Introduction To Calculus of Variations

Calculus of Variations Lesson 1

This is the **first lesson** of Part 3: Calculus of Variations.

In this part of the course, we will be diving into an area of math that shows up everywhere in advanced physics – **calculus of variations**.

Everything discussed in this part of the course is going to build on top of the things we've covered previously, such as *single*- and *multivariable calculus*, so I would recommend refreshing up on those. We'll also be looking at lots of **examples and applications from physics**, as the goal is to give you the tools you'll need to understand physics first and foremost.

In this lesson, we will first begin by going over the underlying goal of what calculus of variations is all about as well as some of the basic concepts.

In the following lessons, we will then go over lots of applications, examples and overall, looking at calculus of variations at a deeper level. Throughout the lessons, we will also discover lots of ways in which variational calculus appears in our current description of modern physics.

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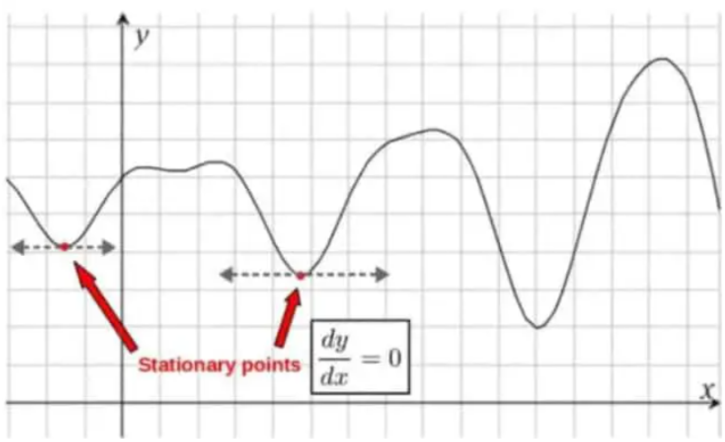
1. Brief Introduction To Calculus of Variations

At this point, we should have **single-variable calculus** covered quite well. The most important part about this in our context is going to be *minimizing* and *maximizing* single-variable functions, so we'll review this briefly. The applications of finding minima or maxima of single-variable functions appear everywhere in physics, so this is certainly an important topic.

We might, for example, have a function that describes the current in an electric circuit as a function of time and finding the maxima of this function would equate to knowing when the current is the biggest – lots of real-world applications with that one!

Generally, if we have a function of some variable, $f(x)$, we find its *extremal points* - either minima, maxima or stationary points in general - by setting its **first derivative equal to zero**, $df/dx = 0$ and solving for the value of x.



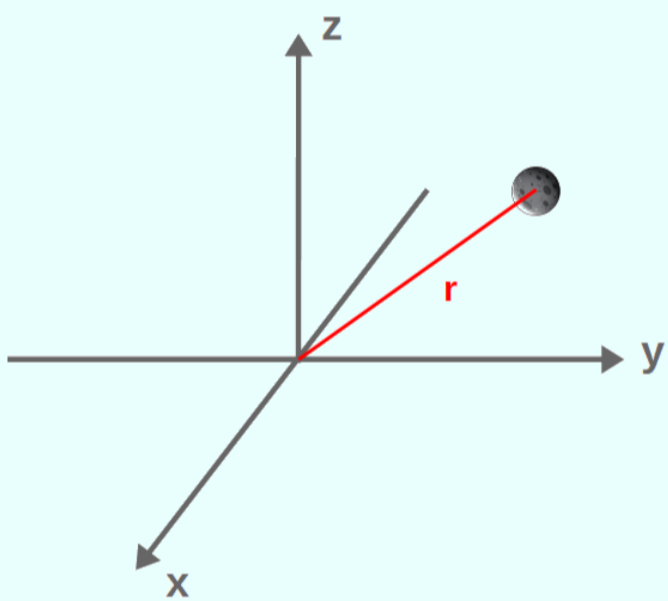
This gives us the value of the variable x at which the function is at an extremal point - in other words, when the function $f(x)$ has a minimum, maximum or stationary value.

The underlying reason for why we want to solve the equation $df(x)/dx = 0$ specifically comes from the fact that the slope of the tangent line of a given function (described by the derivative of that function) is zero at the stationary points.

Example: Finding The Minimum of a Radial Potential Energy Function

A very common physical application of finding extrema of single-variable functions is for finding the minimum of an “effective potential”, as this tells us a lot about the physical behaviour of a system and its stability.

