A#37 THE TRAPEZIUM RULE

AEM questions are taken from past exam papers - they have been carefully chosen to represent a typical exam question at each level of difficulty. If you can do these questions, you're ready to move onto past papers for this topic.

APPRENTICE

a. Use the trapezium rule, with 4 strips each of width 0.2, to find an estimate for $\int_0^{0.8} \cos x \ \mathrm{d}x$ to 3 dp.

b. Explain, with the aid of a sketch, why the value from part a. is an under-estimate.

EXPERT

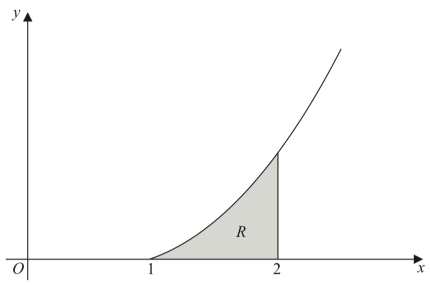


Figure 1 shows a sketch of part of the curve with equation $y = x^2 \ln x$, $x \geq 1$. The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis and the line $x = 2$.

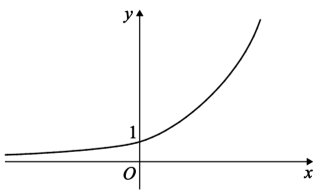
The table below shows corresponding values of x and y for $y = x^2 \ln x$.

a. Complete the table, giving the missing value of y to 4 decimal places.

b. Use the trapezium rule with all the values of y in the completed table to obtain an estimate for the area of R, giving your answer to 3 dp.

x	1	1.2	1.4	1.6	1.8	2
y	0	0.2625		1.2032	1.9044	2.7726

MASTER



The diagram shows a sketch of the curve $y = 2^{3x}$.

a. Use the trapezium rule with five ordinates to find an approximate value for $\int_0^1 2^{3x} \ \mathrm{d}x$. Give your answer to two decimal places.

b. Explain how you could obtain a better approximation of $\int_0^1 2^{3x} \ \mathrm{d}x$ using the trapezium rule.

c. The point $P(1, k)$ lies on the curve $y = 2^{3x}$. Use your answer to part a. to find an approximate value for the area of the region bounded by the curve, the line $x = 0$ and the line $y = k$.
Give your answer to two decimal places.

A#38 CHANGE OF SIGN ARGUMENT

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APPRENTICE

It is given that the curves with equations $y = 6\ln x$ and $y = 8x - x^2 - 3$ intersect at a single point where $x = \alpha$.

Show that α lies between 5 and 6.

EXPERT

The curve with equation $y = \dfrac{3x+4}{x^3 - 4x^2 + 2}$ has a stationary point at P. It is given that P is close to the point with coordinates $(2.4, -1.6)$.

a. Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-6x^3 + 32x + 6}{(x^3 - 4x^2 + 2)^2}$ and that the x-coordinate of P satisfies $x = \sqrt[3]{\dfrac{16}{3}x + 1}$.

b. By first using an iterative process based on the equation in part a., find the coordinates of P, giving each coordinate correct to 3 decimal places.

MASTER

a. By sketching the curves $y = x(2x+5)$ and $y = \cos^{-1} x$ (where y is in radians) in a single diagram, show that the equation $x(2x+5) = \cos^{-1} x$ has exactly one real root.

b. Use the iterative formula $x_{n+1} = \dfrac{\cos^{-1} x_n}{2x_n + 5}$ with $x_1 = 0.25$ to find the root correct to 3 significant figures.
Show the result of each iteration correct to at least 4 significant figures.

c. Two new curves are obtained by transforming each of the curves $y = x(2x+5)$ and $y = \cos^{-1} x$ by the pair of transformations:

reflection in the x-axis followed by reflection in the y-axis.

State an equation of each of the new curves and determine the coordinates of their point of intersection, giving each coordinate correct to 3 significant figures.

A#39 COBWEBS & STAIRCASES

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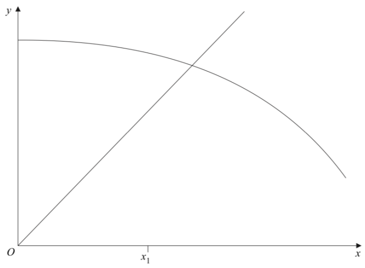
APPRENTICE

$F(x) = 2 + \ln x$. The iteration $x_{n+1} = F(x_n)$ is to be used to find a root, α, of the equation $x = 2 + \ln x$.

a. Taking $x_1 = 3.1$, find x_2 and x_3, giving your answers correct to 5 decimal places.

b. Illustrate the iteration by drawing a sketch of $y = x$ and $y = F(x)$, showing how the values of x_n approach α. State whether the convergence is of the 'staircase' or 'cobweb' type.

EXPERT



The curve $y = x^3 + 4x - 3$ intersects the x-axis at the point A where $x = \alpha$.

a. Show that α lies between 0.5 and 1.0.

