-\cos x)^4}{4}+c$

(h) $3\sqrt{5-2\cos x}+c$

4. **(a)** $\frac{4}{125}(5x+3)^{\frac{3}{2}}(5x-2)$ **(b)** $2(x-8)\sqrt{x+4}+c$ **(c)** $\frac{(x-3)^6(21x^2+114x+169)}{168}+c$

(d) $\frac{1}{3}(x+6)\sqrt{2x-3}+c$ **(e)** $\tan^{-1}x+c$ **(f)** $\frac{1}{6}\tan^{-1}\frac{2x}{3}+c$

5. (a) $27\frac{67}{135}$ (b) $2\frac{1}{3}$ (c) $\tan^{-1}\frac{\pi}{4}$ (d) $\frac{\pi}{60}$

(e) $5\frac{1}{3}$

■ Exercise 14 E

1. 1.36 **2.** 6.70 **3.** $\frac{1}{8}$ **4.** $-\frac{1}{3}$

5. ∞ **6.** 2.5 **7.** ∞ **8.** $-\frac{1}{2}$

■ *Miscellaneous Exercise 14*

1. $\frac{2\sqrt{3}}{3}$ **2.** π

3. (a) (i) $2\ln 5$ or 0.811 (ii) $\frac{3}{4}$ (b) π

4. (a) 2 (b) $\frac{\pi^2}{2}$

5. 1.701 **6.** 11.9

7. (a) 3 (b) (i) 5.183 (ii) 2.82

8. 1; (i) 3.19 (ii) 9.50 **9.** 0.63; $\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\ldots+1$

10. $\frac{4}{1-2x}+\frac{6x}{1+3x^2}$ **11.** (a) $-\frac{1}{2}x\cos 2x+\frac{1}{4}\sin 2x+c$ (b) $x-2\ln|x+1|+c$

12. (a) $\frac{1}{16}e^{4x}(4x-1)+c$ (b) $\frac{1}{9}x^3(3\ln x-1)+c$ **13.** $\ln|x|-\ln|x+1|+c$ **15.** $\frac{2}{3}$

16. $\frac{1}{4}\left(1-\frac{3}{e^2}\right)$ **17.** (a) 1

18. (a) $2\ln|x+4|-\ln|x+1|+c$ (b) (i) 0.721 (ii) 0.719

19. (a) $\frac{1}{3}\ln|1+x|-\frac{1}{3}\ln|2-x|+c$ (b) $\frac{\pi}{2}$

20. (a) $-\frac{1}{4}\ln|x+1|+\frac{3}{4}\ln|x-3|+c$ (b) $\frac{16}{105}$

■ Exercise 15 A

1. 2.59 **2.** 1.92 **3.** 1.106 **4.** 3.15 **5.** 1.31

6. 1.89 **7.** –2, –1. 11 **8.** $k = 1$; $\beta = 1.24$; **9. (a)** not convergent **(b)** not convergent

■ *Miscellaneous Exercise 15*

1. 0.38 **2.** 3.678..., 3.986..., 3. 691..., 3.97...; one **3.** $\theta_2 = 0.249$, $\theta_3 = 0.249$

4. 1.37 **5.** $Xe^{-X} + e^{-X} = 0.1$, 3.89

■ Exercise 16 A

1. (a) $\mathbf{r} = t(\mathbf{i} - 2\mathbf{j} - 3\mathbf{k})$ **(b)** $\mathbf{r} = t(2\mathbf{i} - \mathbf{j} - 5\mathbf{k})$

(c) $\mathbf{r} = (1 + t)\mathbf{i} + (-2 + t)\mathbf{j} + (-3 - 2t)\mathbf{k}$ **(d)** $\mathbf{r} = (-1 + t)\mathbf{i} + (2 - 4t)\mathbf{j} + (-4 + 5t)\mathbf{k}$

(e) $\mathbf{r} = (3 - 8t)\mathbf{i} + (-5 + 6t)\mathbf{j} + (1 + t)\mathbf{k}$ **(f)** $\mathbf{r} = (1 - t)\mathbf{i} + (-2 + t)\mathbf{j} + (-3 + t)\mathbf{k}$

(g) $\mathbf{r} = (2 + 2t)\mathbf{i} + (-1 - t)\mathbf{j} - 5\mathbf{k}$ **(h)** $\mathbf{r} = (-1 + 3t)\mathbf{i} + 2\mathbf{j} + (-4 - 5t)\mathbf{k}$

