De Moivre's Theorem

► from the HL Formula Booklet ◄

de Moivre's Theorem $\left[r\left(\cos\theta + \mathrm{i}\sin\theta\right)\right]^n = r^n\left(\cos n\theta + \mathrm{i}\sin n\theta\right) = r^n \mathrm{e}^{\mathrm{i}n\theta} = r^n \mathrm{cis}\, n\theta$

Exercises – **<u>No</u>** calculator on all questions [**worked solutions** included]

1. Given that $z = 1 - i$, determine z^7 in both polar form and Cartesian form.

2. Compute $\left(1 + \mathrm{i}\sqrt{3}\right)^9$.

3. Find all complex solutions to the equation $w^4 = 16$. Express solutions in Cartesian form.

4. Find the three cube roots of $8\mathrm{i}$; expressing them in modulus-argument form.

5. Find the square roots of $-6 + 6\sqrt{3}\mathrm{i}$ and write them in Cartesian form.

De Moivre's Theorem

WORKED SOLUTIONS

1. $z=1-i$

Write z in polar form: $|z|=\sqrt{1^2+(-1)^2}=\sqrt{2}$

$$\arg(z)=\tan^{-1}\left(\frac{-1}{1}\right)=-\frac{\pi}{4}$$

In polar form: $z=\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right]=\sqrt{2}\mathrm{cis}\left(-\frac{\pi}{4}\right)$

Applying de Moivre's Theorem: $z^7=\left(\sqrt{2}\right)^7\left[\cos\left(7\left(-\frac{\pi}{4}\right)\right)+\mathrm{i}\sin\left(7\left(-\frac{\pi}{4}\right)\right)\right]$

$$z^7=8\sqrt{2}\left[\cos\left(-\frac{7\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{7\pi}{4}\right)\right]$$

$$z^7=8\sqrt{2}\left[\cos\left(\frac{\pi}{4}\right)+\mathrm{i}\sin\left(\frac{\pi}{4}\right)\right]$$

$$z^7=8\sqrt{2}\left[\frac{\sqrt{2}}{2}+\mathrm{i}\frac{\sqrt{2}}{2}\right]$$

$$z^7=8+8\mathrm{i}$$

Thus, in polar form $z^7=8\sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{4}\right)$, and in Cartesian form $z=4+4i$

2. Write $1+\mathrm{i}\sqrt{3}$ in polar form: $|z|=\sqrt{1^2+\left(\sqrt{3}\right)^2}=2$

$$\arg(z)=\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)=\frac{\pi}{3}$$

In polar form: $1+i\sqrt{3}=2\left[\cos\left(\frac{\pi}{3}\right)+\mathrm{i}\sin\left(\frac{\pi}{3}\right)\right]=2\mathrm{cis}\left(\frac{\pi}{3}\right)$

Applying de Moivre's Theorem: $\left(1+\mathrm{i}\sqrt{3}\right)^9=\left[2\mathrm{cis}\left(\frac{\pi}{3}\right)\right]^9$

$$=2^9\mathrm{cis}\left(9\cdot\frac{\pi}{3}\right)$$

$$=512\mathrm{cis}(3\pi)$$

$$=512(-1)$$

$$=-512$$

De Moivre's Theorem

3. Write 16, equivalent to $16+0\text{i}$, in polar form: $|z|=\sqrt{16^2}=16$

$$\arg(z)=\tan^{-1}\left(\frac{0}{16}\right)=0$$

Thus, $16=16\left[\cos(0)+\text{i}\sin(0)\right]=16\,\text{cis}(0)$

Consider $w=r(\cos\theta+\text{i}\sin\theta)=r\,\text{cis}\,\theta$

Then $w^4=16 \quad\Rightarrow\quad [r\,\text{cis}\,\theta]^4=16\,\text{cis}(0+n\cdot 2\pi) \quad\Rightarrow\quad r^4\,\text{cis}(4\theta)=16\,\text{cis}(0+n\cdot 2\pi)$

$r^4=16$ and $4\theta=n\cdot 2\pi \quad\Rightarrow\quad \theta=n\cdot\frac{\pi}{2}$

$r=\pm 2$ when $n=0 \quad\Rightarrow\quad \theta=0$

when $n=1 \quad\Rightarrow\quad \theta=\frac{\pi}{2}$

when $n=2 \quad\Rightarrow\quad \theta=\pi$

when $n=3 \quad\Rightarrow\quad \theta=\frac{3\pi}{2}$ $\left[\text{or } \theta=-\frac{\pi}{2}\right]$

when $n=4 \quad\Rightarrow\quad \theta=2\pi$ $[\text{or } \theta=0]$, but this repeats first solution – stop

Solutions: $w=2(\cos 0+\text{i}\sin 0)=2(1+0)=2$

$w=-2(\cos 0+\text{i}\sin 0)=-2(1+\text{i}\cdot 0)=-2$

$w=2\left(\cos\frac{\pi}{2}+\text{i}\sin\frac{\pi}{2}\right)=2(0+\text{i}\cdot 1)=2\text{i}$

$w=-2\left(\cos\frac{\pi}{2}+\text{i}\sin\frac{\pi}{2}\right)=-2(0+\text{i}\cdot 1)=-2\text{i}$

$w=2(\cos\pi+\text{i}\sin\pi)=2(-1+\text{i}\cdot 0)=-2$

answers begin to repeat; all four solutions have been obtained

The four solutions are: $2,\ -2,\ 2\text{i}$ and -2i

De Moivre's Theorem

4. Write $8\mathrm{i}$, equivalent to $0+8\mathrm{i}$, in polar form: $|z|=\sqrt{8^2}=8$

$$\arg(z)=\tan^{-1}\left(\frac{8}{0}\right) \Rightarrow \arg(z)=\frac{\pi}{2} \text{ because } \tan\frac{\pi}{2} \text{ is undefined}$$

