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Pricing and simulations of catastrophe bonds

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ABSTRACT

The increasing number of natural catastrophes like floods, hurricanes, and earthquakes not only causes many victims, but also leads to severe production, infrastructure, and individual property losses. Classic insurance mechanisms may be inadequate for dealing with such losses because of the dependencies that exist, inter alia, between the sources of the losses, the huge values of claims, and problems with adverse selection and moral hazard. To cope with the dramatic consequences of extreme events, new financial and insurance instruments are required. One example of a catastrophe-linked security is the catastrophe bond (cat bond), also known as the Act-of-God bond. In this paper we price some catastrophe bonds. We apply models of the risk-free spot interest rate under the assumption that the occurrence of the catastrophe is independent of financial market behavior. We then use Monte Carlo simulations to analyze the numerical properties of the pricing formulas thus obtained. We make a twofold contribution to the literature of catastrophe bond pricing. First, we prove a general pricing formula, which can be applied to cat bonds with different payoff functions under the assumption of different models of the risk-free spot interest rate. Second, we price some new types of cat bonds with interest rate dynamics described by the CIR and the Hull–White model.

1. Introduction

The insurance industry faces overwhelming risks from natural catastrophes. Losses from Hurricane Andrew, for example, reached US$30 billion in 1992, while the losses from Hurricane Katrina in 2005 are estimated at $40–60 billion (see Muermann, 2008). To cope with the dramatic impacts of extreme events like these, an integrated policy that combines mitigation measures with diversified ex ante and ex post financial instruments is required.

Classic insurance mechanisms are unsuitable for addressing the extreme losses caused by natural catastrophes. For many insurers, even a single catastrophe can cause problems relating to reserve adequacy or even lead to bankruptcy. For example, after Hurricane Andrew more than 60 insurance companies fell into insolvency (see Muermann, 2008). For traditional insurance models (see, e.g. Borch, 1974) independent risk claims that are small in relation to the value of the whole insurance portfolio are the norm. This classic strategy of selecting an insurance contract portfolio is justified by the law of large numbers and the central limit theorem (see Borch, 1974; Ermoliev et al., 2001).

However, catastrophic risks mean that new approaches are needed to building insurance company portfolios. As the sources of losses caused by natural catastrophes are strongly dependent on time and location, the traditional portfolio-building strategy can only increase the probability of insurer bankruptcy (see Ermoliev et al., 2001).

As mentioned earlier, a single catastrophic event, like an earthquake or a hurricane, could result in damages of $50–100 billion. On the other hand, worldwide financial markets may fluctuate by tens of billions of dollars on a daily basis. This is why securitization of losses, that is, the "packaging" of risks into tradable assets in the form of so-called catastrophe derivatives, may be useful for dealing with the impacts of extreme natural catastrophes (see, e.g. Cummins et al., 2002; Freeman and Kunreuther, 1997; Froot, 2001; Harrington and Niehaus, 2003; Nowak and Romaniuk, 2010a,c,d; Nowak et al., 2012).

The most popular catastrophe-linked security is the catastrophe bond (cat bond) or Act-of-God bond (see, for example, Cox et al., 2000; D'Arcy and France, 1992; Ermolieva et al., 2007; George, 1999; Nowak and Romaniuk, 2009b; O'Brien, 1997; Romaniuk and Ermolieva, 2005; Vaugirard, 2003). In 1993 catastrophe derivatives were introduced by the Chicago Board of Trade (CBoT). These financial derivatives were based on underlying indexes reflecting the insured property losses due to natural catastrophes reported by insurance and reinsurance companies.

The payoff received by the cat bond holder is linked to an additional random variable, namely, that a natural catastrophe occurs in a specified region at a fixed time interval. An event such as this is called the triggering point (see George, 1999). The structure of payments for cat bonds also depends on some primary underlying asset (e.g. the LIBOR).

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The triggering point may be connected, for example, with the issuers actual losses (losses from, say, a flood), losses modeled by special software based on the real parameters of a catastrophe, the insurance industry index, the real parameters of a catastrophe (e.g., earthquake magnitude or wind speeds in case of windstorms), or the hybrid index related to modeled losses (see, e.g. Niedzielski, 1997; Walker, 1997). For some cat bonds (like the Atlas Re II issued for the SCOR Group), the triggering points may be the second or third event during a fixed period of time.

The financial literature with respect to catastrophe bonds and their pricing is not very rich. In a number of papers, authors mainly emphasize the advantages of investing in cat bonds. The pricing methods used, as described in Anderson et al. (2000) and Froot (2001), are very simplified ones. The probabilistic model, proposed in Froot (2001), also has limitations. In Bodoff and Gan (2009) an analysis of empirical data is conducted, and the issuing price of cat bonds is described as a linear function of the expected loss. In Kai et al. (2007) the behavioral finance method is applied, the authors emphasizing its potential for practical use in China. In Wang (2004) probability transforms are used to extend the Sharpe ratio concept to evaluate the risk-adjusted performance of cat bonds.

There are a few approaches using stochastic processes with discrete time. Catastrophe bond pricing in discrete time within the framework of representative agent equilibrium is the subject of Cox and Pedersen (2000). A similar pricing method is used in Reshetar (2008), where the payoff functions are linked to two types of underlying processes: catastrophic property losses and catastrophic mortality.

There are several advanced models with continuous time. In Baryshnikov et al. (1998) compound Poisson processes are used to incorporate various characteristics of the catastrophe process. The authors assume that the arbitrage and the "real-life" measures coincide, which may be seen as a disadvantage in their approach. The paper contains interesting numerical considerations, but no analytical pricing formula is obtained. The authors of Burnecki and Kukla (2003) continue and correct the method from Baryshnikov et al. (1998). The approach in Burnecki and Kukla (2003) is applied in Härdle and Lopez Cabrera (2010) for cat bonds connected with Mexican earthquakes. In Albrecher et al. (2004) the doubly stochastic compound Poisson process is used to model the claim index, and QMC algorithms are applied. Random variables describing claims are independent, and the claims' reporting lags are incorporated into the model. In Egamia and Young (2008) the indifference pricing method is used for the valuation of structured cat bonds. A very important and interesting approach is presented by Vaugirard in Vaugirard (2003). The author was one of the first to apply the arbitrage approach to cat bond pricing. In common with the other valuation methods mentioned here, the main problem with Vaugirard's solution is the market incompleteness caused by natural risk being taken into account. This problem has been considered in many papers devoted to financial derivative pricing (e.g., see Föllmer and Schweizer, 1991; Miyahara, 2005; Schweizer, 1992) and also in the fuzzy framework (see Nowak and Romaniuk, 2010b). Vaugirard overcame the problem of the market incompleteness and non-traded insurance-linked underlyings in the manner of Merton (see Merton, 1976). In Vaugirard (2003) a catastrophe bondholder was deemed to have a short position on an option based upon a risk index. In Lin et al. (2008) the authors applied an approach similar to that in Vaugirard (2003), using the Markov-modulated Poisson process to describe the arrival rate of natural catastrophes.

In Nowak and Romaniuk (2009b,a, 2010d), where a portfolio of financial and insurance instruments were the main subject of analysis, we assumed the simple form of the catastrophe bond payoff function and the Vasicek or the Hull–White model of the spot interest rate. In Nowak and Romaniuk (2010a,c) we considered catastrophe bonds with a piecewise linear payoff function, while the spot interest rate behavior was described by the Merton or the Vasicek stochastic equation. In Nowak et al. (2012) we priced the catastrophe bond with a stepwise payoff function, assuming Vasicek's interest rate dynamics.