(i) $\mathbf{r} = (1 - 3t)\mathbf{i} + (-2 + 3t)\mathbf{j} + (-3 + t)\mathbf{k}$ **(j)** $\mathbf{r} = (-1 + 3t)\mathbf{i} + (2 - 3t)\mathbf{j} - 4\mathbf{k}$

(k) $\mathbf{r} = (3 - 6t)\mathbf{i} + (-5 + 3t)\mathbf{j} + (1 + 5t)\mathbf{k}$ **(l)** $\mathbf{r} = (-5 + t)\mathbf{i} + (1 + t)\mathbf{j} + (2 - 2t)\mathbf{k}$

2. (a) (–5, 6, –8) **(b)** (5, –1, 4)

(c) (5, –4, 1) **(d)** (–13, 20, –1)

(e) lines do not intersect

3. O (0, 0, 0), A(4, 0, 0), B(4, 3, 0), C(0, 3, 0), $O_1(0, 0, 3)$
$A_1(4, 0, 3)$, $B_1(4, 3, 3)$, $C_1(0, 3, 3)$; $\mathbf{r} = 4t\mathbf{i} + 3t\mathbf{j} + 3t\mathbf{k}$; $\mathbf{r} = (4 - 4s)\mathbf{i} + 3s\mathbf{j} + 3s\mathbf{k}$; yes; $\left(2, \frac{3}{2}, \frac{3}{2}\right)$

■ Exercise 16 B

1. (a) skew **(b)** parallel **(c)** skew **(d)** skew

(e) skew **(f)** intersect at (5, 1, 2) **(g)** parallel **(h)** skew

2. (a) 165.2° **(b)** 63.5° **(c)** 124.7° **(d)** 80.4°

(e) 141.8° **(f)** 59.0° **(g)** 65.9°

■ Exercise 16 C

1. (a) $\mathbf{i} + \mathbf{j} - \mathbf{k}$; $x + y - z = 1$ **(b)** $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$; $2x + y - 2z = 1$ **(c)** $\mathbf{i} - 2\mathbf{j} - \mathbf{k}$; $x - 2y - z = 2$

(d) $2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$; $2x - 4y + z = 12$ **(e)** $3\mathbf{i} + \mathbf{j}$; $3x + y = 0$ **3.** $x - 2y + 5z = 0$

4. $2x - y + z = 0$; $2x - y + z = 7$ **5.** $2x + y + z = 1$ **6.** $5x + 3y + 4z = 6$

■ Exercise 16 D

1. (a) intersect at (5, 0, 1) **(b)** parallel **(c)** lies in plane **(d)** intersect at (3, 1, 2)

(e) intersect at (0, 1, 0) **(f)** parallel **(g)** lie in plane **(h)** intersect at $\left(0, \frac{-1}{2}, \frac{-3}{2}\right)$

2. **(a)** $\mathbf{r} = (2 - 2t)\mathbf{i} + (-1 + 3t)\mathbf{j} - t\mathbf{k}$ **(b)** $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + (1 - t)\mathbf{k}$
(c) $\mathbf{r} = (1 + 2t)\mathbf{i} + (1 - t)\mathbf{j} + (2 + 3t)\mathbf{k}$ **(d)** $\mathbf{r} = (1 + 3t)\mathbf{i} + (-2 - t)\mathbf{j} + (2 + 5t)\mathbf{k}$

Exercise 16 E

1. $\left(\frac{7}{3}, -\frac{1}{3}, \frac{5}{3}\right), \frac{1}{3}\sqrt{21}$ **2.** $\left(\frac{4}{3}, \frac{5}{3}, \frac{-5}{3}\right), \frac{2}{3}\sqrt{6}$ **3.** **(a)** $\frac{1}{3}\sqrt{21}$ **(b)** $\frac{2}{11}\sqrt{110}$
(c) $\frac{1}{3}\sqrt{3}$ **(d)** $\frac{1}{6}\sqrt{318}$ **(e)** $\frac{1}{2}\sqrt{6}$ **4.** **(a)** $\left(\frac{5}{3}, -\frac{2}{3}, \frac{2}{3}\right)$
(b) $(2, -2, -12)$ **(c)** $\left(-\frac{4}{5}, 1, -\frac{2}{5}\right)$ **(d)** $(3, -3, 4)$ **(e)** $\left(\frac{1}{3}, \frac{2}{3}, \frac{8}{3}\right)$
5. **(a)** $\sqrt{11}$ **(b)** $2\sqrt{6}$ **(c)** $3\sqrt{29}$ **(d)** $\frac{2}{3}\sqrt{6}$
(e) $\frac{2}{13}\sqrt{26}$

