Part 4: Lagrangian Mechanics In Action

In this part, we'll finally get to what this book is actually about - Lagrangian mechanics. We'll be discussing all the basics of Lagrangian mechanics and how to use it, including generalized coordinates, generalized momenta and constraints.

This part will cover most of the things you need to know about Lagrangian mechanics, as well as some examples - lots of examples - to illustrate the points. You'll find more applications also in Part 7. In the next part, on the other hand, we'll dive much deeper into some of the more theoretical aspects of the Lagrangian formulation.

1. Review of The Lagrangian Formulation

I want to first provide a quick reminder of what Lagrangian mechanics is actually all about. A more detailed "overview" is found back in Part 2.

As we discussed previously, Lagrangian mechanics is all about describing motion and finding equations of motion by analyzing the kinetic and potential energies in a system. We do this by first and foremost, constructing a **Lagrangian** for the system.

The Lagrangian is a function that encodes all the dynamics of a system *locally*, at each point in time. In classical mechanics, the Lagrangian is given by the difference in kinetic and potential energy.

The Lagrangian:

$$L = T - V$$

In the above, $T = \frac{1}{2}mv^2$ is the (total) kinetic energy of the system and V is the (total) potential energy of the system, typically taken to be a function of position.

We then describe the motion (time evolution) of a system by constructing an **action** out of the Lagrangian. The action describes the entire trajectory of a system through time and is given by a functional $S = \int_{t_1}^{t_2} L dt$.

The real, physical trajectory the system takes through time is the one in which the action has a stationary value.

In the previous part, we discussed how we can find the stationary "points" of functionals, which requires the condition that $\delta S = 0$. This is called the **principle of stationary action**.

Then, by applying the tools of variational calculus like we did in the previous part, we can arrive at an equation that describes the condition the action and thus, also the Lagrangian has to satisfy in order to be stationary. This is the **Euler-Lagrange equation**.

The Euler-Lagrange equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

The Euler-Lagrange equation is what gives us the **equations of motion** for a system, any system in fact. This is the central equation in Lagrangian mechanics that we'll be using all throughout the rest of this book.

Now, so far, everything has been very theoretical and in my opinion, the best way to understanding theoretical concepts is by applying them in practice, as counter-intuitive as that may sound.

Therefore, we need to look at some examples to see how all of this actually works. Before we understand how to use Lagrangian mechanics in practice, we cannot dive into any of the more advanced and interesting topics like Noether's theorem.

However, there is just one more thing to discuss first and these are called **generalized coordinates** - they are the q_i's you see above. These are what we use to describe any system and how the system evolves in time.

2. Generalized Coordinates

Solving a problem in Lagrangian mechanics always begins by *choosing* how to describe the system of interest. This is done by choosing a set of variables - coordinates - to describe the system in.

In the Lagrangian formulation, there isn't really any preferred coordinate systems. In contrast, we saw previously that working with Newton's laws in any coordinate system other than the Cartesian system can often get cumbersome.

This is indeed one of the greatest advantages of Lagrangian mechanics - being able to choose any variable as a coordinate to describe a system in.

We could, for example, pick an angle as our coordinate in a problem involving rotational motion or choose our coordinates to be some relative distances between two objects. We can even go as far as choosing the electric charge as a "coordinate" to describe an electric circuit - the point is that we can pick anything we like as a coordinate in the Lagrangian formulation.

The beautiful thing about Lagrangian mechanics is that the problem solving methods work exactly the same, no matter what coordinates you wish to use. This is because the Euler-Lagrange equation has the same form in ALL coordinate system - unlike Newton's second law, $F = ma$.

Now, we call the set of coordinates needed to describe a system **generalized coordinates**.

Simply put, generalized coordinates are a set of parameters that completely specify the configuration of a system at each point in time. Generalized coordinates are used to derive the equations of motion for the system, which then determine how the coordinates change as functions of time - describing all the dynamics of the system.

The “configuration” of a system is just a fancy way of saying the coordinates of each object in the system.

Generalized coordinates are typically denoted by q_i, where the i-index refers to how many coordinates we have - q_1, q_2, q_3 and so on. For a dynamic system, these are going to be functions of time.

The equations of motion for a system are then obtained from the Euler-Lagrange equations, which there are always one of for each generalized coordinate. So, a system with two generalized coordinates, q_1 and q_2, has two Euler-Lagrange equations, resulting in two equations of motion that describe the system (one for each coordinate):

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = 0 \text{ and } \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} = 0$$

The $\dot{q}$'s here are the time derivatives of the generalized coordinates, in other words, the coordinate velocities (sometimes also called generalized velocities).

We then have to solve both of the equations of motion to find how the system as a whole evolves in time.

Now, the ability to pick generalized coordinates certainly sounds like a convenience in favor of the Lagrangian formulation, but is it actually useful in practice? In other words, couldn't we just always choose Cartesian x, y, z -coordinates to describe any system in and be done with it?

Well, the answer is yes, pretty much any system can be described using Cartesian coordinates, but it's definitely NOT always going to be the most practical choice. Here are just a few reasons why:

- Generalized coordinates can be chosen to *implicitly* encode all information about the **constraints** in a system (for example, by choosing polar coordinates (r and θ) for a pendulum, we could automatically include the fact that the pendulum moves in a circle; using only Cartesian coordinates, you'd have to impose certain constraints on the system).

- The "correct" choice of generalized coordinates can **reduce the number of equations** needed to solve a specific problem (as an example, choosing only an angle θ for a pendulum would be enough to find the equations of motion, while in Cartesian coordinates, you'd need both the x and y -coordinates).

- Choosing the "correct" set of generalized coordinates often allows us to find **conserved quantities** in a given system quite easily.

- Generalized coordinates allow different **dynamical quantities** to be expressed easily in a more general form (such as generalized momentum and generalized forces).

In particular, the first and third points here are what really make the use of generalized coordinates stand out compared to standard Newtonian mechanics.

When using Newton's laws, whenever we want to constrain a system to behave in a certain way, we have to add in constraint forces that "force" the system to behave in exactly the way we want it to.

However, when using generalized coordinates, we do not have to add in these constraints by hand; we can, in most cases, choose our generalized coordinates in such a way that they already include the constraints - we essentially eliminate the need for any constraint forces.

Also, the other nice thing is that we can tell directly by looking at the Lagrangian which generalized coordinates have an associated conserved quantity by finding so-called **cyclic coordinates** (we'll talk about this more in the next part). This simply wouldn't be possible in Newtonian mechanics.

2.1. How To Choose Generalized Coordinates

But how do we know which generalized coordinates to choose? Well, the simple answer is that it depends on the system at hand.

Just to be clear, ANY set of generalized coordinates will work (granted that these coordinates are actually sufficient for describing the system), but choosing the "correct" ones may be the difference between making a problem extremely complicated or making a problem seem almost too easy.

This is largely because generalized coordinates are never unique, meaning that there are always multiple “correct” choices of coordinates for any given problem.

Actually, pretty much any set of coordinates that satisfy the **constraints** of the problem will be valid choices of generalized coordinates. This is the power of Lagrangian mechanics.

The topic of actually finding suitable generalized coordinates for a given system is a little tricky and that’s simply because there isn’t really a general method to do it.

In other words, finding generalized coordinates for a system is done on a case-by-case basis and it will be different for different problems. Picking the right coordinates from the get-go requires intuition that only comes from practice.

Now, there are still patterns and certain “rules-of-thumb” that you can use when finding generalized coordinates:

1. It’s often worthwhile to look at **symmetries in a given system** as this allows you to quickly figure out which generalized coordinates may be suitable:

   - For spherically symmetric problems in 3D, *spherical coordinates* (radial distance r and two angles, θ and φ) are often the best choice, such as in the case of a potential that only depends on the radial distance.
   - Equivalently, for rotational symmetry in 2D, *polar coordinates* (r and θ) are often suitable.
   - For axisymmetric problems (symmetry about one particular axis), *cylindrical coordinates* will often be the easiest to use.

2. For **systems with multiple objects**, using *relative distances* between the objects as generalized coordinates will often work well. An example of this is the two-body problem (which we'll look at in another part).

3. Some **helpful questions** you may want to think about when choosing

generalized coordinates are:

- **What variables are actually changing in the system?** A generalized coordinate needs to be something dynamical, something that is changing with time and needs to have a time derivative.
- **Are the coordinates consistent with the specific constraints of the problem?** In other words, do the coordinates already include the constraints? If not, use different coordinates or add the constraints in by hand (with Lagrange multipliers).
- **Are the coordinates you're choosing enough to determine the positions of all objects in the system?** You generally need as many generalized coordinates as there are degrees of freedom in the system.

Often, the best place to begin is simply going to be to figure out how many generalized coordinates you'll need for the given problem in the first place. Luckily, this is quite simple.

The number of **degrees of freedom** in any given system (i.e. the minimum number of variables needed to describe the system) can be calculated by the formula $DN - c$, where D is the number of spacial dimensions (1, 2 or 3), N is the number of objects in the system and c is the number of constraints.

The minimum number of independent generalized coordinates you'll need is then the same as the number of degrees of freedom in the system.

For example, in the case of one object in 2D circular motion, we have $D = 2, N = 1$ and $c = 1$ (there is one constraint, which is that the radius of the circle has to be constant), so we would need $2 \cdot 1 - 1 = 1$ generalized coordinates (the natural choice in this case would be an angle due to rotational symmetry).

Above, I mentioned *independent* generalized coordinates. Generalized coordinates do not necessarily have to be independent. For example, if certain constraints are included in a system, the generalized coordinates can depend on one another.

However, the degrees of freedom in a system always corresponds to the minimum number of independent generalized coordinates.

Now, once you have figured out the generalized coordinates you should use, what do you actually do with them in practice?

Perhaps the best tip I could give you here that I do pretty much 100% of the time is to **begin by writing down the Cartesian coordinates of each object in the system in terms of these generalized coordinates** (I'll show examples of this later).

This is almost always the easiest way to find the Lagrangian, because when you have the x, y, z -coordinates in terms of the generalized coordinates, all you do is take the time derivatives of these and then construct the total kinetic energy (and likewise for the total potential energy) by the formula:

$$T = \frac{1}{2}\sum_N m_N\left(\dot{x}_N^2 + \dot{y}_N^2 + \dot{z}_N^2\right)$$

This N here denotes the number of objects in the system, so you have to sum over the kinetic energies of all the objects.

This will automatically give you the correct kinetic energy (and potential energy) in terms of your generalized coordinates, instead of trying to find the correct expressions for these without first referencing Cartesian coordinates. The rest is then just constructing the Lagrangian and applying the Euler-Lagrange equations.

Now, since generalized coordinates can be almost anything we wish, what about their dimensions or units?

In principle, generalized coordinates can have any dimensions. For example, choosing the radial distance as a generalized coordinate will have dimensions of length, but an angle as a generalized coordinate will be dimensionless.

If you simply use the method of first expressing the Cartesian coordinates of everything in the system in terms of the generalized coordinates, this will ensure you always get the correct units in the Lagrangian.

3. Simple Example Systems

Let's now take a look at various examples to illustrate the process of using Lagrangian mechanics! The basic step-by-step framework we'll use for solving a problem in Lagrangian mechanics is more or less as follows:

1. **Find a set of suitable generalized coordinates for the problem**. These might be the usual Cartesian coordinates (x,y,z), but it is also possible to use coordinate systems such as spherical coordinates (r,θ,φ).

2. **Write down the Lagrangian in terms of the generalized coordinates**. The Lagrangian will be the difference between the kinetic and potential energies of each object in the system:

$$L = \frac{1}{2}\sum_N m_N v_N^2 - \sum_N V_N,$$ where N is the number of objects in the system.

Note; in most cases, it's easier to first express the Lagrangian in Cartesian coordinates and then transform to whatever generalized coordinates you're using!

3. **Apply the Euler-Lagrange equations to the Lagrangian**. You'll have one Euler-Lagrange equation for each coordinate you've chosen for the system.

4. **Simplify and solve the equations of motion**. At this point, you should have one second order differential equation for each coordinate.

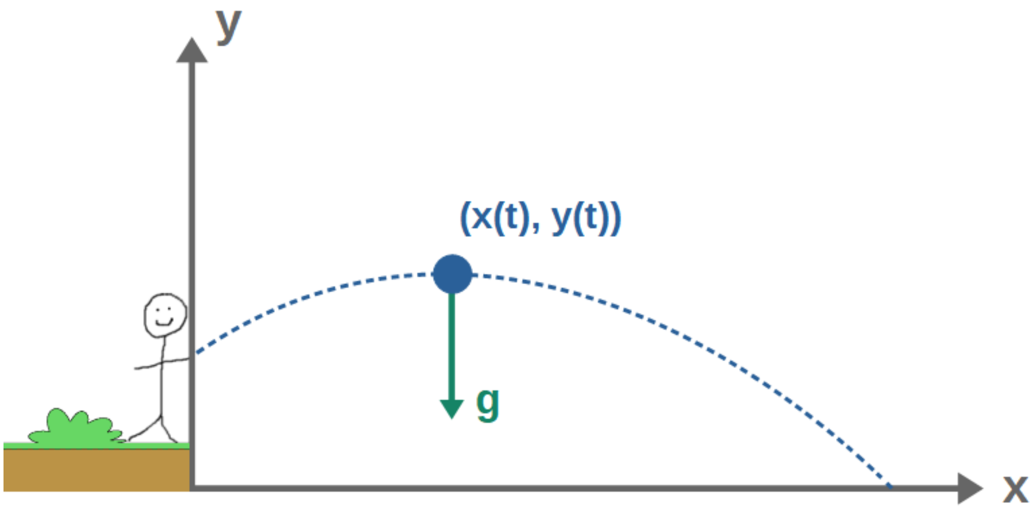
3.1. Projectile Motion

Let's begin with something very simple - projectile motion! We already saw how this is solved using Newtonian mechanics, but let's now see how it's done in the Lagrangian framework.

It's easiest to describe the projectile using the x,y -Cartesian coordinate system here. These will also be our generalized coordinates, so we have $q_1 = x$ and $q_2 = y$.

These are enough to describe the position of the projectile at all times, since we have $2 \cdot 1 - 0 = 2$ degrees of freedom (two dimensions with one object and no constraints) and thus, we need two generalized coordinates in total.

The projectile is under a constant acceleration, g, in the downwards y-direction:



We can desribe the projectile's potential energy as $V = mgh$ (just the basic formula for gravitational potential energy), where the height of the projectile is the y-coordinate, $h = y$.

The kinetic energy of the projectile in terms of our generalized coordinates is simply going to be:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

Here, $\dot{x}$ and $\dot{y}$ are the time derivatives of the coordinates, which are simply the components of the projectile's velocity. The Lagrangian for the projectile is then:

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

This is our full Lagrangian describing this problem in terms of the generalized coordinates x and y.

Now, we'll have two Euler-Lagrange equations, one for each coordinate here. The first one will be for $q_1 = x$:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

Plugging the Lagrangian into this and calculating these partial derivatives, we get (note that we treat x and $\dot{x}$ as independent variables here, just like we treated y and y' back when doing variational calculus):

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)-mgy\right)-\frac{\partial}{\partial x}\left(\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)-mgy\right)=0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}m\cdot 2\dot{x}\right)-0=0$$

$$\Rightarrow \quad \frac{d}{dt}(m\dot{x})=0$$

$$\Rightarrow \quad m\ddot{x}=0$$

$$\Rightarrow \quad \ddot{x}=0$$

This is the equation of motion for the x-coordinate, telling us that there is no acceleration in the x-direction as expected. This equation of motion has the solution $x(t)=v_{0x}t+x_0$ as we saw back in Part 1.

Notice that the Lagrangian did not depend on the x-coordinate explicitly, which resulted in the x-acceleration being zero here - the same as $m\dot{x}$ being constant. This is an example of a cyclic coordinate, but we'll come back to this later.

Now, we also have an Euler-Lagrange equation for the y-coordinate and plugging the Lagrangian into this, we get:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}}-\frac{\partial L}{\partial y}=0$$

$$\Rightarrow \quad \frac{d}{dt}\frac{\partial}{\partial \dot{y}}\left(\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)-mgy\right)-\frac{\partial}{\partial y}\left(\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)-mgy\right)=0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}m\cdot 2\dot{y}\right)-(-mg)=0$$

$$\Rightarrow \quad \frac{d}{dt}(m\dot{y})+mg=0$$

$$\Rightarrow \quad m\ddot{y}+mg=0$$

$$\Rightarrow \quad \ddot{y}=-g$$

So, as expected, we get the equation of motion of the y-coordinate which tells us that the y-acceleration is $-g$. This equation of motion, of course, has the solution $y(t)=y_0+v_{0y}t-\frac{1}{2}gt^2$.