The present paper can be treated as an extension of the Vaugirard approach. Simultaneously it summarizes our earlier results concerning catastrophe bond pricing. We assume (i) no possibility of arbitrage, (ii) the independence of catastrophe occurrence from the behavior of the financial market, and (iii) the replicability of interest rate changes by other financial instruments. This paper contributes a general catastrophe bond pricing formula, which can be applied to different types of payoff functions and different interest rate models. In particular, we price catastrophe bonds using three Gaussian models of the risk-free spot interest rate. We begin with the Vasicek model and subsequently consider two of its alternatives: the Hull–White model and the Cox–Ingersoll–Ross (CIR) model. We also consider two complex forms of catastrophe bond payoff functions: a stepwise one and a piecewise linear one. Using the martingale method of pricing we find the valuation formulas for cat bonds. Catastrophe bonds for the Vasicek interest rate model were priced by us before. In Nowak and Romaniuk (2010d) we also consider the Hull–White interest rate dynamics, but only cat bonds with a simple payoff function. An approach similar to ours for default-free and default-risky cat bonds with a simple form of payoff function and CIR interest rate dynamics was proposed in Lee and Yu (2002). New cases of catastrophic instruments, considered in this paper, are cat bonds with a complex payoff structure, priced under the assumption of the Hull–White and CIR dynamics of the spot interest rate. However, we show that all the formulas for cat bonds presented here can be treated as special cases of the valuation formula proved in our main theorem. We then use Monte Carlo simulations to show how our theoretical results might be applied to compute examples of cat bond pricing, and we analyze the numerical properties of the pricing formulas obtained.

In comparison with other continuous time-pricing models, our approach has several advantages. We give a very general cat bond pricing formula that can be applied to many kinds of catastrophe bonds, especially those with a more complicated payoff structure. Following the Vaugirard method of changing the probability measure, we justify the use of the conditional expected value in the pricing formula. Such a justification cannot be found in the catastrophe bond literature, despite this expectation being used by many authors. We do not incorporate the utility function into the pricing model (as in Cox and Pedersen, 2000; Egamia and Young, 2008; Reshetar, 2008), as choosing a well-suited utility function can, in practice, be an additional problem.

This paper is organized as follows. In Section 2 we introduce pricing formulas for some models of catastrophe bonds and for risk-free interest rates. The formulas obtained are used in simulations in Section 3 to conduct numerical analysis. Section 4 is devoted to some conclusions and final remarks.

2. Cat bond pricing

2.1. Definitions and notations

In this section we introduce a pricing formula for catastrophe bonds, which generalizes our earlier approaches from Nowak and Romaniuk (2009b, 2010a,c,d) and Nowak et al. (2012).

We begin with notations and basic definitions concerning catastrophe bonds and their pricing. We define stochastic processes describing the dynamics of the spot interest rate and aggregated catastrophe losses.

We apply stochastic models with continuous time and time horizon in the form $[0, T']$, where $T' > 0$. The date of maturity of catastrophe bonds T is not later than T', i.e. $T \leq T'$. We consider two probability measures: P and Q and we denote the expectations with respect to them by E^P and E^Q.

We define stochastic processes and random variables with respect to probability P.

Let $(W_t)_{t\in[0,T']}$ be a Brownian motion. It will be used in the stochastic model of the risk-free interest rate.

Let $(U_i)_{i=1}^{\infty}$ be a sequence of independent, identically distributed random variables. We treat U_i as the value of losses during the i-th catastrophic event. The independence of losses is a typical assumption in insurance. In this paper we focus our attention on this; however, it is possible to generalize our approach to the case of dependent losses. For real-life cases the selection of the U_i distribution is a crucial point. Appropriate goodness-of-fit tests and estimation of parameters of the selected distribution is necessary (see, e.g. Albrecher et al., 2004; Chernobai et al., 2005; Hewitt and Lefkowitz, 1979; Papush et al., 2001).

We also define the compound Poisson process by the formula

$$\tilde{N}_t = \sum_{i=1}^{N_t} U_i, \quad t \in [0, T'],$$

where N_t is a homogeneous Poisson process (HPP) with an intensity $\kappa > 0$. In Section 3 we will use the symbol κ_{HPP} in place of κ to emphasize that its value is constant in time. For each $t \in [0, T']$ the value of the process N_t is equal to the number of catastrophic events until the moment t. In particular,

$$N_0 = 0 \quad P\text{-a.s.},$$

$$E^P N_t = \kappa t \quad \text{for } t \in [0, T'] \text{ and}$$

$$P(N_t - N_s = k) = e^{-\kappa(t-s)} \frac{[\kappa(t-s)]^k}{k!}, \quad k = 0, 1, 2, \ldots.$$

Moments of jumps of process $(N_t)_{t\in[0,T']}$ are interpreted as moments of catastrophic events.

For each $t \in [0, T']$ the process $\tilde{N}_t$ describes the aggregated catastrophe losses until the moment t. $\left(\tilde{N}_t\right)_{t\in[0,T']}$ is a nondecreasing stochastic process, with right continuous trajectories of a stepwise form. The heights of its jumps are equal to the values of losses caused by catastrophic events.

All the above processes and random variables are defined on a filtered probability space $\left(\Omega, F, (F_t)_{t\in[0,T']}, P\right)$. The filtration $(F_t)_{t\in[0,T']}$ is given by the formulas

$$F_t = \sigma\left(F_t^0 \cup F_t^1\right), \qquad F_t^0 = \sigma(W_s, s \leq t),$$

$$F_t^1 = \sigma\left(\tilde{N}_s, s \leq t\right), \quad t \in [0, T'].$$

We assume that

$$F_0 = \sigma(\{A \in F : P(A) = 0\}),$$

the Brownian motion $(W_t)_{t\in[0,T']}$ is independent of $(N_t)_{t\in[0,T']}$ and $(U_i)_{i=1}^{\infty}$, and also the sequence $(U_i)_{i=1}^{\infty}$ is independent of $(N_t)_{t\in[0,T']}$. Then the probability space with filtration satisfies standard assumptions (i.e., σ-algebra F is P-complete, filtration $(F_t)_{t\in[0,T']}$ is right continuous and F_0 contains all the sets in F of P-probability zero). Moreover, we assume that random variables $U_i, i = 1, 2, \ldots,$ have the bounded second moment.

We denote by $(B_t)_{t\in[0,T']}$ the banking account, satisfying the following equation:

$$dB_t = r_t B_t dt, \quad B_0 = 1,$$

where $r = (r_t)_{t\in[0,T']}$ is the risk-free spot interest rate.

We assume that zero-coupon bonds are traded on the market. We denote by $B(t, T)$ the price at the time t of a zero-coupon bond with a maturity date $T \leq T'$ and with face value equal to 1.

We price catastrophe bonds under the assumption of no possibility of arbitrage on the market. We also make two additional assumptions. We first assume that investors are neutral toward the nature jump risk (Assumption 1). This assumption has been verified by the market (see, e.g. Anderson et al., 2000; Vaugirard, 2003). Secondly (Assumption 2), we assume routinely that changes in the interest rate r can be replicated by existing financial instruments (especially zero-coupon bonds).

