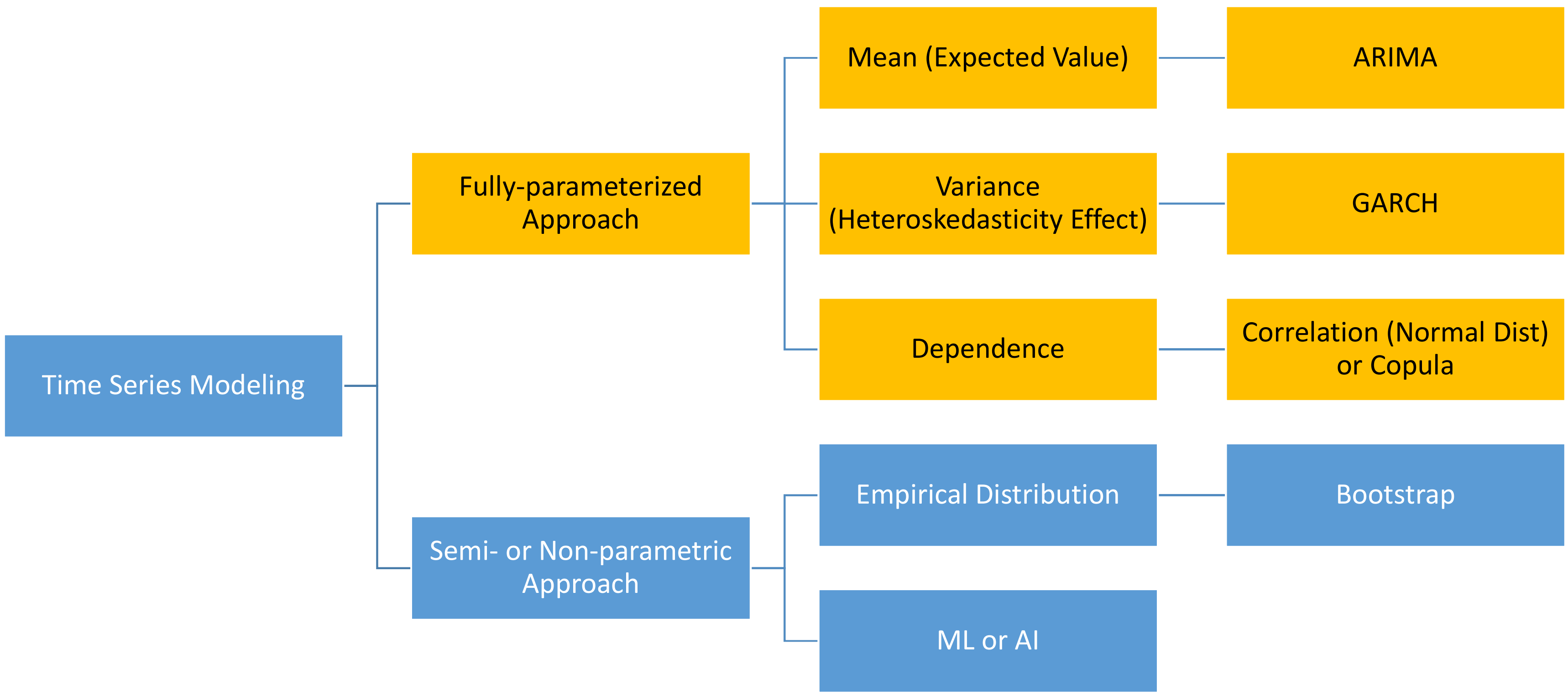
Copula & Time Series Modeling for Risk Analysis Application

Yas Suttakulpiboon

My Approach to Time Series Modeling



Mean Equation

- The mean equation of an univariate time series x_t can be described by the process

$$x_t = E(x_t \mid F_{t-1}) + \varepsilon_t$$

- where $E(\cdot \mid \cdot)$ denotes the conditional expectation operator, F_{t-1} the information set at time $t - 1$, and ε_t the innovations of the time series.