For this example, we'll look at a particle or object of mass m moving around in a central potential $V(r)$ (meaning that the potential, or the force acting on the particle, only depends on the radial distance to the center – for example, gravitational potentials are of this form.



The effective potential, in this case, will be of the form:

$$V_{eff}(r) = \frac{L^2}{2mr^2} + V(r)$$

Here, L is the angular momentum of the particle and it is a constant (i.e. conserved).

The effective potential is an extremely useful tool for qualitative analysis of how a given system behaves. The reason we often analyze effective potentials in many problems involving rotation in 3D is because it nicely incorporates both of the “radial” as well as the “angular” motion of a system.

The first term in the effective potential above describes the rotational part of the particle's motion, as it involves the angular momentum L. The second term describes the radial part of the potential. The effective potential then incorporates both of these into one "effective" potential – in a sense, it describes the balance between the radial and the angular forces that determine the orbits of the particle.

Anyway, for our purpose, we want to find the minima of this effective potential as an example. The minimum of a radial effective potential generally corresponds to an orbit of constant radius (i.e. a circular orbit) for the particle. Therefore, the radius r of a circular orbit is found by setting the derivative of the effective potential to zero:

$$\frac{dV_{eff}(r)}{dr} = 0$$

$$\Rightarrow \quad \frac{d}{dr}\left(\frac{L^2}{2mr^2} + V(r)\right) = 0$$

$$\Rightarrow \quad \frac{L^2}{2mr^3} \cdot (-2) + \frac{dV(r)}{dr} = 0$$

$$\Rightarrow \quad \frac{dV(r)}{dr} = \frac{L^2}{mr^3}$$

So, given a *specific* central potential $V(r)$ - here, we just have one in a general form $V(r)$ - we would solve this equation to find the minimum of the effective potential, which describes the possible circular orbits a particle can have under this particular central potential. As an example, we could look at a gravitational potential (near a central mass M) of the form:

$$V(r) = -\frac{GMm}{r}$$

Plugging this into the equation above and solving for r, we find:

$$\frac{d}{dr}\left(-\frac{GMm}{r}\right) = \frac{L^2}{mr^3} \quad \Rightarrow \quad r = \frac{L^2}{GM}$$

So, we find that a particle with a given constant angular momentum L can have a circular orbit with radius L^2 / GM around a gravitating central mass M.

This is just one particular example of finding extrema of single-variable functions, but an extremely useful one. Finding minima of effective potential functions is used, for example, to find circular orbits of a charged particle in an electric field or even relativistic circular orbits around a black hole - but those are topics for another day!

Calculus of variations is based on a similar idea. It is the study of finding extremal points of something called **functionals**, which are essentially **functions of other functions**. In this sense, calculus of variations is just a generalization of finding extremal points of single-variable functions.

The perhaps difficult thing to conceptualize is that the extremal "points" of a functional are not really points, but rather *entire functions* – functions that either maximize, minimize or make stationary a given functional. So, the entire goal of variational calculus is to find the stationary "points" (extrema) of functionals. These extrema themselves are some kind of functions that depend on the particular problem at hand.

1.1. Why Should You Care About Calculus of Variations?

Before we discuss calculus of variations in detail, it's worth discussing why we would want to do so in the first place. Well, here are just a couple applications of variational calculus:

- Finding paths of shortest distance, called **geodesics**.
- Finding surfaces of minimal surface area.
- Describing the motion of objects under gravity in **general relativity**.
- Deriving equations of motion for systems in **Lagrangian mechanics**.
- Modeling the dynamics of fields in **field theories**.
- Describing the motion of light rays in a material in the field of **optics**.

In the context of *modern physics*, calculus of variations is actually one of the most important areas of math to master if you want to understand topics like quantum field theory or general relativity.

The main reason for this is the fact that almost all our modern theories of physics are described by an **action principle**. In short, this means that the dynamics of any given theory (such as the dynamics of the electromagnetic field) can be encoded into a quantity called the **action** – which surprise, surprise, is a functional.

Then, the actual dynamics of the given theory (the "field equations" if we're talking about a quantum field theory, for example) are found by making the action functional *stationary* by applying something called the **principle of stationary action** – again, requiring the tools of variational calculus. This is generally the way in which all modern field theories are formulated.