We assume that the dynamics of $B(t, T), t \in [0, T]$, are described by the equation

$$dB(t, T) = B(t, T)(\mu_t^T dt + \sigma_t^T dW_t),$$

where $\mu^T = (\mu_t^T)_{t\in[0,T]}$ is the drift and $\sigma^T = (\sigma_t^T)_{t\in[0,T]}$ is the volatility of the bond price process. The process $\bar{\lambda} = (\bar{\lambda}_t)_{t\in[0,T]}$, where

$$\bar{\lambda}_t = \frac{\mu_t^T - r_t}{\sigma_t^T}, \quad t \in [0, T],$$

is called the market price of risk. In a no-arbitrage market all bonds, regardless of maturity time T, have the same market price of risk. For further details we refer the reader to Kwok (2008). We assume that $\bar{\lambda}$ satisfies the Novikov condition

$$E^P\left[\exp\left(\frac{1}{2}\int_0^T \bar{\lambda}_t^2 dt\right)\right] < \infty.$$

2.2. General result

In Vaugirard (2003) a simple form of the cat bond payoff function is considered. It assumes that if the triggering point does not occur, the bondholder is paid the face value Fv; and if the triggering point does occur, the bondholder receives the face value minus a coefficient in percentage w, i.e. $Fv(1 - w)$. The triggering point is the first passage in time through a level K of risk index I, driven by a Poisson jump-diffusion process. Therefore, bondholders are deemed to be in a short position on a one-touch, up-and-in digital option on I. Thanks to such an approach the appropriate martingale method, similar to that used for option pricing, can be applied to the case of cat bonds and a conditional expectation with respect to the equivalent martingale measure can be used in the pricing formula. Although we do not explicitly define the risk index, we follow the steps of the Vaugirard method of pricing. However, our theorem in this section is more general. In Vaugirard (2003) only the Vasicek spot interest rate model is applied and only the above-mentioned cat bond payoff function is considered. The main theorem, proved below, enables the user to price many types of catastrophe bond with different payoff functions, under different models of interest rate dynamics.

For a date of maturity and payoff T and a face value Fv we consider a catastrophe bond $IB_{\mathrm{cat}}(T, Fv)$ with a payoff function $\nu_{IB_{\mathrm{cat}}(T,Fv)}$ dependent on T, Fv and the compound Poisson process $\tilde{N}$. The following theorem shows the general valuation expression for this bond.

Theorem 1. *Let $IB(t)$ be the price of a $IB_{\mathrm{cat}}(T, Fv)$ at time t. Then*

$$IB(t) = E^Q\left(\exp\left(-\int_t^T r_u du\right) \nu_{IB_{\mathrm{cat}}(T,Fv)} | F_t\right). \tag{1}$$

In particular,

$$IB(0) = E^Q\left(\exp\left(-\int_0^T r_u du\right)\right) E^Q \nu_{IB_{\mathrm{cat}}(T,Fv)}. \tag{2}$$

Proof. Using the above assumptions, we obtain the unique probability measure Q using arguments similar to Vaugirard (2003). If $\bar{\lambda}$ is the market price of risk process, then the following Radon–Nikodym derivative defines the measure Q:

$$\frac{dQ}{dP} = \exp\left(-\int_0^T \bar{\lambda}_t dW_t - \frac{1}{2}\int_0^T \bar{\lambda}_t^2 dt\right) \quad P\text{-a.s.}$$

For Q the family $B(t,T)$, $t \le T \le T'$, is an arbitrage-free family of zero-coupon bond prices with respect to r. That is, for each $T \in [0, T']$, $B(T,T) = 1$ and the process of discounted zero-coupon bond price

$$B(t,T)/B_t, \quad t \in [0,T],$$

is a martingale with respect to Q. We then have the following pricing formula for a zero-coupon bond:

$$B(t,T) = E^Q\left(e^{-\int_t^T r_u du}|F_t\right), \quad t \in [0,T].$$

Using arguments similar to Vaugirard (2003), we obtain analogous equality for the catastrophe bond:

$$IB(t) = E^Q\left(\exp\left(-\int_t^T r_u du\right) \nu_{IB_{\text{cat}}(T,Fv)}|F_t\right).$$

From Assumption 1, $\exp\left(-\int_t^T r_u du\right)$ and $\nu_{IB_{\text{cat}}(T,Fv)}$ are independent under Q. Therefore, for $t = 0$, formula (1) can be written in the form (2). □

2.3. Some particular cases

In this subsection we consider some particular catastrophe bond cases with different payoff functions and using different interest rate models. Their valuation is a consequence of Theorem 1.

We start by defining a catastrophe bond with a stepwise payoff function. Let

$$0 < K_1 < \cdots < K_n, \quad n > 1$$

be a sequence of constants. Let $\tau_i : \Omega \to [0, T']$, $1 \le i \le n$ be a sequence of stopping times defined as follows

$$\tau_i(\omega) = \inf_{t \in [0,T']}\left\{\tilde{N}(t)(\omega) > K_i\right\} \wedge T', \quad 1 \le i \le n.$$

Let

$$w_1 < w_2 < \cdots < w_n$$

be a sequence of nonnegative constants, for which $\sum_{i=1}^n w_i \le 1$.

Definition 1. We denote by $IB_s(T, Fv)$ a catastrophe bond satisfying the following assumptions:

(a) If the catastrophe does not occur in the period $[0,T]$, i.e. $\tau_1 > T$, the bondholder is paid the face value Fv.
(b) If $\tau_n \le T$, the bondholder receives the face value minus the sum of write-down coefficients in the percentage $\sum_{i=1}^n w_i$.
(c) If $\tau_{k-1} \le T < \tau_k$, $1 < k \le n$, the bondholder receives the face value minus the sum of write-down coefficients in the percentage $\sum_{i=1}^{k-1} w_i$.
(d) Cash payments are made at the date of maturity T.

The following lemma, proved in Nowak et al. (2012), gives the form of the cumulative distribution functions of τ_i and can be applied to computations of the catastrophe bond price.

Lemma 1. *The value of the cumulative distribution function Φ_i, $1 \le i \le n$, at the moment T, has the form*

$$\Phi_i(T) = 1 - \sum_{j=0}^{\infty} \frac{(\kappa T)^j}{j!} e^{-\kappa T} \Phi_{\tilde{U}_j}(K_i),$$

where $\Phi_{\tilde{U}_j}$ is the cumulative distribution function of the sum $\tilde{U}_j = \sum_{p=0}^j U_p$. In the above formula we assume that $U_0 \equiv 0$.

As previously noted, the choice of the distribution of U_p is the key in real-life cases. Additionally, for many distributions $\Phi_{\tilde{U}_j}$ is analytically intractable because of the complex nature of the convolution that is obtained. Numerical simulations may therefore be useful in such cases.

The next type of catastrophe bond is a bond with a piecewise linear payoff function defined below.

Let $0 \le K_0 < K_1 < \cdots < K_n$ and $w_1 < w_2 < \cdots < w_n$ with $\sum_{j=1}^n w_j \le 1$.

