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SISTEM PERSAMAAN LINEAR

Systems of Linear Algebraic Equations

Sistem Persamaan Linear

- Acuan
  - Chapra, S.C., Canale R.P., 1990, *Numerical Methods for Engineers*, 2nd Ed., McGraw-Hill Book Co., New York.
    - Chapter 7, 8, dan 9, hlm. 201-290.

Sistem Persamaan Linear

- Serangkaian *n* persamaan linear:

$$
\left.
\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = c_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = c_2 \\
. \\
. \\
. \\
a_{1n}x_1 + a_{2n}x_2 + \ldots + a_{nn}x_n = c_n
\end{array}
\right\}
$$

Sejumlah *n* persamaan linear ini harus diselesaikan secara simultan untuk mendapatkan $x_1, x_2, \ldots, x_n$ yang memenuhi setiap persamaan tsb.

Metode Penyelesaian

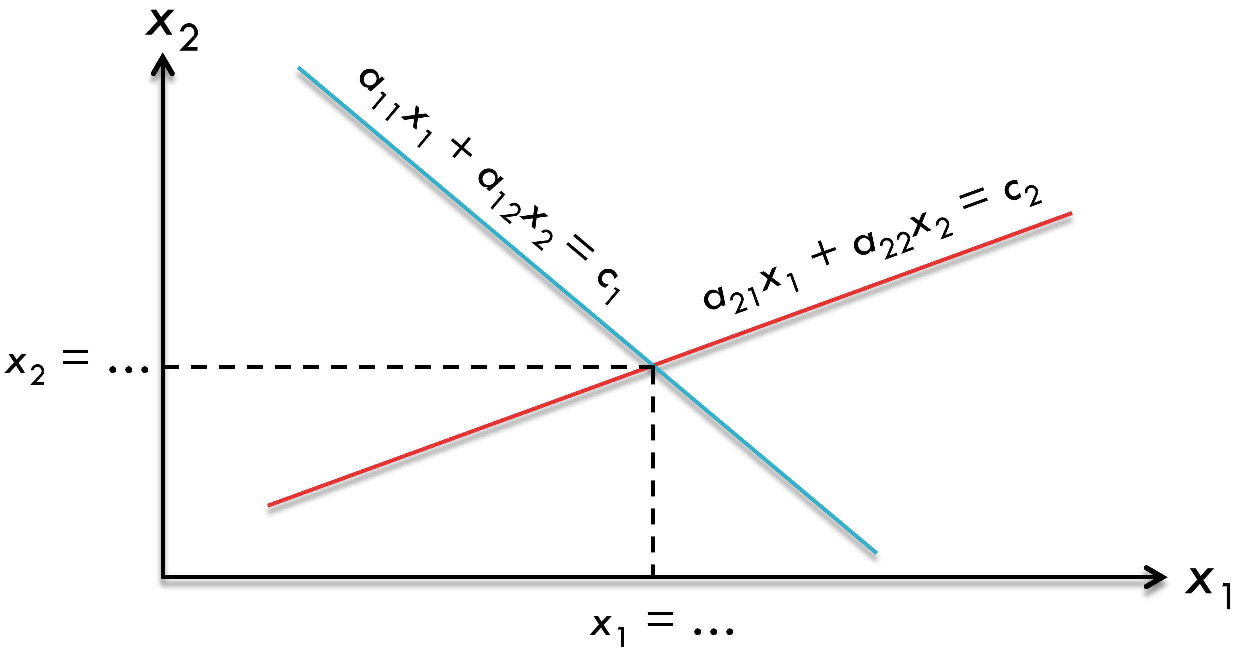
Jml. pers. sedikit, *n* «

- Penyelesaian
  - Grafis
  - Cramer
  - Eliminasi

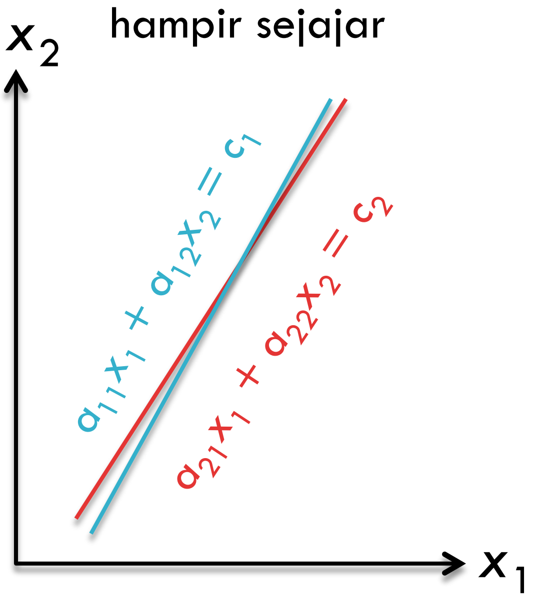
Jml. pers. banyak, *n* »

- Penyelesaian langsung
  - Eliminasi Gauss
  - Gauss-Jordan
- Iteratif
  - Jacobi
  - Gauss-Seidel
  - *Successive Over Relaxation*

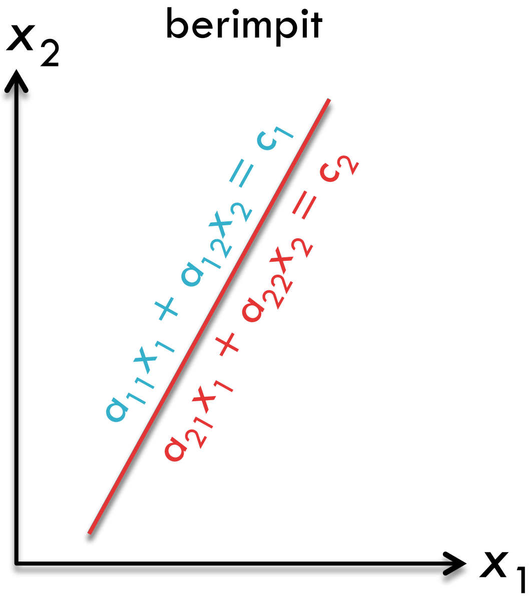
Metode Grafis



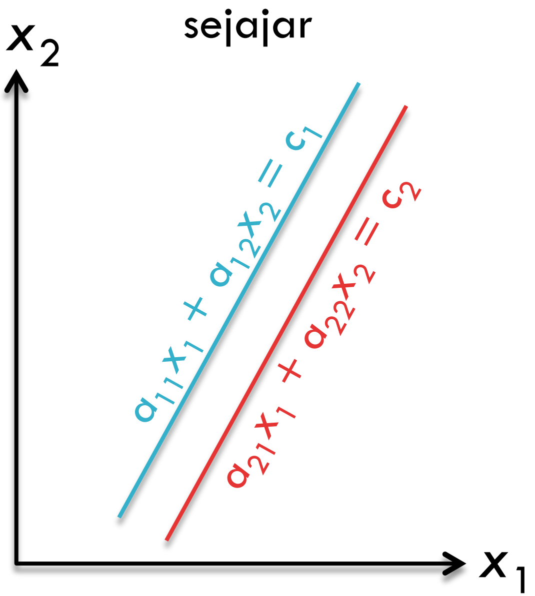
Metode Grafis



ill-conditioned system



singular system



singular system

Metode Cramer

- Variabel tak diketahui, x_i, merupakan perbandingan dua determinan matriks
  - Penyebut : determinan, *D*, matriks koefisien sistem persamaan
  - Pembilang : determinan matriks koefisien sistem persamaan seperti penyebut, namun koefisien kolom ke *i* diganti dengan koefisien c_i
- Contoh
  - 3 persamaan linear

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= c_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= c_3
\end{aligned}
$$

Metode Cramer

$$[A] = \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$x_1 = \frac{\begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{23} & a_{33} \end{vmatrix}}{D} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix}}{D} \qquad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{23} & a_3 \end{vmatrix}}{D}$$

Determinan Matriks

- Matriks bujur sangkar: $n \times n$
- Mencari determinan matriks
  - Hitungan manual
  - MSExcel, dengan fungsi =MDETERM()
- Contoh hitungan determinan matriks 2×2 dan 3×3

$$[A] = \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$[B] = \mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Determinan Matriks

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$D = \det \mathbf{B} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right)$$