However, calculus of variations comes up even in classical mechanics in a formulation called **Lagrangian mechanics**. In Lagrangian mechanics, we describe the dynamics of a mechanical system by finding stationary solutions to an action functional – using calculus of variations. These solutions are the solutions to the equations of motion for the system, which are the *same solutions* we would get by using Newton's laws.

So, calculus of variations allows us to describe all of classical mechanics using the Lagrangian formulation instead of Newton's laws. It turns out that the Lagrangian formulation is much more powerful and more general than Newton's laws, making this an extremely useful application of variational calculus.

Calculus of variations also shows up quite frequently in various **geometric applications**. A typical example of where variational calculus comes up is in finding the *minimum distance* between any two points, say, in the xy-plane (or on some other, more complicated surface).

In this case, the distance itself would be described by a functional called the *arc length functional*. To minimize this arc length functional then means to find curves or functions $y(x)$ that minimize (or more generally, make *stationary*) the arc length functional. We, of course, do this using calculus of variations.

More generally, this problem of finding minimal distances between two points is the problem of finding **geodesics** in various geometries and it is what a large portion of *differential geometry* is about. Therefore, variational calculus is extremely important for differential geometry and also for **general relativity**, since general relativity is based on the mathematics of differential geometry!

Hopefully all of this gave you a bit of motivation for why you should want to learn calculus of variations. We'll look at lots of examples and physics applications later, which should make it even more clear that calculus of variations is an area of math you *really* should want to learn.

2. Functionals

In calculus of variations, the central mathematical objects of interest are called **functionals**. In the simplest sense, a functional is a "function of a function" – that is, a thing that takes in an **entire function** as its input and **returns a single number**. This number describes the value of the functional for that particular input function.

So, a functional is a more general object than an ordinary function. An ordinary function would take in just a single number (the value of a variable x, for example) and returns another number that describes the value of the function for that input value.

Sidenote; a functional is more generally defined as a *mapping* from a general mathematical space to the real or complex numbers. So, the input of a functional technically doesn't have to be just the space of functions.

For example, we could even think about the **dot product** with a given vector as a functional - this would take in a vector and return a number. In this case, the dot product would be thought of as a functional that maps a vector from an inner product space to the real (or complex) numbers. We'll revisit in the tensor calculus - lessons later on when discussing the *metric tensor*.

However, in the context of variational calculus, we will only consider functionals that take in a function and return a number (in most cases, a real number). That is, functionals that are mappings from the space of functions to the real numbers - *functions of functions*.

So, we're interested in functionals that take in some function – usually a single-variable function, such as a curve y(x) – and return a number. But how do we express such a thing mathematically?

Well, perhaps your first guess might be something of the form:

$$F(y) = (y(x))^2$$

This would indeed be a "function of a function" – it takes in an entire function $y(x)$ as its input. However, this does NOT return a number, instead it returns another function of the variable x. For example, if you were to plug in $y(x) = x^2$, you'd get $F(x) = x^4$. So, this is not a valid functional.

Generally, a valid functional that takes in a full function and always returns just a single number can be obtained by writing the functional as a **definite integral**. For example, something of the following form would be a valid functional:

$$F(y) = \int_0^1 (y(x))^2 dx$$

This would now take in a function y(x) and return a single number instead of a new function of x. We can see this by plugging in, for example $y(x) = x^2$:

$$F = \int_0^1 \left(x^2\right)^2 dx = \int_0^1 x^4 dx = \frac{1}{5}\Big/_0^1 x^5 = \frac{1}{5}$$

In general, we write functionals in the form of a definite integral. For a general functional, its integrand (the expression inside the integral) does not have to involve just $y(x)$, it can also also involve x itself, the derivative of $y(x)$ - $dy(x)/dx$ - or any number of higher derivatives of $y(x)$.

However, in physics, we're most often interested in functionals with their integrands being some function that involve just x, $y(x)$ or $dy(x)/dx$ and not other, higher order derivatives. These are the most common types of functionals encountered in physics and geometry.

So, the **general form of a functional** we're interested in here can be written as:

$$F(y) = \int_{x_1}^{x_2} f\left(x, y(x), \frac{dy(x)}{dx}\right) dx$$

The reason such functionals are interesting to us is because of reasons related to an incredibly important area of physics called *Lagrangian mechanics*. In Lagrangian mechanics (and field theory as well), we have something called **action functionals**, which are of the form:

$$S(q) = \int_{t_1}^{t_2} L\left(t, q_i(t), \frac{dq_i(t)}{dt}\right) dt$$

We won't go over this in more detail now, but the noteworthy point is that this functional is exactly of the form shown above - it involves only up to *first derivatives*.