Definition 2. We denote by $IB_p(T, Fv)$ a catastrophe bond with face value Fv, maturity and payoff date T and a payoff function in the form

$$\nu_{IB_p(T,Fv)} = Fv\left[1 - \sum_{j=0}^{n-1} \frac{\tilde{N}_T \wedge K_{j+1} - \tilde{N}_T \wedge K_j}{K_{j+1} - K_j} w_{j+1}\right].$$

$IB_p(T, Fv)$ has the following properties

1. The payoff function is a piecewise linear function of losses $\tilde{N}_T$.
2. If the catastrophe does not occur ($\tilde{N}_T < K_0$), the bondholder receives a payoff equal to the bond's face value Fv.
3. If $\tilde{N}_T \ge K_n$, the bondholder receives a payoff equal to $Fv(1 - \sum_{i=1}^n w_i)$.
4. If $K_j \le \tilde{N}_T \le K_{j+1}$ for $j = 0, 1, 2, \ldots, n-1$, the bondholder receives a payoff equal to
$$Fv\left[1 - \sum_{0 \le i < j} w_{i+1} - \frac{\tilde{N}_T \wedge K_{j+1} - \tilde{N}_T \wedge K_j}{K_{j+1} - K_j} w_{j+1}\right]$$
and when $\tilde{N}_T$ increases in the interval $[K_j, K_{j+1}]$ the payoff changes linearly from value $Fv\left[1 - \sum_{0 \le i < j} w_{i+1}\right]$ to value $Fv\left[1 - \sum_{0 \le i \le j} w_{i+1}\right]$.

Lemma 2 (proved in Nowak and Romaniuk, 2010c) is very useful in the pricing procedure of catastrophe bonds discussed in this subsection.

Lemma 2. *Let*

$$\varphi_m = P(\tilde{N}_T \le K_m), \quad m = 0, 1, 2, \ldots, n$$

and let

$$e_m = E\tilde{N}_T I_{K_m < \tilde{N}_T \le K_{m+1}}, \quad m = 0, 1, 2, \ldots, n-1.$$

The following equality holds

$$E^Q \nu_{IB_p(T,Fv)} = Fv\Bigg[1 - (1 - \varphi_n)\sum_{j=1}^n w_j - \sum_{m=0}^{n-1}\Bigg\{(\varphi_{m+1} - \varphi_m)\sum_{0 \le j < m} w_{j+1} + \frac{e_m - (\varphi_{m+1} - \varphi_m) K_m}{K_{m+1} - K_m} w_{m+1}\Bigg\}\Bigg].$$

We further denote by $IB_s(0)$ the price of catastrophe bond $IB_s(T, Fv)$ and by $IB_p(0)$ the price of catastrophe bond $IB_p(T, Fv)$ at the moment 0.

2.3.1. The Vasicek model

We consider the Vasicek model of the risk-free spot interest rate r. The interest rate process satisfies the following equation

$$dr_t = a(b - r_t)\,dt + \sigma dW_t \tag{3}$$

for positive constants a, b and σ. We also assume that $\bar{\lambda}_t = \lambda$, $t \in [0, T']$, is constant.

Theorem 2. *Let the risk-free spot interest rate r be described by the Vasicek model. Let*

$$\Phi = \sum_{i=1}^{n} w_i \Phi_i, \tag{4}$$

where Φ_i are the cumulative distribution functions of τ_i. Then

$$IB_s(0) = Fve^{-T \cdot R(T, r_0)} \{1 - \Phi(T)\} \tag{5}$$

and

$$IB_p(0) = e^{-T \cdot R(T, r_0)} E^Q \nu_{IB_p(T, Fv)}, \tag{6}$$

where

$$R(\theta, r) = R_\infty - \frac{1}{a\theta}\left\{(R_\infty - r)\left(1 - e^{-a\theta}\right) - \frac{\sigma^2}{4a^2}\left(1 - e^{-a\theta}\right)^2\right\}$$

and

$$R_\infty = b - \frac{\lambda\sigma}{a} - \frac{\sigma^2}{2a^2}.$$

Proof. We apply Theorem 1. Formulas (5) and (6) follow from (2), since for the zero-coupon bond, under the Vasicek interest rate dynamics, the following equality holds (see, e.g. Vaugirard, 2003)

$$E^Q\left(\exp\left(-\int_0^T r_u du\right)\right) = e^{-T \cdot R(T, r_0)}. \quad \square$$

Formula (5) was proved by us in Nowak et al. (2012) and in Nowak and Romaniuk (2009b) and Nowak and Romaniuk (2009a) for $n = 1$. In turn, equality (6) was introduced and proved in Nowak and Romaniuk (2010a,c).

2.3.2. The Hull–White model

We denote by $f^M(t, T)$ the market instantaneous forward rate at time t for maturity T. In particular, $f^M(0, T)$ is associated with the zero-bond curve by the formula

$$f^M(0, T) = -\frac{\partial \ln P^M(0, T)}{\partial T}.$$

We assume the Hull–White (extended Vasicek) model of the risk-free spot interest rate r. Its dynamics are described by the following stochastic equation

$$dr_t = (\vartheta(t) - ar_t)\,dt + \sigma dW_t \tag{7}$$

for constants $a, \sigma > 0$ and function ϑ which is exactly fitted into the term structure of current market interest rates, given by the formula

$$\vartheta(t) = \frac{\partial f^M(0, t)}{\partial t} + af^M(0, t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right) + \lambda\sigma.$$

In Nowak and Romaniuk (2010d) we considered catastrophe bonds for the Hull–White interest rate model, but their payoff function had an easy, nonlinear form.

We assume the constant form of process $\bar{\lambda}_t = \lambda$, $t \in [0, T']$. The next theorem, which is also a consequence of Theorem 1, gives the pricing formula for the stepwise and the piecewise linear payoff functions.

Theorem 3. *Let the risk-free spot interest rate r be described by the Hull–White (extended Vasicek) model. Let Φ be defined by (4). Then*

$$\begin{aligned} IB_s(0) &= P^M(0, T)\exp\left(B(0, T) f^M(0, 0)\right) \\ &\quad \times e^{-B(0,T)r_0} Fv\{1 - \Phi(T)\} \end{aligned} \tag{8}$$

and

$$\begin{aligned} IB_p(0) &= P^M(0, T)\exp\left(B(0, T) f^M(0, 0)\right) \\ &\quad \times e^{-B(0,T)r_0} E^Q \nu_{IB_p(T, Fv)}, \end{aligned} \tag{9}$$

where

$$B(t, T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right).$$

Proof. Similar to the previous case, (8) and (9) follow from (2), as

$$E^Q\left(\exp\left(-\int_t^T r_u du\right) | F_t\right) = A(t, T)\, e^{-B(t,T)r_t}, \tag{10}$$

where

$$\begin{aligned} A(t, T) = \frac{P^M(0, T)}{P^M(0, t)} \exp\Bigg(& B(t, T) f^M(0, t) \\ & - \frac{\sigma^2}{4a}\left(1 - e^{-2at}\right) B(t, T)^2\Bigg). \end{aligned}$$

For the equality (10) we refer the reader to Brigo and Mercurio (2006). $\square$

2.3.3. CIR model

We assume the CIR model of the risk-free spot interest rate. The dynamics of r are described by the following stochastic equation

$$dr(t) = a(b - r_t)\,dt + \sigma\sqrt{r_t} dW_t \tag{11}$$

for constants $a, b, \sigma > 0$ such that $2ab > \sigma^2$.

The Cox–Ingersoll–Ross model (the CIR model) was introduced by Cox, Ingersoll, and Ross as an extension of the Vasicek model. It is often used for modeling the risk-free interest rate on the market. As the volatility part in the CIR equation is proportional to $\sqrt{r}$, to obtain a convenient analytical pricing formula after the change of measure (see Brigo and Mercurio, 2006; Carmona and León, 2007), we assume the stochastic form of process $\bar{\lambda}$ given by

$$\bar{\lambda}_t = \frac{\lambda}{\sigma}\sqrt{r_t}, \quad t \in [0, T'],$$

where λ is a constant. Such an approach is frequently applied in the zero-coupon bond pricing. Under this choice of $\bar{\lambda}$, we receive the interest rate process which is tractable under measures P and Q. It can be helpful for estimation purposes (see Brigo and Mercurio, 2006).

