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Author(s): Marie-Odile Albizzati and Hélyette Geman

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Marie-Odile Albizzati
Hélyette Geman

ABSTRACT

The valuation of the prepayment option embedded in mortgages attracts the attention of practitioners and academics (see Schwartz and Torous, 1989) both because of its direct negative effect on the financial value of a bank balance sheet in case of drop in interest rates and also because of its impact on the design and pricing of mortgage-backed securities. In the same manner, life insurance policyholders may surrender their contracts and take advantage of higher yields available in the financial markets; this is a source of concern for life insurers, especially during periods of highly volatile interest rates such as have prevailed in recent years. We address the surrender option pricing problem as the valuation of a contingent claim for the insurer, where the contingency is closely related to the level of interest rates, and directly price by arbitrage the surrender option embedded in life insurance policies. A closed-form solution is derived in the case of a single-premium policy when the investment portfolio consists of a fixed-term zero-coupon bond, and the dynamics of stochastic interest rates are driven by the Heath-Jarrow-Morton (1992) model. The price of the option is computed in the case of French contracts using both the closed-form expression and Monte Carlo simulations.

Introduction

The economic and financial developments over the last 15 years have brought to life insurers both new opportunities and new challenges. On one hand, the uncertainty of pay-as-you-go retirement schemes generated by longer life expectancies and lower birth rates has entailed a shift of savings in the French economy toward life insurance policies. Nonmortality-related contracts account today for more than 80 percent of French life insurance contracts. They represent more than half of the national savings and amounted to Fr 250 billion in 1993, climbing from Fr 49 billion in 1983.

Marie-Odile Albizzati is a doctoral student at the ESSEC and Université Paris VII. She gratefully acknowledges support from Groupama Gestion and Laure Elie on this research. Hélyette Geman is Profesor of Finance at the University of Reims and at the ESSEC Graduate School of Management. She gratefully acknowledges support from Ceressec.

On the other hand, volatile interest rates, disintermediation, and competition from banks and financial institutions offering similar types of products have forced life insurers to promise and guarantee higher rates of return on the savings components of life insurance and annuity contracts and, hence, to assume investment risks associated with higher book yields. Moreover, if the guaranteed return is not high enough compared to other forms of investment, mainly in the case of a rise in interest rates, policyholders may decide on early termination of their existing policies and choose a higher yield alternative offered in the capital markets (e.g., money market funds). It is the valuation of this surrender option in the context of stochastic interest rates that this article addresses, after an overview of interest rate risk management for life insurers.

Asay, Bouyaoucos, and Marciano (1989) have offered an option-adjusted-spread approach to estimating the financial value of outstanding policies. Their methodology is not given explicitly, however, because it is the proprietary approach used by Goldman Sachs for the valuation of callable bonds or mortgage-backed securities to take into account the prepayment option. Moreover, as pointed out by Babbel and Zenios (1992), the spread is difficult to estimate and strongly depends on the volatility assumption in the model. Lastly, whether one looks at a single contract or at a pool of contracts, the property of path-dependency must be taken into account since several lapses cannot be observed on the same policy. This article looks at the problem the other way around and directly calculates the value of the surrender option embedded in life insurance policies.

This surrender option, which is indeed an exchange option, cannot be priced by the formula provided in the seminal paper by Margrabe (1978), who assumed deterministic interest rates (as did Black and Scholes, 1973). Our problem, by definition, is set in the framework of stochastic interest rates. Moreover, Margrabe's exchange option was a European option; the surrender option has an exercise date that is an optimal stopping time. We partly solve the latter difficulty by considering a pool of homogeneous life insurance policies and using the fundamental averaging effect of the insurance mechanism. This leads to an evaluation of the option by arbitrage under the risk-neutral probability as the expectation of random cash flows occurring at well-defined dates and discounted with stochastic interest rates. We use under a generalized form the forward neutral probability measure introduced by Geman (1989) and Jamshidian (1989), which proved very powerful in pricing interest rate derivative instruments such as floating-rate notes and interest rate swaps (see El Karoui and Geman, 1991, 1993).

The next section presents some elements of life insurer asset-liability management. Then, we provide a closed-form expression of the surrender option value in the case of a single-premium policy when its dollar amount is invested in a fixed-term zero-coupon bond of the asset portfolio, and the dynamics of the term structure of interest rates are assumed to be driven by a one-factor model with a deterministic term structure of volatilities. The option price is computed under different interest rate volatilities, using both the closed-form expression and Monte Carlo simulations.

Managing Interest Rate Risk in European Life Insurers

For decades, European insurers were managed quite differently than banks. Real estate located in expensive urban districts consistently appreciated over time, enhancing insurers' assets. Technical management of the company was essentially the responsibility of managers having solid statistical and actuarial backgrounds but not necessarily familiar with modern finance. Only in recent years has a more financial component been introduced—by choice or by necessity—for a number of converging reasons. First, the developing reality of the European economic community is resulting in more vigorous competition among banks, insurers, and other financial institutions, and across countries. Second, the process of mergers and acquisitions has widely developed among insurers, not only in Europe but also worldwide. The market value of the target firm must be estimated as precisely as possible. And the same holds true when mutual companies convert to the stock form of ownership. Third, the value of real estate has decreased, although less dramatically than in the United States. Finally, the gloomy prospects for pay-as-you-go retirement schemes in Europe have generated opportunities for private pension funds.[1]

Competition from banks and other financial institutions has led insurers concerned about the yields they can offer their clients to market these *contrats à taux garanti,* which are similar to U.S. guaranteed investment contracts (GICs) and require the same financial skills in their management. These instruments are accumulation savings vehicles with essentially no life insurance content (upon death, the policyholder's estate receives the invested money plus interest) and with a major tax advantage if there is no early surrender of the policy. In the United States, insurers met the difficulties posed by competition and made the subsequent necessary adaptation earlier. Very high volatilities of interest rates were observed in the 1970s. Insurers faced disintermediation from other sectors of the financial services industry and responded by offering guaranteed yields (e.g., GICs), sometimes incompatible with a desirable equilibrium of the balance sheet.