Perhaps after this example, you're wondering "how is this any better than Newtonian mechanics?". Well, the answer is that it isn't any better. For the simplest possible systems like this one, Lagrangian mechanics won't be any better than Newtonian mechanics.

However, as we will see soon with the pendulum example, the Lagrangian approach will turn out much more straightforward than what we had to do with Newtonian mechanics.

The point is that Lagrangian mechanics only really starts to show its power over the Newtonian formulation for more complicated systems and especially for systems that are more suitable to describe in coordinates other than the Cartesian coordinate system.

3.2. Simple Pendulum

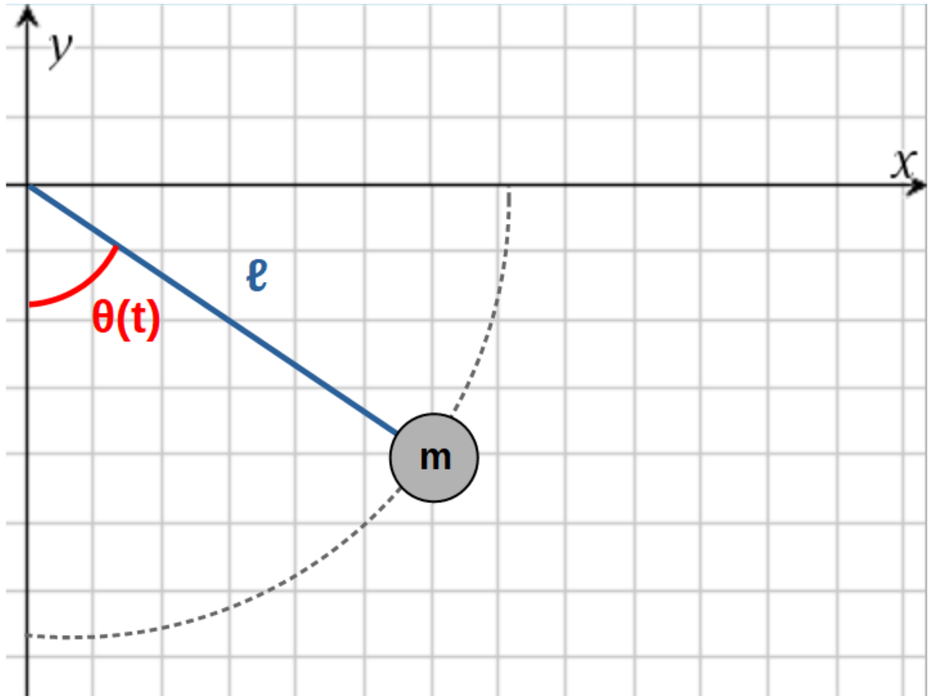
As a quick reminder, the simple pendulum consists of a mass at the end of a rigid rod that's swinging back and forth in a plane under the influence of gravity (a constant acceleration g pointing downwards). The mass of the "pendulum bob" is m and the length of the rod l.

We'll begin by placing the pendulum in an x,y -coordinate system. Now, while you could pick the x- and y -coordinates of the bob as your generalized coordinates, this is not going to be the best choice. The reason for this is that there is a constraint (the length of the rod, i.e. distance from the origin, is a constant l) and describing this constraint in Cartesian coordinates is a bit cumbersome.

If we calculate the degrees of freedom in this system, we have two spacial dimensions, one object and one constraint, so $2 \cdot 1 - 1 = 1$ degree of freedom. Therefore, we only actually need one generalized coordinate to fully describe this pendulum.

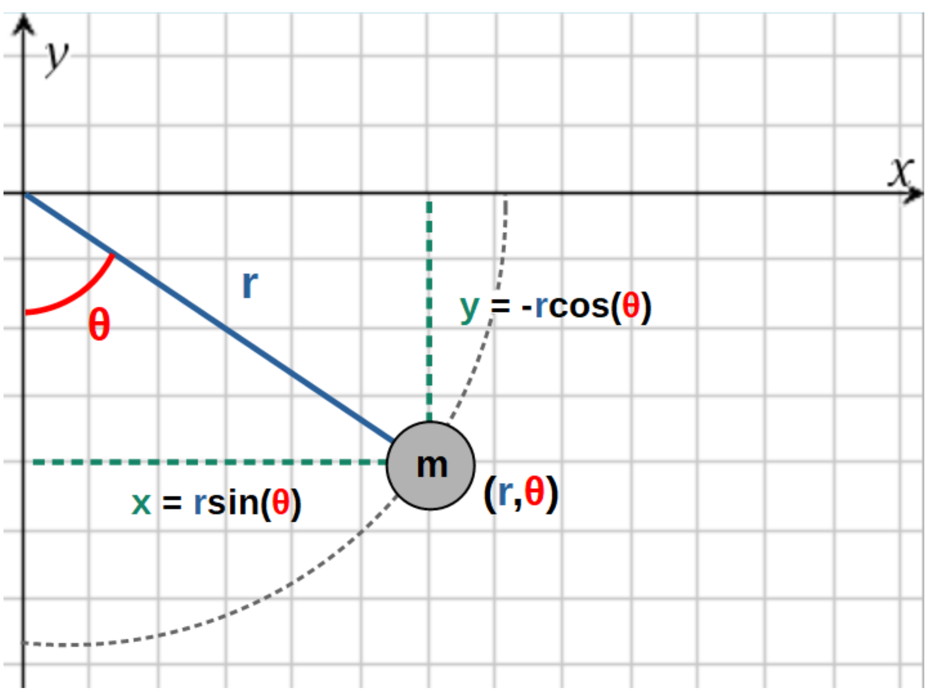
Since this is a rotational motion problem, polar coordinates are naturally going to be the most convenient choice. Moreover, since the distance to the origin is constant at all times, we have $r = l$ and indeed, the polar angle θ (which is a function of time) will be enough to determine the position of the pendulum at all times.

We will therefore pick $\theta(t)$ as our generalized coordinate for this problem:



Now, how do we then get the kinetic and potential energies of the pendulum bob in polar coordinates? Well, like I stated earlier, the easiest course of action is most often to first express everything in Cartesian coordinates and then transform the expressions to be in terms of our generalized coordinate θ.

The relationship between the Cartesian coordinates of the bob and the polar coordinates r and θ are given by (see the Appendices for more on this):



Notice that the y-coordinate has a negative value here. Now, since the radial distance is a constant, we have $r = l$ and the coordinates of the bob are given by:

$$x = l\sin\theta$$
$$y = -l\cos\theta$$

Now, constructing the kinetic and potential energies from these relations is very easy. We know that the kinetic energy in Cartesian coordinates is given by:

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

All we do is take the time derivatives of the expressions above and plug them into here. So, let's do that (note that we have to use the chain rule here since θ depends on time):

$$\dot{x} = \frac{d}{dt}(l\sin\theta) \quad \Rightarrow \quad \dot{x} = l\dot{\theta}\cos\theta$$
$$\dot{y} = \frac{d}{dt}(-l\cos\theta) \quad \Rightarrow \quad \dot{y} = l\dot{\theta}\sin\theta$$

These are the coordinate velocities in terms of the generalized coordinate θ. Plugging these into the kinetic energy formula, we have:

$$\begin{aligned} T &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \\ &= \frac{1}{2}m\left(l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta\right) \\ &= \frac{1}{2}ml^2\dot{\theta}^2\left(\cos^2\theta + \sin^2\theta\right) \\ &= \frac{1}{2}ml^2\dot{\theta}^2 \end{aligned}$$

Here, we've used the fact that $\cos^2\theta + \sin^2\theta = 1$. This is the kinetic energy of the pendulum bob, expressed in terms of our generalized coordinate θ.

Now, the potential energy of the bob is simply the usual gravitational potential energy $V = mgh$. The height h here is just the y-coordinate of the bob, which is given by $y = -l\cos\theta$ in terms of the generalized coordinate θ. So, the potential energy is:

$$V = -mgl\cos\theta$$

We then have all the pieces we need to construct the Lagrangian for the pendulum bob:

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

This is the Lagrangian for this system in terms of our generalized coordinate θ. Since we have only one coordinate here, we also have just one Euler-Lagrange equation - an equation for the coordinate θ, of course:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

Plugging in the Lagrangian, we get:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta\right) - \frac{\partial}{\partial \theta}\left(\frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta\right) = 0$$
$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}ml^2 \cdot 2\dot{\theta}\right) - mgl\frac{\partial}{\partial \theta}(\cos\theta) = 0$$
$$\Rightarrow \quad ml^2\frac{d}{dt}(\dot{\theta}) - mgl(-\sin\theta) = 0$$
$$\Rightarrow \quad ml^2\ddot{\theta} + mgl\sin\theta = 0$$

Moving the $\sin\theta$-term to the right and dividing by ml^2, we get the final equation of motion:

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

We already discussed some of the implications of this equation and how to solve it using a linear appoximation back in Part 1, so we won't go over it again.

The main point here is... look how easy it was to obtain this compared to using $F = ma$! We didn't have to deal with any time derivatives of basis vectors or any other complicated vector stuff - all we did was construct the Lagrangian in terms of the coordinate θ and the equation of motion popped out automatically.

Now, you may wonder what happened to the constraint force (tension) we had to introduce when using Newton's laws. Well, we didn't need it *at all.* This is because we already chose our coordinates (namely, the r-coordinate) such that the constraint - the length of the rod remaining a constant l - was satisfied right from the beginning.

This illustrates the power of generalized coordinates - if we choose them right, we can implicitly encode any constraints into them and get the equations of motion with very little work compared to using Newton's laws and having to include constraint forces.

Now, if you wanted to find what the tension force is using Lagrangian mechanics, that's also possible (but again, not necessary if we do not care about it) using the Lagrange multiplier method. We'll discuss this later.

I really hope this example showed you the power of Lagrangian mechanics. However, this was only the start - it gets even better for much more complicated systems, such as a three-dimensional pendulum. Good luck working that one out using $F = ma$!

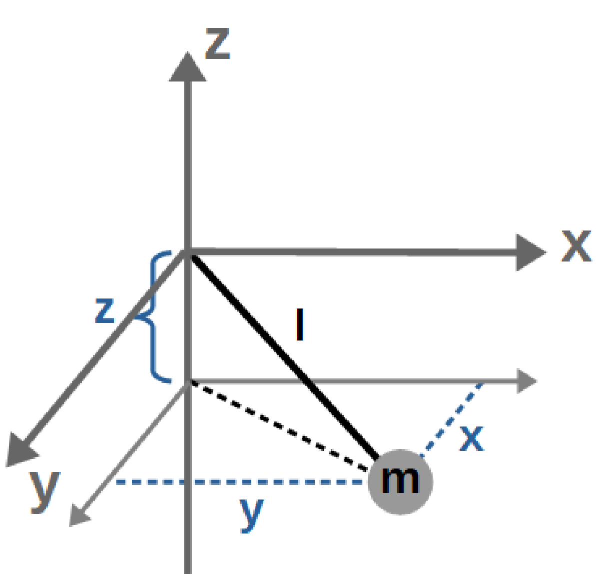
3.3. Spherical Pendulum

Our next example will be quite a bit more complicated. This one will hopefully illustrate the power of Lagrangian mechanics even better.

Now, this example will be a pendulum (mass m and length l again), but one that is now allowed to swing not just in a plane, but in all three dimensions. We call this a spherical pendulum. The motion of the pendulum bob can get very complicated, but also has some interesting properties.

We'll first derive the equations of motion for the spherical pendulum and after that, we'll do some analysis of its motion.

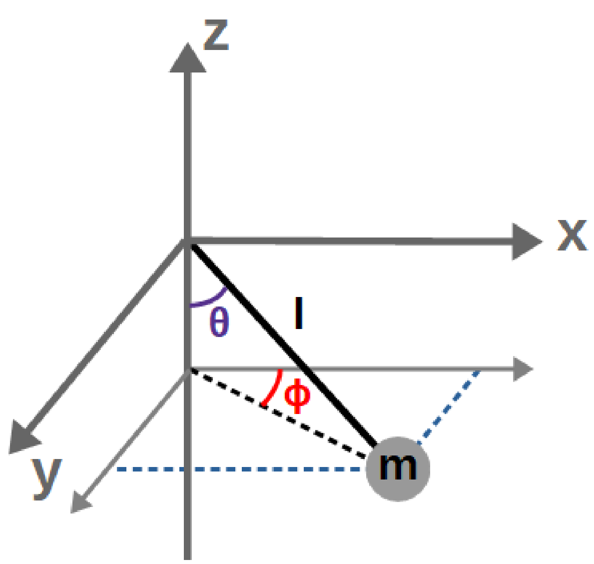
To begin with, we can once again, describe the position of the mass m by placing the pendulum in a Cartesian coordinate system (this time a 3D coordinate system with the x-, y- and z-axes):



Now, Cartesian coordinates are certainly not the best choice here. Let's think about the degrees of freedom we have in this system. We have one object in three dimensions with one constraint (the distance to the origin has to be a constant l again), so $3 \cdot 1 - 1 = 2$ degrees of freedom. So, we need two generalized coordinates.

The natural choice here will be a spherical coordinate system (with a radial distance r and two angles, θ and ϕ) due to the rotational nature of the pendulum.

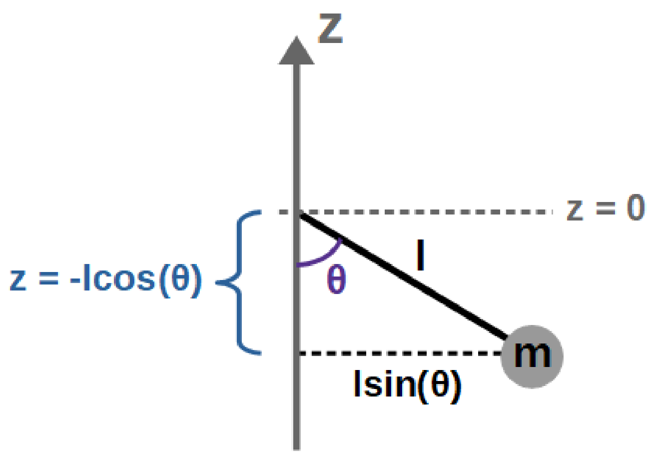
We will choose the two angles, θ and ϕ, as our generalized coordinates (see picture below). These are indeed the only necessary coordinates we need to specify the pendulum bob's position since its distance from the origin is fixed, so we have $r = l$.



The angle ϕ here "rotates" in the x,y-plane and θ is the angle between the rod and the vertical z-axis (it's essentially the same angle as in the simple pendulm case).

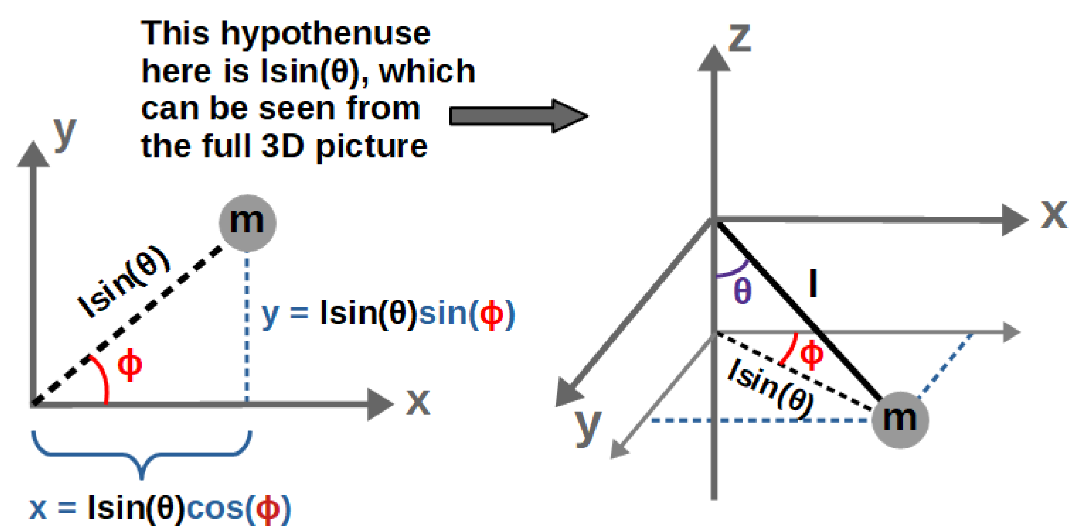
Now, let's try to express the x, y, z -coordinates of the bob in terms of these two generalized coordinates. We can do this with a little bit of trigonometry.