The next theorem gives the pricing formula for catastrophe bonds, under assumption of the CIR form of the stochastic process describing the interest rate.

Theorem 4. *Let the risk-free spot interest rate r be described by the Cox–Ingersoll–Ross model. Let Φ be defined by (4). Then*

$$IB_s(0) = P(r, 0)\, Fv\{1 - \Phi(T)\}, \tag{12}$$

$$IB_p(0) = P(r, 0)\, E^Q \nu_{IB_p(T, Fv)}, \tag{13}$$

where

$$P(r,0) = A(T)\, e^{-r_0 B(T)},$$

$$A(T) = \left[\frac{\theta_1 e^{\theta_2 T}}{\theta_2 \left(e^{\theta_1 T} - 1\right) + \theta_1} \right]^{\theta_3},$$

$$B(T) = \frac{e^{\theta_1 T} - 1}{\theta_2 \left(e^{\theta_1 T} - 1\right) + \theta_1},$$

$$\theta_1 = \sqrt{(a+\lambda)^2 + 2\sigma^2}, \qquad \theta_2 = \frac{a+\lambda+\theta_1}{2} \quad \text{and}$$

$$\theta_3 = \frac{2ab}{\sigma^2}.$$

Proof. We apply Theorem 1. To prove Theorem 4 we use the zero-coupon bond pricing formula for the Cox–Ingersoll–Ross interest rate model (see, e.g. Wu, 2000), from which it follows that

$$E^Q \left(\exp\left(-\int_0^T r_u du \right) \right) = P(r,0). \quad \square$$

3. Numerical experiments

To price the catastrophe bonds and analyze the features of pricing formulas presented in Section 2, we conducted the appropriate Monte Carlo simulations. Our main aim is to illustrate the possibility of pricing cat bonds via numerical computation, despite the complex nature of equations introduced in Section 2.3.

We analyze the price of a catastrophe bond when interest rates are described by the Vasicek model (in the case of Model I and Model II), the Hull–White model (Model III) and the CIR model (Model IV). This section thus has a similar structure to that of Section 2.3. The parameters of the models of interest rates applied in our experiments were fitted in Episcopos (2000) and Hull and White (1993) for real-life data.

We assume that the losses generated are of a catastrophic nature, that is, they are rare, but each loss has a high value. Therefore the quantity of losses is modeled by HPP and the value of each loss is given by a random variable with a relatively high expected value and variance (i.e., high risk with high variability). We assume that the value of the loss is modeled by a lognormal distribution or Weibull distribution which are commonly used in simulations of risk events in insurance. The intensity of HPP and the parameters of the applied distributions were fitted in Chernobai et al. (2005) for data describing natural catastrophic events in the United States provided by the Property Claim Services (PCS) of the Insurance Service Office Inc. (ISO).

Apart from the lognormal and Weibull distributions, other types of complex probabilistic distributions (e.g., gamma, Burr, generalized Pareto—see Chernobai et al., 2005; Furman, 2008; Hogg and Klugman, 1983; Hewitt and Lefkowitz, 1979; Melnick and Tenenbein, 2000; Papush et al., 2001; Rioux and Klugman, 2004) or simulations based on historical records (see Ermolieva and Ermoliev, 2005; Pekárová et al., 2005) are possible.

We assume that the face value of the bond in each experiment is set to 1 (one monetary unit assumption) and the trading horizon of the catastrophe bond is set to 1 year. In the analysis we apply the stepwise (see Definition 1) payoff function (in the case of Model I) or the piecewise (see Definition 2) linear payoff function (in the case of Model II, Model III, and Model IV).

The overall characterization of the models is summarized in Table 1.

For each model we start with pricing the catastrophe bond for the given sets of parameters. We then alter one or two parameters and the other set as constants. In each experiment we generate $N = 1\,000\,000$ simulations.

Table 1
Characterization of models applied in numerical experiments.

Model	Model of interest rates	Distribution of losses	Payoff function
I	Vasicek	Lognormal	Stepwise
II	Vasicek	Weibull	Piecewise
III	Hull–White	Lognormal	Piecewise
IV	CIR	Lognormal	Piecewise

Table 2
Parameters of Model I.

	Parameters
Vasicek model	$a = 0.0235,\ b = 0.0055,\ \sigma = 0.0,\ r_0 = 0.0614$
Intensity of HPP	$\kappa_{HPP} = 31.7143$
Lognormal distribution	$\mu_{LN} = 17.3570, \sigma_{LN} = 1.7643$
Triggering points	$K_1 = Q_{HPP-LN}(0.75),\ K_2 = Q_{HPP-LN}(0.95)$
Values of losses coefficients	$w_1 = 0.2, w_2 = 0.3$

3.1. The Vasicek model

Model I: We analyze the price of the catastrophe bond when interest rates are described by the Vasicek model. The value of each loss is modeled by lognormal distribution and the payoff function is the simple stepwise function (see (5)). The parameters of this model can be found in Table 2.

For the Vasicek interest rate model, we use the parameters described in Episcopos (2000) based on a one-month interbank rate for the United States. To assess the intensity of losses (driven by HPP) and the lognormal distribution of each loss we apply the parameters described in Chernobai et al. (2005). The triggering points for the payment function are connected with exceeding the limits given by quantiles of the cumulated value of losses for the HPP process (number of losses) and lognormal distribution (value of each loss). This x-th quantile is denoted further by $Q_{HPP-LN}(x)$. The parameters of HPP and lognormal distribution for these quantiles are also described in Chernobai et al. (2005) (i.e., they are the same as those for the simulated process of catastrophic events). The values of losses coefficients for the bondholder are also set.

In this case the price of the catastrophe bond is equal to 0.879891.

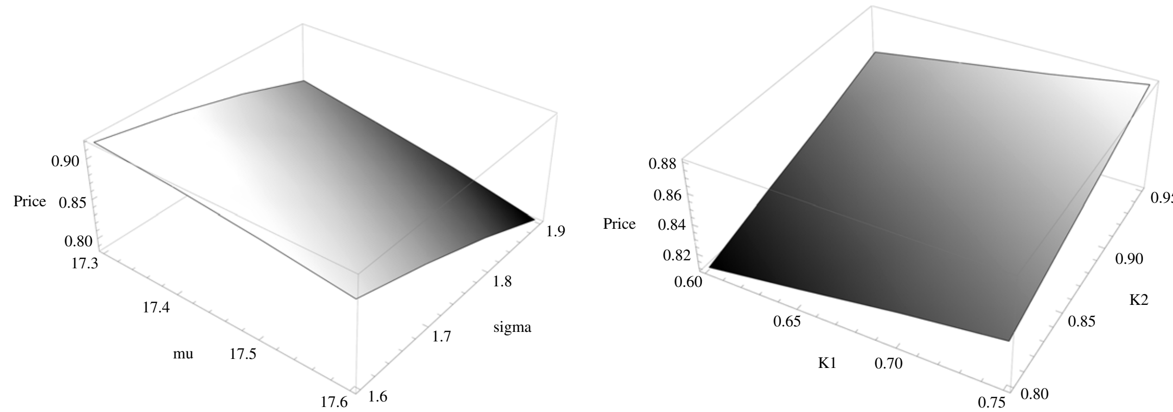
Model I, *Analysis* I: We analyze the price of the catastrophe bond as the function of μ_{LN} and σ_{LN} (see Table 3(a)), with the other parameters being set as in Table 2 (see Fig. 1(a)). The situation we analyze is thus one where the parameters of distribution describing the value of losses are slightly changed compared with those fitted into historical data. Taking into account only one variable and with the other set to constant, the appropriate cut of this graph is a hyperbolic-type function. However, the relative changes in price are quite subtle, as can be seen in Table 3(b) for a set of prices where $\mu_{LN} = 17.4$ and $\sigma_{LN} = 1.7$.