Interest rate volatility generates for insurers a "surplus volatility" that must be estimated. When interest rates go up, the insurer's portfolio, which consists mainly of bonds, decreases in value. At the same time, customers may surrender their existing policies to buy new contracts offering higher yields. Both the assets and liabilities are negatively affected. Bankers experience the analogous difficult situation when interest rates go down and rational customers exercise the prepayment option of their mortgages (see, e.g., Schwartz and Torous, 1989). When interest rates go down, life insurance policies become hardly more attractive than savings passbooks. If this decline occurs after a return has been guaranteed to the policyholders and a proper hedge has not

[1] The Netherlands has the highest amount of capital per inhabitant invested in pension funds; in France, there has been over the last few years an explosion of demand for life insurance contracts with no mortality-related component (A. P. Information Services Limited, 1993).

been established, the asset portfolio may not be adequate to satisfy the firm's liabilities.

Consequently, in order to prevent failures, U.S. governmental authorities have imposed more severe regulation on equity requirements (e.g., risk-based capital). Insurers have become more aware of asset-liability management techniques. Macaulay-type risk indicators measure price responses to shape-preserving (parallel) shocks to the term structure of interest rates. Beyond duration (Macaulay or modified duration), which measures the first-order sensitivity to interest rate fluctuations, convexity and even higher-order derivatives are used to account for the fact that interest rate fluctuations are no longer infinitesimal in either developing or developed countries. Moreover, to recognize nonparallel shifts of the yield curve, most insurers are implementing two complementary methods (A. P. Information Services Limited, 1993): simulations of interest rate variations and their impact on asset and liability values, and a stochastic modeling of the term structure dynamics incorporating the current yield curve and *allowing only for arbitrage-free interest rate fluctuations*. Obviously, this stochastic modeling of the yield curve is necessary to price any optional feature embedded in the balance sheet, such as the surrender option which is considered below.

In Europe, a similar evolution is being observed. The same or more severe regulatory rules are being established both within each country and at the European Economic Community level. Most well known is the Cooke ratio (defined by the July 1988 Basle Agreement on Banking Regulations and Supervisory Practices), which imposes an equity requirement for all European banks—namely a minimum value of 8 percent for the ratio of capital divided by a weighted sum of the assets, the weights being an increasing function of default risk. The analogous constraints would not be binding for most insurers because German, Swiss, and French insurers hold a high amount of equity and reserves; in France, though, the profits have decreased, and insurers need to enhance their competitiveness and the quality of their management. In particular, life insurers are concerned by the possibility of experiencing a wave of early termination of policies in case interest rates rise. This surrender option is studied at length in the remainder of this article.

Valuation of the Surrender Option

This section describes the characteristics of the insurance policy under analysis as well as the tax rules applying to it. We then introduce a stochastic modeling of the term structure dynamics allowing only for arbitrage-free movements and price in this setting the surrender option when exercised at a well defined date. Using the pooling-of-risks principle, we can take into account the random nature of the time of occurrence of policy lapses and provide a closed-form expression for the surrender option.

Description of the Contract

The insurance policy may be purchased in a lump sum or through a series of payments. As in Asay, Bouyaoucos, and Marciano (1989), this article focuses on single premium deferred annuities. We assume that a single premium is paid at inception of the contract and denote by K_0 the upfront proceeds for the insurer at time zero after the upfront fees have been paid. The lifetime of the policy is T = 8 years if there is neither early termination nor rollover.

Corresponding to this liability, French insurers invest in a portfolio consisting mainly of bonds. For simplicity, we will assume that the assets associated with these policies are zero-coupon bonds having the same maturity as the contract. For a contract that is held for eight years, there is a guaranteed minimum return paid to the policyholder. We represent the return effectively paid to the policyholder as $\lambda R(0,T)$, where λ is a positive constant not greater than one, and R(0,T) denotes the yield on a zero-coupon bond maturing at time T. $\lambda = 0.9$ is an accurate description of French insurers' practice. Consequently, at maturity, the value of the policy is $K(T) = K_0 e^{\lambda T R(0,T)}$, and, for simplicity, we will take $K_0 = 1$.

The contract under analysis (*contrat à taux garanti*) is a tax-advantaged savings product offered by life insurers. Interest income on a policy is tax-free in France if the policy is held for eight years. In the case of early surrender, the tax rate depends on whether surrender occurs before or after four years. A proper representation of this tax rate at time t is

$$x(t) = 0.381 \; I_{(t < 4)} + 0.181 \; I_{(4 \leq t < 8)},$$

where I denotes the indicator function; I = 1 if the corresponding inequality (e.g., t < 4) is satisfied, and I = 0 otherwise.

Because of competition and regulation, the penalty on early surrender is very small in practice. For simplicity, we will assume there is none. Consequently, the capital received by a policyholder who terminates an insurance contract at time t, which is also called the cash surrender value, amounts to $V_s(t) = e^{\lambda t R(0,T)}$. After taxes, the payoff for the policyholder reduces to $K(t) = 1 + (e^{\lambda t R(0,T)} - 1)(1-x(t))$.

The interest rate that would prevail on a new contract of the same nature bought at time t is R(t,T), yield on a zero-coupon bond maturing at time t + T, whose market value is $B(t,T+t) = e^{-TR(t,T)}$.

Valuation of a European Surrender Option

For the insurer, the market value of the zero-coupon bonds (in number $\frac{1}{B(0,T)}$ when the upfront proceeds amount to one dollar) associated with this contract in the asset portfolio is $V_m(t) = \frac{B(t,T)}{B(0,T)}$. We can write

$$V_s(t) = e^{\lambda R(0,T)t} = V_m(t) + h(t), \tag{1}$$

where the cash flow h(t) (positive or negative) is guaranteed by the insurer. The rational policyholder compares at any time t the terminal value of his or her contract held to maturity, K(T), with the terminal value of a new contract initiated at time t under the same financial conditions. That is, $K(t)(1-\beta)e^{\lambda(T-t)R(t,T)}$, where ß represents the upfront management fees if starting a new contract.