Exercise 16 F

1. **(a)** 70.5° **(b)** 9.6° **(c)** 28.1° **(d)** 60.8°
(e) 49.8° **(f)** 73.2° **(g)** 38.0° **(h)** 65.8°
2. **(a)** 90° **(b)** 70.9° **(c)** 70.9° **(d)** 123.1°
(e) 118.6°

Miscellaneous Exercise 16

1. **(i)** $-\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$ **(ii)** $\frac{2}{3}$ **(iii)** $\frac{2}{3}$
2. **(i)** $-\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ **(ii)** $3\mathbf{j} - \mathbf{k}$ **(iii)** 55° **(iv)** 35°
4. **(b)** $(2, -3, 7)$ **5.** **(a)** $2\sqrt{14}$ **(b)** $\frac{13}{5\sqrt{7}}$ **(c)** $\mathbf{r} = \begin{pmatrix} 1+t \\ 2+t \\ 3+t \end{pmatrix}$
6. **(a)** 45.2° **(b)** $\frac{68}{13}$ **(c)** $\begin{pmatrix} 1 \\ -7 \\ -\frac{3}{2} \end{pmatrix}; \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix}$ **(d)** $21x + 2y - 8z = 19$
7. B (2, 1, 3); C (–4, 2, 7); $\begin{pmatrix} 10 \\ 16 \\ 11 \end{pmatrix}$
8. **(a)** $\begin{pmatrix} -2 \\ \frac{1}{2} \\ 0 \end{pmatrix}$ **(b)** $2x + y + 2z = 19$ **(c)** 153.0° **(d)** $a = 4, b = -3, c = 0$

9. **(a)** $79.1°$ **(b)** $\begin{pmatrix} \frac{2}{7} \\ \frac{4}{7} \\ -\frac{6}{7} \end{pmatrix}$ **(c)** $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ **(d)** $\frac{4}{7}\sqrt{21}$

10. $\begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$; $1 + t = s$, $6 - 2t = 2s$, $1 - 3t = -s$; $(2, 4, -2)$; $\sqrt{30}$

11. **(b)** 1.59 m **(c)** $\mathbf{r} = 12t\mathbf{i} + 4t\mathbf{j} + (2 + 3t)\mathbf{k}$

12. $A(-2, 4, 3)$; $x - 2y - z + 13 = 0$; $\mathbf{r} = (2t - 2)\mathbf{i} + (4 - t)\mathbf{j} + (4t + 3)\mathbf{k}$

13. $\mathbf{r} = \begin{pmatrix} 2+t \\ 3 \end{pmatrix}$; $\mathbf{r} = \begin{pmatrix} 1+3s \\ 1+5s \end{pmatrix}$; $\begin{pmatrix} \frac{11}{5} \\ 3 \end{pmatrix}$; $\frac{1}{2}\sqrt{6}$ **14.** **(b)** $\frac{3}{2}\sqrt{14}$ **(c)** $\sqrt{42}$

Exercise 17 A

1. **(a)** $|y| = Ae^x$ **(b)** $|y| = k|x|$ **(c)** $y^2 = 2x + k$

(d) $y = -\ln|x + c|$ **(e)** $y = -\ln|e^x + c|$ **(f)** $y = \frac{ke^x}{1 - ke^x}$

(g) $|\sin y| = k|\sin x|$ **(h)** $(y - 2)^2 = (x + 2)^2 + k$ **(i)** $|y - 2| = k|x + 2|$

(j) $-\cos y = \sin\left(x + \frac{\pi}{3}\right) + c$ **(k)** $|\sec y| = ke^{\sin x}$ **(l)** $|\sec y| = k|x|$

(m) $y^2 = \ln(x^2 - 1) + k$ **(n)** $y = \frac{2}{15}(x + 1)^{\frac{3}{2}}(3x - 2) + c$ **(o)** $s = \frac{1}{1 - ke^t}$