First, let's look at the situation from the "side". In other words, let's look at the triangle formed with the z-axis, from which we can determine the z-coordinate.



So, from this, we get the z-coordinate of the bob expressed in terms of our generalized coordinates as $z = -l\cos\theta$.

Now let's look at the situation from above. In other words, let's look at what happens in the x, y -plane to determine the x and y coordinates:



From these relations, we get the x- and y-coordinates of the bob also in terms of our generalized coordinates. So, we have then have the coordinates of the mass m as:

$$x = l\sin\theta\cos\phi$$
$$y = l\sin\theta\sin\phi$$
$$z = -l\cos\theta$$

Most of the difficult work has been done already - now all we do is find the velocities from these and then the Lagrangian.

We can take the time derivatives of these expressions (note that since both θ and ϕ

are functions of time, we have to use both the product and chain rules when differentiating x and y):

$$\dot{x} = \frac{d}{dt}(l \sin \theta \cos \phi)$$
$$= l \cos \phi \frac{d}{dt} \sin \theta + l \sin \theta \frac{d}{dt} \cos \phi$$
$$= l\dot{\theta} \cos \phi \cos \theta - l\dot{\phi} \sin \theta \sin \phi$$

$$\dot{y} = \frac{d}{dt}(l \sin \theta \sin \phi)$$
$$= l \sin \phi \frac{d}{dt} \sin \theta + l \sin \theta \frac{d}{dt} \sin \phi$$
$$= l\dot{\theta} \sin \phi \cos \theta + l\dot{\phi} \sin \theta \cos \phi$$

$$\dot{z} = \frac{d}{dt}(-l \cos \theta) = l\dot{\theta} \sin \theta$$

Now, the kinetic energy of the pendulum bob is then the sum of the squares of these velocities, $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$. Writing out all these squares is not difficult but would take too much space on this page, so I won't write it out here. You can do this by hand as an exercise if you wish.

What you end up with after A LOT of terms cancelling and applying some trigonometric identities a few times (namely, $\cos^2\theta + \sin^2\theta = 1$ and the same for ϕ) is the following kinetic energy:

$$T = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)$$

Now, the potential energy is much simpler. It is again just $V = mgh$, with the height being the z-coordinate ($z = -l \cos \theta$):

$$V = mgz = -mgl \cos \theta$$

We then have our Lagrangian for the spherical pendulum:

$$L = T - V = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + mgl \cos \theta$$

In this case, since we have two generalized coordinates, we will also have two equations of motion and thus, two Euler-Lagrange equations. The first one is the Euler-Lagrange equation for the coordinate θ. From that, we get:

$$
\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \\
&\Rightarrow \quad \frac{d}{dt}\left(ml^2\dot{\theta}\right) - \frac{1}{2}ml^2\dot{\phi}^2\frac{\partial}{\partial \theta}\left(\sin^2\theta\right) - mgl\frac{\partial}{\partial \theta}\cos\theta = 0 \\
&\Rightarrow \quad ml^2\ddot{\theta} - ml^2\dot{\phi}^2\sin\theta\cos\theta + mgl\sin\theta = 0 \\
&\Rightarrow \quad \boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta + \dot{\phi}^2\sin\theta\cos\theta}
\end{aligned}
$$

This is the equation of motion for the θ-coordinate. Notice that the first term on the right-hand side is the same as for the simple pendulum, but the second term comes from the fact that the pendulum is also swinging in the ϕ-direction (meaning a non-zero $\dot{\phi}$).

Let's now construct the equation of motion for the ϕ-coordinate. Since nothing in the Lagrangian depends explicitly on the ϕ-coordinate, we have $\partial L / \partial \phi = 0$ and we get:

$$
\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0 \quad \Rightarrow \quad \boxed{\frac{d}{dt}\left(ml^2\dot{\phi}\sin^2\theta\right) = 0}
$$

Now, I've purposefully left this equation of motion in this form. We don't actually have to calculate this time derivative. We can simply just integrate this on both sides, which results in this expression inside the parentheses being a constant (which I'll call a):

$$
\begin{aligned}
&\int \frac{d}{dt}\left(ml^2\dot{\phi}\sin^2\theta\right)dt = \int 0 dt \\
&\Rightarrow \quad ml^2\dot{\phi}\sin^2\theta = a
\end{aligned}
$$

It turns out that this constant, a, is actually a conserved quantity - the generalized momentum associated with the ϕ-coordinate. For now, however, it's not so important (we'll talk about generalized momenta soon).

So, we basically have two equations of motion describing the system right now:

$$
\ddot{\theta} = -\frac{g}{l}\sin\theta + \dot{\phi}^2\sin\theta\cos\theta
$$

$$ml^2\dot{\phi}\sin^2\theta = a$$

Both of these have to be satisfied simultaneously for us to find a solution describing the spherical pendulum. Therefore, what we can do is combine both of these equations into one, which automatically takes care of the fact that they should both be satisfied.

We can do this by solving the second equation for $\dot{\phi}$, which gives us:

$$ml^2\dot{\phi}\sin^2\theta = a \quad \Rightarrow \quad \dot{\phi} = \frac{a}{ml^2\sin^2\theta}$$

Then, inserting this expression in the place of $\dot{\phi}$ into the θ-equation, we get:

$$\ddot{\theta} = -\frac{g}{l}\sin\theta + \dot{\phi}^2\sin\theta\cos\theta$$

$$\Rightarrow \quad \ddot{\theta} = -\frac{g}{l}\sin\theta + \left(\frac{a}{ml^2\sin^2\theta}\right)^2\sin\theta\cos\theta$$

$$\Rightarrow \quad \boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta + \frac{a^2}{m^2l^4}\frac{\cos\theta}{\sin^3\theta}}$$

We now have an equation of motion completely in terms of the θ-coordinate that determines the motion of this spherical pendulum! In principle, this could be used to, for example, simulate the pendulum's motion using a computer or numerically solve for its motion.

Also, note that we were still able to obtain such a complicated equation of motion using a fairly straightforward approach - all we did was construct the Lagrangian and calculate the Euler-Lagrange equations.

Doing the same thing with Newtonian mechanics, while it is possible, would be A LOT more complicated. You'd have to first transform Newton's second law, a vector equation, into spherical coordinates and then proceed from there - it would be a lot of work!

Now, in general, the above equation of motion is an incredibly difficult differential equation to actually solve. However, we can still try to find out what it tells us about the behaviour of the spherical pendulum.

How Does The Spherical Pendulum Behave?

Let's now look at some aspects related to the motion of the spherical pendulum. In particular, we'll construct a so-called **effective potential** for the spherical pendulum, which allows us to look at some interesting details.

The way we'll do this is by finding the **total energy** of the pendulum first. Simply put, the total energy (which is conserved) is just the sum of the kinetic and potential energy, $T + V$. Inserting these (which we calculated earlier), the energy is:

$$E = T + V = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) - mgl\cos\theta$$

We can now use the expression we found from the equations of motion that $\dot{\phi} = \dfrac{a}{ml^2\sin^2\theta}$ and insert this:

$$E = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \frac{a^2}{m^2l^4\sin^4\theta}\sin^2\theta\right) - mgl\cos\theta$$

$$\Rightarrow \quad E = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{a^2}{2ml^2\sin^2\theta} - mgl\cos\theta$$

Now, here we essentially have a kinetic energy term, $\frac{1}{2}ml^2\dot{\theta}^2$, and two terms that don't have any velocity-dependence anymore and only depend on the position θ.

Generally speaking, potentials are often taken to be position-dependent, so we could say that the energy can be expressed as the sum of a kinetic term and an "effective" potential that is a function of θ only (again, this only makes sense since we've eliminated the $\dot{\phi}$-dependence and replaced it with something that only involves θ):

$$E = \frac{1}{2}ml^2\dot{\theta}^2 + V_{eff}(\theta), \text{ where } V_{eff}(\theta) = \frac{a^2}{2ml^2\sin^2\theta} - mgl\cos\theta.$$

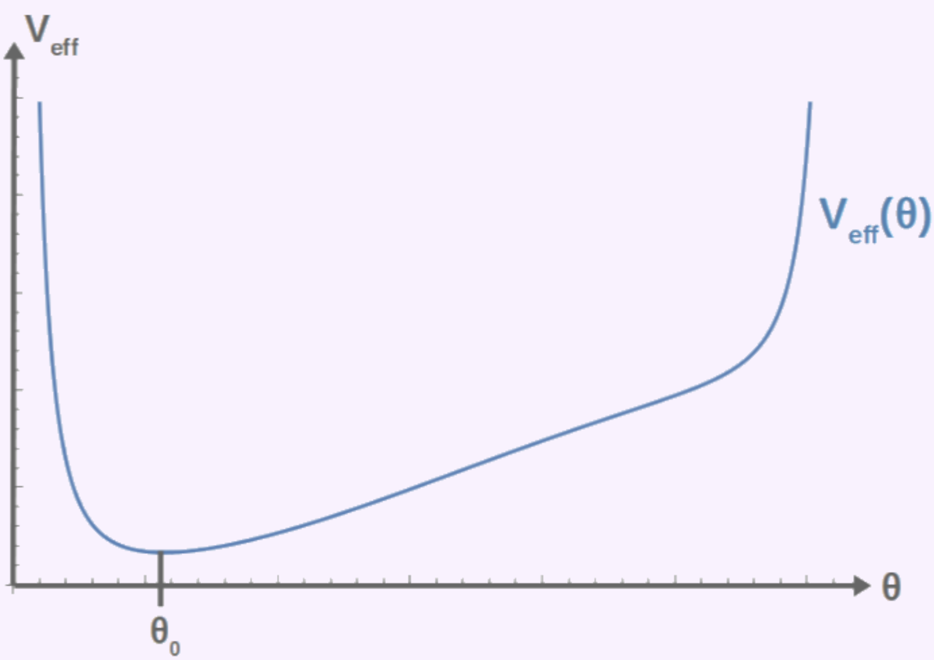
Now, what's the point of this exactly? Well, the effective potential is an extremely useful tool for qualitative analysis of how a given system behaves. For example, by plotting the effective potential, we can literally *see* how a system may behave.

The reason we often analyze effective potentials in many problems involving rotation in 3D is because the effective potential nicely incorporates both of the "radial" as well as the "angular" motion of a system.

We may not be able to see it directly here, but the first term in the effective potential above, the term involving a^2, describes the rotational part of the potential (and therefore the forces involved as well!) and the second term describes the fact that there is also a tension force affecting the radial motion of the pendulum.

The effective potential then incorporates both of these into one "effective" potential - in a sense, it describes the balance between the radial and the angular forces that determine the motion of the pendulum.

Now, by plotting this effective potential for the spherical pendulum, we can see that there is a minimum point on the graph (here I've plotted the values for $0 \leqslant \theta \leqslant \pi$):



Now, what does the minimum of a graph mean in terms of the function $V_{eff}(\theta)$? Well, the derivative $dV_{eff} \,/\, d\theta$ is zero at the minimum and the derivative of a potential generally corresponds to a *force* - an "effective" force in this case.

So, at the minimum of the effective potential, the "effective force" is zero, which physically means that the radial and the angular parts of the force are perfectly balanced - at this minimum, the pendulum will remain at a constant value of the θ-coordinate (which is the θ_0 here)!

If we have a constant θ-coordinate, let's see what the coordinate ϕ does in that case. Go back to the expression for $\dot{\phi}$ and plug in $\theta = \theta_0$:

$$\dot{\phi} = \frac{a}{ml^2 \sin^2\theta_0}$$

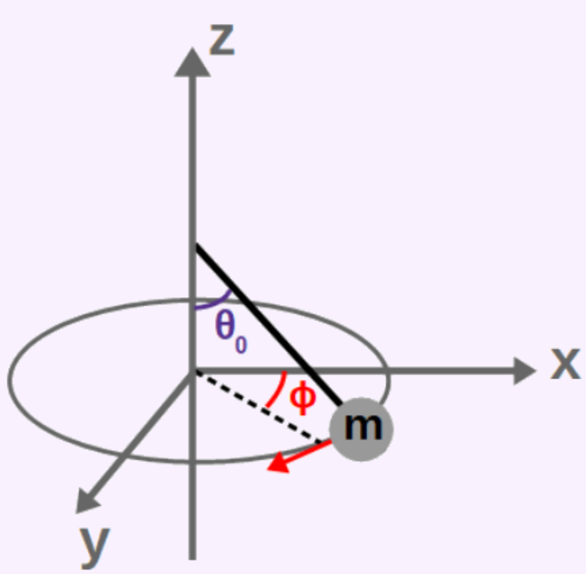
Everything here is a constant - in other words, the pendulum is swinging with a constant angular speed $\omega = \frac{a}{ml^2 \sin^2\theta_0}$ in the ϕ-direction. We can integrate this to get $\phi(t)$:

$$\phi(t) = \omega t + \phi_0$$

We've found a possible solution to the equations of motion of the spherical pendulum! In particular, the solution describes both of the angular coordinates of the pendulum:

$$\phi(t) = \omega t + \phi_0$$
$$\theta(t) = \theta_0$$

This solution describes the pendulum at a constant value of the θ-coordinate, while it swings at a constant angular speed (i.e. in a circle) in the x, y -plane. Such a system actually has its own name and is called a *conical pendulum*:



To find the actual value of the θ-coordinate at which this happens (θ_0), you need to differentiate the effective potential, set it equal to zero and solve for the value of θ:

$$\frac{dV_{eff}}{d\theta} = 0$$

Inserting V_{eff} and differentiating it, we get:

$$\frac{d}{d\theta}\left(\frac{a^2}{2ml^2 \sin^2\theta} - mgl\cos\theta\right) = -\frac{a^2\cos\theta}{ml^2\sin^3\theta} + mgl\sin\theta$$

Setting this equal to zero, so $\theta = \theta_0$, we get:

$$-\frac{a^2\cos\theta_0}{ml^2\sin^3\theta_0} + mgl\sin\theta_0 = 0$$
$$\Rightarrow \quad a^2\cos\theta_0 = m^2gl^3\sin^4\theta_0$$

Now, actually solving this for θ_0 is incredibly messy and will not give you any kind of nice looking or illuminating result. The actual value of θ_0 is not so important here, but rather what it implies in terms of the properties of the motion.

So, we can instead solve for the parameter a here (you'll see why soon):

$$a = m\sin^2\theta_0\sqrt{\frac{gl^3}{\cos\theta_0}}$$

Now, since the pendulum bob is moving in circular motion at this particular coordinate value θ_0, its angular speed ω is a constant and for circular motion this can be expressed in terms of the period T (time it takes to move once around the circle) as $\omega = \frac{2\pi}{T}$.

Earlier, we defined the angular speed as $\omega = \frac{a}{ml^2\sin^2\theta_0}$.

If we now plug in what we found the value of a to be, $a = m\sin^2\theta_0\sqrt{\frac{gl^3}{\cos\theta_0}}$, we can express the angular speed only in terms of θ_0:

$$\omega = \frac{m\sin^2\theta_0\sqrt{\dfrac{gl^3}{\cos\theta_0}}}{ml^2\sin^2\theta_0} = \sqrt{\frac{g}{l\cos\theta_0}}$$

Then, by using $\omega = 2\pi/T$, we can solve for the circular motion period of the pendulum:

$$\omega = \sqrt{\frac{g}{l\cos\theta_0}} = \frac{2\pi}{T} \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{l\cos\theta_0}{g}}$$

This is the period of the conical pendulum.

3.4. Rope Sliding Down a Table

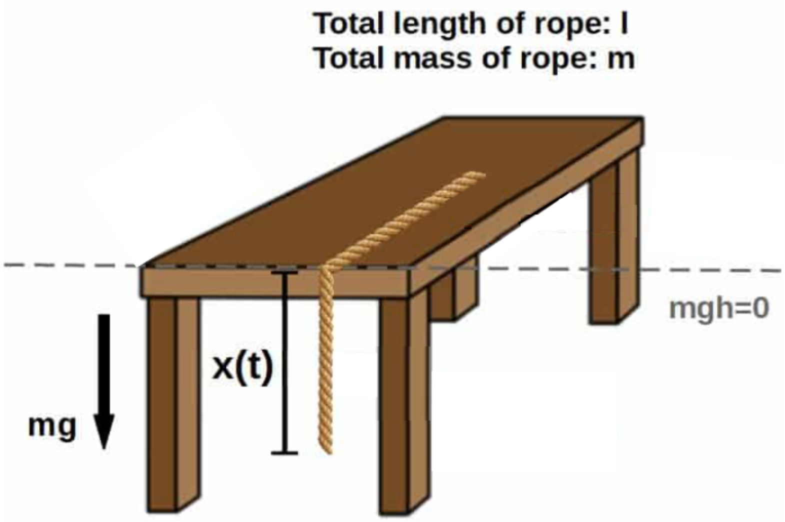
This example is quite interesting as it illustrates the fact that we can really choose any variables we wish as our generalized coordinates and the Lagrangian framework still gives us the equations of motion using exactly the same procedure as with any other coordinates.