Remembering that $K(T) = e^{\lambda TR(0,T)}$, we see that a necessary condition for surrender at time t is $D(t) > 1$, where

$$D(t) = \frac{(1-\beta)K(t)e^{\lambda(T-t)R(t,T)}}{e^{\lambda TR(0,T)}} \tag{2}$$

will be called the decision criterion and also be denoted D_t. The option will be exercised at time t by the rational policyholder only if $D_t > 1$. We can observe that $D(t) > 1 \Leftrightarrow R(t,T) > \gamma(t)$, where

$$\gamma(t) = \frac{T}{T-t}R(0,T) - \frac{1}{\lambda(T-t)}\ln[(1-\beta)K(t)]. \tag{3}$$

Figure 1

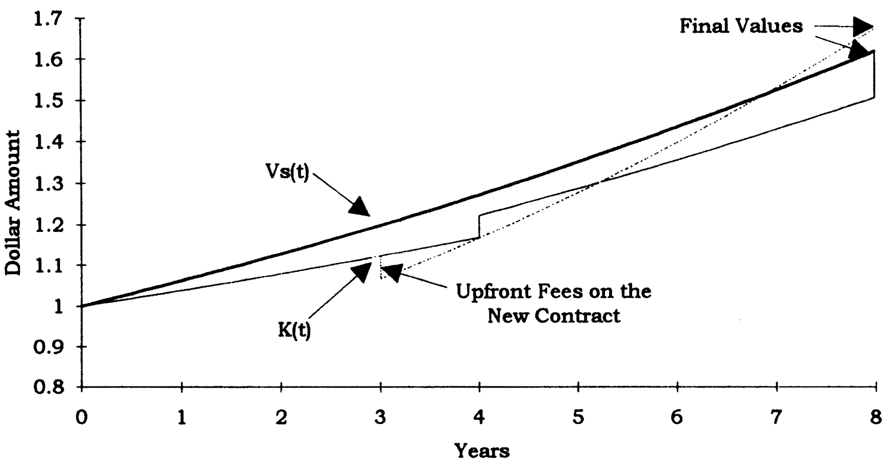


Figure 1, which plots over time the cash surrender value $V_s(t)$ and the after-tax payoff to the policyholder K(t), illustrates the case where $D_t > 1$ for $t = 3$. Figure 2 plots over time the values of R(t,T) for which $D(t) = 1$ when the guaranteed return at time zero equals 6 percent, the upfront management fees ß when starting a new contract equal 5 percent, and there are no surrender charges. Remembering that the surrender option is in fact an American option since its maturity is unknown at inception of the contract, Figure 3 represents the random cash flow provided by the exercise of the American option, namely the quantity $h(t)I_{(D(t)>1)}$, where $t \in [0,T]$. We first assume in the valuation

Figure 2

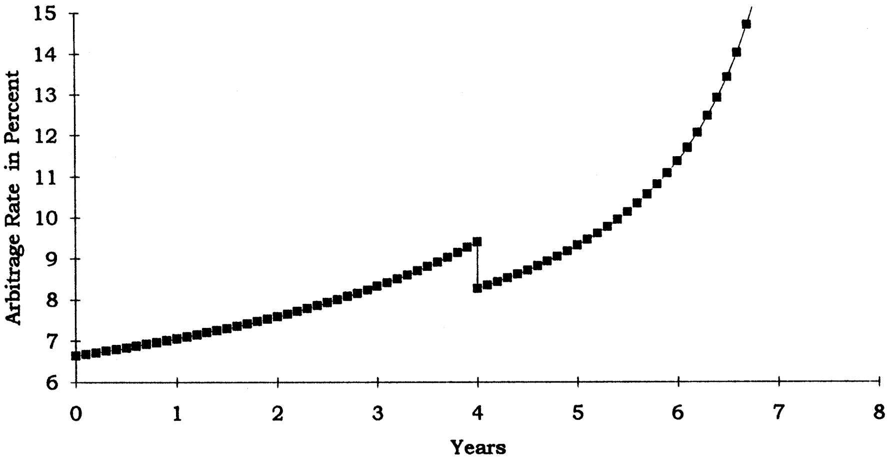
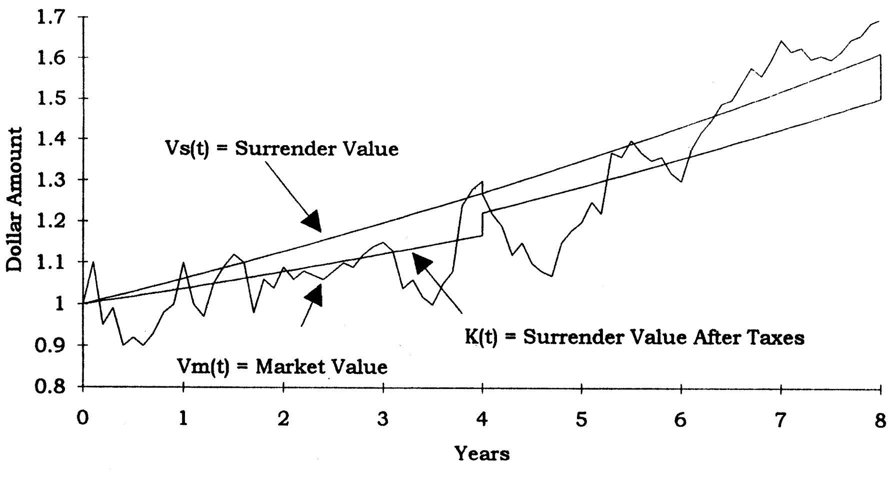


Figure 3



methodology that this option can only be exercised at a well-defined time t and will justify later how this assumption fits into the real situation of an insurer managing a pool of contracts.

The uncertainty in the economy is represented by a probability space (Ω, F, P). The accruing information available to all agents is represented by the filtration $(F_t)_{t\geq 0}$ satisfying the usual conditions. Assuming the surrender option could only be exercised at time t (t deterministic and belonging to the interval [0,T]), its value at date zero that we denote by $C^t(0)$ even though it is a put option (to avoid any confusion with a probability), is

$$C^{t}(0) = E_Q\left[h(t)\ I_{(D(t)>1)}\ e^{-\int_0^t r(s)ds}\right], \tag{4}$$

where $(r(s))_{s\geq 0}$ = the short-term interest rate process; $r(t)$ is supposed to be F_t-measurable.

Q = the risk-adjusted probability measure equivalent to P under which basic securities discounted prices are martingales (see Harrison and Kreps, 1979, and Harrison and Pliska, 1981). Moreover, we assume market completeness, which ensures the existence of a price for any contingent claim. (In our model described below, this is not in fact an assumption, since we have one source of randomness and certainly two nonredundant securities are traded in the market.)

$h(t)$ $= V_s(t) - V_m(t) = e^{\lambda t R(0,T)} - \dfrac{B(t,T)}{B(0,T)}$.