(p) $e^y = e^x(x - 1) + c$

2. **(a)** $y = -e^{-x}$ **(b)** $y = \frac{1}{2}\ln|2x + 1| + 1$ **(c)** $y^2 = 2x^2 + 1$

(d) $(y + 1)^2 = (x - 1)^2 + 1$ **(e)** $y = 2x - 3$ **(f)** $y = x$

(g) $\cos y = -\sin\left(x + \frac{\pi}{4}\right)$ **(h)** $\sec y = \sqrt{2}e^{\sin x + 1}$ **(i)** $y = 1 - \cos x$

(j) $y = -\frac{1}{4}\cos\left(2x - \frac{\pi}{3}\right) + \left(\frac{1}{2} - \frac{\sqrt{3}}{4}\right)x + \frac{9}{8} + \left(\frac{\sqrt{3} - 2}{12}\right)\pi$

Exercise 17 B

1. **(a)** $\frac{dn}{dt} = kn$ **(b)** $\frac{ds}{dt} = ks$ **(c)** $\frac{dx}{dt} = -kx$ **(d)** $\frac{dx}{dt} = -kx^2$

(e) $\frac{dx}{dt} = -kx$ **(f)** $\frac{dp}{dt} = ap - bp^2$ **(g)** $\frac{dh}{dt} = -k\sqrt{h}$

2. $\frac{dh}{dt} = \frac{k}{h^2}$, $\frac{1}{3}h^3 = kt + c$; 29.16 minutes

3. $x = \frac{e^{kt}}{e^{kt} + 99}$

4. $v = 600\,000 \times 0.85^t$; V = R 564 000

5. $p = \frac{2a}{b(e^{-at} + 2)}$; $t \to \infty$; $p \to \frac{a}{b}$

6. $\frac{dn}{dt} = kn$, $n = Ae^{kt}$, 817; 280

7. $\theta = 20 + 60\left(\frac{5}{6}\right)^{\frac{1}{5}t}$

■ *Miscellaneous Exercise 17*

1. **(a)** $x = Ae^{kt}$ **(b)** $x = \frac{a}{e^{-akt} + 1}$

2. $y = 2xe^{1-x}$

3. $y = kx^2$; $y = \frac{1}{2}x^2$; $y = -x^2$

4. $y = \left(1 + \frac{1}{2}\ln|\sec x|\right)^2$

6. 2.04

7. 34.7 days; $\frac{dN}{dt} = \frac{1}{50}N - F$

8. **(a)** £ 745 **(b)** $\frac{400\sqrt{5}}{3}$

9. $\frac{dx}{dt} = -kx$

10. $\frac{dm}{dt} = -km$, $m \to A$

■ Exercise 18 A

1. **(a)** $-i$ **(b)** $3i$ **(c)** -2 **(d)** -5 **(e)** $-3i$ **(f)** $2i$

2. **(a)** $6 - 3i$ **(b)** $9 + \sqrt{3}i$ **(c)** $2 + 4i$ **(d)** 1 **(e)** $7\sqrt{3} - 2\sqrt{3}i$ **(f)** $3 - 3i$

3. **(a)** $-1 + 18i$ **(b)** $29 + 29i$ **(c)** $\frac{5}{2} + \sqrt{2}i$ **(d)** $28 + 11\sqrt{3}i$ **(e)** $8x^2 + 15y^2 + 2xyi$ **(f)** $10 + 6i$

4. **(a)** $\frac{3}{5} - \frac{4}{5}i$ **(b)** $\frac{12}{41} + \frac{15}{41}i$ **(c)** $-\frac{14}{25} + \frac{23}{25}i$ **(d)** $\frac{11}{14} + \frac{5}{14}\sqrt{3}i$ **(e)** $1 + 3i$ **(f)** $\frac{73 - 14i}{221}$ **(g)** $\frac{25 + 5i}{17}$ **(h)** $\frac{6}{5} - \frac{7}{5}i$

5. **(a)** $-21 - 20i$ **(b)** $-40 + 42i$ **(c)** $2 - 11i$ **(d)** $52 - 47i$ **(e)** $\frac{9}{250} - \frac{13}{250}i$ **(f)** $\cos\theta - i\sin\theta$ **(g)** $\frac{1}{2} + \frac{1}{2}i\tan\frac{\theta}{2}$ **(h)** $\cos 2\theta + i\sin 2\theta$ **(i)** $\frac{1}{2} - \frac{1}{2}i\tan\frac{\theta}{2}$