Now, this problem essentially consists of a rope (or chain or whatever) placed on the edge of a table, so that its other end is hanging off the table. Since gravity is pulling the hanging end of the rope downwards, it will begin to slide down the table and we'll want to find the equations describing how the rope slides.

A picture of the situation can be seen down below with all of the information we'll use. Essentially, we're dividing the rope into two parts; the part hanging off the table and the part on top of the table.

What makes this interesting is the fact that when the rope slides, the lengths of these individual parts will both change with time (although the total length of the rope does not!).

We'll call $x(t)$ the total length of the part of the rope that's hanging off the table, which will be a function of time, of course. The total mass of the rope is m and the total length is l.



We're assuming the rope to be at rest to begin with (i.e. it has zero initial velocity) and that the table is frictionless (since we haven't looked at how friction works in the Lagrangian formulation yet).

Now, let's think about our generalized coordinates here. This is technically a two-dimensional problem since the rope can move either along the table or down vertically. The rope as a whole is just one object and we also have a constraint - the total length of the rope must be a constant, l.

Therefore, we only need $2 \cdot 1 - 1 = 1$ generalized coordinate (since we have two dimensions, one object and one constraint). The natural question now would be "what do we actually pick as our generalized coordinate?".

Well, let's think about what is actually changing with time in this problem - at least the length of the hanging part of the rope, which we called $x(t)$, is changing. Therefore, it makes sense to pick this as our generalized coordinate for the problem.

With this in mind, let's begin constructing the Lagrangian.

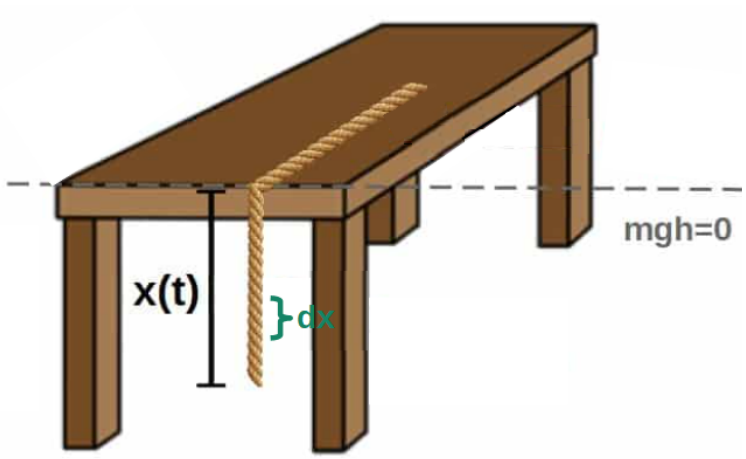
First, the kinetic energy - this is pretty simple since the velocity of every part of the rope is the same, given by the rate of change of $x(t)$, $\dot{x}$. This is because the part of the rope that's on the table is also moving with this same velocity (otherwise the rope would get stretched, which we assume does not happen here).

The kinetic energy of the rope is therefore just:

$T = \frac{1}{2} m \dot{x}^2$, where m is the total mass of the rope.

Now, the potential energy is a bit trickier. First of all, only the part hanging off the table has potential energy, the other part of the rope is just sitting on the table and its height is not changing due to gravity (we've chosen the potential energy to be zero at the table).

So, the total potential energy of the rope is just the potential energy of this hanging part. Now, how do we find what this potential energy is? Well, we've already done a similar thing in the last part with the catenary problem - we divide the rope into small parts, each of mass dm, length dx and with potential energy dV.



The key is that we're only considering the part of the rope that is hanging off the table, which of course, is changing with time.

Now, since we're taking the reference level of the potential energy at the table, the height of the hanging part (relative to the table) is given by the function $-x(t)$, with a negative sign since it's measured below the reference level. So, each small part of the chain has a potential energy:

$$dV = -gxdm$$

If we assume a uniform rope, its mass density is the same at all points and is just given by the total mass of the rope divided by the total length, $\rho = m/l$.

However, we can also express the mass density ρ of any given "small piece" of the rope as the mass of that small piece, dm, divided by the length of that piece, dx, which gives us:

$$\rho = \frac{dm}{dx} \quad \Rightarrow \quad dm = \frac{m}{l}dx$$

So, the potential energy dV now becomes:

$$dV = -gxdm = -\frac{mg}{l}xdx$$

To find the total potential energy of the piece that's hanging off the table, we simply integrate this from 0 to x (since the length of the hanging piece is given by $x(t)$):

$$V = \int_0^x dV = -\frac{mg}{l}\int_0^x xdx = -\frac{mg}{l}\Big|_0^x\left(\frac{1}{2}x^2\right) = -\frac{mg}{2l}x^2$$

We now have both the kinetic energy and the potential energy for the rope, from which we can construct the Lagrangian:

$$L = T - V = \frac{1}{2}m\dot{x}^2 + \frac{mg}{2l}x^2$$

To get the equation of motion, we plug this Lagrangian into the Euler-Lagrange equation for our generalized coordinate x:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$
$$\Rightarrow \quad \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2 + \frac{mg}{2l}x^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{2}m\dot{x}^2 + \frac{mg}{2l}x^2\right) = 0$$
$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}m \cdot 2\dot{x}\right) - \frac{mg}{2l} \cdot 2x = 0$$
$$\Rightarrow \quad m\ddot{x} - \frac{mg}{l}x = 0$$
$$\Rightarrow \quad \ddot{x} = \frac{g}{l}x$$

This is the equation of motion governing the sliding of the rope. Now, this looks quite similar to the simple harmonic motion differential equation (of the form $\ddot{x} = -kx$), but not quite - it's missing a minus sign.

This minus sign is actually quite crucial and because we have a plus sign here, the solution is not a sine function, but actually a hyperbolic cosine function (the full solution can be found in the "Longer Calculations" -section):

$x(t) = x_0 \cosh\left(\sqrt{\frac{g}{l}}t\right)$, where x_0 is the initial length of the rope that is hanging off the table.

From this, we could for example, find the time t_s it takes for the rope to completely fall off the table by plugging in $x(t_s) = l$ (at that point, the whole length l of the rope is hanging off the table, i.e. it has fallen off) and then solve for the sliding time t_s:

$$x(t_s) = x_0 \cosh\left(\sqrt{\frac{g}{l}} t_s\right) = l \quad \Rightarrow \quad t_s = \sqrt{\frac{l}{g}} \operatorname{arccosh}\left(\frac{l}{x_0}\right)$$

Hopefully these examples illustrate the general process of using the Lagrangian formulation. Next, let's dive deeper into various aspects of Lagrangian mechanics!

4. Generalized Momentum

As you start using Lagrangian mechanics and the Euler-Lagrange equation more, there are certain things that are very useful to know - the most notable one probably being the concept of **generalized momentum**.

When you go through calculations in Lagrangian mechanics, there always seems to be a clear pattern - in particular, when taking derivatives of the Lagrangian with respect to the generalized velocities, $\dot{q}_i$, a "momentum-like thing" shows up in the Euler-Lagrange equation.

For example, take a particle moving in the x-direction under some potential $V(x)$. From its Lagrangian, $L = \frac{1}{2} m\dot{x}^2 - V(x)$, we would get:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad \Rightarrow \quad \frac{d}{dt}(m\dot{x}) + \frac{\partial V}{\partial x} = 0$$

Notice the quantity inside the parentheses - it has exactly the form of a linear momentum, $p = mv$.

In fact, this term in the Euler-Lagrange equation - the derivative of the Lagrangian with respect to $\dot{q}_i$ - is how we define momentum in Lagrangian mechanics. This term is the definition of a quantity called **generalized momentum**, which we label as p_i:

$$\frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot{q}_i}} - \frac{\partial L}{\partial q_i} = 0$$

$$\boxed{p_i = \frac{\partial L}{\partial \dot{q}_i}}$$

The index i here refers to which generalized coordinate the momentum is associated with. For example, if we have generalized coordinates x and y in the Lagrangian, we would also have generalized momenta associated with both of these coordinates, defined as:

$$p_x = \frac{\partial L}{\partial \dot{x}}$$
$$p_y = \frac{\partial L}{\partial \dot{y}}$$

> *For every generalized coordinate q_i that the Lagrangian has a time derivative of, there exists a generalized momentum p_i associated with that particular coordinate.*

Now, what's the point of defining such a quantity? Well, first of all, it gives us a much more general definition of what momentum is, instead of $p = mv$.

In Lagrangian mechanics, momentum is a measure of how much the Lagrangian of a system changes for a small change in the velocity in some direction. This is what the derivative tells us.

A nice way to state this can be found in the book No-Nonsense Classical Mechanics by Jakob Schwichtenberg - generalized momentum is a "measure of how responsive a Lagrangian is to changes in velocity".

In many cases, the generalized momenta in a system will indeed be the same as in Newtonian mechanics. However, not always.

When you get into more advanced mechanics, the usual naive form of momentum, $p = mv$, is not going to be enough anymore in certain cases. We'll look at examples of this soon.

The most useful thing about the concept of generalized momentum is that it allows us to define the notion of momentum in more complicated situations where the usual $p = mv$ simply does not work anymore.

Moreover, generalized momentum - the quantity $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$ - encodes all forms of momentum in the same expression; linear momentum, angular momentum and even the momentum of an electromagnetic field.

In particular, when the generalized coordinate q_i we're looking at happens to be an angle, then the generalized momentum associated with this coordinate will be a component of angular momentum.

Now, another useful thing is that defining generalized momentum in the way we did also allows us to generalize the notion of **momentum conservation**. We'll look at this in the part on symmetries and conservation laws.

4.1. Examples of Generalized Momenta

Below I've included some examples of what these generalized momenta may look like in different cases and how they are obtained from the Lagrangian.

Note that the definition of generalized momentum, $p_i = \partial L / \partial \dot{q}_i$ is valid outside of standard mechanical systems too and some of the examples show that.

4.1.1. Generalized Momentum In Curvilinear Coordinates

In Lagrangian mechanics, we will be using **curvilinear coordinates** - coordinate systems in which the basis vectors are not constant - very often. Polar coordinates and spherical coordinates are both examples of curvilinear coordinates.

Now, the question is how the generalized momenta look like and behave in these curvilinear coordinates.

To understand this, let's take a look at polar coordinates. In polar coordinates, since we have two coordinates, we will also have two generalized momenta. It turns out that one of these is the standard **linear momentum**, of the form $p = mv$, and the other is a component of **angular momentum**, $\vec{L} = \vec{r} \times \vec{p}$.

In polar coordinates, r and θ, the Lagrangian for a free particle has the following form (note that we're not including the potential energy here since it's generally a function of position and so it doesn't play any role in the generalized momenta anyway):

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

You'll find the full derivation of where this Lagrangian comes from in the "Longer Calculations" -section.

Now, let's consider the generalized momenta. First, the momentum associated with the r-coordinate:

$$p_r = \frac{\partial L}{\partial \dot{r}} = \frac{1}{2}m \cdot 2\dot{r} = m\dot{r}$$

This is simply the linear momentum, $p = mv$, in the radial direction - nothing too surprising about that! Now, let's consider the momentum associated with the angular coordinate, θ:

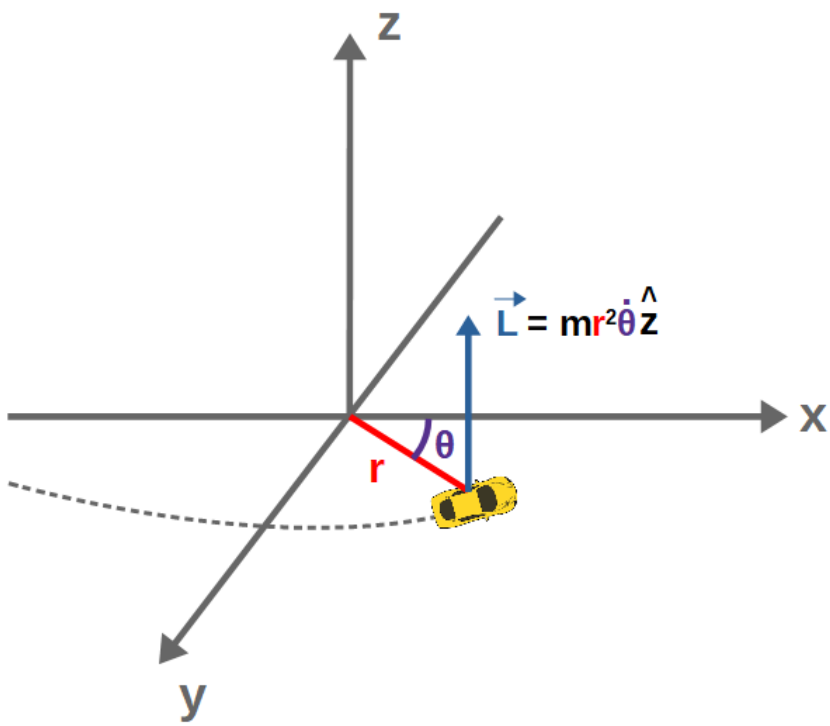
$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}mr^2 \cdot 2\dot{\theta} = mr^2\dot{\theta}$$

First of all, you might notice that this has the units of mass times distance times velocity or in other words, the units of distance times (linear) momentum - exactly the same units as angular momentum.

Indeed, this quantity, $mr^2\dot{\theta}$, *is* the angular momentum or one of the components of angular momentum (relative to the point $r = 0$) - it depends on both the radial distance and the angular speed, just like angular momentum would. But in which direction is this angular momentum component in?

Perhaps not surprisingly, it's the z-component of the angular momentum vector, $\vec{L} = \vec{r} \times \vec{p}$. Now, polar coordinates are technically only defined on a plane, so this component of angular momentum would point "outside of your screen". Moreover, the z-component is the only component of angular momentum in this case.

An intuitive way to understand this is that the plane of rotation is the x, y -plane when using polar coordinates and the angular momentum always points perpendicular to the plane of rotation - therefore, it must point in the z-direction.



We can also understand this mathematically by converting this quantity, $p_\theta = mr^2\dot{\theta}$, to Cartesian coordinates. We already know that $r^2 = x^2 + y^2$ when expressed in terms of Cartesian coordinates, but what would $\dot{\theta}$ be in terms of x and y?

Well, let's use the relations between Cartesian and polar coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

In particular, let's divide y by x to get:

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \quad \Rightarrow \quad \theta = \arctan\left(\frac{y}{x}\right)$$

We can take the time derivative of this by using the fact that $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$ as well as the chain and product rules:

$$\dot{\theta} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{d}{dt}\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad \dot{\theta} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\frac{d}{dt}y + y\frac{d}{dt}\frac{1}{x}\right)$$

$$\Rightarrow \quad \dot{\theta} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{\dot{y}}{x} - y\frac{\dot{x}}{x^2}\right)$$

Let's now factor out a $1/x^2$-from the denominator:

$$\dot{\theta} = \frac{x^2}{x^2+y^2}\left(\frac{\dot{y}}{x} - \frac{y\dot{x}}{x^2}\right)$$

$$\Rightarrow \quad \dot{\theta} = \frac{1}{x^2+y^2}(x\dot{y} - y\dot{x})$$

The angular momentum $p_\theta = mr^2\dot{\theta}$ will then be:

$$p_\theta = m\left(x^2+y^2\right)\frac{1}{x^2+y^2}(x\dot{y} - y\dot{x}) = m(x\dot{y} - y\dot{x})$$

Do you recognize what this represents? It's the z-component of the cross product between the (Cartesian) position and velocity vectors, $\vec{r} = (x, y, z)$ and $\vec{v} = (\dot{x}, \dot{y}, \dot{z})$, which gives us the z-component of the angular momentum vector in the end:

$$p_\theta = m(\vec{r} \times \vec{v})_z = (\vec{r} \times m\vec{v})_z = (\vec{r} \times \vec{p})_z = L_z$$

Therefore, p_θ is the generalized momentum associated with the θ-coordinate and physically, it represents the z-component of the angular momentum. It also turns out that this quantity is conserved, so:

$$\frac{dp_\theta}{dt} = 0$$

We'll come back to where this comes from when discussing cyclic coordinates.

So, in polar coordinates, the generalized momenta describe both a linear momentum (p_r) as well as an angular momentum (p_θ).