Model I, *Analysis* II: We analyze the price of the catastrophe bond as the function of triggering points, where K_1 and K_2 are given by appropriate quantiles Q_{HPP-LN} (see Table 3(a)). The other parameters are the same as in Table 2. As we use only two triggering points, a satisfactory graph can be created (see Fig. 1(b)). In this case the cuts of the graph (i.e., the functions of only one variable, with the other set to constant) seem to be almost linear.

Some exact prices for $K_1 = Q_{HPP-LN}(0.7)$ and $K_2 = Q_{HPP-LN}(0.8)$ are given in Table 3(c). Even for such a wide set of quantiles, the relative changes in price are quite subtle.

Model II: We analyze the price of the catastrophe bond when interest rates are described by the Vasicek model, the value of each loss is modeled by the Weibull distribution, and the payoff function is the piecewise linear one (see (6)). The parameters of this model can be found in Table 4.

For the Vasicek model of interest rates we use the parameters specified in Episcopos (2000) for a one-month interbank rate for



(a) Analysis I: price of the bond (Z axis) as the function of μ_{LN} (X axis) and σ_{LN} (Y axis).

(b) Analysis II: price of the bond (Z axis) as the function of K_1 (measured in $Q_{\mathrm{HPP-LN}}(\cdot)$, X axis) and K_2 (measured in $Q_{\mathrm{HPP-LN}}(\cdot)$, Y axis).

Fig. 1. Graphs for numerical analysis of Model I.

Table 3
Parameters and data for numerical analysis of Model I.

(a) Variables applied in numerical analysis				
	Variables			
Model I, Analysis I	$\mu_{\mathrm{LN}} \in [17.3, 17.6]$, $\sigma_{\mathrm{LN}} \in [1.6, 1.9]$			
Model I, Analysis II	$K_1 \in [0.6, 0.75]$, $K_2 \in [0.8, 0.95]$			
(b) Model I, Analysis I: price of the bond as the function of μ_{LN} and σ_{LN}				
	$\mu_{\mathrm{LN}} = 17.4$			
σ_{LN}	1.6	1.7	1.8	1.9
Price	0.911603	0.889852	0.86275	0.83049
Difference	0	−0.021751	−0.027102	−0.03226
	$\sigma_{\mathrm{LN}} = 1.7$			
μ_{LN}	17.3	17.4	17.5	17.6
Price	0.901956	0.889852	0.875786	0.858252
Difference	0	−0.012104	−0.014066	−0.017534
(c) Model I, Analysis II: price of the bond as the function of K_1 and K_2				
	$K_2 = Q_{\mathrm{HPP-LN}}(0.8)$			
$K_1(Q_{\mathrm{HPP-LN}})$	0.6	0.65	0.7	0.75
Price	0.809312	0.818697	0.82863	0.837118
Difference	0	0.009385	0.009933	0.008488
	$K_1 = Q_{\mathrm{HPP-LN}}(0.7)$			
$K_2(Q_{\mathrm{HPP-LN}})$	0.8	0.85	0.9	0.95
Price	0.82863	0.841553	0.856099	0.869869
Difference	0	0.012923	0.014546	0.01377

the United Kingdom. We use a market other than that in the previous case so as to have a wider scope of characterizations. For the intensity of losses (driven by HPP) and the Weibull distribution of each loss, we apply the parameters described in Chernobai et al. (2005).

We set $n = 2$, so we use three triggering points and two shifts in the payoff function (see Definition 2). We are thus able to prepare appropriate graphs to illustrate our simulations. There are also catastrophe bonds in existence which have payment functions that depend on the second or third catastrophe in any given year (like Atlas Re) or on there being a certain surplus above the value of the first catastrophe. This means our example is similar to such cat bonds.

The triggering points for the payment function are connected with exceeding the limits given by the quantiles of the cumulated value of losses for the HPP process (number of losses) and the Weibull distribution (value of each loss) with the parameters mentioned above. This x-th quantile is denoted further by $Q_{\mathrm{HPP-W}}(x)$. Values of losses coefficients for the bond's holder are also set.

Table 4
Parameters of Model II.

	Parameters
Vasicek model	$a = 0.0263$, $b = 0.0988593$, $\sigma = 0.01$, $r_0 = 0.1039$
Intensity of HPP	$\kappa_{\mathrm{HPP}} = 31.7143$
Weibull distribution	$\beta_W = 0.0187$, $\tau_W = 0.2656$
Triggering points	$K_0 = Q_{\mathrm{HPP-W}}(0.75)$, $K_1 = Q_{\mathrm{HPP-W}}(0.85)$, $K_2 = Q_{\mathrm{HPP-W}}(0.95)$
Values of loss coefficients	$w_1 = 0.2$, $w_2 = 0.3$

In such a case the price of the catastrophe bond is equal to 0.842215.

Model II, *Analysis* I: We analyze the price of the cat bond as a function of the parameters of the Weibull distribution β_W and τ_W (see Table 5(a)). Other parameters are as in Table 4. As we could see in Fig. 2(a), the associated cuts of the graph seem to be hyperbolic. Some prices for $\beta_W = 0.0185$ and $\tau_W = 0.29$ may be found in Table 5(b).

Model II, *Analysis* II: We analyze the price of the catastrophe bond as the function of the values of losses coefficients w_1 and w_2 (see Table 5(a)). Other parameters are the same as in Table 4. As we assume that $n = 2$, an appropriate graph may be constructed (see Fig. 2(b)). As we could see, the cuts of the graph seem to be almost linear. Some prices for $w_1 = 0.1$ and $w_2 = 0.1$ can be found in Table 5(c).

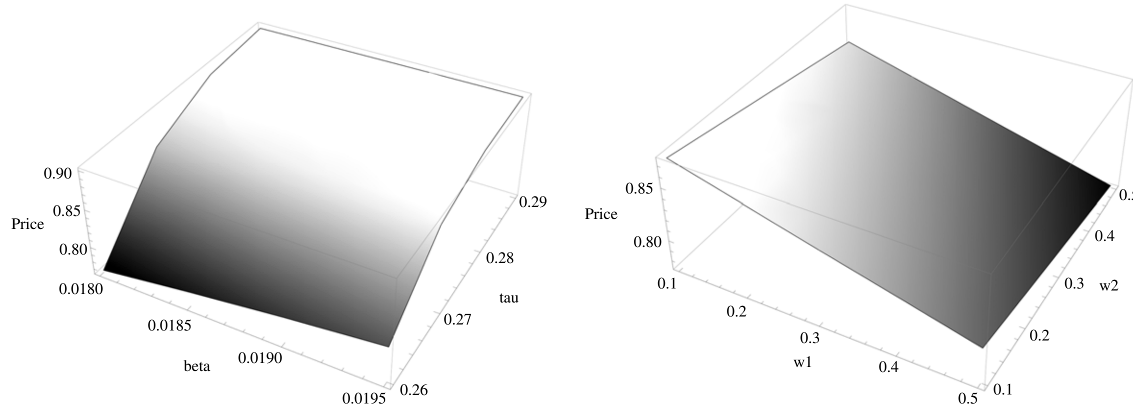
3.2. The Hull–White model

Model III: We analyze the price of the catastrophe bond when interest rates are described by the Hull–White model, the value of each loss is modeled by a lognormal distribution and the payoff function is the piecewise linear one (see (9)). The parameters of this model can be found in Table 6.