6. **(a)** $-2 \pm i$ **(b)** $\frac{-3 \pm i\sqrt{3}}{2}$ **(c)** $\frac{-1 \pm i\sqrt{11}}{6}$ **(d)** $\frac{-5 \pm i\sqrt{39}}{4}$

■ Exercise 18 B

2. **(a)** $2, \frac{\pi}{6}$ **(b)** $\sqrt{2}, \frac{3\pi}{4}$ **(c)** $2, -\frac{2\pi}{3}$ **(d)** 5, –0.644 rad

(e) 13, 0.395 rad **(f)** 10, 2.50 rad **(g)** 13, −1.97 rad **(h)** $\sqrt{13}$, −0.588 rad

(i) $3, \frac{\pi}{2}$ **(j)** $4, \frac{\pi}{2}$ **(k)** $5, -\frac{\pi}{2}$ **(l)** $\sqrt{3}, \pi$

(m) $3, \frac{\pi}{4}$ **(n)** $2, -\frac{\pi}{3}$ **(o)** $4, -\frac{5\pi}{6}$ **(p)** $2, \frac{2\pi}{3}$

3. **(a)** $2\left(\cos\frac{\pi}{3} + \text{i}\sin\frac{\pi}{3}\right)$ **(b)** $2\sqrt{2}\left(\cos\frac{3\pi}{4} + \text{i}\sin\frac{3\pi}{4}\right)$

(c) $6\left(\cos\frac{2\pi}{3} - \text{i}\sin\frac{2\pi}{3}\right)$ **(d)** 13(cos 1.18 − isin 1.18)

(e) 10(cos 0.927 + isin 0.927) **(f)** $\sqrt{41}$ (cos 2.25 + isin 2.25)

(g) $\sqrt{58}$ (cos 1.98 − isin 1.98) **(h)** $\sqrt{73}$ (cos 1.21 − isin 1.21)

(i) $2\left(\cos\frac{\pi}{2} + \text{i}\sin\frac{\pi}{2}\right)$ **(j)** 5(cos π + isin π)

(k) $6\left(\cos\frac{\pi}{2} - \text{i}\sin\frac{\pi}{2}\right)$ **(l)** $8\left(\cos\frac{\pi}{2} + \text{i}\sin\frac{\pi}{2}\right)$

4. **(a)** $\frac{3}{2} - 3\text{i}\frac{\sqrt{3}}{2}$ **(b)** $2\sqrt{2} - 2\text{i}$ **(c)** $2\sqrt{2} - 2\sqrt{2}\text{i}$ **(d)** $-3 + 3\text{i}$

■ Exercise 18 C

1. **(a)** $2e^{\frac{-\text{i}11\pi}{12}}, -1.93 - 0.518\text{i}\ ;\ 2e^{\frac{\text{i}5\pi}{12}}, 0.518 + 1.93\text{i}\ ;\ \frac{1}{2}e^{-\frac{\text{i}5\pi}{12}}, 0.129 - 0.483\text{i}$

(b) $6e^0, 6;\ \frac{3}{2}e^{\frac{\text{i}2\pi}{3}}, -\frac{3}{4} + \frac{3\sqrt{3}\text{i}}{4}\ ;\ \frac{2}{3}e^{\frac{-\text{i}2\pi}{3}}, -\frac{1}{3} - \frac{\sqrt{3}}{3}\text{i}$ **(c)** $e^{\text{i}\pi}, -1;\ 2e^{\frac{\text{i}\pi}{2}}, 2\text{i};\ \frac{1}{2}e^{-\frac{\text{i}\pi}{2}}, -\frac{1}{2}\text{i}$

(d) $8e^{\frac{\text{i}\pi}{6}}, 4\sqrt{3} + 4\text{i};\ 2e^{\frac{-\text{i}\pi}{2}}, -2\text{i};\ \frac{1}{2}e^{\frac{\text{i}\pi}{2}}, \frac{1}{2}\text{i}$ **(e)** $3e^{-\text{i}\pi}, -3;\ 3e^{\frac{\text{i}\pi}{2}}, 3\text{i};\ \frac{1}{3}e^{\frac{-\text{i}\pi}{2}}, -\frac{1}{3}\text{i}$