ARMA mean equation

- The ARMA(m,n) process of autoregressive order m and moving average order n can be described as

$$x_t = \mu + \sum_{i=1}^{m} a_i x_{t-i} + \sum_{j=1}^{n} b_j \varepsilon_{t-j} + \varepsilon_t \,,$$

- with mean μ, autoregressive coefficients ai and moving average coefficients bi .

ARIMA mean equation

- Let Y denote the *original* series
- Let y denote the *differenced* (stationarized) series

No difference (d=0): $y_t = Y_t$

First difference (d=1): $y_t = Y_t - Y_{t-1}$

Second difference (d=2): $y_t = (Y_t - Y_{t-1}) - (Y_{t-1} - Y_{t-2})$

$= Y_t - 2Y_{t-1} + Y_{t-2}$

ARIMA mean equation

$$\hat{y}_t = \overbrace{\mu}^{\text{constant}} + \overbrace{\phi_1 y_{t-1} + \ldots + \phi_p y_{t-p}}^{\text{AR terms (lagged values of } y\text{)}}$$

$$\underbrace{-\theta_1 e_{t-1} \ldots - \theta_q e_{t-q}}_{\text{MA terms (lagged errors)}}$$

By convention, the AR terms are + and the MA terms are –

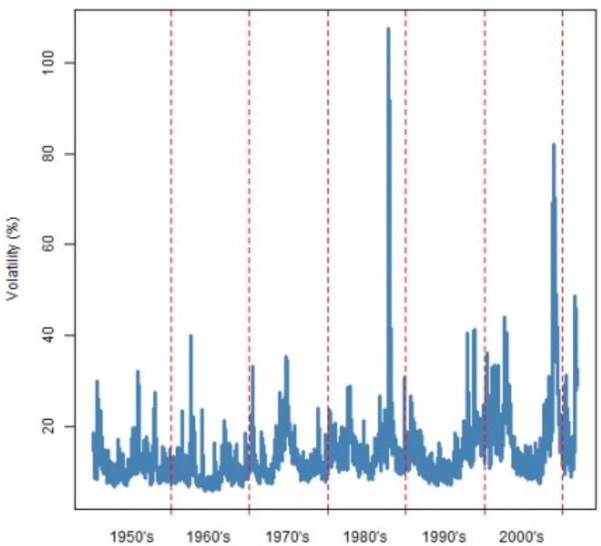
Not as bad as it looks! Usually $p+q \leq 2$ and either $p=0$ or $q=0$ (pure AR or pure MA model)

ARIMA models we've already met

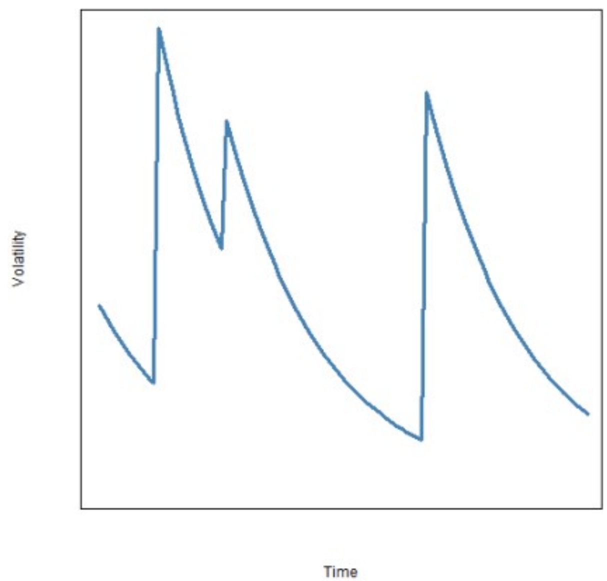
- ARIMA(0,0,0)+c = mean (constant) model
- ARIMA(0,1,0) = RW model
- ARIMA(0,1,0)+c = RW with drift model
- ARIMA(1,0,0)+c = regress Y on Y_LAG1
- ARIMA(1,1,0)+c = regr. Y_DIFF1 on Y_DIFF1_LAG1
- ARIMA(2,1,0)+c = " " plus Y_DIFF_LAG2 as well
- ARIMA(0,1,1) = SES model
- ARIMA(0,1,1)+c = SES + constant linear trend
- ARIMA(1,1,2) = LES w/ damped trend (leveling off)
- ARIMA(0,2,2) = generalized LES (including Holt's)