Metode Cramer

- Contoh: 3 persamaan linear

$$\begin{aligned} 3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \\ 0.1x_1 + 7x_2 - 0.3x_3 &= -19.3 \\ 0.3x_1 - 0.2x_2 + 10x_3 &= 71.4 \end{aligned}$$

$$\mathbf{A} \cdot \mathbf{X} = \mathbf{C}$$

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

$$\begin{aligned} \det \mathbf{A} &= (3)[(7)(10) - (-0.3)(-0.2)] - (-0.1)[(0.1)(10) - (-0.3)(0.3)] + \\ &\quad (-0.2)[(0.1)(-0.2) - (7)(0.3)] \\ &= 210.353 \end{aligned}$$

Metode Cramer

$$[A_1] = \begin{bmatrix} 7.85 & -0.1 & -0.2 \\ -19.3 & 7 & -0.3 \\ 71.4 & -0.2 & 10 \end{bmatrix}$$

$$\det \mathbf{A_1} = |A_1| = 631.059$$

$$x_1 = \frac{\det \mathbf{A_1}}{\det \mathbf{A}} = \frac{631.059}{210.353} = 3$$

$$[A_2] = \begin{bmatrix} 3 & 7.85 & -0.2 \\ 0.1 & -19.3 & -0.3 \\ 0.3 & 71.4 & 10 \end{bmatrix}$$

$$\det \mathbf{A_2} = |A_2| = -525.883$$

$$x_2 = \frac{\det \mathbf{A_2}}{\det \mathbf{A}} = \frac{-525.883}{210.353} = -2.5$$

$$[A_3] = \begin{bmatrix} 3 & -0.1 & 7.85 \\ 0.1 & 7 & -19.3 \\ 0.3 & -0.2 & 71.4 \end{bmatrix}$$

$$\det \mathbf{A_3} = |A_3| = 1472.471$$

$$x_3 = \frac{\det \mathbf{A_3}}{\det \mathbf{A}} = \frac{1472.471}{210.353} = 7$$

Metode Eliminasi

- Contoh: 2 persamaan linear

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 &= c_1 \quad \Rightarrow a_{21}\left[a_{11}x_1 + a_{12}x_2 = c_1\right] \quad \Rightarrow a_{11}a_{21}x_1 + a_{12}a_{21}x_2 = c_1a_{21} \\
a_{21}x_1 + a_{22}x_2 &= c_1 \quad \Rightarrow a_{11}\left[a_{21}x_1 + a_{22}x_2 = c_2\right] \quad \Rightarrow a_{21}a_{11}x_1 + a_{22}a_{11}x_2 = c_2a_{11}
\end{aligned}
$$

$$a_{22}a_{11}x_2 - a_{12}a_{21}x_2 = c_2a_{11} - c_1a_{21}$$

$$x_2 = \frac{c_2a_{11} - c_1a_{21}}{a_{22}a_{11} - a_{12}a_{21}}$$

$$x_1 = \frac{c_2a_{22} - c_1a_{12}}{a_{22}a_{11} - a_{12}a_{21}}$$

Eliminasi Gauss

- Strategi
  - *Forward elimination*
  - *Back substitution*
- Contoh
  - 3 persamaan linear

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1 \qquad (1)$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2 \qquad (2)$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3 \qquad (3)$$

Eliminasi Gauss

- *Forward elimination #1*
  - Hilangkan x_1 dari pers. kedua dan ketiga dengan operasi perkalian koefisien dan pengurangan dengan pers. pertama.

pivot coefficient *pivot equation*

$$\boxed{a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1}$$

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \left(a_{23} - \frac{a_{21}}{a_{11}}a_{13}\right)x_3 = c_2 - \frac{a_{21}}{a_{11}}c_1$$

$$\left(a_{32} - \frac{a_{31}}{a_{11}}a_{12}\right)x_2 + \left(a_{33} - \frac{a_{31}}{a_{11}}a_{13}\right)x_3 = c_3 - \frac{a_{31}}{a_{11}}c_1$$

$$\Rightarrow$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1 & (1) \\ a'_{22}x_2 + a'_{23}x_3 &= c'_2 & (2') \\ a'_{32}x_2 + a'_{33}x_3 &= c'_3 & (3') \end{aligned}$$

