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Cambridge International AS & A Level

Further Mathematics

Further Pure Mathematics 1

Sophie Goldie

Rose Jewell

Series editor: Roger Porkess

DYNAMIC LEARNING

HODDER EDUCATION

Cambridge
International AS & A Level

Further Mathematics
Further Pure Mathematics 1

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DYNAMIC LEARNING
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Much of the material in this book was published originally as part of the MEI Structured Mathematics series. It has been carefully adapted for the Cambridge International AS & A level Mathematics syllabus. The original MEI author team for Pure Mathematics comprised Catherine Berry, Val Hanrahan, Terry Heard, David Martin, Jean Matthews, Roger Porkess and Peter Secker.

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Introduction

This is the first in a series of four books supporting the Cambridge International AS & A Level Further Mathematics 9231 syllabus for examination from 2020. It is preceded by five books supporting Cambridge International AS & A Level Mathematics 9709. The seven chapters in this book cover the further pure mathematics required for the Paper 1 examination. This part of the series also contains a more advanced book for further pure mathematics, and a book each for further mechanics and further probability and statistics.

These books are based on the highly successful series for the Mathematics in Education and Industry (MEI) syllabus in the UK but they have been redesigned and revised for Cambridge International students; where appropriate, new material has been written and the exercises contain many past Cambridge International examination questions. An overview of the units making up the Cambridge International syllabus is given in the following pages.

Throughout the series, the emphasis is on understanding the mathematics as well as routine calculations. The various exercises provide plenty of scope for practising basic techniques; they also contain many typical examination-style questions.

The original MEI author team would like to thank Sophie Goldie and Rose Jewell who have carried out the extensive task of presenting their work in a suitable form for Cambridge International students and for their many original contributions. They would also like to thank Cambridge Assessment International Education for its detailed advice in preparing the books and for permission to use many past examination questions.

Roger Porkess

Series editor

How to use this book

The structure of the book

This book has been endorsed by Cambridge Assessment International Education. It is listed as an endorsed textbook for students taking the Cambridge International AS & A Level Further Mathematics 9231 syllabus. The Further Pure Mathematics 1 syllabus content is covered comprehensively and is presented across seven chapters, offering a structured route through the course.

The book is written on the assumption that you have covered and understood the work in the Cambridge International AS & A Level Mathematics 9709 syllabus.

Each chapter is broken down into several sections, with each section covering a single topic. Topics are introduced through **explanations**, with **key terms** picked out in red. These are reinforced with plentiful **worked examples**, punctuated with commentary, to demonstrate methods and illustrate application of the mathematics under discussion.

Regular **exercises** allow you to apply what you have learned. They offer a large variety of practice and higher-order question types that map to the key concepts of the Cambridge International syllabus. Look out for the following icons.

PS **Problem-solving questions** will help you to develop the ability to analyse problems, recognise how to represent different situations mathematically, identify and interpret relevant information, and select appropriate methods.

M **Modelling questions** provide you with an introduction to the important skill of mathematical modelling. In this, you take an everyday or workplace situation, or one that arises in your other subjects, and present it in a form that allows you to apply mathematics to it.

CP **Communication and proof questions** encourage you to become a more fluent mathematician, giving you scope to communicate your work with clear, logical arguments and to justify your results.

Exercises also include questions from real Cambridge Assessment International Education past papers, so that you can become familiar with the types of questions you are likely to meet in formal assessments.

Answers to exercise questions, excluding long explanations and proofs, are available online at www.hoddereducation.com/cambridgeextras, so you can check your work. It is important, however, that you have a go at answering the questions before looking up the answers if you are to understand the mathematics fully.

ACTIVITY

In addition to the exercises, **Activities** invite you to do some work for yourself, typically to introduce you to ideas that are then going to be taken further. In some places, activities are also used to follow up work that has just been covered.

Other helpful features include the following.

? This symbol highlights points it will benefit you to **discuss** with your teacher or fellow students, to encourage deeper exploration and mathematical communication. If you are working on your own, there are answers available online at www.hoddereducation.com/cambridgeextras.

! This is a **warning** sign. It is used where a common mistake, misunderstanding or tricky point is being described to prevent you from making the same error.

A variety of notes are included to offer advice or spark your interest:

Note

Notes expand on the topic under consideration and explore the deeper lessons that emerge from what has just been done.

Historical note

Historical notes offer interesting background information about famous mathematicians or results to engage you in this fascinating field.

Technology note

Although graphical calculators and computers are not permitted in the examinations for this Cambridge International syllabus, we have included **Technology notes** to indicate places where working with them can be helpful for learning and for teaching.

Finally, each chapter ends with the **key points** covered, plus a list of the **learning outcomes** that summarise what you have learned in a form that is closely related to the syllabus.

Digital support

Comprehensive online support for this book, including further questions, is available by subscription to MEI's Integral® online teaching and learning platform for AS & A Level Mathematics and Further Mathematics, integralmaths.org. This online platform provides extensive, high-quality resources, including printable materials, innovative interactive activities, and formative and summative assessments. Our eTextbooks link seamlessly with

Integral, allowing you to move with ease between corresponding topics in the eTextbooks and Integral.