Thus, $8\mathrm{i}=8\left[\cos\frac{\pi}{2}+\mathrm{i}\sin\frac{\pi}{2}\right]=8\,\mathrm{cis}\frac{\pi}{2}$

Let $z=r(\cos\theta+\mathrm{i}\sin\theta)=r\,\mathrm{cis}\,\theta$

Solve $z^3=8\mathrm{i} \Rightarrow [r\,\mathrm{cis}\,\theta]^3=8\,\mathrm{cis}\left(\frac{\pi}{2}+n\cdot 2\pi\right) \Rightarrow r^3\,\mathrm{cis}(3\theta)=8\,\mathrm{cis}\left(\frac{\pi}{2}+n\cdot 2\pi\right)$

$r^3=8$ and $3\theta=\frac{\pi}{2}+n\cdot 2\pi \Rightarrow \theta=\frac{\pi}{6}+n\cdot\frac{2\pi}{3}$

$r=2$ when $n=0 \Rightarrow \theta=\frac{\pi}{6}$

when $n=1 \Rightarrow \theta=\frac{5\pi}{6}$

when $n=2 \Rightarrow \theta=\frac{3\pi}{2}$ $\left[\text{or } \theta=-\frac{\pi}{2}\right]$

when $n=3 \Rightarrow \theta=\frac{13\pi}{6}$; but $\frac{13\pi}{6}>2\pi$, so no further solutions

Solutions: $z=2\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)=2\mathrm{cis}\frac{\pi}{6}$

$z=2\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right)=2\mathrm{cis}\frac{5\pi}{6}$

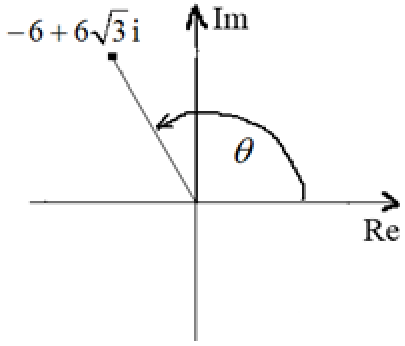
$z=2\left[\cos\left(-\frac{\pi}{2}\right)+\mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right]=2\mathrm{cis}\left(-\frac{\pi}{2}\right)$

The three cube roots of $8\mathrm{i}$ are $2\mathrm{cis}\frac{\pi}{6}$, $2\mathrm{cis}\frac{5\pi}{6}$, $2\mathrm{cis}\left(-\frac{\pi}{2}\right)$

De Moivre's Theorem

5. Write $-6+6\sqrt{3}\,\mathrm{i}$ in polar form: $|z|=\sqrt{(-6)^2+\left(6\sqrt{3}\right)^2}=\sqrt{36+108}=\sqrt{144}=12$

$\theta=\arg(z)=\tan^{-1}\left(\dfrac{6\sqrt{3}}{-6}\right)=\tan^{-1}\left(-\sqrt{3}\right)\ \Rightarrow\ \arg(z)=-\dfrac{\pi}{3}$



But $-6+6\sqrt{3}\,\mathrm{i}$ is in 2nd quadrant in complex plane,

So, $\arg(z)=\dfrac{2\pi}{3}$

Thus, $-6+6\sqrt{3}\,\mathrm{i}=12\left[\cos\left(\dfrac{2\pi}{3}\right)+\mathrm{i}\sin\left(\dfrac{2\pi}{3}\right)\right]=12\operatorname{cis}\left(\dfrac{2\pi}{3}\right)$

Let $z=r\left(\cos\theta+\mathrm{i}\sin\theta\right)=r\operatorname{cis}\theta$

Solve $z^2=-6+6\sqrt{3}\,\mathrm{i}\ \Rightarrow\ \left[r\operatorname{cis}\theta\right]^2=12\operatorname{cis}\left(\dfrac{2\pi}{3}+n\cdot2\pi\right)\ \Rightarrow\ r^2\operatorname{cis}(2\theta)=12\operatorname{cis}\left(\dfrac{2\pi}{3}+n\cdot2\pi\right)$

$r^2=12$ and $2\theta=\dfrac{2\pi}{3}+n\cdot2\pi\ \Rightarrow\ \theta=\dfrac{\pi}{3}+n\cdot\pi$

$r=\pm2\sqrt{3}$ when $n=0\ \Rightarrow\ \theta=\dfrac{\pi}{3}$

when $n=1\ \Rightarrow\ \theta=\dfrac{4\pi}{3}$

when $n=2\ \Rightarrow\ \theta=\dfrac{7\pi}{3}$ … but $\dfrac{7\pi}{3}>2\pi$, so not valid - stop

Solutions: $z=2\sqrt{3}\left(\cos\left(\dfrac{\pi}{3}\right)+\mathrm{i}\sin\left(\dfrac{\pi}{3}\right)\right)=2\sqrt{3}\left(\dfrac{1}{2}+\mathrm{i}\dfrac{\sqrt{3}}{2}\right)=\sqrt{3}+3\mathrm{i}$

$z=2\sqrt{3}\left(\cos\left(\dfrac{4\pi}{3}\right)+\mathrm{i}\sin\left(\dfrac{4\pi}{3}\right)\right)=2\sqrt{3}\left(-\dfrac{1}{2}+\mathrm{i}\left(-\dfrac{\sqrt{3}}{2}\right)\right)=-\sqrt{3}-3\mathrm{i}$

Thus, the two square roots of $-6+6\sqrt{3}\,\mathrm{i}$ are $\sqrt{3}+3\mathrm{i}$ and $-\sqrt{3}-3\mathrm{i}$