The larger point is that the same applies to other curvilinear coordinates as well - any time we have a coordinate that describes a "distance" of some sorts, the generalized momentum associated with this coordinate will be a linear momentum, while the generalized momenta associated with angular coordinates will be components of angular momentum.

Just to really hammer the point home, let's take a quick look at how the same kind of thing appears when working with spherical coordinates (r,θ,φ). The Lagrangian for a free particle (no potential energy) is given by:

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2 + r^2\dot{\theta}^2 \sin^2\varphi\right)$$

Since we have three coordinates, we will have three generalized momenta. These are given by:

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$
$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}\sin^2\varphi$$
$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi}$$

The first two of these represent the same things as in polar coordinates, the linear momentum in the r-direction as well as the z-component of the angular momentum. Note, however, that z-component of angular momentum, p_θ, has an extra factor of $\sin^2\varphi$.

Moreover, the angular momentum p_θ is also conserved in this case (again, we'll be able to understand why that is in a later part).

What is new here is the generalized momentum associated with the φ-coordinate, p_φ. This looks like it has the same form as the z-component of angular momentum we saw in the polar coordinate case, but it depends on the angular speed $\dot{\varphi}$ instead.

The quantity p_φ is NOT the z-component of angular momentum (p_θ is, like explained above), it actually turns out to be a combination of the x- and y-components of angular momentum. You can prove this if you wish by converting the expression $p_\varphi = mr^2\dot{\varphi}$ to Cartesian coordinates.

So, in spherical coordinates, the generalized momenta are the linear momentum in the r-direction (p_r), the z-component of angular momentum (p_θ) as well as a combination of the x- and y-components of angular momentum (p_φ). Only one of these, the z-component of angular momentum, p_θ, is conserved.

4.1.2. Relativistic Momentum

Our next example is going to be perhaps simpler to understand than angular momentum, but it requires some concepts from special relativity.

In special relativity, the Lagrangian for a free particle (of mass m) has a slightly different form than the one in classical mechanics, namely:

$L = -mc^2\sqrt{1 - \frac{1}{c^2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)}$, where c is the speed of light.

Here, I've written the Lagrangian in Cartesian coordinates. It's possible to prove that this Lagrangian actually reduces to the classical free Lagrangian $\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$ in the limit that the velocity components are small compared to c. We won't, however, do this here. If you want to know more about where this Lagrangian comes from, you can check out this article: https://profoundphysics.com/special-relativity-for-dummies-an-intuitive-introduction/.

Now, what we're interested in here are the generalized momenta. The generalized momentum associated with the x-coordinate would be:

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} \\ &= -mc^2 \frac{1}{2\sqrt{1 - \frac{1}{c^2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)}} \cdot \frac{\partial}{\partial \dot{x}}\left(1 - \frac{1}{c^2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)\right) \\ &= -mc^2 \frac{1}{2\sqrt{1 - \frac{1}{c^2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)}} \cdot \left(-\frac{2\dot{x}}{c^2}\right) \\ &= \frac{m\dot{x}}{\sqrt{1 - \frac{1}{c^2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)}} \end{aligned}$$

In special relativity, we often call this square root factor by a special name - the Lorentz factor, which is defined as:

$$\gamma = \frac{1}{\sqrt{1 - \frac{1}{c^2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)}} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Using this, the x-momentum would be:

$$p_x = \gamma m \dot{x}$$

Similarly, we can find the y- and z-components of momentum to be:

$$p_y = \gamma m \dot{y}$$
$$p_z = \gamma m \dot{z}$$

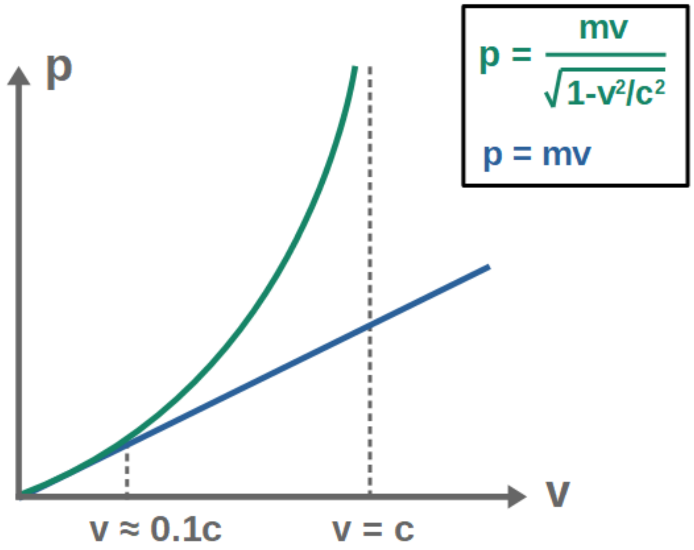
We could indeed represent these as a single vector:

$\vec{p} = \gamma m \vec{v}$, where $\vec{v}$ is the usual velocity vector $\vec{v} = (\dot{x}, \dot{y}, \dot{z})$ in Cartesian coordinates.

Now, what does this result tell us? Well, it tells us that in special relativity (i.e. for objects with velocities close to the speed of light), the momentum of an object is not $\vec{p} = m\vec{v}$, but contains a "correction factor" γ, which crucially also *increases* as a function of the object's velocity.

So, in special relativity, the momentum and velocity of an object do NOT have a simple *linear* relation $p = mv$, but rather a more complicated *non-linear* relation of the form $p = mv / \sqrt{1 - v^2 / c^2}$.

In fact, the relativistic momentum begins to increase much more rapidly than the classical momentum as the velocity of an object gets large. You can see a graph of this below:



The velocity above which the relativistic momentum starts to increase more rapidly than the classical momentum is around 10 % of the speed of light, as shown in the graph.

Moreover, the relativistic momentum blows up to infinity as the velocity of an object approaches the speed of light c - highlighting the fact that no massive object can ever reach the speed of light as this would require an infinite amount of momentum (which, of course, is unphysical).

For our purposes, the point here is that we were able to derive the relativistic momentum formula from the definition of generalized momentum.

This highlights an even more general point - in more advanced areas of physics, the definition of generalized momentum $p_i = \partial L / \partial \dot{q}_i$ *is the **fundamental definition of momentum*** (although in quantum mechanics and field theories this works slightly differently).

4.1.3. Generalized Momentum In an Electromagnetic Field

Our last example of generalized momenta will highlight the fact that generalized momentum is NOT always the same as the linear momentum we see in Newtonian mechanics.

In particular, if we have a charged particle (with mass m and electric charge q) in an electromagnetic field, its interaction with the electromagnetic field will be described by the following Lagrangian:

$$L = \frac{1}{2}mv^2 - q\phi + q\vec{v} \cdot \vec{A}$$

Here, ϕ and $\vec{A}$ are the electric scalar potential and the magnetic vector potential - these encode how the particle interacts with the electric and magnetic fields. We'll take a much deeper look at where this Lagrangian comes from in Part 7.

In this Lagrangian, we basically see a kinetic energy term and two peculiar, velocity-dependent potential terms (we'll learn how to deal with velocity-dependent potentials in more detail in the next part).

The interesting feature for us here is indeed the fact that the potential is velocity-dependent. In particular, the magnetic vector potential seems to "couple" to the particle's velocity - highlighting the fact that magnetic fields only affect moving charged particles.

However, this also leads to interesting consequences for the generalized momenta in such a system. The components of generalized momenta (p_x, p_y and p_z) in this case would be:

$$
\begin{aligned}
p_x &= \frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x \\
p_y &= \frac{\partial L}{\partial \dot{y}} = m\dot{y} + qA_y \\
p_z &= \frac{\partial L}{\partial \dot{z}} = m\dot{z} + qA_z
\end{aligned}
$$

We could write these as a vector:

$$\vec{p} = m\vec{v} + q\vec{A}$$

Now, because the magnetic vector potential term in the Lagrangian couples to the particle's velocity, this leads to the fact that the generalized momenta will contain terms that involve the magnetic vector potential.

Perhaps this isn't entirely surprising - remember, it was stated earlier that generalized momentum is basically a quantity that encodes the "sensitivity of the Lagrangian to changes in velocity".

Well, since the effect of a magnetic field on a charged particle is stronger the faster the particle is moving - this is how magnetic forces generally work - then we would also expect these effects to somehow show up when calculating the "sensitivity of the Lagrangian to changes in velocity" (generalized momenta). This is indeed what happens.

Interestingly, the extra vector potential term in the generalized momentum, $q\vec{A}$, actually describes the momentum of the electromagnetic field itself (yes, an electromagnetic field also carries momentum!).

The key point here was to show you that generalized momenta and ordinary momenta are not always the same. The concept of generalized momentum is, well, a more general definition that describes the components of ALL the momenta in a system, not just a particle's linear momentum components.

We can clearly see this from the example shown here, since the components of generalized momenta contained both the momentum of the particle itself, but also the momentum of the electromagnetic field itself - describing components of the *total* momentum in the system.

5. Constrained Dynamics

In Part 3, we explored constrained optimization and the **Lagrange multiplier method** in the context of variational calculus. Since most of Lagrangian mechanics is based on variational calculus, it's natural to ask what role constraints and Lagrange multipliers play specifically in Lagrangian mechanics. This is what we'll explore next.

5.1. Constraints In The Lagrangian Formulation

In variational calculus, we used constraints to find the extrema of a functional under some particular **constraint**. In Lagrangian mechanics, solutions to the equations of motion of a system are the extrema of the action functional.

Therefore, constraints are used in the Lagrangian formulation to constrain the solutions to an equation of motion - to restrict the dynamics of a physical system in some way.

Now, in Newtonian mechanics, we have to introduce **constraint forces** to constrain the description of a system - to make sure that our mathematical models of a given physical system behave in exactly the way they should (such as an object not falling through the ground even though gravity is pulling it that way).

The key thing to realize is that in Lagrangian mechanics, we do not need constraint forces to constrain systems - by a clever choice of generalized coordinates, we can **implicitly encode any constraints into the coordinates** themselves and thus, not need to manually introduce constraint forces.

We already discussed generalized coordinates earlier, but we didn't really get into how we can choose them such that they are consistent with the constraints in a particular system.

We do this by constructing a **constraint function** involving our generalized coordinates of the form $g(q_1, q_2, q_3, ..., q_i, t) = 0$, similar to what we did in variational calculus. We'll talk more about this soon.

We can then either use this relation to solve for one of the coordinates in terms of the others and plug that straight into the Lagrangian (thus, implicitly encoding the constraint into the Lagrangian) or we can use the constraint function when applying the Lagrange multiplier method. In the end, both will give us the same, constrained equations of motion.

Now, in variational calculus, the Euler-Lagrange equation with constraints had the form $\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = \lambda\left(\frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'}\right)$, where λ was a parameter called a **Lagrange multiplier**.

In Lagrangian mechanics, we've replaced the f with the Lagrangian L and $y(x)$'s with the coordinates $q_i(t)$. Note, however, that we will only consider constraints here that do not depend on velocity (these are called holonomic), so $\partial g / \partial \dot{q}_i = 0$. Thus, the **Euler-Lagrange equation with constraints** that we'll be using in Lagrangian mechanics is:

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \lambda\frac{\partial g}{\partial q_i}}$$

With multiple constraints, we simply sum over all the constraint equations and Lagrange multipliers on the right-hand side, i.e. $\sum_j \lambda_j \frac{\partial g_j}{\partial q_i}$.

Before we look at constrained dynamics in more detail, I want to really highlight the big picture here. In Lagrangian mechanics, there are generally two ways to constrain a system:

1. **Pick the generalized coordinates for a system in such a way that they implicitly encode the constraints by using the constraint equation**. This way you end up with a constrained Lagrangian that you can then use to find the

equations of motion without needing to add any constraint forces.

2. **Derive the equations of motion for the system using an unconstrained Lagrangian but by adding additional Lagrange multiplier terms to the Euler-Lagrange equation**. This will give you the equations of motion as well as expressions for the necessary constraint forces.

Most often, the first way is going to be easiest if we only find the equations of motion (which are consistent with our constraints, of course), but do not care about the constraint forces themselves. If this is the case, you don't need any Lagrange multipliers or anything additional - just follow the procedure we've used previously.

However, there are situations where finding explicit expressions for the constraint forces themselves will be useful (we'll see some examples of this later), for example, if we wish to know when a constraint in a system breaks down. For this, you have to use the Lagrange multiplier method.

5.2. Constraint Equations & Types of Constraints

In Lagrangian mechanics, whenever we want to constrain a system in some way, we will generally begin by writing down a **constraint equation**. We need a constraint equation both in order to choose generalized coordinates that are consistent with the constraints, but also for the Lagrange multiplier method to work.

Constraint equations are simply equations that relate the coordinates describing a given system to the constraints that the system must obey. Constraint equations can be used to find relations between coordinates and simplify the choice of generalized coordinates.

We usually want to encode the constraint into a function of the generalized coordinates q_i, and possibly time, and this is called a **constraint function** g (more precisely, this would be called a holonomic constraint; I'll explain what this means later).

The constraint itself is then given by setting this function equal to zero, which gives us an equation - called a *constraint equation* - that relates the coordinates together in some way:

$$g(q_1, q_2, q_3, \ldots, q_i, t) = 0$$

This can always be done for any equation by just moving every term to one side. We'll see why this is useful a bit later.

An example of a constraint equation might be a particle constrained to move on the surface of a sphere, in which case the coordinates (x, y, z) of the particle must satisfy the relation (R is the radius of the sphere):

$$x^2 + y^2 + z^2 = R^2 \quad \Rightarrow \quad \underbrace{x^2 + y^2 + z^2 - R^2}_{\text{This is the "constraint function"}} = 0$$

This is the "constraint function"
$g(x,y,z) = x^2 + y^2 + z^2 - R^2$

If we chose our generalized coordinates as spherical coordinates (r, θ, φ) instead of Cartesian, the constraint function would be as simple as $g(r, \theta, \varphi) = r - R$ and the constraint equation simply $g(r, \theta, \varphi) = 0$, in other words, $r = R$.

Now, what do you actually do with these constraint equations?

Well, simply put, these are equations you can use to simplify your choice of generalized coordinates for the system. If you use the constraint equations cleverly, you may be able to choose your generalized coordinates in such a way that they *implicitly* contain all the constraints.

You could then use these cleverly chosen generalized coordinates to find the equations of motion for the system, which will automatically obey the constraints.

The way to do this is quite straightforward; if you have a constraint equation that relates the coordinates in some way, you can use that equation to express one of the coordinates in terms of the others. In other words, if we have constraints in a system, not all generalized coordinates will be independent of one another.

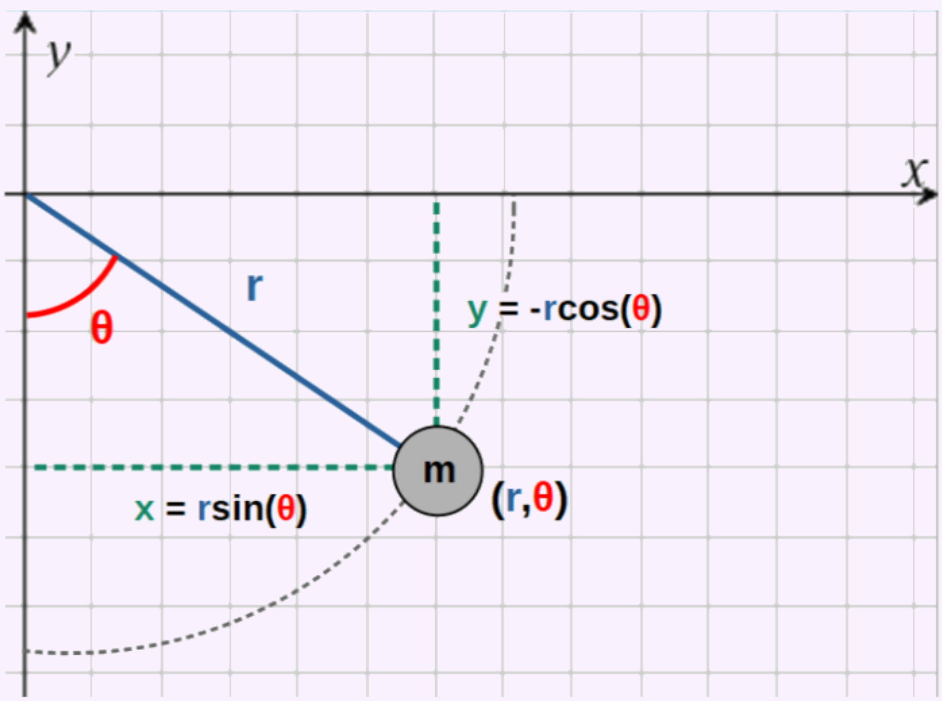
You can then eliminate one of the coordinates using the constraint equation, which will make sure that the constraint is then implicitly encoded into your description of the system - then, the Lagrangian you write down using these coordinates you've chosen will automatically be consistent with the constraints and so will the equations of motion.