Stochastic modeling of the term structure dynamics. We describe the interest rate movements through the zero-coupon bond dynamics under Q and assume the latter are driven by a particular case of the Heath-Jarrow-Morton model

$$\frac{dB(t,T)}{B(t,T)} = r(t)dt + \sigma(t,T)dW_t, \tag{5}$$

where $(W(t))_{t\geq 0}$ = a Q-Brownian motion and where we assume deterministic volatilities of the form $\sigma(t,T) = \dfrac{\sigma(1-e^{-a(T-t)})}{a}$ with a and σ being positive constants. This specification of the volatility term structure leads to Gaussian interest rates and closed-form expressions for most interest-rate derivative instrument prices. The superiority of two (or more) factors over one-factor models of interest rates is still debated in the financial literature (see Cohen and Heath, 1992). Consequently, overcoming the difficulties involved in the surrender option pricing and providing a closed-form solution that is easy to compute may be quite useful, even in the context of a one-factor model.

The first difficulty in the calculation of C in equation (4) is to pull the discount factor out of the expectation operator. For that, we introduce the forward neutral probability measure Q_t relative to time t, defined by its Radon-Nikodym derivative with respect to Q

$$\frac{dQ_t}{dQ} = \frac{e^{-\int_0^t r(s)ds}}{B(0,t)} = \exp\left\{\int_0^t \sigma(s,t)dW_s - \frac{1}{2}\int_0^t [\sigma(s,t)]^2 ds\right\}$$

(see Geman, 1989), and, from Girsanov's theorem, the relationship

$$dW_s^t = dW_s - \sigma(s,t)ds \tag{6}$$

defines $(W_s^t)_{s\geq 0}$ as a Q_t-Brownian motion. We can observe that this change of probability measure, meant to absorb the stochastic nature of interest rates up to time t, consists in fact in taking as a new numéraire the zero-coupon bond maturing at time t.

From equation (5) and the relationship $B(t,T+t) = e^{-TR(t,T)}$, we derive

$$R(t,T) = f(0,t,T) - \int_0^t \frac{\sigma(s,T+t)-\sigma(s,t)}{T} dW_s + \frac{1}{2}\int_0^t \frac{\sigma^2(s,T+t)-\sigma^2(s,t)}{T} ds,$$

where f(0,t,T) represents the forward rate observed at time zero relative to the period [t,t+T]. Using equation (6), we can also write

$$R(t,T) = f(0,t,T) - \int_0^t \frac{\sigma(s,T+t)-\sigma(s,t)}{T} dW_s^t + \frac{T}{2}\mathrm{VarR}(t,T), \tag{7}$$

$$\text{where } \mathrm{VarR}(t,T) = \int_0^t \frac{[\sigma(s,T+t)-\sigma(s,t)]^2}{T^2} ds = \frac{\sigma^2}{2T^2}\left(\frac{1-e^{-aT}}{a}\right)^2 \left(\frac{1-e^{-2at}}{a}\right).$$

Coming back to equation (4) and using the expression of h(t) derived from equation (3), we obtain

$$C^t(0) = E_Q\left[e^{\lambda tR(0,T)} e^{-\int_0^t r(s)ds} I_{(D(t)>1)}\right] - E_Q\left[\frac{B(t,T)}{B(0,T)} e^{-\int_0^t r(s)ds} I_{(D(t)>1)}\right].$$

Introducing the new probability measure Q_t, the first expectation can easily be written as $B(0,t)e^{\lambda tR(0,T)}E_{Q_t}\left(I_{(D(t)>1)}\right)$. The second one, in the same manner, is equal to $\frac{B(0,t)}{B(0,T)}E_{Q_t}\left(B(t,T)\, I_{(D(t)>1)}\right)$. Using the general change of numéraires formula established in Geman, El Karoui, and Rochet (1992), $X(0)E_{Q_X}[Y(T)\phi] = Y(0)E_{Q_Y}[X(T)\phi]$, where X and Y are two arbitrary securities and ϕ an F_T-measurable random cash flow, the second expectation simply reduces to $E_{Q_T}\left(I_{(D(t)>1)}\right)$, and the put option price $C^t(0)$ appears as

$$C^t(0) = e^{\lambda tR(0,T)}B(0,t)E_{Q_t}\left(I_{(D(t)>1)}\right) - E_{Q_T}\left(I_{(D(t)>1)}\right). \tag{8}$$

We observed that $D(t) > 1 \Leftrightarrow R(t,T) > \gamma(t)$. Using the dynamics of R(t,T) described in equation (7) and the fact that $(R(s,T))_{s\geq 0}$ is a Gaussian process under Q_t, it is easy to show with standard probability arguments that

$$E_{Q_t}\left(I_{(D(t)>1)}\right) = N(d_1^t),$$

where N denotes the cumulative function of the normal distribution, and

$$d_1^t = \frac{-\gamma(t)+f(0,t,T)+\frac{T}{2}\mathrm{Var}R(t,T)}{\sqrt{\mathrm{Var}R(t,T)}}.$$

The hypothesis of deterministic volatilities in equation (5) induces Gaussian interest rates and a significant tractability in the calculations.

We now introduce for u > t a similar Brownian motion change as in equation (6): $dW_s^u = dW_s - \sigma(s,u)ds$, and can write

$$R(t,T) = f(0,t,T) + \frac{T}{2}\mathrm{Var}R(t,T) - (u-t)\mathrm{Cov}(R(t,T),R(t,(u-t))) - \int_0^t \frac{\sigma(s,T+t)-\sigma(s,t)}{T} dW_s^u.$$

Consequently, for u > t,

$$E_{Q_u}[R(t,T)] = E_{Q_t}[R(t,T)]-(u-t)\mathrm{Cov}(R(t,T),R(t,u-t)),$$

where

$$\mathrm{Cov}(R(t,T),R(t,u-t)) = \frac{T}{u-t}\mathrm{Var}R(t,T)\frac{1-e^{-a(u-t)}}{1-e^{-aT}}.$$

Coming back to formula (8), we can write

$$E_{Q_T}\left(I_{(D(t)>1)}\right) = N(d_2^t),$$

$$\text{where } d_2^t = d_1^t-\frac{(T-t)\mathrm{cov}(R(t,T-t),R(t,T))}{\sqrt{\mathrm{Var}R(t,T)}},$$

which finally gives

$$C^t(0) = e^{\lambda tR(0,T)}B(0,t)N(d_1^t)-N(d_2^t). \tag{8'}$$

Formula (8′) now looks like the Black and Scholes formula, or more precisely like Margrabe's formula generalized to stochastic interest rates, which is not surprising since the surrender option is nothing but an exchange option in a context of interest rates necessarily nondeterministic (otherwise, the option would not exist).