2. $e^{\frac{\text{i}3\pi}{4}}, -\text{i}$ **3.** $e^{\frac{\text{i}\pi}{3}};\ e^{\frac{\text{i}\pi}{6}};\ \text{i};\ \frac{\sqrt{3}}{2} + \frac{1}{2}\text{i};\ \frac{\sqrt{3}}{2} - \frac{1}{2}\text{i}$

4. $2e^{\frac{-\text{i}\pi}{3}}\ ;\ 2e^{\frac{\text{i}\pi}{6}}\ ;\ 2\sqrt{3} - 2\text{i};\ -\text{i};\ \text{i}$

5. $\sqrt{2}^{\frac{\text{i}\pi}{4}}\ ;\ 2e^{\frac{-\text{i}\pi}{3}}\ ;\ -2 + 2\text{i};\ -2 - 2\text{i}\sqrt{3}\ ;\ 10.9 + 2.93\text{i};\ -0.183 - 0.683\text{i};\ -0.366 + 1.37\text{i}$

6. **(a)** 2 − i, −2 + i **(b)** $\sqrt{6}$ (1 + 2i), $-\sqrt{6}$ (1 + 2i)

(c) 5 − 2i, −5 + 2i **(d)** 4 − 3i, −4 + 3i

(e) 2 − 3i, −2 + 3i **(f)** 5 + 3i, −5 − 3i

10. r^2; 1; 0, 2θ

Exercise 18 D

1. **(a)** $1 + 2i$; $Z^3 - 5Z^2 + 11Z - 15 = 0$
   (b) $3 + 2i, 2 - 5i$; $Z^4 - 10Z^3 + 66Z^2 - 226z + 377 = 0$
   (c) $-i, \sqrt{3} - i$; $Z^5 - \left(2\sqrt{3} + 1\right) Z^4 + \left(5 + 2\sqrt{3}\right) Z^3 - \left(5 + 2\sqrt{3}\right) Z^2 + \left(4 - 2\sqrt{3}\right) Z - 4 = 0$
   (d) $2 - 3i, 3 + 2i, 4 + i$; $(Z^2 - 4Z + 13)(Z^2 - 6Z + 13)(Z^2 - 8Z + 17) = 0$

Miscellaneous Exercise 18

1. **(a)** $a = \frac{3 + \sqrt{21}}{2}$, $b = \frac{-3 + \sqrt{21}}{2}$ **(b)** real part -1, imaginary part $-\sqrt{3}$

2. **(a)** $z^2 - 4z + 9 = 0$ **(b)** $-1 \pm i\sqrt{3}$

3. **(a)** $\frac{2}{5}$; $x = -0.9, y = 2.5$

5. **(a)** $\frac{5}{13} - \frac{14}{13}i$; $\sqrt{34}$, -0.54 rad

6. **(a) (i)** $\frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$ **(b) (i)** $2, \frac{\pi}{4}$ **(ii)** $\frac{1}{2}\sqrt{2}$; $2\sqrt{2}$

7. $W = \frac{13}{2} + \frac{5}{2}i, Z = -\frac{1}{2} - \frac{i}{2}$

8. **(a)** $-\frac{1}{2} + \frac{i}{2}$ **(b)** $\frac{1}{\sqrt{2}}, \frac{3\pi}{4}$

9. **(a)** Real, -1; Im, $-\sqrt{3}$ **(b)** $\frac{5}{4}, \frac{\pi}{12}$

10. **(a)** $\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} - \frac{1}{2}i$ **(b)** $\frac{1 - w}{w}$

11. **(a)** $1 - 3i$ **(b)** $Z^3 - 4Z^2 + 14Z - 20 = 0$
12. $3 - 9i$

13. $5, -0.927$ rad **(a)** $\frac{5}{3}$ **(b)** 0.120 rad

14. 6

15. $i\sqrt{3}, -i\sqrt{3}$; $\frac{2\pi}{3}, -\frac{2\pi}{3}$; $\frac{1}{\sqrt{2}} + i\frac{\sqrt{3}}{\sqrt{2}}$; $-\frac{1}{\sqrt{2}} - i\frac{\sqrt{3}}{\sqrt{2}}$