Example: Constraint Equation For The Simple Pendulum

You might have noticed that we already used the notion of a constraint equation in our examples earlier. In fact, we used constraint equations in both the simple pendulum example as well as the spherical pendulum example.

We indeed found the equations of motion that were consistent with the constraints of the pendulum motion, without needing any constraint forces. Let's quickly revisit how we did this to illustrate the point.

Let's again place the pendulum in an x, y-coordinate system and express the coordinates of the bob using polar coordinates (r and θ):



Now, we have a constraint in this example - the distance of the pendulum bob to the origin (in other words, the coordinate r) has to be a constant l. Therefore, we have a constraint equation:

$$r = l$$

This can be expressed using a constraint function as $g(r, \theta) = r - l = 0$.

If we were to just write down the coordinates of the pendulum bob without taking into account this constraint, we'd have:

$$x = r\sin\theta$$
$$y = -r\cos\theta$$

However, we now have a constraint equation we can use to eliminate one of our coordinates, namely the r-coordinate. Using this constraint ($r = l$), the coordinates of the pendulum bob now become:

$$x = r\sin\theta \quad \Rightarrow \quad x = l\sin\theta$$
$$y = -r\cos\theta \quad \Rightarrow \quad y = -l\cos\theta$$

The nice thing is that we've now eliminated the r-coordinate completely and we have the coordinates of the pendulum expressed in terms of only one generalized coordinate, θ.

But even better, this choice of generalized coordinates is now also consistent with the constraint (the length of the rod must be l), because the distance from the center for all values of θ is:

$$\sqrt{x^2 + y^2} = \sqrt{l^2\sin^2\theta + l^2\cos^2\theta} = \sqrt{l^2\left(\sin^2\theta + \cos^2\theta\right)} = l$$

Thus, the equations of motion we get by using these coordinates will also obey the constraint.

Now, this is of course, what we did previously already. However, this simple example just illustrates a more general point - we can use constraint equations to eliminate coordinates and encode the constraints themselves into our choice of generalized coordinates.

The beautiful thing about this is that we are then able to obtain the correct equations of motion for the system (that automatically obey the constraints) simply by choosing our generalized coordinates such that they implicitly contain the constraints.

This then allows us to find the equations of motion without ever needing to introduce any constraint forces (such as the tension in the pendulum example).

Now, this method of using the constraint equations when choosing the generalized coordinates for a system is one way to include constraints in Lagrangian mechanics.

As mentioned earlier, this method works great if you only care about finding the correct equations of motion, but not necessarily if we wanted to know something about the **constraint forces** themselves. This method will completely eliminate the need for any constraint forces, so you won't be able to get information about those this way.

You'll simply get the correct equations of motion WITHOUT needing to introduce any constraint forces, which is often what we want in order to describe the system.

However, we can also find constraint forces using the Lagrangian formulation if we want to. This is explained later, but first we need to discuss the different types of constraints that are possible.

5.2.1. Holonomic vs Non-Holonomic Constraints

Generaly, there are two types of constraints we'll have in Lagrangian mechanics: **holonomic constraints** and **non-holonomic constraints**.

Holonomic constraints are constraints that can be written as an equality between coordinates and time, in other words, as a constraint equation of the form:

$$g(q_1, q_2, q_3, \ldots, q_i, t) = 0$$

An example of a holonomic constraint would be a particle moving in a circle, in which case its coordinates would be constrained by $x^2 + y^2 = R^2$. This is the type of constraint we saw previously with the pendulum as well.

Any constraint that cannot be expressed as an equality between the coordinates (and possibly time) is a non-holonomic constraint.

Now, holonomic constraints are always expressed as equalities involving the coordinates and possibly time, however, NOT the velocities.

Moreover, holonomic constraints are sometimes categorized into two "sub-types":

- **Scleronomic constraint** - the constraint equation does not explicitly depend on time, so it has the form $g(q_1, q_2, q_3, \ldots, q_i) = 0$.
- **Rheonomic constraint** - the constraint equation does explicitly depend on time, so it has the form $g(q_1, q_2, q_3, \ldots, q_i, t) = 0$.

Now, this is just terminology you should be aware of in case you ever encounter it, but there really isn't much difference between these two types of holonomic constraints in practice - the most important thing is that they are both holonomic.

An example of a non-holonomic constraint, on the other hand, would be a particle allowed to move anywhere inside a circle, in which case its coordinates would be constrained by $x^2 + y^2 \leq R^2$.

So, non-holonomic constraints can be inequalities.

Also, any constraint that involves the velocities or differentials of the coordinates (but cannot be integrated to obtain the "full constraint equation" that involves the coordinates themselves) is a non-holonomic constraint.

You'll find an example of where such a constraint comes up in the case of describing the physics of a unicycle.

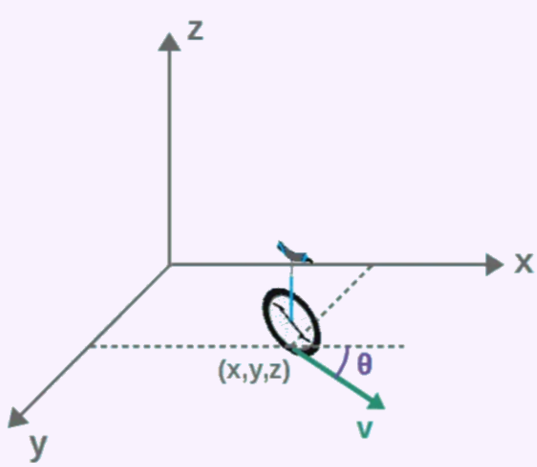
Non-Holonomic Constraint Example: Motion of a Unicycle

The motion of a unicycle is a classic example of where non-holonomic constraints show up in the form of constraints for the velocities, but not for the coordinates themselves.

Let's place the unicycle in an x, y, z -coordinate system, so that as it moves, its position is going to be described by some coordinates (x, y, z).

Now, the special thing about a unicycle is that it can also have an "orientation", meaning that at any point, the unicycle can be pointed in a different direction.

We'll label this orientation by an angle θ. This is an extra degree of freedom in the motion of the unicycle, so at this point, we actually need four degrees of freedom (however, we can reduce this by using some constraints).



The vector $\vec{v}$ here describes the velocity of the unicycle.

If the unicycle is not allowed to "jump" up and down, we have $z = 0$ at all times (this is a constraint), which reduces the degrees of freedom to just three - the position coordinates x and y as well as the orientation θ. These will be our generalized coordinates, which are all functions of time.

However, while the angle θ can point in any direction at different points (meaning that there is no straightforward relation or constraint between θ, x and y), the components of the velocity cannot just be anything.

This is because the direction of the velocity $\vec{v}$ is determined by the orientation of the unicycle, in other words, by the angle θ.

Thus, the components of the velocity, $\dot{x}$ and $\dot{y}$, are given by:

$$\begin{aligned} \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \end{aligned}$$

, where v is the magnitude of $\vec{v}$.

If we divide the second equation by the first, we get:

$$\frac{\dot{y}}{\dot{x}} = \frac{v\sin\theta}{v\cos\theta} \quad \Rightarrow \quad \dot{y} = \dot{x}\tan\theta$$

Here we basically have a constraint for the velocities. This is an example of a non-holonomic constraint, since it can be expressed in the form of a function of velocities (and coordinates):

$g(\theta, \dot{x}, \dot{y}) = 0$, with $g(\theta, \dot{x}, \dot{y}) = \dot{y} - \dot{x}\tan\theta$

Now, the thing about such a constraint is that in general, it cannot be used to reduce the degrees of freedom in the system (i.e. solve for one of the coordinates in terms of the others), since we cannot write this in the form of a function of the coordinates only.

You can certainly try to express the above non-holonomic constraint in terms of the coordinates only by integrating both sides with respect to time:

$$\dot{y} = \dot{x}\tan\theta \quad \Rightarrow \quad y = \int \dot{x}\tan\theta dt$$

The problem here is that both $\dot{x}$ and θ are independent functions of time. We therefore cannot do the integral on the right-hand side without knowing $x(t)$ and $\theta(t)$ beforehand and thus, we cannot get a relation between the coordinates from this.

This is one of the key points about non-holonomic constraints; they do not allow us to reduce the degrees of freedom even though they technically do constrain the system in some way.

Now, the situation would change if the angle θ is assumed to NOT depend on time. In other words, the angle θ just has some constant value θ_0 at all times (physically, this means that the unicycle cannot turn its direction).

In this case, we could integrate the constraint to get:

$$y = \int \dot{x} \tan \theta_0 dt = \tan \theta_0 \int \frac{dx}{dt} dt = \tan \theta_0 \int dx = x \tan \theta_0$$

I've set the integration constant to zero here.

So, we do get a relation between the only coordinates in this case, so actually, our non-holonomic constraint turns into a holonomic constraint for this special case:

$g(x, y) = 0$, with $g(x, y) = y - x \tan \theta_0$

Note that in this case, $\theta = \theta_0$ at all times means that θ is not a coordinate anymore as it cannot depend on time.

In this special case, we would only need one generalized coordinate to describe the system, the coordinate x, since we can express y in terms of x using the constraint equation.

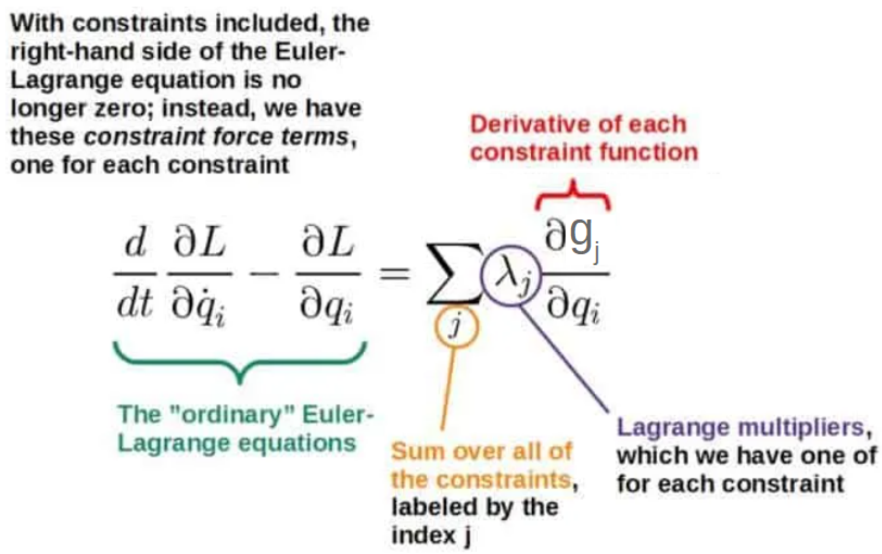
For the rest of this part, we'll only be discussing holonomic constraints as these are the most common types of constraints and also the most useful ones, generally speaking.

5.3. How To Find Constraint Forces

So far, we've only discussed how to implicitly encode constraints into the generalized coordinates describing a system and how doing this leads to the correct equations of motion that obey the constraints.

However, doing it this way does not actually give you the **constraint forces** themselves. So, if we wanted to find the actual constraint forces, what would we do in that case?

Well, we do this by using the Lagrange multiplier method we discussed already in Part 3. When using this method, we have to use the modified Euler-Lagrange equations with additional Lagrange multiplier terms on the right-hand side:



Intuitively, the right-hand side here represents the constraint forces, so these Lagrange multipliers will give you information about how the constraint forces behave. We'll understand this better in the next part when we discuss generalized forces in detail.

Before we discuss Lagrange multipliers in more detail, the rough step-by-step process of how constraint forces can be found in Lagrangian mechanics goes as follows:

1. Define the constraints by writing down a constraint equation for each constraint.

2. Define a set of generalized coordinates *without* applying the constraints yet.

3. Write down the Lagrangian (again, with no constraints applied).

4. Apply the modified Euler-Lagrange equations involving Lagrange multipliers.

5. You should now have the unconstrained equations of motion for each coordinate with the equations involving the Lagrange multipliers.

6. At this point, apply your constraint equations to the equations of motion and solve for the Lagrange multipliers, which will give you the constraint forces.

Essentially, the most important part in this process is step #2 and the fact that we do NOT use the constraints yet when first defining our generalized coordinates. Essentially, we leave our generalized coordinates unconstrained for a little bit.

This kind of a “trick” allows us to find the **unconstrained equations of motion** with Lagrange multipliers, to which we will then apply the constraints later on. Doing it this way allows us to actually find the constraint forces (this will become clear later when we look at various examples).

In practice, when we write down the *unconstrained* Lagrangian in step #3, we will essentially get “additional” equations of motion (which would not be there if we chose our generalized coordinates to implicitly contain the constraints).

We can then use these “additional” equations of motion to solve for the Lagrange multipliers and find the constraint forces that way.

So, we’re basically adding in extra degrees of freedom by introducing these Lagrange multipliers, but since we get these “additional” unconstrained equations of motion, we also have more equations. However, we will still have as many equations as degrees of freedom, so the problem is still well-defined.

Now, the constraints themselves are defined by the constraint equations, which we discussed before.

These are again equations that set relationships between the coordinates and we write these equations, generally, in the form $g(q_1, q_2, q_3, \dots, q_i, t) = 0$.

It’s important to write the equation in this form (which you can always do by simply moving everything to one side), because this gives you the correct constraint function, g.

Example: Finding The Tension In a Pendulum

Let's consider the pendulum example again (if the pendulum system is getting a little old at this point, don't worry - we'll see different examples soon).

Earlier, we found the following relations between the polar coordinates (r, θ) and the Cartesian coordinates (x, y) of the pendulum bob:

$$x = r \sin\theta$$
$$y = -r \cos\theta$$

We then used our constraint equation ($r = l$) to essentially eliminate the r-coordinate completely and describe the pendulum with only the θ-coordinate, which did lead us to the correct equations of motion.

We won't do that now. Instead, we'll construct the "unconstrained" Lagrangian for this system (meaning that the radial distance, the r-coordinate is allowed to vary, for now). We'll then use our constraints later, but in this process, we'll be able to find the tension force as well.

If we take the time derivatives of these "unconstrained" x- and y-coordinates, we get (note that r is now also a function of time, so we have to use the product rule):

$$\dot{x} = \dot{r} \sin\theta + r\dot{\theta} \cos\theta$$
$$\dot{y} = -\dot{r} \cos\theta + r\dot{\theta} \sin\theta$$

We can then get the kinetic energy by squaring these. Doing this, you'll end up with the Lagrangian in polar coordinates (see the "Longer Calculations" for this):

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

The potential energy is going to be $V = mgy$, which in terms of our generalized coordinates is $V = -mgr\cos\theta$. Our "unconstrained" Lagrangian is then:

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta$$

Notice that we have two coordinates here, r and θ, so we'll get an Euler-Lagrange equation for each of them. We have to use the modified Euler-Lagrange equations here with the added stuff on the right-hand side:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_j \lambda_j \frac{\partial g_j}{\partial q_i}$$

In our case, we only have one constraint, so this sum over j has just one term:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \lambda \frac{\partial g}{\partial q_i}$$

For the r-coordinate, we have the Euler-Lagrange equation as:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \lambda \frac{\partial g}{\partial r}$$

Let's now recall what our constraint equation was for the pendulum: $r = l$. Therefore, we can write this constraint in terms of the constraint function as:

$g(r) = 0$, where $g(r) = r - l$

The partial derivative of this is then just $\partial g / \partial r = 1$. So, the Euler-Lagrange equation for r becomes:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \lambda$$

By inserting the Lagrangian, we get the equation of motion for the r-coordinate:

$$\frac{d}{dt}(m\dot{r}) - \frac{\partial}{\partial r}\left(\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta\right) = \lambda$$
$$\Rightarrow \quad m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta = \lambda$$
$$\Rightarrow \quad \ddot{r} = \frac{\lambda}{m} + r\dot{\theta}^2 + g\cos\theta$$

Let's then look at the Euler-Lagrange equation for θ:

$d \;\; \partial L \quad \partial L \quad \partial g$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda \frac{\partial g}{\partial \theta}$$

The right-hand side here goes to zero, since our constraint function g does not depend on θ. We then have:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$
$$\Rightarrow \quad \frac{d}{dt}\left(mr^2\dot{\theta}\right) - \frac{\partial}{\partial \theta}\left(\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta\right) = 0$$

Note that we have to use the product rule on the first term here when taking the time derivative. Also, when taking the time derivative of r^2, we have to use the chain rule (so, $dr^2 / dt = 2r\dot{r}$):

$$2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} + mgr\sin\theta = 0$$
$$\Rightarrow \quad \ddot{\theta} = -\frac{g}{r}\sin\theta - \frac{2}{r}\dot{r}\dot{\theta}$$

Notice what we have now - two unconstrained equations of motion:

$$\ddot{r} = \frac{\lambda}{m} + r\dot{\theta}^2 + g\cos\theta$$
$$\ddot{\theta} = -\frac{g}{r}\sin\theta - \frac{2}{r}\dot{r}\dot{\theta}$$

We've essentially obtained the unconstrained equations of motion in order to get this additional equation for the Lagrange multiplier λ. We can NOW apply our constraint ($r = l$), which means that $\dot{r} = \ddot{r} = 0$. Our equations of motion will then become:

$$0 = \frac{\lambda}{m} + l\dot{\theta}^2 + g\cos\theta$$
$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

Do you see what just happened? We first obtained the unconstrained equations of motion and after that, we applied our constraint. This results in the right equation of motion (the one for θ) we've already seen multiple times previously, but we also got an additional equation that we can now solve for λ:

$$0 = \frac{\lambda}{m} + l\theta^2 + g\cos\theta \quad \Rightarrow \quad \lambda = -ml\theta^2 - mg\cos\theta$$

If you now look at the expression we found for the tension force using Newton's laws in Part 1, it is exactly this (apart from a minus sign, which we get here because the tension is "opposing" the radial direction), so:

$$T = -\lambda = ml\dot{\theta}^2 + mg\cos\theta$$

This hopefully illustrates the process of finding constraint forces in Lagrangian mechanics as well as how the Lagrange multipliers describe these constraint forces.