For the Hull–White model we use the parameters described in Hull and White (1993). For the intensity of losses (driven by HPP) and the lognormal distribution of each loss we apply the parameters described in Chernobai et al. (2005). We assume that $n = 2$ and, as previously, the triggering points are connected with exceeding the limits given by quantiles $Q_{\mathrm{HPP-LN}}(x)$.

In this case the price of the catastrophe bond is equal to 0.829764.

Model III, *Analysis* I: We set $K_0 = Q_{\mathrm{HPP-LN}}(0.45)$ and analyze the price of the catastrophe bond as the function of K_1 and K_2 (see Table 7(a)). Other parameters are the same as in Table 6. Only two triggering points are used as variables to enable us to prepare an appropriate graph (see Fig. 3(a)), but a similar analysis



(a) Analysis I: price of the bond (Z axis) as the function of β_W (X axis) and τ_W (Y axis).

(b) Analysis II: price of the bond (Z axis) as the function of w_1 (X axis) and w_2 (Y axis).

Fig. 2. Graphs for numerical analysis of Model II.

Table 5
Parameters and data for numerical analysis of Model II.

(a) Variables applied in numerical analysis					
	Variables				
Model II, Analysis I	$\beta_W \in [0.018, 0.0195]$, $\tau_W \in [0.26, 0.29]$				
Model II, Analysis II	$w_1 \in [0.1, 0.5]$, $w_2 \in [0.1, 0.5]$				
(b) Model II, Analysis I: price of the bond as the function of β_W and τ_W					
	$\tau_W = 0.29$				
β_W	0.018	0.0185	0.019	0.0195	
Price	0.900366	0.900654	0.900834	0.900977	
Difference	0	0.000288	0.00018	0.000143	
	$\beta_W = 0.0185$				
τ_W	0.26	0.27	0.28	0.29	
Price	0.782262	0.866747	0.894999	0.900654	
Difference	0	0.084485	0.028252	0.005655	
(c) Model II, Analysis II: price of the bond as the function of w_1 and w_2					
	$w_1 = 0.1$				
w_2	0.1	0.2	0.3	0.4	0.5
Price	0.875782	0.867806	0.859902	0.851882	0.843813
Difference	0	−0.007976	−0.007904	−0.00802	−0.008069
	$w_2 = 0.1$				
w_1	0.1	0.2	0.3	0.4	0.5
Price	0.875782	0.858121	0.840316	0.822898	0.805151
Difference	0	−0.017661	−0.017805	−0.017418	−0.017747

can be made for all K_i. Some exact prices and some differences in prices for $K_1 = Q_{\text{HPP-LN}}(0.5)$ and $K_2 = Q_{\text{HPP-LN}}(0.95)$ can be found in Table 7(b).

Model III, *Analysis* II: We analyze the price of the catastrophe bond as the function of μ_{LN} and σ_{LN} (see Table 7(a)) with other parameters set as in Table 6. The graph (see Fig. 3(b)) has the same hyperbolic-like cuts that can be seen in Fig. 1(a). Some exact prices and differences for $\mu_{\text{LN}} = 17.4$ and $\sigma_{\text{LN}} = 1.7$ can be found in Table 7(c).

3.3. The CIR model

Model IV: We analyze the price of the catastrophe bond when interest rates are described by the CIR model, the value of each loss is modeled by lognormal distribution and the payoff function is the piecewise linear one (see (13)). The parameters of this model can be found in Table 8.

For the CIR model of interest rates we use the parameters described in Episcopos (2000) for a one-month interbank rate for

Table 6
Parameters of Model III.

	Parameters
Hull–White model	$a = 0.1$, $\sigma = 0.014$, $r_0 = 0.095$, $r(1) = 0.1$
Intensity of HPP	$\kappa_{\text{HPP}} = 31.7143$
Lognormal distribution	$\mu_{\text{LN}} = 17.3570$, $\sigma_{\text{LN}} = 1.7643$
Triggering points	$K_0 = Q_{\text{HPP-LN}}(0.75)$, $K_1 = Q_{\text{HPP-LN}}(0.85)$, $K_2 = Q_{\text{HPP-LN}}(0.95)$
Values of losses coefficients	$w_1 = 0.3$, $w_2 = 0.3$

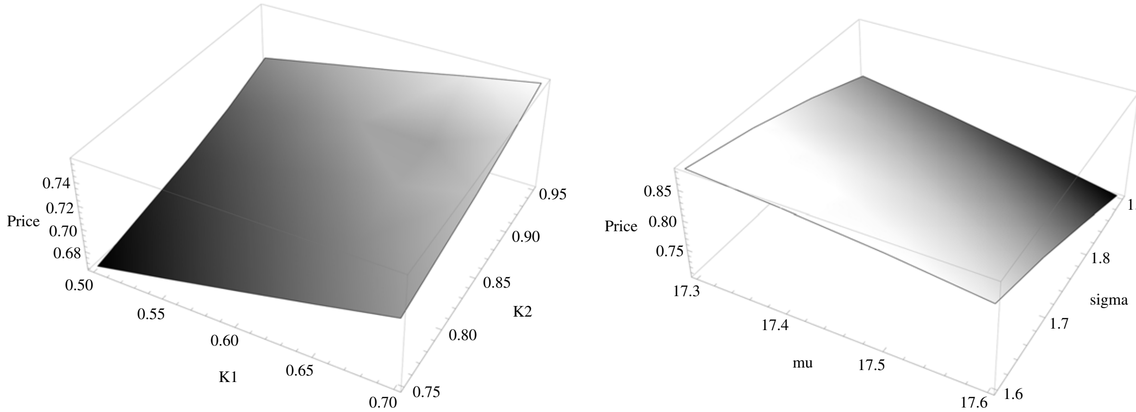
Table 7
Parameters and data for numerical analysis of Model III.

(a) Variables applied in numerical analysis					
	Variables				
Model III, Analysis I	$K_1 \in [0.5, 0.7]$, $K_2 \in [0.75, 0.95]$				
Model III, Analysis II	$\mu_{\text{LN}} \in [17.3, 17.6]$, $\sigma_{\text{LN}} \in [1.6, 1.9]$				
(b) Model III, Analysis I: price of the bond as the function of K_1 and K_2					
	$K_2 = Q_{\text{HPP-LN}}(0.95)$				
$K_1(Q_{\text{HPP-LN}})$	0.5	0.55	0.6	0.65	0.7
Price	0.713708	0.724512	0.73516	0.746877	0.757744
Difference	0	0.010804	0.010648	0.011717	0.010867
	$K_1 = Q_{\text{HPP-LN}}(0.5)$				
$K_2(Q_{\text{HPP-LN}})$	0.75	0.8	0.85	0.9	0.95
Price	0.665199	0.675093	0.684722	0.69736	0.713708
Difference	0	0.009894	0.009629	0.012638	0.016348
(c) Model III, Analysis II: price of the bond as the function of μ_{LN} and σ_{LN}					
	$\mu_{\text{LN}} = 17.4$				
σ_{LN}	1.6	1.7	1.8	1.9	
Price	0.871577	0.843729	0.807277	0.764421	
Difference	0	−0.027848	−0.036452	−0.042856	
	$\sigma_{\text{LN}} = 1.7$				
μ_{LN}	17.3	17.4	17.5	17.6	
Price	0.858967	0.843729	0.824886	0.801844	
Difference	0	−0.015238	−0.018843	−0.023042	

the United States. For the intensity of losses (driven by HPP) and the lognormal distribution of each loss we apply the parameters described in Chernobai et al. (2005). We assume that $n = 2$ and, as previously, the triggering points are connected with exceeding the limits given by quantiles $Q_{\text{HPP-LN}}(x)$.