Numerical values are calculated for the European options with maturities t = 1, 2,..., 7 for different parameter values. The initial term structure of interest rates is taken on June 25, 1993, which is date zero. The parameter a, proved in earlier studies to be stable over long periods of time (El Karoui and Geman, 1991, 1993), is set at the value 0.1; from the study of liquid interest rate derivative securities at different dates, the second parameter σ, crucial in the term structure of volatilities, is shown by the same authors to fall between 2 and 3 percent. The value ß = 5 percent represents the standard upfront management fees in France. We run the calculations for $\lambda = 0.9$, which is a fair representation of the real situation and also, for the sake of comparison in two other cases, $\lambda = 0.75$ and $\lambda = 1$. Figures 4a and 4b plot the value at time zero of the surrender option as a function of the date of exercise for the volatility parameter σ set equal to 2 percent and 3 percent.

At this point, we make two important observations, one practical, the other theoretical. First, at variance with a belief shared by a number of insurers preoccupied by this problem, the maximum value of the European option is

Figure 4a

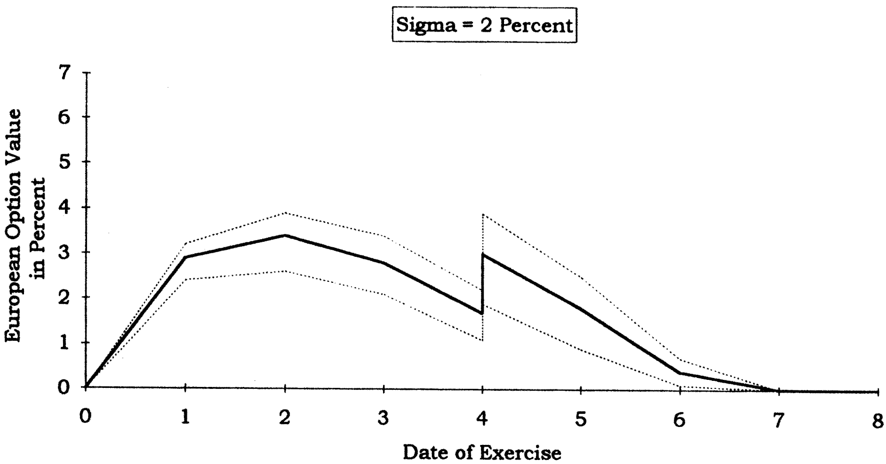
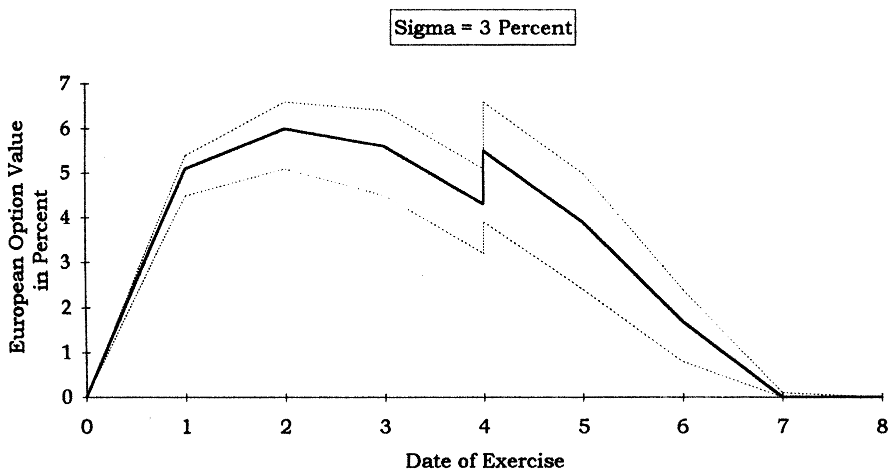


Figure 4b

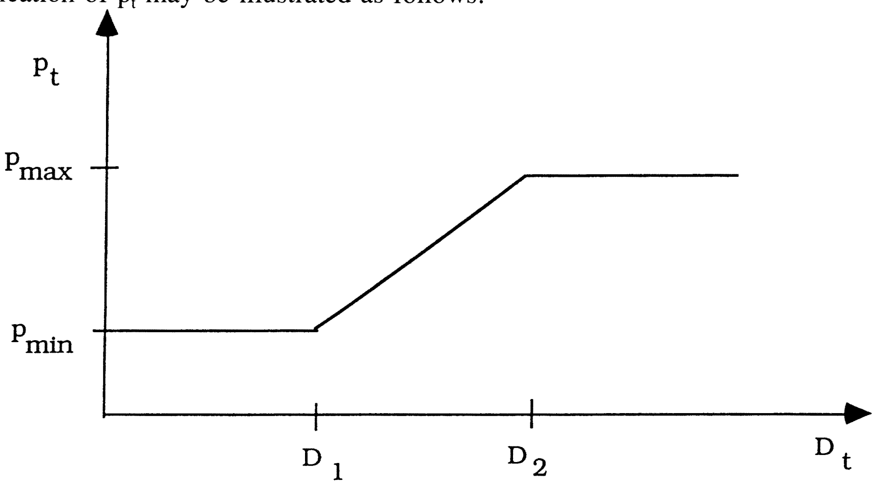


observed for maturities reached two to four years after inception of the contract and not for maturities greater than four years. Thus, the tax effect is dominated by the time value of the option. Second, one must keep in mind that, as with American options, the surrender option on a policy has a greater value than the supremum of the values of the corresponding European option for different possible exercise dates.

The Case of a Life Insurance Policy Pool

The fundamental feature of insurance is the pooling of risks, and, from the insurer's viewpoint, the surrender option on a pool of life insurance policies can be represented by the frequency p(t) of the policies that are early terminated at time t, where t = 1, 2,..., (T-1) since policy covenants permit only lapses at the end of the calendar year. This property, typical of the insurance mechanism, allows us to address the difficulty raised by the American nature of the surrender option and is consequently crucial to exhibit a closed-form expression for the surrender option.

Lapse behavior. We denote by p(t) or p_t the proportion of lapses at date t among the contracts still active in the pool and express p(t) as a deterministic function f of the decision criterion D_t. Moreover, as often assumed in mortgage prepayment models (see d'Andria, Boulier, and Elie, 1991, for instance), we represent f as a nondecreasing piece-wise linear function of the variable D_t. The existence of lapses reflecting policyholders' personal circumstances (including death) and independent of financial considerations accounts for a nonzero value for p_t even when D_t is below the value of D_1 that corresponds to the minimum threshold worth of the paperwork of early surrender. These "irrational" lapses are analogous to noneconomic prepayments on low-rate mortgages. In the same manner, some rational lapses never occur, and a reasonable specification of p_t may be illustrated as follows:



where $D_1 = 1$, $D_2 = 1.5$, $p_{min} = 0.03$, and $p_{max} = 0.6$.