Again, I want to stress that this method only works because we first find the unconstrained equations of motion. Only by doing this will we get enough equations to solve for the constraint forces (Lagrange multipliers).

On the other hand, if you apply the constraints *before* finding the equations of motion, you'll only obtain the equations of motion themselves (which are perfectly correct if these are the only thing you wish to find), but not expressions for the actual constraint forces.

The example above might give you an idea of how Lagrange multipliers describe the constraint forces, but let's discuss this in more general terms next.

5.4. Physical Meaning of The Lagrange Multipliers

Lagrange multipliers, as we saw earlier, are these coefficients we add to the Euler-Lagrange equations generally when dealing with constrained optimization problems. Physically, it seems that Lagrange multipliers are somehow connected to the constraint forces in a system.

The way to intuitively understand this is to once again, connect everything back to what we learned in the part on variational calculus - we learned that Lagrange multipliers essentially tell us how much the extremal values of a given functional (in other words, solutions to the Euler-Lagrange equations) change as the constraint is changed by a small amount.

Now, making this analogy between variational calculus and Lagrangian mechanics, we also gain a very nice physical interpretation of the Lagrange multipliers.

Namely, the Lagrange multipliers physically tell us how much the extremal values of the action functional changes with respect to a small change in the constraint.

Since the extremal values of the action *are* the solutions to the equations of motion for a system, the Lagrange multipliers then essentially describe how much the solutions to the equations of motion, the coordinates $q_i(t)$, change when changing the constraint by a small amount.

But how exactly does this relate to constraint forces? Well, since the constraint function is only a function of position and not velocity (assuming holonomic constraints), changing it would only result in the Lagrangian changing through the potential energy term, but not the kinetic energy term.

So, if the constraint changes and as a result, the potential energy changes, then there must be a force that would change the system's potential energy (since a force can be expressed as derivatives of potential energy) - this is exactly the constraint force.

This is what gives the interpretation of the Lagrange multipliers as being closely related to the constraint forces in a system. Of course, this isn't a very rigorous way of coming to this conclusion but it might give some useful intuition - take what you will of this argument, however, in the next part we'll be able to understand this more precisely.

Now, while Lagrange multipliers indeed contain information about the constraint forces in a given system, an extremely important point to note is that the Lagrange multipliers themselves are NOT the actual, physical constraint forces.

Yes, they do contain information about the constraint forces but by themselves, they are not the constraint forces. In fact, by themselves, the Lagrange multipliers are just some parameters that may or may not have any physical meaning.

The actual constraint forces are given by ***generalized*** **constraint forces**, which is the term appearing on the right-hand side of the Euler-Lagrange equations. We'll take a deeper look at this in the next part when discussing generalized forces.

Generalized constraint forces:

$$Q_i^c = \sum_j \lambda_j \frac{\partial g_j}{\partial q_i}$$

In other words, to find constraint forces, we first have to find the Lagrange multipliers and then "combine" them with this partial derivative of the constraint function -term and this is what gives us the actual constraint forces.

For each generalized coordinate q_i, this Q_i^c will give you the component of the total constraint force in the direction of the coordinate q_i, which may or may not be zero.

As an example (which you'll find more of later), consider the Lagrange multiplier we found for the simple pendulum system:

$$\lambda = -ml\dot{\theta}^2 - mg\cos\theta$$

The constraint force components for the pendulum in both the r- and θ-directions would then be (as a reminder, the constraint function here is $g = r - l$):

$$Q_r^c = \lambda\frac{\partial g}{\partial r} = \lambda = -ml\dot{\theta}^2 - mg\cos\theta$$
$$Q_\theta^c = \lambda\frac{\partial g}{\partial \theta} = 0$$

These tell us that there is no constraint force component in the θ-direction, which is to be expected since the motion of the pendulum is not constrained in this direction in any way - the constraint force (tension, in this case) only acts in the radial direction.

5.5. Examples of Constraints

We've discussed a lot of theoretical aspects of constrained dynamics so far. It's now time to look at some more examples of constraints and finding constraint forces in practice using the Lagrangian formulation.

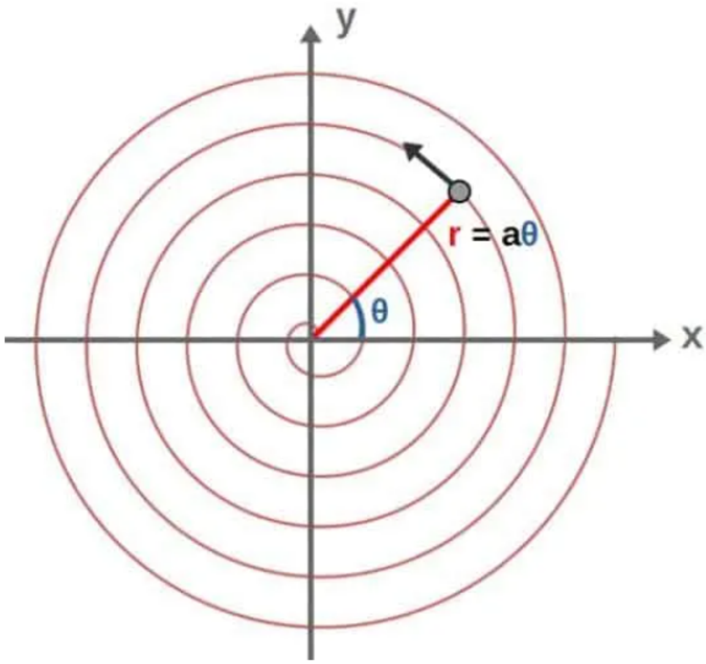
5.5.1. Particle Moving on a Spiral

This example essentially consists of finding the equations of motion and the necessary constraint forces for a particle moving along a spiral trajectory.

Mathematically, we can describe a spiral in polar coordinates by saying that the radial coordinate increases linearly with the angle (this is called an Archimedean spiral). So, in polar coordinates, the spiral is described by the following curve:

$$r(\theta) = a\theta$$

This a here is some proportionality constant with units of length. The trajectory of the particle then looks more or less as follows:



Let's begin by finding the unconstrained Lagrangian of the particle. Since there is no external potential in this case, the Lagrangian just consists of the kinetic energy, which we can write using the standard form of a free Lagrangian in polar coordinates:

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Note that we haven't used the constraint at all yet, so this would be the Lagrangian for a freely moving particle in polar coordinates (not related to any spiral motion). Our constraint function and constraint equation, in this case, can be obtained from the relation $r = a\theta$:

$$g(r,\theta) = r - a\theta = 0$$

If you wanted to only find the equations of motion, you could simply plug $r = a\theta$ into the Lagrangian and find the equations of motion for the θ-coordinate. However, we want to also find the constraint forces in this case.

So, we'll have then two equations of motion, one for the r-coordinate and one for θ. Later on, when we apply our constraint, we can reduce these to just an equation of motion for θ as well as an expression for the constraint force.

Now, the equation of motion for r will be (note that we have only one constraint):

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \lambda\frac{\partial g}{\partial r}$$

$$\Rightarrow \quad \frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 = \lambda$$

$$\Rightarrow \quad \ddot{r} = \frac{\lambda}{m} + r\dot{\theta}^2$$

On the other hand, the equation of motion for θ will be:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda \frac{\partial g}{\partial \theta}$$

$$\Rightarrow \quad \frac{d}{dt}\left(mr^2\dot{\theta}\right) = \lambda(-a)$$

$$\Rightarrow \quad 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} = -a\lambda$$

$$\Rightarrow \quad \ddot{\theta} = -\frac{a\lambda}{mr^2} - \frac{2}{r}\dot{r}\dot{\theta}$$

So, we have the following two *unconstrained* equations of motion:

$$\ddot{r} = \frac{\lambda}{m} + r\dot{\theta}^2$$

$$\ddot{\theta} = -\frac{a\lambda}{mr^2} - \frac{2}{r}\dot{r}\dot{\theta}$$

We will now apply the constraint $r = a\theta$, which means that $\dot{r} = a\dot{\theta}$ and $\ddot{r} = a\ddot{\theta}$. Using these, our two equations become:

$$a\ddot{\theta} = \frac{\lambda}{m} + a\theta\dot{\theta}^2 \quad \Rightarrow \quad \ddot{\theta} = \frac{\lambda}{ma} + \theta\dot{\theta}^2$$

$$\ddot{\theta} = -\frac{a\lambda}{ma^2\theta^2} - \frac{2}{a\theta}a\dot{\theta}\dot{\theta} \quad \Rightarrow \quad \ddot{\theta} = -\frac{\lambda}{ma\theta^2} - \frac{2}{\theta}\dot{\theta}^2$$

Now, since both of these are equations for $\ddot{\theta}$, they must be equal and thus, we have:

$$\frac{\lambda}{ma} + \theta\dot{\theta}^2 = -\frac{\lambda}{ma\theta^2} - \frac{2}{\theta}\dot{\theta}^2$$

We can now solve this for the Lagrange multiplier λ:

$$\lambda = -\frac{2\theta + \theta^3}{1 + \theta^2} ma\dot{\theta}^2$$

Then, we can insert this into either of the equations of motion for $\ddot{\theta}$ to obtain:

$$\ddot{\theta} = \frac{\lambda}{ma} + \theta\dot{\theta}^2$$
$$\Rightarrow \quad \ddot{\theta} = -\frac{1}{ma}\left(\frac{2\theta + \theta^3}{1+\theta^2} ma\dot{\theta}^2\right) + \theta\dot{\theta}^2$$
$$\Rightarrow \quad \ddot{\theta} = \left(-\frac{2\theta + \theta^3}{1+\theta^2} + \theta\right)\dot{\theta}^2$$
$$\Rightarrow \quad \ddot{\theta} = -\frac{\theta}{1+\theta^2}\dot{\theta}^2$$

We then have the equation of motion for the particle as well as an expression for the Lagrange multiplier:

$$\ddot{\theta} = -\frac{\theta}{1+\theta^2}\dot{\theta}^2$$
$$\lambda = -\frac{2\theta + \theta^3}{1+\theta^2} ma\dot{\theta}^2$$

The components of the constraint forces in both the r- and θ-directions are then:

$$Q_r^c = \lambda\frac{\partial g}{\partial r} = -\frac{2\theta + \theta^3}{1+\theta^2} ma\dot{\theta}^2$$
$$Q_\theta^c = \lambda\frac{\partial g}{\partial \theta} = \frac{2\theta + \theta^3}{1+\theta^2} ma^2\dot{\theta}^2$$

This basically solves the problem - we have the equations of motion as well as an expression for what the constraint force has to be in order for the particle to stay on the spiral. This wasn't too complicated using Lagrangian mechanics, but good luck doing the same with Newton's laws!

The unfortunate thing is that the equation of motion for this system does not have a useful closed-form solution. In fact, we can manipulate and integrate the equation of motion for θ to get the following expression:

$$\theta\sqrt{1+\theta^2} + \text{arcsinh}\,\theta = At + B$$

However, this cannot be solved to find $\theta(t)$. The more important thing in Lagrangian mechanics (and all of classical mechanics, more generally) usually is to just find the equations of motion, not necessarily solve them.

Solving an equation of motion analytically, even if that would be nice, is not really that important since we can solve any differential equation numerically or construct a simulation of the system with a computer anyway.

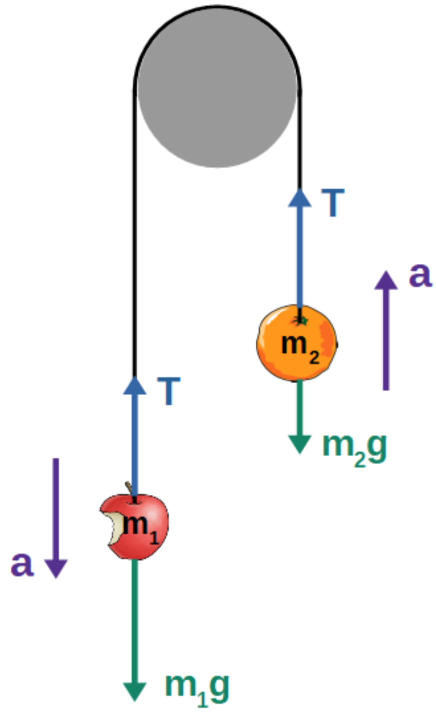
5.5.2. The Atwood Machine

The Atwood machine is a classic example of using Newton's laws and in fact, it's pretty easy to solve using Newton's laws. Keeping with the theme of the book, we're going to solve the Atwood machine using Lagrangian mechanics and the method of Lagrange multipliers just to illustrate the point that both of the formulations do give us the same results in the end.

First, however, let's take a look at how the Atwood machine would be solved using Newton's laws. This allows us to then verify later on that the results we get from Lagrangian mechanics is correct.

Essentially, the Atwood machine consists of two masses (m_1 and m_2) connected by a cable on a pulley. We then want to find out how these masses behave when gravity pulls both of them downwards.

The forces in question (assuming no friction) will be the force of gravity, given by mg, as well as the tension in the cable. The key here is that the tension acting on both of the masses as well as the acceleration of both masses will be the same since they are connected by the same cable (this is a constraint on the system, in some sense, and we'll formulate it into one when we get to solving this problem with Lagrangians):



As a sidenote, don't ask why these masses hanging on the pulley are fruits here.

Anyway, if we write down Newton's laws in the vertical direction for both the masses, choosing upwards as the positive direction, we get two equations (notice that the accelerations will have different signs):

$$T - m_1 g = -m_1 a$$
$$T - m_2 g = m_2 a$$

Let's take the first equation, subtract the second equation from it and then solve for a:

$$T - m_1 g - (T - m_2 g) = -m_1 a - m_2 a$$
$$\Rightarrow \quad -m_1 g + m_2 g = -m_1 a - m_2 a$$
$$\Rightarrow \quad a = \frac{m_1 - m_2}{m_1 + m_2} g$$

This is the acceleration of both masses (it's a constant), which we could integrate twice to get the positions of the masses. Now, if we substitute this into one of our Newton's force equations, we can solve for the tension as well:

$$T - m_2 g = m_2 a$$
$$\Rightarrow \quad T - m_2 g = m_2 \frac{m_1 - m_2}{m_1 + m_2} g$$

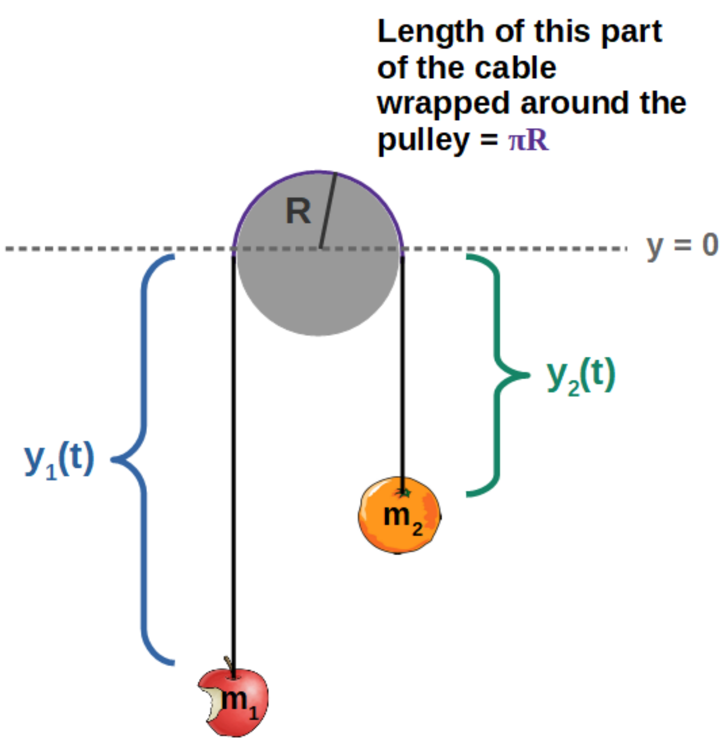
$$\Rightarrow \quad T = m_2 \frac{m_1 - m_2}{m_1 + m_2} g + m_2 g$$

$$\Rightarrow \quad T = \frac{2 m_1 m_2}{m_1 + m_2} g$$

So, we've obtained the acceleration and tension using Newton's laws, which pretty much tell us anything we want about the system. Now, let's see how we would solve the problem using Lagrangian mechanics.

