Parallel, intersecting, skew and perpendicular lines

To determine whether two lines are parallel, intersecting, skew or perpendicular, we will need to perform a number of tests on the two lines.

Given two lines,

$$L_1: \quad x_1 = a_1 + b_1 t \qquad y_1 = c_1 + d_1 t \qquad z_1 = e_1 + f_1 t$$

$$L_2: \quad x_2 = a_2 + b_2 s \qquad y_2 = c_2 + d_2 s \qquad z_2 = e_2 + f_2 s$$

then the lines are

parallel if the ratio equality is true.

$$\frac{b_1}{b_2} = \frac{d_1}{d_2} = \frac{f_1}{f_2}$$

intersecting if the lines are not parallel or if you can solve them as a system of simultaneous equations.

perpendicular if the lines are intersecting and their dot product is 0.

$$L_1 \cdot L_2 = 0$$

skew if the lines are not parallel and not intersecting.

Example

Say whether the lines are parallel, intersecting, perpendicular or skew.

$$L_1: \quad x_1 = 1 + 5t \qquad y_1 = -3 + 2t \qquad z_1 = 1 + t$$

$$L_2: \quad x_2 = 2 + 3s \qquad y_2 = 3 + 4s \qquad z_2 = 3 - 2s$$

We'll start by testing the lines to see if they're parallel by pulling out the coefficients

$$\frac{b_1}{b_2} = \frac{d_1}{d_2} = \frac{f_1}{f_2}$$

$$\frac{5}{3} = \frac{2}{4} = \frac{1}{-2}$$

$$\frac{5}{3} = \frac{1}{2} = \frac{1}{-2}$$

Since $5/3 \neq 1/2 \neq -1/2$, we know the lines are not parallel.

Because they're not parallel, we'll test to see whether or not they're intersecting. We'll set the equations for x, y, and z from each line equal to each other. If we can find a solution set for the parameter values s and t, and this solution set satisfies all three equations, then we've proven that the lines are intersecting.

Setting $x_1 = x_2$, we get

$$x_1 = x_2$$

$$1 + 5t = 2 + 3s$$

$$5t = 1 + 3s$$

[1] $t = \frac{1}{5} + \frac{3}{5}s$

Setting $y_1 = y_2$, we get

$$y_1 = y_2$$

[2] $-3 + 2t = 3 + 4s$

Plugging **[1]** into **[2]** gives

$$-3 + 2\left(\frac{1}{5} + \frac{3}{5}s\right) = 3 + 4s$$

$$-3+\frac{2}{5}+\frac{6}{5}s=3+4s$$

$$-6+\frac{2}{5}=4s-\frac{6}{5}s$$

$$-\frac{28}{5}=\frac{14}{5}s$$

$$s=-\frac{28}{14}$$

[3] $s=-2$

Plugging **[3]** into **[1]** gives

$$t=\frac{1}{5}+\frac{3}{5}(-2)$$

$$t=\frac{1}{5}-\frac{6}{5}$$

[4] $t=-1$

Setting $z_1=z_2$, we get

$$z_1=z_2$$

[5] $1+t=3-2s$

Plugging **[3]** and **[4]** into **[5]** gives

$$1+t=3-2s$$

$$1+(-1)=3-2(-2)$$

$$0=7$$

Since $0\neq 7$, the lines are not intersecting.

Because L_1 and L_2 are not parallel and not intersecting, by definition they must be skew.

In the previous example, we didn't test for perpendicularity because only intersecting lines can be perpendicular, and we found that the lines were not intersecting. If we had found that L_1 and L_2 were in fact perpendicular, we would have needed to test for perpendicularity by taking the dot product, like this:

$$L_1 \cdot L_2 = (1 + 5t)(2 + 3s) + (-3 + 2t)(3 + 4s) + (1 + t)(3 - 2s)$$

$$L_1 \cdot L_2 = 2 + 3s + 10t + 15st - 9 - 12s + 6t + 8st + 3 - 2s + 3t - 2st$$

$$L_1 \cdot L_2 = 21st - 11s + 19t - 4$$

$$0 \neq 21st - 11s + 19t - 4$$

Since the dot product isn't 0, we've proven that the lines are not perpendicular.