Eliminasi Gauss

- *Forward elimination #2*
  - Hilangkan x_2 dari pers. ketiga dengan operasi perkalian koefisien dan pengurangan dengan pers. kedua.

pivot coefficient

pivot equation

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1 \\
\boxed{a'_{22}x_2 + a'_{23}x_3 = c'_2} & \\
\left(a'_{33} - \frac{a'_{32}}{a'_{22}}a'_{23}\right)x_3 &= c'_3 - \frac{a'_{32}}{a'_{22}}c'_2
\end{aligned}
$$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1 & (1) \\
a'_{22}x_2 + a'_{23}x_3 &= c'_2 & (2') \\
a''_{33}x_3 &= c''_3 & (3'')
\end{aligned}
$$

Eliminasi Gauss

- *Back substitution*
  - Hitung x_3 dari pers. (3"), hitung x_2 dari pers. (2'), dan x_1 dari pers. (1)

$$x_3 = \frac{c_3''}{a_{33}''} \Rightarrow x_2 = \frac{c_2' - a_{23}' x_3}{a_{22}'} \Rightarrow x_1 = \frac{c_1 - a_{12} x_2 - a_{13} x_3}{a_{11}}$$

Eliminasi Gauss

- *Forward elimination*

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= c_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= c'_2 \\
a''_{23}x_3 + \ldots + a''_{2n}x_n &= c''_2 \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
a_{nn}^{n-1}x_n &= c_n^{n-1}
\end{aligned}
$$

- *Back substitution*

$$x_n = \frac{c_n^{n-1}}{a_{nn}^{n-1}}$$

$$x_i = \frac{c_i^{i-1} - \sum_{j=i+1} a_{ij}^{i-1} x_j}{a_{ii}^{i-1}}, \quad i = n-1, n-2, \ldots, 1$$

Eliminasi Gauss

- Contoh: 3 persamaan linear

$$
\begin{aligned}
&(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \\
&(2) \quad 0.1x_1 + 7x_2 - 0.3x_3 = -19.3 \\
&(3) \quad 0.3x_1 - 0.2x_2 + 10x_3 = 71.4
\end{aligned}
$$

Eliminasi Gauss

- *Forward elimination*
  - Eliminasi x_2 dari Pers. 2 dan 3, Pers. 1 sebagai pivot

$$
\begin{array}{ll}
(1) & 3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \\
(2') & 0x_1 + 7.0033x_2 - 0.2933x_3 = -19.5617 \\
(3'') & 0x_1 - 0.19x_2 + 10.02x_3 = 70.615
\end{array}
$$

  - Eliminasi x_3 dari Pers. 3, Pers. 2 sebagai pivot

$$
\begin{array}{ll}
(1) & 3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \\
(2') & 0x_1 + 7.0033x_2 - 0.2933x_3 = -19.5617 \\
(3'') & 0x_1 + 0x_2 + 10.0120x_3 = 70.0843
\end{array}
$$

Eliminasi Gauss

- *Back substitution*
  - Menghitung x_3 dari Pers. 3"

$$x_3 = \frac{70.0843}{10.0120} = 7$$

  - Substitusi x_3 ke Pers. 2' untuk menghitung x_2

$$x_2 = \frac{-19.5617 + 0.2933(7)}{7.0033} = -2.5$$

  - Substitusi x_3 dan x_2 ke Pers. 1 untuk menghitung x_1

$$x_1 = \frac{7.85 + 0.2(7) + 0.1(-2.5)}{3} = 3$$

Metode Eliminasi

- Strategi
  - Eliminasi variabel tak diketahui, x_i, dengan penggabungan dua persamaan.
  - Hasil eliminasi adalah satu persamaan yang dapat diselesaikan untuk mendapatkan satu variabel x_i.

Kelemahan Metode Eliminasi

- Pembagian dengan nol
  - *Pivot coefficient* sama dengan nol ataupun sangat kecil.
  - Pembagian dengan nol dapat terjadi selama proses eliminasi ataupun substitusi.
- *Round-off errors*
  - Selama proses eliminasi maupun substitusi, setiap langkah hitungan bergantung pada langkah hitungan sebelumnya dan setiap kali terjadi kesalahan; kesalahan dapat berakumulasi, terutama apabila jumlah persamaan sangat banyak.
- *Ill-conditioned systems*
  - *Ill-condition* adalah situasi dimana perubahan kecil pada satu atau beberapa koefisien berakibat perubahan yang besar pada hasil hitungan.