As observed above, we are led to the valuation of a set of European options with different dates of exercise. The cash flows associated with the pool of insurance policies (taking into account the possible surrenders at times 1, 2,..., T-1 are the following:

Dates	0	1...	t	T
Liabilities	0	$p_1V_s(1)$	$a_tp_tV_s(t)$	$a_TV_s(T)$
Assets	0			$\frac{1}{B(0,T)}$

where a_t denotes the proportion of policies in the pool still alive at time t. If there was no surrender, the cash flows on the pool would be

Dates	0	1...	t	T
Liabilities	0	0	0	$a_TV_s(T)$
Assets	0			$\frac{a_T}{B(0,T)}$

and we obtain by difference the flows corresponding to the surrender option:

Dates	0	1...	t	T
Liabilities	0	$p_1V_s(1)$	$a_tp_tV_s(t)$	0
Assets	0			$(1-a_T)\frac{1}{B(0,T)}$

where $V_s(t) = e^{\lambda t R(0,T)}$.

The cash flows in the last table—which correspond to the cost for the insurer of the surrender option—depend on the insurer's management policy, both through the choice of the asset portfolio associated with the contracts under analysis (in our model, zero-coupon bonds) and through the coefficient λ, which defines how much of the portfolio yield goes to the policyholder.

The value C at time zero of the surrender option is obtained by arbitrage arguments from the series of cash flows described above:

$$\begin{aligned} C &= E_Q\left[\sum_{t=1}^{T-1} e^{-\int_0^t r(s)ds} p_t a_t V_s(t) - e^{-\int_0^T r(s)ds}(1-a_T)\frac{1}{B(0,T)}\right] \\ &= \sum_{t=1}^{T-1} B(0,t)E_{Q_t}[p_t a_t V_s(t)] - E_{Q_T}(1-a_T). \end{aligned} \tag{9}$$

This formula features the fact that the surrender option on a pool of policies is a portfolio of Black and Scholes-type options and is very similar to the expression of a coupon-bond option under Gaussian interest rates.

Observing that $a_T + \sum_{j=1}^{T-1} p_j a_j = 1$, we can also write

$$C = \sum_{t=1}^{T-1} E_{Q_t}\left[e^{-\int_0^t r(s)ds} p_t a_t \left(V_s(t) - \frac{B(t,T)}{B(0,T)} \right) \right].$$

This latter expression exhibits the loss at each time t due to the early termination of the policies, namely $p_t a_t\ (V_s(t) - V_m(t))$, where, as was defined before, $V_s(t)$ is the value of the policy when terminated at date t, and $V_m(t)$ is the market value of the asset portfolio.

Numerical Valuation of the Surrender Option

We first compute the option value using the closed-form expression derived above. From formula (9),

$$C = \sum_{t=1}^{T-1} B(0,t) E_{Q_t}[p_t a_t V_s(t)] - E_{Q_T}(1 - a_T).$$

We must observe that, for $2 \le t \le T$, $a_t = \prod_{k=1}^{t-1}(1 - p_k)$. Because $V_s(t) = e^{\lambda t R(0,T)}$ is deterministic, the difficulty remaining after the introduction of the new set of probability measures Q_t is the computation of $E_{Q_t}\left[p_t \prod_{k=1}^{t-1}(1 - p_k) \right]$. Since p_t depends on the interest rates prevailing at time t, and p_k, for $k < t$, depends on the interest rates at time k prior to t, these quantities are not independent. Assuming this independence—and again, this approximation can be justified by the insurance mechanism—we can write for $u \ge t$

$$E_{Q_u}(p_u a_u) = E_{Q_u}(p_u) \prod_{t=1}^{u-1} E_{Q_u}(1 - p_t),$$

$$\text{where } p_t = f(D_t) = p_{min}\, I_{(D(t)<D_1)} + \left(\frac{p_{max} - p_{min}}{D_2 - D_1} D_t + \frac{p_{max} D_1 - p_{min} D_2}{D_1 - D_2} \right) I_{(D_1 \le D_t < D_2)} + p_{max}\, I_{(D(t) \ge D_2)}.$$

Using the same calculations as those following equation (8), we can determine real numbers e_u^t and ε_u^t such that

$$E_{Q_u}\left(I_{(D(t)\geq D_1)}\right) = N(e_u^t)$$

and

$$E_{Q_u}\left(I_{(D(t)\geq D_2)}\right) = N(\varepsilon_u^t).$$

The only remaining difficulty now comes from the computation of $E_{Q_u}\left(D_t\, I_{D(t)\geq D_1}\right)$ and the analogous expression with D_2. Remembering that $D(t) = (1-\text{ß})e^{-\lambda TR(0,T)}K(t)e^{\lambda(T-t)R(t,T)}$, we can factor the deterministic part $g(t) = (1-\text{ß})e^{-\lambda TR(0,T)}K(t)$ and write

$$D(t) = g(t)e^{\lambda(T-t)R(t,T)}.$$

Moreover, we can observe that $D(t) \geq D_1 \Leftrightarrow R(t,T) \geq h(t)$,

$$\text{where } h(t) = \frac{1}{\lambda(T-t)}\ln\left(\frac{D_1}{g(t)}\right),$$

$$\text{and } E_{Q_u}\left(D(t)\, I_{(D(t)\geq D_1)}\right) = g(t)E_{Q_u}\left[e^{\lambda(T-t)R(t,T)}\, I_{(R(t,T)\geq h(t))}\right].$$

Introducing a final change of probability measure Q_u^* and the Q_u^*-Brownian motion (W_s^{u*}) defined through Girsanov's theorem by $dW_s^{u*} = dW_s^{u} + \lambda\left(\frac{T-t}{T}\right)(\sigma(s,T+t)-\sigma(s,t))ds$, we obtain

$$E_{Q_u}\left(D(t)\, I_{(D(t)\geq D_1)}\right)$$

$$= g(t)\exp[\lambda(T-t)E_{Q_u}(R(t,T))]E_{Q_u}\left[\exp\left(\frac{\lambda^2(T-t)^2}{2}\operatorname{Var}R(t,T)\, I_{(R(t,T)\geq h(t))}\right)\right]$$

$$= g(t)\exp\left[\frac{\lambda^2(T-t)^2}{2}\operatorname{Var}R(t,T)+\lambda(T-t)E_{Q_u}R(t,T)\right]E_{Q_{u^*}}\left[I_{(R(t,T)\geq h(t))}\right].$$

We observe again that the assumption of deterministic interest rates allowed us to provide an explicit formula.