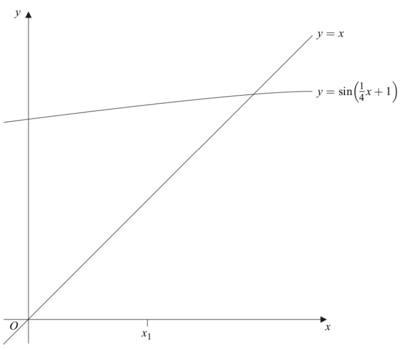
b. Show that $x^3 + 4x - 3 = 0$ can be rearranged into the form $x = \dfrac{3 - x^2}{4}$.

c. Use the iteration $x_{n+1} \dfrac{3 - x_n^3}{4}$ with $x_1 = 0.5$ to find x_3 to 2 dp.

d. The sketch shows parts of the graphs of $y = \dfrac{3 - x^3}{4}$ and $y = x$, and the position of x_1.

On the sketch, draw a cobweb or staircase diagram to show how convergence takes place, indicating the positions of x_2 and x_3 on the x-axis.

MASTER



a. i. Sketch the graph of $y = \sin^{-1} x$, where y is in radians.
State the coordinates of the end points of the graph.

ii. By drawing a suitable straight line on your sketch, show that the equation $\sin^{-1} x = \dfrac{1}{4}x + 1$ has only one solution.

b. The root of the equation $\sin^{-1} x = \dfrac{1}{4}x + 1$ is α.

Show that $0.5 < \alpha < 1$.

c. The equation $\sin^{-1} x = \dfrac{1}{4}x + 1$ can be rewritten as $x = \sin\left(\dfrac{1}{4}x + 1\right)$.

i. Use the iteration $x_{n+1} = \sin\left(\dfrac{1}{4}x_n + 1\right)$ with $x_1 = 0.5$ to find the values of x_2 and x_3, to 3 dp.

ii. The sketch shows parts of the graphs of $y = \sin\left(\dfrac{1}{4}x + 1\right)$ and $y = x$, and the position of x_1.

Add a cobweb or staircase diagram to show how convergence takes place, indicating the positions of x_2 and x_3 on the x-axis.

A#40 THE NEWTON-RAPHSON METHOD

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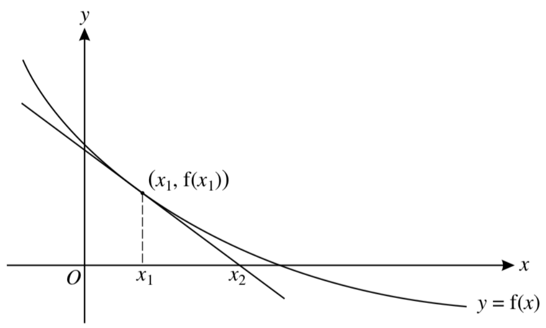
APPRENTICE

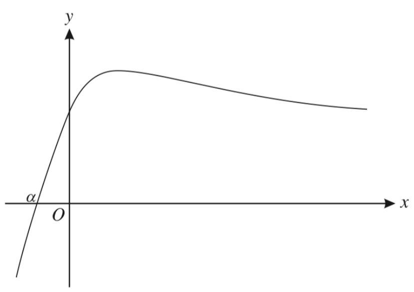


A curve with no stationary points has equation $y = f(x)$. The equation $f(x) = 0$ has one real root α, and the Newton-Raphson method is to be used to find α. The tangent to the curve at the point $(x_1, f(x_1))$ meets the x-axis where $x = x_2$ (see diagram).

a. Show that $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$.

b. Describe briefly, with the help of a sketch, how the Newton-Raphson method, using an initial approximation $x = x_1$, gives a sequence of approximations approaching α.

EXPERT



The diagram shows the graph of $y = xe^{-x} + 1$.
The curve crosses the x-axis at $x = \alpha$.

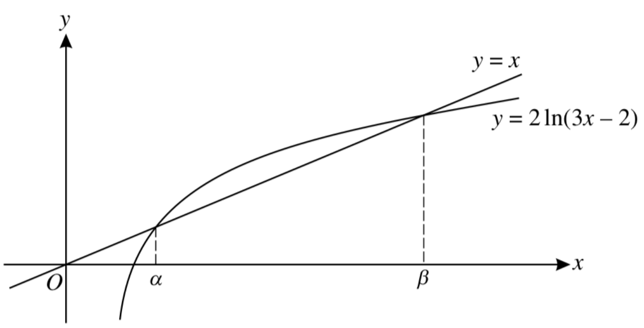
a. Use differentiation to show that the x-coordinate of the stationary point is 1.

α is to be found using the Newton-Raphson method, with $f(x) = xe^{-x} + 1$.

b. Explain why this method will not converge to α if an initial approximation x_1 is chosen such that $x_1 > 1$

c. Use this method, with a first approximation $x_1 = 0$, to find the next three approximations x_2, x_3 and x_4.

MASTER



The line $y = x$ and the curve $y = 2\ln(3x - 2)$ meet where $x = \alpha$ and $x = \beta$, as shown in the diagram.

a. Show that the equation $x = 2\ln(3x - 2)$ can be rewritten as $x = \dfrac{1}{3}\left(e^{\frac{1}{2}x} + 2\right)$.

b. Use the Newton-Raphson method, with $f(x) = \dfrac{1}{3}\left(e^{\frac{1}{2}x} + 2\right) - x$ and $x_1 = 1.2$, to find x_2 correct to 2 dp.

b. If $x_1 = \ln 36$, explain why the Newton-Raphson method would not converge to a root of $f(x) = 0$.