R

Variance Equation: GARCH



=



+

Volatility Noise + MA Component

Variance Equation: GARCH

- The mean equation does not take into account heteroskedastic effects typically observed in financial time series. Engle [1982] introduced the Autoregressive Conditional Heteroskedastic model, named ARCH, later generalised by Bollerslev [1986], named GARCH.

$$
\begin{aligned}
\varepsilon_t &= z_t \sigma_t \,, \\
z_t &\sim \mathcal{D}_\vartheta(0,1) \,, \\
\sigma_t^2 &= \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \,,
\end{aligned}
$$

Variance Equation: GARCH

- Ding [1993] introduced the APARCH(p,q) variance that can be expressed as

$$
\begin{aligned}
\varepsilon_t &= z_t \sigma_t \, , \\
z_t &\sim \mathcal{D}_\vartheta(0,1) \, , \\
\sigma_t^\delta &= \omega + \sum_{i=1}^{p} \alpha_i (|\varepsilon_{t-i}| - \gamma_i \varepsilon_{t-i})^\delta + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^\delta \, ,
\end{aligned}
$$

- where δ > 0 and −1 < γi < 1. This model adds the flexibility of a varying exponent with an asymmetry coefficient γi to take the leverage effect into account and the varying power δ to consider the Taylor effect.

Multivariate GARCH: DCC-GARCH

The Dynamic Conditional Correlation (DCC-) GARCH belongs to the class "Models of conditional variances and correlations" as discussed in Section 3.3. It was introduced by Engle and Sheppard in 2001 [11]. The idea of the models in this class is that the covariance matrix, $\boldsymbol{H}_t$, can be decomposed into conditional standard deviations, $\boldsymbol{D}_t$, and a correlation matrix, $\boldsymbol{R}_t$. In the DCC-GARCH model both $\boldsymbol{D}_t$ and $\boldsymbol{R}_t$ are designed to be time-varying.

Suppose we have returns, $\boldsymbol{a}_t$, from n assets with expected value 0 and covariance matrix $\mathbf{H}_t$. Then the Dynamic Conditional Correlation (DCC-) GARCH model is defined as:

$$\boldsymbol{r}_t = \boldsymbol{\mu}_t + \boldsymbol{a}_t \tag{24}$$

$$\boldsymbol{a}_t = \boldsymbol{H}_t^{1/2} \boldsymbol{z}_t \tag{25}$$

$$\boldsymbol{H}_t = \boldsymbol{D}_t \boldsymbol{R}_t \boldsymbol{D}_t \tag{26}$$

Notation:

$\mathbf{r}_t$:	$n \times 1$ vector of log returns of n assets at time t.
$\boldsymbol{a}_t$:	$n \times 1$ vector of mean-corrected returns of n assets at time t, i.e. $\mathrm{E}[\boldsymbol{a}_t]$=0. $\mathrm{Cov}[\boldsymbol{a}_t] = \boldsymbol{H}_t$.
$\boldsymbol{\mu}_t$:	$n \times 1$ vector of the expected value of the conditional $\boldsymbol{r}_t$.
$\boldsymbol{H}_t$:	$n \times n$ matrix of conditional variances of $\boldsymbol{a}_t$ at time t.
$\boldsymbol{H}_t^{1/2}$:	Any $n \times n$ matrix at time t such that $\boldsymbol{H}_t$ is the conditional variance matrix of a_t. $\boldsymbol{H}_t^{1/2}$ may be obtained by a Cholesky factorization of $\boldsymbol{H}_t$.
$\boldsymbol{D}_t$:	$n \times n$, diagonal matrix of conditional standard deviations of $\boldsymbol{a}_t$ at time t.
$\boldsymbol{R}_t$:	$n \times n$ conditional correlation matrix of $\mathbf{a}_t$ at time t.
$\boldsymbol{z}_t$:	$n \times 1$ vector of iid errors such that $\mathrm{E}[\boldsymbol{z}_t]$=0 and $\mathrm{E}[\boldsymbol{z}_t \boldsymbol{z}_t^T] = I$.