At this point, we can compute the different quantities obtained in this expression. We use again the term structure observed on June 25, 1993 (date zero), and the parameters a and σ described earlier. As mentioned before, $D_1 = 1$, $D_2 = 1.5$, $p_{min} = 0.03$, and $p_{max} = 0.06$. We obtain for the surrender option values the following results: σ = 2 percent, C = 0.7; σ = 3 percent, C = 2.9.

An alternative method that avoids the approximation in the expectation of the product of the probabilities p_k uses Monte Carlo simulations in equation (9) that fully take into account the fact that the cash flows associated with the pool of insurance policies under analysis are path-dependent: the cash flow in any given period depends not only upon the current level of interest rates through p_t, but also upon the entire history of interest rates relevant to the pool through the numbers $a_t = \prod_{(k<t)} (1-p_k)$.

As observed earlier, for k = 1, 2,..., T-1, p_k is a function of R(k,T). Starting from the yield curve observed at time zero (which provides in particular the forward rates), we simulate the quantities

$$R(k,T) = f(0,k,T) - \int_0^k \frac{\sigma(s,T+k)-\sigma(s,k)}{T} dW_s + \frac{1}{2}\int_0^k \frac{\sigma^2(s,T+k)-\sigma^2(s,k)}{T} ds.$$

We calculate an approximate value for the stochastic integral involved in the second term and, using 500 draws, the simulations provide the following intervals of values: σ = 2 percent, [0.65; 0.76]; σ = 3 percent, [2.2; 2.6]. The width of these intervals may be explained by the long time to maturity (eight years) of the surrender option.

Sensitivity of the Option Price to Different Factors

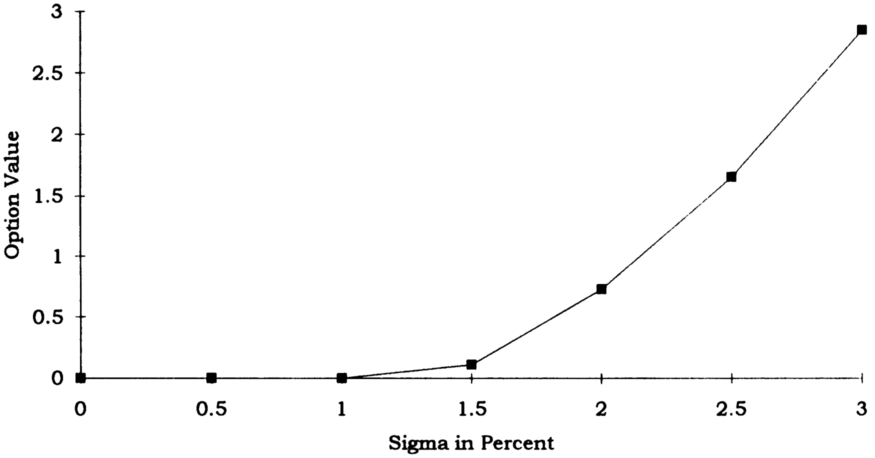
Influence of the initial yield curve. The initial yield curve is a significant parameter in the evaluation of the option. The price of the surrender option increases with the slope of the initial yield curve and decreases with the level of the initial yield curve (see Table 1).

Table 1
Sensitivity to the Slope and Level of the Initial Yield Curve

	σ = 2 *Percent*	σ = 3 *Percent*
Yield Curve Slope		
Negative	C = 0	C = 1.69
Null	C = 0.29	C = 2.16
Positive	C = 0.73	C = 2.85
Yield Curve Level		
Yield Curve of June 25, 1993	C = 0.73	C = 2.85
Downward Parallel Move of 1 Percent	C = 0.91	C = 3.11
Upward Parallel Move of 1 Percent	C = 0.57	C = 2.61

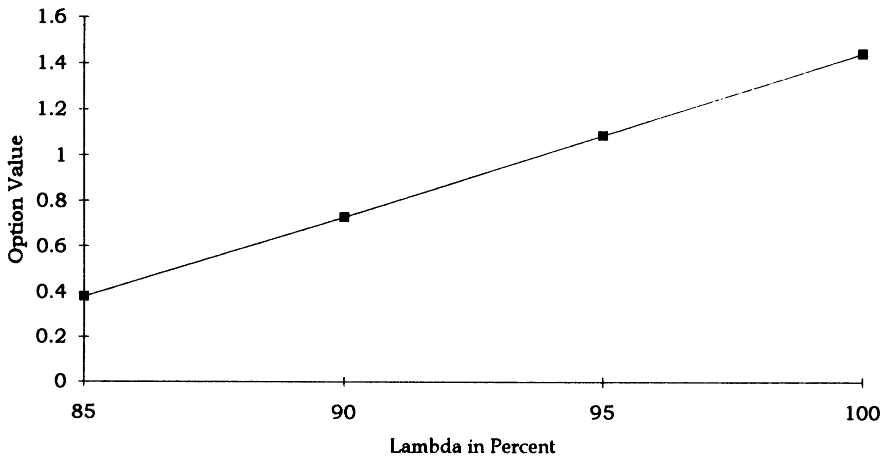
Sensitivity to the volatility of interest rates. The two parameters involved in the term structure of volatilities are a and σ. We observed above that the parameter a is very stable over time, and we set it equal to 0.1. Figure 5 plots the values of the surrender option on the pool of contracts for different values of σ between 0 and 3 percent.

Figure 5



Sensitivity to the choice of λ. Figure 6 represents the values of the option empirically obtained for different values of λ. Interestingly, the value of the surrender option on the pool of contracts appears as an affine function of the coefficient λ, at least for realistic values of this parameter.

Figure 6



Sensitivity to the specification of the function f. The choice of the surrender rate function f plays an essential role. The price of the option increases with the slope of this function, which reflects the "intensity" of rational behavior in the pool. The price of the option also naturally increases with the width of the interval $[p_{min}, p_{max}]$ (which expresses again a more rational behavior of the policyholders on both ends).

Conclusion

The possibility of early surrender of life insurance policies is a systemic risk for insurers since the option value is a significant percentage of the policy value which amounts in France to several billion francs. Insurers face a dilemma: either experience early terminations of life insurance contracts or guarantee a high yield to avoid these surrenders, in which case the management of the corresponding asset portfolio becomes difficult.