Perbaikan

- Pemilihan *pivot* (*pivoting*)
  - Urutan persamaan dipilih sedemikian hingga yang menjadi *pivot equation* adalah persamaan yang memberikan *pivot coefficient* terbesar.

Metode Penyelesaian

- Matriks Inversi
  - Gauss-Jordan
- Metode Iteratif
  - Jacobi
  - Gauss-Seidel

Metode Gauss-Jordan

- Mirip dengan metode eliminasi Gauss, tetapi tidak diperlukan *back substitution*.
- Contoh
  - 3 persamaan linear

$$
\begin{aligned}
3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \\
0.1x_1 + 7x_2 - 0.3x_3 &= -19.3 \\
0.3x_1 - 0.2x_2 + 10x_3 &= 71.4
\end{aligned}
$$

Metode Gauss-Jordan

$$\left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.1 & 10 & 71.4 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 3/3 & -0.1/3 & -0.2/3 & 7.85/3 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.1 & 10 & 71.4 \end{array}\right]$$

$$\Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.1 & 10 & 71.4 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0 & 7.0033 & -0.2933 & -19.5617 \\ 0 & -0.1900 & 10.0200 & 70.6150 \end{array}\right]$$

Metode Gauss-Jordan

$$\left[\begin{array}{ccc|c} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0 & 7.0033 & -0.2933 & -19.5617 \\ 0 & -0.1900 & 10.0200 & 70.6150 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0/7.0033 & 7.0033/7.0033 & -0.2933/7.0033 & -19.5617/7.0033 \\ 0 & -0.1900 & 10.0200 & 70.6150 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0 & 1 & -0.0419 & -2.7931 \\ 0 & -0.1900 & 10.0200 & 70.6150 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -0.0681 & 2.5236 \\ 0 & 1 & -0.0419 & -2.7931 \\ 0 & 0 & 10.0120 & 70.0843 \end{array}\right]$$

Metode Gauss-Jordan

$$\left[\begin{array}{ccc|c} 1 & 0 & -0.0681 & 2.5236 \\ 0 & 1 & -0.0419 & -2.7931 \\ 0/10.0120 & 0/10.0120 & 10.0120/10.0120 & 70.0843/10.0120 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -0.0681 & 2.5236 \\ 0 & 1 & -0.0419 & -2.7931 \\ 0 & 0 & 1 & 7 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2.5 \\ 0 & 0 & 1 & 7 \end{array}\right] \Rightarrow \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 3 \\ -2.5 \\ 7 \end{Bmatrix}$$

Gauss-Jordan vs Eliminasi Gauss

- Metode Gauss-Jordan
  - Jumlah operasi lebih banyak (50%)
  - Memiliki kelemahan yang sama dengan eliminasi Gauss
    - Pembagian dengan nol
    - *Round-off error*

Matriks Inversi

$$[A] \cdot \{X\} = \{C\} \Rightarrow \{X\} = [A]^{-1} \cdot \{C\}$$

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{31} & a_{33} & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & a_{11}^{-1} & a_{12}^{-1} & a_{13}^{-1} \\ 0 & 1 & 0 & a_{21}^{-1} & a_{22}^{-1} & a_{23}^{-1} \\ 0 & 0 & 1 & a_{31}^{-1} & a_{32}^{-1} & a_{33}^{-1} \end{array}\right]$$

Matriks Inversi

- Contoh: 3 persamaan linear

$$
\begin{aligned}
&(1) \quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \\
&(2) \quad 0.1x_1 + 7x_2 - 0.3x_3 = -19.3 \\
&(3) \quad 0.3x_1 - 0.2x_2 + 10x_3 = 71.4
\end{aligned}
$$