In this case the price of the catastrophe bond is equal to 0.862881.

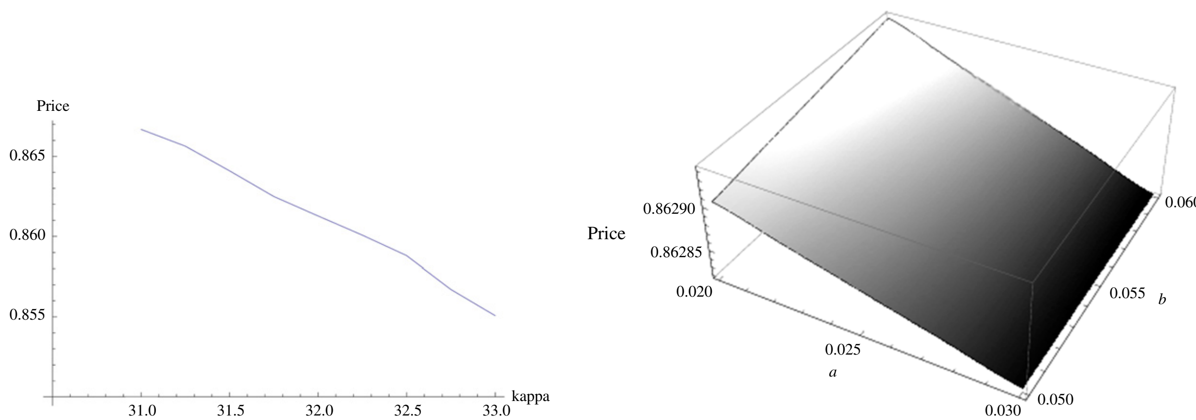
Model IV, *Analysis* I: We analyze the price of the catastrophe bond as the function of intensity of HPP κ_{HPP} (see Table 9(a)), with other parameters set as in Table 8. We can therefore observe



(a) Analysis I: price of the bond (Z axis) as the function of K_1 (measured in $Q_{\mathrm{HPP-LN}}(\cdot)$, X axis) and K_2 (measured in $Q_{\mathrm{HPP-LN}}(\cdot)$, Y axis).

(b) Analysis II: price of the bond (Z axis) as the function of μ_{LN} (X axis) and σ_{LN} (Y axis).

Fig. 3. Graphs for numerical analysis of Model III.



(a) Analysis I: price of the bond (Y axis) as the function of HPP intensity κ_{HPP} (X axis).

(b) Analysis II: price of the bond (Z axis) as the function of a (X axis) and b (Y axis) for the CIR model.

Fig. 4. Graphs for numerical analysis of Model IV.

Table 8
Parameters of Model IV.

	Parameters
CIR model	$a = 0.0241$, $b = 0.0539419$, $\sigma = 0.0141421$, $r_0 = 0.0614$
Intensity of HPP	$\kappa_{\mathrm{HPP}} = 31.7143$
Lognormal distribution	$\mu_{\mathrm{LN}} = 17.3570$, $\sigma_{\mathrm{LN}} = 1.7643$
Triggering points	$K_0 = Q_{\mathrm{HPP-LN}}(0.75)$, $K_1 = Q_{\mathrm{HPP-LN}}(0.85)$, $K_2 = Q_{\mathrm{HPP-LN}}(0.95)$
Values of losses coefficients	$w_1 = 0.2$, $w_2 = 0.3$

how the price behaves if catastrophes are more or less frequent compared with the parameter fitted into historical data (see Fig. 4(a)). Some exact prices and differences in prices can be found in Table 9(b).

Model IV, *Analysis* II: We analyze the price as the function of a and b (see Table 9(a)) for the CIR model with other parameters set as in Table 8. As we have seen, the appropriate cuts of the graph seem to be close to linear (see Fig. 4(b)). Some exact prices and some differences in prices for $a = 0.02$ and $b = 0.054$ can be found in Table 9(c).

4. Conclusions

The insurance industry faces overwhelming risks caused by natural catastrophes, but classical insurance mechanisms are

Table 9
Parameters and data for numerical analysis of Model IV.

(a) Variables applied in numerical analysis

	Variables
Model IV, Analysis I	$\kappa_{\mathrm{HPP}} \in [31, 33]$
Model IV, Analysis II	$a \in [0.02, 0.03]$, $b \in [0.05, 0.06]$

(b) Model IV, Analysis I: price of the bond as the function of HPP intensity κ_{HPP}

κ_{HPP}	31	31.25	31.5	31.75	32
Price	0.866683	0.865637	0.864093	0.862489	0.861274
Difference	0	−0.001046	−0.001544	−0.001604	−0.001215
κ_{HPP}	32.25	32.5	32.75	35	
Price	0.860076	0.858799	0.856708	0.855075	
Difference	−0.001198	−0.001277	−0.002091	−0.001633	

(c) Model IV, Analysis II: price of the bond as the function of a and b

$a = 0.02$

b	0.05	0.052	0.054	0.056	0.058	0.06
Price	0.862905	0.862888	0.862871	0.862855	0.862838	0.862822
Difference	0	−0.000017	−0.000017	−0.000016	−0.000017	−0.000016

$b = 0.054$

a	0.02	0.022	0.024	0.026	0.028	0.03
Price	0.862871	0.862876	0.86288	0.862885	0.862889	0.862893
Difference	0	0.000005	0.000004	0.000005	0.000004	0.000004

not appropriate for dealing with such extreme losses. Even a single catastrophe could cause problems with reserve adequacy for many insurers or even the bankruptcy of insurance firms. Traditional insurance models deal with independent risks that generate proportionately small claims in terms of the value of the whole insurance portfolio. New approaches are needed for insuring against catastrophic risks, as the sources of losses caused by natural catastrophes are strongly dependent on time and localization. Additionally, in the wake of such events enormous financial claims are made.

A single catastrophic event, for example, an earthquake or a hurricane, could result in damage worth perhaps tens of billions of dollars, reaching the same scale as the daily fluctuations on worldwide financial markets. Because of this, securitization of losses (in the form of the so-called catastrophe derivatives) may be helpful in dealing with the results of extreme natural catastrophes. An example of a catastrophe-linked security is the catastrophe bond.

In this paper we price some catastrophe bonds applying models of the risk-free spot interest rate under the assumption of no arbitrage, independence of the catastrophe occurrence from the behavior of the financial market, and the possibility of replication of interest rate changes by other existing financial instruments. We use the martingale method of pricing. We describe examples of catastrophe bonds with two types of payoff function (the stepwise payoff function and the piecewise linear payoff function) for three models of interest rates (the Vasicek model, the Hull–White model and the CIR model).

The pricing formulas obtained are then used in Monte Carlo simulations to analyze some numerical properties. We describe the behavior of the cat bond price, taking into account, inter alia, variables like the shape parameter and scale parameter for the value of loss distribution, the value of triggering points, and the value of percentage loss for the payment function.

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