Sum & Product of the Roots of a Polynomial Equation

If the **quadratic equation** $ax^2+bx+c=0,\ a\neq 0$ has roots α and β, then the **sum of the roots,** $\alpha+\beta=-\frac{b}{a}$ and the **product of the roots**, $\alpha\beta=\frac{c}{a}$

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sum & product of the roots of any **polynnomial equation**

For the polynomial equation of degree n given by $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0=0,\ a_n\neq 0$

the **sum of the roots** is $-\frac{a_{n-1}}{a_n}$ and the **product of the roots** is $\frac{(-1)^n a_0}{a_n}$

Exercises – <u>No</u> calculator on all questions [**worked solutions** included]

1. The equation $x^2-5x-2=0$ has roots α and β.

(a) Write down the value $\alpha+\beta$ and the value of $\alpha\beta$.

(b) Find the value of $\alpha^2\beta+\alpha\beta^2$.

(c) Find a quadratic equation which has roots $\alpha^2\beta$ and $\alpha\beta^2$.

2. If α and β are the roots of the equation $2x^2+3x-7=0$ has roots, find the quadratic equation with integral coefficients whose roots are:

(a) $2\alpha,\ 2\beta$ (b) $\frac{2}{\alpha},\ \frac{2}{\beta}$

3. Consider the polynomial $f(x)=2x^3+3x^2-6x-18,\ x\in\mathbb{R}$.

(a) For the polynomial equation $f(x)=0$, state

(i) the sum of the roots;

(ii) the product of the roots.

A new polynomial equation is defined to be $g(x)=f(x-5)$

(b) Find the sum of the roots of the equation $g(x)=0$.

4. Consider the equation $2x^4-13x^3+27x^2-13x-15=0$. Given that one of the zeros of the equation is $x_1=2-\mathrm{i}$, find the other three zeros x_2, x_3 and x_4.

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Worked Solutions

1. (a) $\alpha+\beta=5,\ \alpha\beta=-2$

(b) $\alpha^2\beta+\alpha\beta^2=\alpha\beta(\alpha+\beta)=5(-2)=-10$

(c) sum of roots $=\alpha^2\beta+\alpha\beta^2=-10 \Rightarrow -\frac{b}{a}=-10 \Rightarrow \frac{b}{a}=10$

product of roots $=(\alpha^2\beta)(\alpha\beta^2)=(\alpha\beta)^3=(-2)^3=-8 \Rightarrow \frac{c}{a}=-8$

general quadratic equation: $ax^2+bx+c=0 \Rightarrow x^2+\frac{b}{a}x+\frac{c}{a}=0$

thus, quadratic equation with roots $\alpha^2\beta$ and $\alpha\beta^2$ is $x^2+10x-8=0$

2. (a) $\alpha+\beta=-\frac{3}{2},\ \alpha\beta=-\frac{7}{2} \Rightarrow 2\alpha+2\beta=2(\alpha+\beta)=2\left(-\frac{3}{2}\right)=-3 \Rightarrow -\frac{b}{a}=-3 \Rightarrow \frac{b}{a}=3$

$$\Rightarrow (2\alpha)(2\beta)=4\alpha\beta=4\left(-\frac{7}{2}\right)=-14 \Rightarrow \frac{c}{a}=-14$$

thus, quadratic equation with roots 2α and 2β is $x^2+3x-14=0$

(b) $\frac{2}{\alpha}+\frac{2}{\beta}=\frac{2\alpha+2\beta}{\alpha\beta}=\frac{2(\alpha+\beta)}{\alpha\beta}=\frac{2\left(-\frac{3}{2}\right)}{-\frac{7}{2}}=\frac{-3}{-\frac{7}{2}}=\frac{6}{7} \Rightarrow -\frac{b}{a}=\frac{6}{7} \Rightarrow \frac{b}{a}=-\frac{6}{7}$

$$\left(\frac{2}{\alpha}\right)\left(\frac{2}{\beta}\right)=\frac{4}{\alpha\beta}=\frac{4}{-\frac{7}{2}}=-\frac{8}{7} \Rightarrow \frac{c}{a}=-\frac{8}{7}$$

thus, quadratic equation with roots $\frac{2}{\alpha}$ and $\frac{2}{\beta}$ is $x^2-\frac{6}{7}x-\frac{8}{7}=0 \Rightarrow 7x^2-6x-8=0$

3. (a) sum of roots $=-\frac{a_{n-1}}{a_n}=-\frac{3}{2}$; product of roots $=(-1)^n\frac{a_0}{a_n}=-\left(-\frac{18}{2}\right)=9$

(b) If y satisfies $f(y)=0$ then $g(y+5)=0$. Thus, the roots of $g(x)$ are each 5 more that the roots of $f(x)$. There are 3 roots, so need to add $3\cdot 5=15$ to sum of the roots for $f(x)=0$. Therefore, the sum of the roots of $g(x)=0$ is $-\frac{3}{2}+15=\frac{27}{2}$.

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4. $2x^4 - 13x^3 + 27x^2 - 13x - 15 = 0$

$x_2 = 2 + i$ (imaginary zeros come in conjugate pairs)

sum of roots $= \frac{13}{2} = 2+i+2-i+x_3+x_4$

$\frac{13}{2} = 4 + x_3 + x_4 \Rightarrow x_3 + x_4 = \frac{5}{2}$ (1)

product of roots $= -\frac{15}{2} = (2+i)(2-i)x_3x_4$

$-\frac{15}{2} = (4 - i^2)x_3x_4$

$-\frac{15}{2} = 5x_3x_4 \Rightarrow x_3x_4 = -\frac{3}{2}$ (2)

(1) $x_3 = \frac{5}{2} - x_4$

substituting into (2), gives $\left(\frac{5}{2} - x_4\right)x_4 = -\frac{3}{2}$

$-x_4^2 + \frac{5}{2}x_4 + \frac{3}{2} = 0 \Rightarrow 2x_4^2 - 5x_4 - 3 = 0$

$(2x_4 + 1)(x_4 - 3) = 0$

$x_4 = -\frac{1}{2}$ OR $x_4 = 3$

if $x_4 = -\frac{1}{2}$, then $x_3 = 3$

if $x_4 = 3$, then $x_3 = -\frac{1}{2}$

therefore, the other three zeros are:

$x_2 = 2 + i$

$x_3 = -\frac{1}{2}$

$x_4 = 3$