Matriks Inversi

$$[A] = \left[\begin{array}{ccc|ccc} 3 & -0.1 & -0.2 & 1 & 0 & 0 \\ 0.1 & 7 & -0.3 & 0 & 1 & 0 \\ 0.3 & -0.2 & 10 & 0 & 0 & 1 \end{array}\right] \Rightarrow [A] = \left[\begin{array}{ccc|ccc} 1 & -0.0333 & -0.0667 & 0.3333 & 0 & 0 \\ 0.1 & 7 & -0.3 & 0 & 1 & 0 \\ 0.3 & -0.2 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$\Downarrow$$

$$[A] = \left[\begin{array}{ccc|ccc} 1 & -0.0333 & -0.0667 & 0.3333 & 0 & 0 \\ 0 & 7.0033 & -0.2933 & -0.0333 & 1 & 0 \\ 0 & -0.1900 & 10.0200 & -0.0999 & 0 & 1 \end{array}\right]$$

Matriks Inversi

$$[A]=\left[\begin{array}{ccc|ccc} 1 & -0.0333 & -0.0667 & 0.3333 & 0 & 0 \\ 0 & 1 & -0.0417 & -0.0047 & 0.1422 & 0 \\ 0 & -0.1900 & 10.0200 & -0.0999 & 0 & 1 \end{array}\right]$$

$$\Downarrow$$

$$[A]=\left[\begin{array}{ccc|ccc} 1 & 0 & -0.0681 & 0.3318 & 0.0047 & 0 \\ 0 & 1 & -0.0417 & -0.0047 & 0.1422 & 0 \\ 0 & 0 & 10.0121 & -0.1009 & 0.0270 & 1 \end{array}\right]$$

Matriks Inversi

$$[A]=\left[\begin{array}{ccc|ccc} 1 & 0 & -0.0681 & 0.3318 & 0.0047 & 0 \\ 0 & 1 & -0.0417 & -0.0047 & 0.1422 & 0 \\ 0 & 0 & 1 & -0.0101 & 0.0027 & 0.0999 \end{array}\right]$$

$$[A]=\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.3325 & 0.0049 & 0.0068 \\ 0 & 1 & 0 & -0.0052 & 0.1423 & 0.0042 \\ 0 & 0 & 1 & -0.0101 & 0.0027 & 0.0999 \end{array}\right]$$

$$\underbrace{\begin{matrix} 0.3325 & 0.0049 & 0.0068 \\ -0.0052 & 0.1423 & 0.0042 \\ -0.0101 & 0.0027 & 0.0999 \end{matrix}}_{[A]^{-1}}$$

Matriks Inversi

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$$\{X\} = [A]^{-1} \cdot \{C\}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 0.3325 & 0.0049 & 0.0068 \\ -0.0052 & 0.1423 & 0.0042 \\ -0.0101 & 0.0027 & 0.0999 \end{bmatrix} \cdot \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 3.0004 \\ -2.4881 \\ 7.0002 \end{Bmatrix}$$

Metode Iteratif: Jacobi

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= c_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= c_3
\end{aligned}
$$

$$
x_1^0 = 0 \qquad x_2^0 = 0 \qquad x_3^0 = 0
$$

nilai awal, biasanya $x_i^0 = 0$

iterasi diteruskan sampai konvergen $x_i^{n+1} \approx x_i^n, \forall x_i$

$$
x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3}{a_{11}} \qquad
x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3}{a_{22}} \qquad
x_3 = \frac{c_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}
$$

$$
x_1^1 = \frac{c_1 - a_{12}x_2^0 - a_{13}x_3^0}{a_{11}} \qquad
x_2^1 = \frac{c_2 - a_{21}x_1^0 - a_{23}x_3^0}{a_{22}} \qquad
x_3^1 = \frac{c_3 - a_{31}x_1^0 - a_{32}x_2^0}{a_{33}}
$$

$$
x_1^{n+1} = \frac{c_1 - a_{12}x_2^n - a_{13}x_3^n}{a_{11}} \qquad
x_2^{n+1} = \frac{c_2 - a_{21}x_1^n - a_{23}x_3^n}{a_{22}} \qquad
x_3^{n+1} = \frac{c_3 - a_{31}x_1^n - a_{32}x_2^n}{a_{33}}
$$

Metode Iteratif: Jacobi

- Contoh: 3 persamaan linear

$$(1)\quad 3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$(2)\quad 0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

$$(3)\quad 0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Metode Iteratif: Gauss-Seidel

