CAPE UNIT 1 VECTORS

1. Determine the unit vector in the direction of $b = i + 2j - 3k$.

$$\left[\frac{1}{\sqrt{14}}(i + 2j - 3k)\right]$$

2. If $p = 4i + 3j - 2k$ and $q = 2i + j + 11k$ find
   (a) $p.q$
   (b) $q.p$
   (c) $p.p$
   (d) $q.q$

3. If $a = \begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ -14 \\ 1 \end{pmatrix}$ determine:
   (a) $a.b$
   (b) $b.a$
   (c) $a.a$
   (d) $b.b$

Ans: (a) -11 (b) -11 (c) 29 (d) 126

(a) 40 (b) 40 (c) 26 (d) 199

4. Find the angle between the vectors $i - 2j + 3k$ and $2i + j + k$.

[70.9°]

5. Calculate to the nearest degree, the angle between the vectors $4i + k$ and $4i + 3j + 2k$.

[36°]

6. The points A and B have position vectors a and b relative to an origin O, where $a = 4i + 3j - 2k$ and $b = -7i + 5j + 4k$
   (i) Find the length of AB.
   (ii) Use a scalar product to find angle OAB.

[(i) $\sqrt{161}$ (ii) 43°]

7. Points A, B and C have position vectors $-5i - 10j + 12k$, $i + 2j - 3k$ and $3i + 6j + pk$ respectively, where p is a constant.
   (i) Given that angle $ABC = 90°$, find the value of p.
   (ii) Given instead that ABC is a straight line, find the value of p.

[(i) 1 (ii) -8]

8. Determine the equation of the line which passes through the point
   (i) $A(-1, 1, -3)$ and is parallel to the vector $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$.
   (ii) $B(4, 3, -2)$ and is parallel to the vector $\begin{pmatrix} -9 \\ -2 \\ 1 \end{pmatrix}$.

(iii) $C(-3, 4, 3)$ and is parallel to the vector $\begin{pmatrix} 8 \\ -3 \\ -7 \end{pmatrix}$.

(iv) $D(5, 2, -3)$ and is parallel to the vector $\begin{pmatrix} -4 \\ -5 \\ 7 \end{pmatrix}$.

(v) $E(3, 5, -5)$ and is parallel to the vector $\begin{pmatrix} -8 \\ -8 \\ 3 \end{pmatrix}$.

Ans: (i) $r = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ (ii) $r = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -9 \\ -2 \\ 1 \end{pmatrix}$ (iii) $r = \begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ -3 \\ -7 \end{pmatrix}$

(iii) $r = \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ -5 \\ 7 \end{pmatrix}$ (v) $r = \begin{pmatrix} 3 \\ 5 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -8 \\ -8 \\ 3 \end{pmatrix}$

9. Find the equation of the line which passes through the points A and B with position vectors

(i) $\begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -3 \\ -4 \end{pmatrix}$ respectively.

(ii) $\begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ -4 \end{pmatrix}$ respectively.

(iii) $\begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix}$ respectively.

(iv) $\begin{pmatrix} 3 \\ 0 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$ respectively.

(v) $\begin{pmatrix} 5 \\ 5 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 5 \\ 0 \end{pmatrix}$ respectively.

Ans: (i) $r = \begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ -3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 1 \\ -7 \end{pmatrix}$ (ii) $r = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ 0 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ -2 \\ -6 \end{pmatrix}$

(iii) $r = \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}$ or $\begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}$ (iv) $r = \begin{pmatrix} 3 \\ 0 \\ -5 \end{pmatrix}$ or $\begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$ (v) $r = \begin{pmatrix} 5 \\ 5 \\ -5 \end{pmatrix}$ or $\begin{pmatrix} -4 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -9 \\ 0 \\ 5 \end{pmatrix}$

10. Determine the vector equation of the line which passes through $(3, -2, 3)$ and is parallel to the vector $\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$ in

(i) Vector form

(ii) Parametric form and

(iii) Cartesian form.

Ans: (i) $r = 3i - 2j + 3k + \lambda(i + 3j + 4k)$ (ii) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 + \lambda \\ -2 + 3\lambda \\ 3 + 4\lambda \end{pmatrix}$ (iii) $12x - 36 = 4y + 8 = 3z - 9$

11. The line L_1 passes through the points $(2, -3, 1)$ and $(-1, -2, -4)$. The line L_2 passes through the point $(3, 2, -9)$ and is parallel to the vector $4i - 4j + 5k$.

(i) Find an equation for L_1 in the form $r = a + tb$.

(ii) Prove that L_1 and L_2 are skewed.

$[r = (2i + 3j + k \text{ or } -i - 2j - 4k) + t(3i - j + 5k)]$

12. Two lines have vector equations

$$r = \begin{pmatrix} 4 \\ 2 \\ -6 \end{pmatrix} + t\begin{pmatrix} -8 \\ 1 \\ 2 \end{pmatrix} \text{and} r = \begin{pmatrix} -2 \\ a \\ -2 \end{pmatrix} + s\begin{pmatrix} -9 \\ 2 \\ -5 \end{pmatrix}$$

where a is a constant.

(i) Calculate the acute angle between the lines. [Hint: This is the angle between the two direction vectors]

(ii) Given that these two lines intersect, find a and the point of intersection.

$$\left[\text{(i)}15° \quad \text{(ii)}a = 1, \begin{pmatrix} -20 \\ 5 \\ -12 \end{pmatrix}\right]$$

13. Find the equation of the plane, in vector form and Cartesian form, through the point $(-4, 3, 1)$ that is perpendicular to the vector $a = -4i + 7j - 2k$.

Ans: $-4x + 7y - 2z = 35$

14. Find an equation of the plane through the point $(6, 3, 2)$ and perpendicular to the vector $(-2, 1, 5)$. Check if $(2, -1, 0)$ and $(1, -2, 1)$ are in that plane.

Ans: $r.\begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} = 1$, Does not lie on the line, Lies on the line