Even and odd functions

Sometimes we can simplify a definite integral if we recognize that the function we're integrating is an even function or an odd function. If the function is neither even nor odd, then we proceed with integration like normal.

To find out whether the function is even or odd, we'll substitute $-x$ into the function for x. If we get back the original function $f(x)$, the function is even. If we get back the original function multiplied by -1, the function is odd. In other words,

- If $f(-x) = f(x)$, the function is even
- If $f(-x) = -f(x)$, the function is odd

If we discover that the function is even or odd, the next step is to check the limits of integration (the interval over which we're integrating). In order to use the special even or odd function rules for definite integrals, our interval must be in the form $[-a, a]$. In other words, the limits of integration have the same number value but opposite signs, like $[-1,1]$ or $[-5,5]$.

If the function is even or odd and the interval is $[-a, a]$, we can apply these rules:

When $f(x)$ is even,

$$\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx$$

When $f(x)$ is odd,

$$\int_{-a}^{a} f(x)\ dx = 0$$

Example

Integrate.

$$\int_{-2}^{2} 3x^2 + 2 \ dx$$

First we'll check to see if the function meets the criteria for an even or odd function. To see if it's even, we'll substitute $-x$ for x.

$$f(x) = 3x^2 + 2$$

$$f(-x) = 3(-x)^2 + 2$$

$$f(-x) = 3x^2 + 2$$

After substituting $-x$ for x, we were able to get back to the original function, which means we can say that

$$f(x) = f(-x)$$

and therefore that the function is even.

Looking at the given interval $[-2,2]$, we see that it's in the form $[-a, a]$.

Since we know that our function is even and that our interval is symmetric about the y-axis, we can calculate our answer using the formula

$$\int_{-a}^{a} f(x) \ dx = 2\int_{0}^{a} f(x) \ dx$$

$$\int_{-2}^{2} 3x^2 + 2 \ dx = 2\int_{0}^{2} 3x^2 + 2 \ dx$$

Now, instead of integrating the left-hand side, we can instead integrate the right-hand side, and evaluating over the new interval will be a little easier.

$$\int_{-2}^{2} 3x^2 + 2 \ dx = 2\left(\frac{3}{3}x^3 + 2x\right)\Bigg|_{0}^{2}$$

$$\int_{-2}^{2} 3x^2 + 2\ dx = \left(2x^3 + 4x\right)\Big|_0^2$$

$$\int_{-2}^{2} 3x^2 + 2\ dx = \left[2(2)^3 + 4(2)\right] - \left[2(0)^3 + 4(0)\right]$$

$$\int_{-2}^{2} 3x^2 + 2\ dx = 24$$

Let’s try another example.

Example

Integrate.

$$\int_{-7}^{7} 3x^7 + 4\sin x\ dx$$

First we’ll check to see if the function meets the criteria for an even or odd function. Let’s start by testing it to see if it’s an even function by substituting $-x$ for x.

$$f(x) = 3x^7 + 4\sin x$$

$$f(-x) = 3(-x)^7 + 4\sin(-x)$$

$$f(-x) = -3x^7 - 4\sin x$$

$$f(x) \neq f(-x)$$

In order for the function to be even, $f(-x) = f(x)$. Since $f(x) \neq f(-x)$, this function is not even.

So we'll check to see if the function is odd. Remember that an odd function requires $f(-x) = -f(x)$. We can test this by substituting $-x$ for x.

$$f(x) = 3x^7 + 4\sin x$$

$$f(-x) = 3(-x)^7 + 4\sin(-x)$$

$$f(-x) = -3x^7 - 4\sin x$$

$$f(-x) = -\left(3x^7 + 4\sin x\right)$$

$$f(-x) = -f(x)$$

Because $f(-x)$ becomes $-f(x)$, we can say that the function is odd. Looking at the given interval $[-7,7]$, we see that it's in the form $[-a, a]$.

Since we know that our function is odd and that our interval is symmetric about the y-axis, we can calculate the answer using the formula

$$\int_{-a}^{a} f(x)\ dx = 0$$

$$\int_{-7}^{7} 3x^7 + 4\sin x\ dx = 0$$