Consequently, it is necessary for insurers to estimate the value of the existing surrender options in the portfolio of insurance policies and to hedge the risk they represent. This hedge can be achieved by incorporating floating-rate notes in the asset portfolio and other financial instruments, particularly interest rate caps that are commonly available for 7- to 10-year maturities and would suit the insurer's needs. These caps should be tied to the interest rate index most closely related to the policy lapses that need to be hedged—in our example, the *taux moyen obligataire*, an average T-bond yield. At variance with reinsurance, early termination of the coverage is easy to achieve when this one is no longer necessary; the insurer can recoup part of the initial premium either by selling back to the issuer the remaining portion of the cap or by taking an opposite position in a new cap with characteristics identical to the remaining part of the first one. Obviously, these caps would only be an efficient hedge against interest rate-related policy surrenders; for other lapses (whether due to mortality or other factors), the appropriate funding is determined through actuarial expertise. In either case, the coverage cost must be incorporated in the coefficient λ and in the upfront fees when defining the insurer's marketing policy.

Appendix

Numerical Valuation of the Surrender Option

The lifetime of the option (and of the policy, if no surrender occurs) is T = 8 years. Lapses can only occur at times t = 1, 2,... , 7. The set of parameters is defined as follows: To describe the function $p_t = f(D_t)$, p_{min} = 3 percent, p_{max} = 60 percent, d_1 = 1, and d_2 = 1.5. To describe the term structure dynamics, the yield curve at time zero has the following increasing shape.

Θ	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
R(0,Θ) (in %)	6	6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8	6.9	7	7.1	7.2	7.3	7.4	7.5

The parameter a is set equal to 0.1, and the parameter σ observed at time zero is equal to 2 percent but we also run the calculations for a higher volatility σ = 3 percent. To describe the management fees, λ = 0.9, and ß = 5 percent.

Computation of $E_{Q_u}(p_t)$ for $u \geq t$

σ = 2 Percent	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7
u = 1	0.047						
u = 2	0.047	0.050					
u = 3	0.047	0.049	0.045				
u = 4	0.046	0.048	0.044	0.046			
u = 5	0.046	0.048	0.043	0.047	0.038		
u = 6	0.046	0.047	0.043	0.046	0.038	0.031	
u = 7	0.045	0.047	0.042	0.046	0.037	0.031	0.030
u = 8	0.045	0.047	0.042	0.045	0.037	0.031	0.030

σ = 3 Percent	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7
u = 1	0.070						
u = 2	0.068	0.076					
u = 3	0.067	0.074	0.069				
u = 4	0.066	0.072	0.067	0.072			
u = 5	0.065	0.070	0.065	0.070	0.055		
u = 6	0.064	0.069	0.063	0.070	0.053	0.038	
u = 7	0.063	0.067	0.061	0.065	0.051	0.037	0.030
u = 8	0.062	0.066	0.060	0.064	0.050	0.036	0.030

$$P_{Q_u}(D_t < 1) = N(D(t,u,1))$$

σ = 2 Percent	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7
u = 1	0.736						
u = 2	0.742	0.744					
u = 3	0.747	0.752	0.796				
u = 4	0.752	0.758	0.804	0.726			
u = 5	0.756	0.765	0.811	0.737	0.832		
u = 6	0.760	0.770	0.818	0.746	0.830	0.950	
u = 7	0.763	0.775	0.823	0.754	0.838	0.953	1
u = 8	0.766	0.780	0.828	0.762	0.845	0.956	1

σ = 3 Percent	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7
u = 1	0.647						
u = 2	0.657	0.647					
u = 3	0.666	0.661	0.686				
u = 4	0.674	0.673	0.701	0.627			
u = 5	0.681	0.684	0.714	0.645	0.703		
u = 6	0.688	0.693	0.726	0.661	0.720	0.844	
u = 7	0.693	0.702	0.736	0.675	0.736	0.856	0.994
u = 8	0.699	0.709	0.746	0.688	0.749	0.867	0.995

$$E_{Q_u}(D_t \; I_{D_t<1})$$

σ = 2 Percent	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7
u = 1	0.668						
u = 2	0.673	0.665					
u = 3	0.677	0.671	0.707				
u = 4	0.681	0.676	0.713	0.815			
u = 5	0.684	0.681	0.718	0.819	0.751		
u = 6	0.687	0.685	0.722	0.823	0.758	0.869	
u = 7	0.690	0.689	0.726	0.825	0.764	0.871	0.900
u = 8	0.693	0.692	0.730	0.829	0.769	0.872	0.899

σ = 3 Percent	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7
u = 1	0.570						
u = 2	0.578	0.559					
u = 3	0.585	0.570	0.590				
u = 4	0.591	0.579	0.601	0.688			
u = 5	0.597	0.587	0.611	0.699	0.630		
u = 6	0.602	0.594	0.620	0.708	0.644	0.765	
u = 7	0.606	0.600	0.627	0.716	0.655	0.774	0.898
u = 8	0.610	0.605	0.633	0.723	0.666	0.782	0.897

Var (R(t,T)) in Percent

σ = 2 Percent	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7
T = 8	0.017	0.031	0.043	0.052	0.060	0.066	0.071

σ = 3 Percent	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7
T = 8	0.039	0.070	0.096	0.117	0.135	0.149	0.161

E_{Q_u}(R(t,T)) in Percent, T = Eight Years

σ = 2 Percent	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7
u = 1	7.1						
u = 2	7	7.3					
u = 3	7	7.3	7.6				
u = 4	7	7.2	7.5	7.8			
u = 5	7	7.2	7.5	7.7	8		
u = 6	7	7.2	7.4	7.7	8	8.3	
u = 7	7	7.1	7.4	7.6	7.9	8.2	8.5
u = 8	6.9	7.1	7.3	7.6	7.8	8.1	8.4

σ = 3 Percent	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7
u = 1	7.2						
u = 2	7.1	7.5					
u = 3	7.1	7.4	7.8				
u = 4	7	7.3	7.7	8.1			
u = 5	7	7.2	7.5	7.9	8.3		
u = 6	6.9	7.1	7.4	7.8	8.2	8.6	
u = 7	6.9	7.1	7.3	7.6	8	8.4	8.8
u = 8	6.9	7	7.2	7.5	7.8	8.2	8.6

Numerical value of the surrender option: σ = 2 percent, C = 0.76 percent; σ = 3 percent, C = 2.9 percent.

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