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$$x_1^1 = \frac{c_1 - a_{12}x_2^0 - a_{13}x_3^0}{a_{11}}$$

$$x_2^1 = \frac{c_2 - a_{21}x_1^1 - a_{23}x_3^0}{a_{22}}$$

$$x_3^1 = \frac{c_3 - a_{31}x_1^1 - a_{32}x_2^1}{a_{33}}$$

$$x_1^{n+1} = \frac{c_1 - a_{12}x_2^n - a_{13}x_3^n}{a_{11}}$$

$$x_2^{n+1} = \frac{c_2 - a_{21}x_1^{n+1} - a_{23}x_3^n}{a_{22}}$$

$$x_3^{n+1} = \frac{c_3 - a_{31}x_1^{n+1} - a_{32}x_2^{n+1}}{a_{33}}$$

iterasi diteruskan sampai konvergen $x_i^{n+1} \approx x_i^n, \forall x_i$

Metode Iteratif: Gauss-Seidel

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$$
\begin{array}{ll}
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\end{array}
$$

Jacobi vs Gauss-Seidel

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Jacobi

$$\boxed{\begin{aligned}x_1^1\\x_2^1\\x_3^1\end{aligned}}\begin{aligned}&= \left(c_1 - a_{12}x_2^0 - a_{13}x_3^0\right)/a_{11}\\&= \left(c_2 - a_{21}x_1^0 - a_{23}x_3^0\right)/a_{22}\\&= \left(c_3 - a_{31}x_1^0 - a_{32}x_2^0\right)/a_{33}\end{aligned}$$

$$\begin{aligned}x_1^2 &= \left(c_1 - a_{12}x_2^1 - a_{13}x_3^1\right)/a_{11}\\x_2^2 &= \left(c_2 - a_{21}x_1^1 - a_{23}x_3^1\right)/a_{22}\\x_3^2 &= \left(c_3 - a_{31}x_1^1 - a_{32}x_2^1\right)/a_{33}\end{aligned}$$

Gauss-Seidel

$$\begin{aligned}x_1^1 &= \left(c_1 - a_{12}x_2^0 - a_{13}x_3^0\right)/a_{11}\\x_2^1 &= \left(c_2 - a_{21}x_1^1 - a_{23}x_3^0\right)/a_{22}\\x_3^1 &= \left(c_3 - a_{31}x_1^1 - a_{32}x_2^1\right)/a_{33}\end{aligned}$$

$$\begin{aligned}x_1^2 &= \left(c_1 - a_{12}x_2^1 - a_{13}x_3^1\right)/a_{11}\\x_2^2 &= \left(c_2 - a_{21}x_1^2 - a_{23}x_3^1\right)/a_{22}\\x_3^2 &= \left(c_3 - a_{31}x_1^2 - a_{32}x_2^2\right)/a_{33}\end{aligned}$$

Successive Over-relaxation Method

- Dalam setiap iterasi, nilai variabel terbaru (yang baru saja dihitung), x_{n+1}, tidak langsung dipakai pada iterasi selanjutnya
- Pada iterasi selanjutnya, nilai tsb dimodifikasi dengan memasukkan pengaruh nilai variabel lama (pada iterasi sebelumnya), x_n

$$x_1^{new} = \lambda\, x_i^{n+1} + (1-\lambda)\, x_i^n$$

  - faktor relaksasi λ dimaksudkan untuk mempercepat konvergensi hitungan (iterasi)
  - *under-relaxation*: $0 < \lambda < 1$
  - *over-relaxation*: $1 < \lambda < 2$

Successive Over-relaxation Method

$$x_1^{n+1} = \frac{c_1 - a_{12}x_2^n - a_{13}x_3^n}{a_{11}}$$

$$x_2^{n+1} = \frac{c_2 - a_{21}\left[\lambda\, x_1^{n+1} + \left(1-\lambda\right)x_1^n\right] - a_{23}x_3^n}{a_{22}}$$

$$x_3^{n+1} = \frac{c_3 - a_{31}\left[\lambda\, x_1^{n+1} + \left(1-\lambda\right)x_1^n\right] - a_{32}\left[\lambda\, x_2^{n+1} + \left(1-\lambda\right)x_2^n\right]}{a_{33}}$$

Successive Over-relaxation Method

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Sekian