Functionals of the above form are also encountered in **geometric applications** quite often. For example, we'll come to see that the length of a curve $y(x)$ in the xy-plane between two points ($x = a$ and $x = b$) is described by the *arc length functional*, which can be explicitly written as:

$$F(y) = \int_a^b \sqrt{1 + \left(\frac{dy(x)}{dx}\right)^2}\, dx$$

The arc length functional takes in some curve y(x) in the xy-plane and returns the length along the curve between the points a and b. We'll also discuss this in more detail in a later lesson.

In the arc length functional, we have the integrand as:

$$f\left(x, y, \frac{dy(x)}{dx}\right) = \sqrt{1 + \left(\frac{dy(x)}{dx}\right)^2}$$

Important piece of notation: In calculus of variations, we will often be denoting the derivative of a single-variable function as:

$$\frac{dy(x)}{dx} = y'$$

So, a prime-symbol (′) above a function $y(x)$ means the derivative of y with respect to its argument x. With this notation, our arc length functional above, for example, would be represented as:

$$F(y) = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\,dx$$

This is a piece of notation we'll be using all throughout the upcoming lessons.

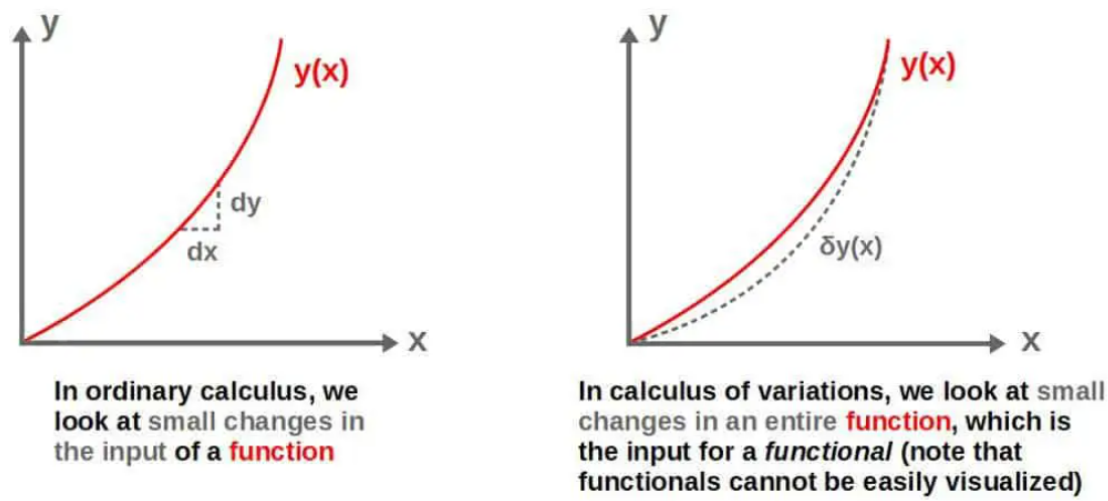
3. Calculus of Variations vs Ordinary Calculus

Calculus of variations is fundamentally based on the same mathematical tools as ordinary calculus (derivatives and integrals), but is quite a bit more complicated than ordinary calculus. Here, we'll take a brief look at comparing these two, so you may get a better idea of what we are actually doing.

First of all, calculus of variations considers the optimization of these objects called **functionals** as opposed to ordinary functions.

In ordinary calculus, we look at how the values of a function change with a small change in its input variable. In variational calculus, on the other hand, we look at how the values of a functional change with a small change in its input, which itself is an *entire function* as opposed to a single variable.

So, we're essentially looking at how a "function of a function" (i.e. functional) changes with a small change in its input function.



For a quick recap, here is a table comparing the main differences between ordinary functions and functionals:

Function	**Functional**
Takes in a number as input	Takes in a function as input
Returns a number	Returns a number
No general form	Generally expressed as a definite integral
Changes described by ordinary derivatives	Changes described by functional derivatives

Here, you'll see the mention of *functional derivative*. This is essentially the analogue of an ordinary derivative but for functionals and will be the topic of the next lesson.