In fact, this turns out to be a constraint problem when we use Lagrangian mechanics. Can you see what the constraint here would be? Well, it's the fact that the total length of the cable must be a constant at all times - we're assuming that the cable doesn't get stretched when the masses move up or down.

But first, let's choose some generalized coordinates to describe the system with. We can place the Atwood machine in an x, y-coordinate system and describe the vertical coordinates of the masses as y_1 and y_2, which are both functions of time, of course:



We're also calling the radius of the pulley thing as some constant R. The value of this constant is not really important, but we need it to describe the length of this part of the cable that is wrapped around the pulley (which is just the circumference of a half-circle, πR).

So, we have our two generalized coordinates, $y_1(t)$ and $y_2(t)$. But we also have a constraint - the total length of the whole cable must be constant, which we'll call l.

This l is going to be the sum of both of the hanging parts of the cable (given by y_1 and y_2) as well as the length of the part that's wrapped around the pulley, πR:

$$l = y_1 + y_2 + \pi R$$

This is our constraint equation, which relates the two coordinates y_1 and y_2. Our constraint function (which is defined by $g(y_1, y_2) = 0$) is therefore:

$$g(y_1, y_2) = y_1 + y_2 + \pi R - l$$

Let's now find the *unconstrained* Lagrangian for this system. The total kinetic energy is simply the sum of the kinetic energies of both the masses:

$$T = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2$$

The total potential energy will be given by the sum of the gravitational potential energies of both masses (note the minus signs since we've chosen the coordinates y_1 and y_2 to be measured below the $y = 0$ level):

$$V = -m_1 g y_1 - m_2 g y_2$$

The Lagrangian is therefore:

$$L = T - V = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2 + m_1 g y_1 + m_2 g y_2$$

We will have two equations of motion from the Euler-Lagrange equations (with constraints involving a Lagrange multiplier λ, which is the same for both equations) - one for y_1 and one for y_2. These are given by:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{y}_1} - \frac{\partial L}{\partial y_1} = \lambda \frac{\partial g}{\partial y_1}$$
$$\Rightarrow \quad m_1 \ddot{y}_1 - m_1 g = \lambda$$
$$\Rightarrow \quad \ddot{y}_1 = \frac{\lambda}{m_1} + g$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_2} - \frac{\partial L}{\partial y_2} = \lambda\frac{\partial g}{\partial y_2}$$

$$\Rightarrow \quad m_2\ddot{y}_2 - m_2 g = \lambda$$

$$\Rightarrow \quad \ddot{y}_2 = \frac{\lambda}{m_2} + g$$

These are the two unconstrained equations of motion. We will now use our constraint equation to express the y_2-coordinate in terms of y_1:

$$l = y_1 + y_2 + \pi R \quad \Rightarrow \quad y_2 = l - y_1 - \pi R$$

From this, we can see that $\ddot{y}_2 = -\ddot{y}_1$ simply by differentiating the constraint equation twice on both sides. Inserting this into the second equation of motion, we get:

$$\ddot{y}_2 = \frac{\lambda}{m_2} + g \quad \Rightarrow \quad -\ddot{y}_1 = \frac{\lambda}{m_2} + g$$

We can then insert the equation $\ddot{y}_1 = \lambda/m_1 + g$ into this and solve for λ:

$$-\left(\frac{\lambda}{m_1} + g\right) = \frac{\lambda}{m_2} + g$$

$$\Rightarrow \quad \lambda = -\frac{2g}{\frac{1}{m_1} + \frac{1}{m_2}}$$

$$\Rightarrow \quad \lambda = -\frac{2m_1m_2g}{m_1 + m_2}$$

We can now insert this into the equation of motion for y_1:

$$\ddot{y}_1 = \frac{\lambda}{m_1} + g$$

$$\Rightarrow \quad \ddot{y}_1 = -\frac{1}{m_1}\frac{2m_1m_2g}{m_1 + m_2} + g$$

$$\Rightarrow \quad \ddot{y}_1 = \left(1 - \frac{2m_2}{m_1 + m_2}\right)g$$

$$\Rightarrow \quad \ddot{y}_1 = \frac{m_1 - m_2}{m_1 + m_2}g$$

Notice that this is the same expression we got for the acceleration when using Newton's laws. The whole thing on the right-hand side is just a constant, so we can simply integrate this twice to get (note; we're assuming both of the masses to be at rest initially):

$$y_1(t) = \frac{1}{2}\frac{m_1 - m_2}{m_1 + m_2} g t^2 + y_0$$

Here y_0 is the initial position of mass m_1. If we plug this into our constraint equation $l = y_1 + y_2 + \pi R$, we can solve for $y_2(t)$ as well:

$$l = y_1 + y_2 + \pi R$$
$$\Rightarrow \quad l = \frac{1}{2}\frac{m_1 - m_2}{m_1 + m_2} g t^2 + y_0 + y_2 + \pi R$$
$$\Rightarrow \quad y_2(t) = \frac{1}{2}\frac{m_2 - m_1}{m_1 + m_2} g t^2 + l - y_0 - \pi R$$

So, the final solutions for the coordinates as functions of time as well as the Lagrange multiplier are:

$$y_1(t) = \frac{1}{2}\frac{m_1 - m_2}{m_1 + m_2} g t^2 + y_0$$
$$y_2(t) = \frac{1}{2}\frac{m_2 - m_1}{m_1 + m_2} g t^2 + l - y_0 - \pi R$$
$$\lambda = -\frac{2m_1 m_2 g}{m_1 + m_2}$$

The constraint force, i.e. the tension T (which is the same for both of the masses) is given by:

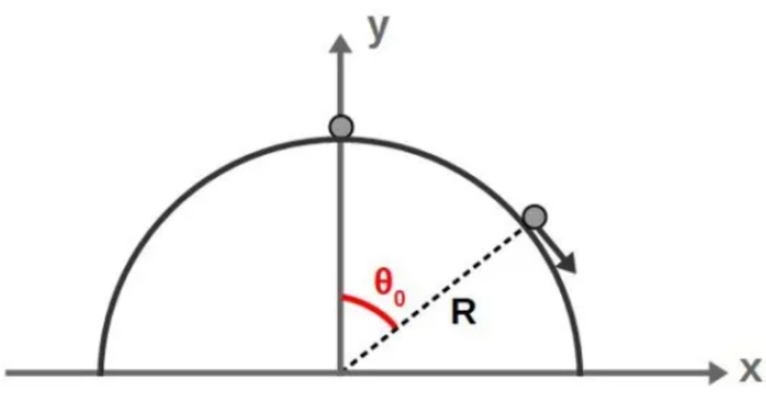
$$T = Q^c_{y_1} = \lambda \frac{\partial g}{\partial y_1} = -\frac{2m_1 m_2 g}{m_1 + m_2}$$

This is the same result we obtained from Newton's laws but with a negative sign instead. This simply comes from the discrepancy between how we chose the direction of acceleration in the Newtonian mechanics case and how we chose our generalized coordinates here. The tension has the same magnitude regardless, which is the important part here.

5.5.3. Particle Sliding off a Sphere

The following example is a particle that begins sliding (initially from rest) down a half-sphere or a dome of some sort. At some point, it will fall off the surface of the sphere and we want to know exactly when the particle leaves the sphere.

In particular, we can describe the particle's position by an angle θ (relative to the vertical) as a generalized coordinate. We therefore want to find the specific angle θ_0 at which the particle falls off the sphere:



This problem can, in fact, be done using Newtonian mechanics as well and the way to do this is to find the normal force of the surface and then look at when that normal force goes to zero, which will give you the angle at which the particle loses contact with the sphere.

However, using Lagrangian mechanics, we have to find the Lagrange multiplier, which turns out to describe the normal force. So, the basic idea to solve this problem is to find an expression for the constraint force (normal force of the surface), we just do it with different methods in the Lagrangian formulation compared to using Newton's laws.

This example also highlights an important application of the Lagrange multiplier method - to be able to not just find the constraint forces, but also analyze when these constraint forces may break down and not apply anymore.

This is because of the fact that in many systems (like our example here), the constraints will only apply up to a certain point, after which they may no longer describe the dynamics of the system accurately and it's often important to know when this happens.

Now, let's begin by finding the Lagrangian for the particle. The Lagrangian turns out to be exactly the same as the Lagrangian for the (unconstrained) simple pendulum, but with the gravitational potential energy having a different sign:

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}\right) - mgr\cos\theta$$

This just comes from the fact that the height in $V = mgh$ is now the $+y$-coordinate, instead of $-y$ like we had with the pendulum earlier. In fact, this whole problem is very similar to an inverted simple pendulum.

Now, above we have the "unconstrained" Lagrangian, so we haven't used any constraints yet.

Our constraint for this system is that in the portion of the motion we care about, which is the part where the particle is ON the sphere still, the particle's distance from the origin (r) is just the radius of the sphere (R), a constant. We can use this to get our constraint function and constraint equation:

$$r = R \quad \Rightarrow \quad g(r) = r - R = 0$$

Let's now look at the equations of motion, which we have two of. First, the equation of motion for the r-coordinate gives us:

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \lambda\frac{\partial g}{\partial r} \\
&\Rightarrow \quad \frac{d}{dt}(m\dot{r}) - \left(mr\dot{\theta}^2 - mg\cos\theta\right) = \lambda \\
&\Rightarrow \quad \ddot{r} = \frac{\lambda}{m} + r\dot{\theta}^2 - g\cos\theta
\end{aligned}$$

For the θ-coordinate, we have:

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda\frac{\partial g}{\partial \theta} \\
&\Rightarrow \quad \frac{d}{dt}\left(mr^2\dot{\theta}\right) - mgr\sin\theta = 0 \\
&\Rightarrow \quad \frac{d}{dt}\left(r^2\dot{\theta}\right) = gr\sin\theta
\end{aligned}$$

It's useful for us to leave the time derivative in this form, as we'll see soon. Now, these are the two unconstrained equations of motion.

If we now use our constraint, $r = R$, we have $\dot{r} = \ddot{r} = 0$ and our equations of motion become:

$$0 = \frac{\lambda}{m} + R\dot{\theta}^2 - g\cos\theta \quad \Rightarrow \quad \dot{\theta}^2 = \frac{g}{R}\cos\theta - \frac{\lambda}{mR}$$
$$\frac{d}{dt}\left(R^2\dot{\theta}\right) = gR\sin\theta \quad \Rightarrow \quad \ddot{\theta} = \frac{g}{R}\sin\theta$$

Now, there is actually a clever trick we can take advantage of here and that is to think of the Lagrange multiplier λ as a function of time.

In fact, it is a function of time in this problem, because when the particle slides off the sphere, the Lagrange multiplier representing the constraint force will go to zero as the particle is not in contact with the sphere's surface anymore (and λ represents the normal force of the surface).

We can see this from the constraint force components, which are given by:

$$Q_r^c = \lambda\frac{\partial g}{\partial r} = \lambda$$
$$Q_\theta^c = \lambda\frac{\partial g}{\partial \theta} = 0$$

In other words, the Lagrange multiplier is precisely the total constraint force in this case and it must be a normal force, since that's the only constraint force we could possibly have in this problem.

Therefore, in a sense, we can think of the Lagrange multiplier as a function of time (we could in fact, represent it as a piecewise function of time, but it's not necessary here).

If we think of the Lagrange multiplier as a function of time, we can then take the time derivative of the first equation (note that we have to use the chain rule on the left-hand side):

$$\frac{d}{dt}\dot{\theta}^2 = \frac{d}{dt}\left(\frac{g}{R}\cos\theta - \frac{\lambda}{mR}\right)$$
$$\Rightarrow \quad 2\dot{\theta}\ddot{\theta} = -\frac{g}{R}\dot{\theta}\sin\theta - \frac{\dot{\lambda}}{mR}$$

$$\Rightarrow \left(2\ddot{\theta} + \frac{g}{R}\sin\theta\right)\dot{\theta} = -\frac{\dot{\lambda}}{mR}$$

We now use our second equation ($\ddot{\theta} = \frac{g}{R}\sin\theta$) and insert it into this one to get:

$$\left(2\frac{g}{R}\sin\theta + \frac{g}{R}\sin\theta\right)\dot{\theta} = -\frac{\dot{\lambda}}{mR}$$
$$\Rightarrow \frac{3g}{R}\sin\theta\dot{\theta} = -\frac{\dot{\lambda}}{mR}$$
$$\Rightarrow \frac{3g}{R}\sin\theta d\theta = -\frac{1}{mR}d\lambda$$

Here, I've basically "cancelled" the dt's from the time derivatives on both sides. This is now in a nice form to be integrated on both sides.

We'll choose our integration limits for θ to go from 0 to the angle at which the particle comes off the sphere, θ_0. For the right-hand side, the integration limits for λ will be from some initial λ_0 to 0, since the Lagrange multiplier (constraint force) should go to zero when $\theta = \theta_0$.

With these integration limits, we then get:

$$\frac{3g}{R}\int_0^{\theta_0}\sin\theta d\theta = -\frac{1}{mR}\int_{\lambda_0}^{0}d\lambda$$
$$\Rightarrow \frac{3g}{R}\Big|_0^{\theta_0}(-\cos\theta) = -\frac{1}{mR}\Big|_{\lambda_0}^{0}\lambda$$
$$\Rightarrow \frac{3g}{R}(1-\cos\theta_0) = \frac{\lambda_0}{mR}$$
$$\Rightarrow \cos\theta_0 = \frac{1}{3mg}(3mg - \lambda_0)$$

This equation defines the angle θ_0, at which the particle will slide off the sphere. Now we just have to figure out what the value λ_0 of the Lagrange multiplier is initially. To do this, let's go back to the equation for $\dot{\theta}^2$ from earlier:

$$\dot{\theta}^2 = \frac{g}{R}\cos\theta - \frac{\lambda}{mR}$$

Let's look at what happens to this equation at $t = 0$. At $t = 0$, the Lagrange multiplier will have the value $\lambda = \lambda_0$, which is what we want to figure out. The θ-coordinate will simply be $\theta = 0$ initially.

Now, we said earlier that the particle begins to slide down the sphere from rest initially. Therefore, at $t = 0$, we have $\dot{\theta} = 0$. With all of these, we can find λ_0 from the above equation:

$$0 = \frac{g}{R}\cos(0) - \frac{\lambda_0}{mR} \quad \Rightarrow \quad \lambda_0 = mg$$

This may not be a surprising result - in fact, since λ corresponds to the normal force of the sphere's surface, then initially when the particle is at the top of the sphere, the force of gravity (mg) will be exactly equal to the normal force.

In any case, let's now plug this into our equation for θ_0 to get:

$$\cos\theta_0 = \frac{1}{3mg}(3mg - \lambda_0)$$
$$\Rightarrow \quad \cos\theta_0 = \frac{1}{3mg}(3mg - mg)$$
$$\Rightarrow \quad \theta_0 = \arccos\left(\frac{2}{3}\right)$$

So, the particle falls off the surface of the sphere at approximately the angle $\theta_0 \approx 48°$. Notice that this result does not depend on any of the properties of the sphere nor of the particle itself - any particle would fall off an arbitrarily big sphere at exactly the same angle of 48°. The only thing that would affect the value of this angle would be if the particle had some initial velocity and was not at rest initially.