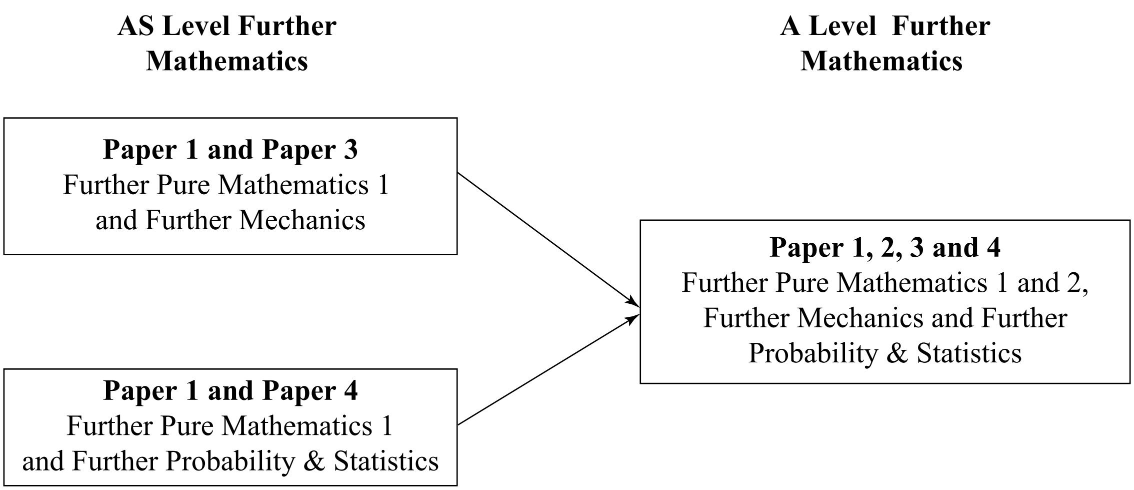
MEI's Integral® material has not been through the Cambridge International endorsement process.

The Cambridge International AS & A Level Further Mathematics 9231 syllabus

The syllabus content is assessed over four examination papers.

Paper 1: Further Pure Mathematics 1 • 2 hours • 60% of the AS Level; 30% of the A Level • Compulsory for AS and A Level	**Paper 3: Further Mechanics** • 1 hour 30 minutes • 40% of the AS Level; 20% of the A Level • Offered as part of AS; compulsory for A Level
Paper 2: Further Pure Mathematics 2 • 2 hours • 30% of the A Level • Compulsory for A Level; not a route to AS Level	**Paper 4: Further Probability & Statistics** • 1 hour 30 minutes • 40% of the AS Level; 20% of the A Level • Offered as part of AS; compulsory for A Level

The following diagram illustrates the permitted combinations for AS Level and A Level.



Prior knowledge

It is expected that learners will have studied the majority of the Cambridge International AS & A Level Mathematics 9709 syllabus content before studying Cambridge International AS & A Level Further Mathematics 9231.

The prior knowledge required for each Further Mathematics component is shown in the following table.

Component in AS & A Level Further Mathematics 9231	Prior knowledge required from AS & A Level Mathematics 9709
9231 Paper 1: Further Pure Mathematics 1	9709 Papers 1 and 3
9231 Paper 2: Further Pure Mathematics 2	9709 Papers 1 and 3
9231 Paper 3: Further Mechanics	9709 Papers 1, 3 and 4
9231 Paper 4: Further Probability & Statistics	9709 Papers 1, 3, 5 and 6

Command words

The table below includes command words used in the assessment for this syllabus. The use of the command word will relate to the subject context.

Command word	What it means
Calculate	work out from given facts, figures or information
Deduce	conclude from available information
Derive	obtain something (expression/equation/value) from another by a sequence of logical steps
Describe	state the points of a topic / give characteristics and main features
Determine	establish with certainty
Evaluate	judge or calculate the quality, importance, amount, or value of something
Explain	set out purposes or reasons / make the relationships between things evident / provide why and/or how and support with relevant evidence
Identify	name/select/recognise
Interpret	identify meaning or significance in relation to the context
Justify	support a case with evidence/argument
Prove	confirm the truth of the given statement using a chain of logical mathematical reasoning
Show (that)	provide structured evidence that leads to a given result
Sketch	make a simple freehand drawing showing the key features
State	express in clear terms
Verify	confirm a given statement/result is true

Key concepts

Key concepts are essential ideas that help students develop a deep understanding of mathematics.

The key concepts are:

Problem solving

Mathematics is fundamentally problem solving and representing systems and models in different ways. These include:

- Algebra: this is an essential tool which supports and expresses mathematical reasoning and provides a means to generalise across a number of contexts.
- Geometrical techniques: algebraic representations also describe a spatial relationship, which gives us a new way to understand a situation.
- Calculus: this is a fundamental element which describes change in dynamic situations and underlines the links between functions and graphs.
- Mechanical models: these explain and predict how particles and objects move or remain stable under the influence of forces.
- Statistical methods: these are used to quantify and model aspects of the world around us. Probability theory predicts how chance events might proceed, and whether assumptions about chance are justified by evidence.

Communication

Mathematical proof and reasoning is expressed using algebra and notation so that others can follow each line of reasoning and confirm its completeness and accuracy. Mathematical notation is universal. Each solution is structured, but proof and problem solving also invite creative and original thinking.

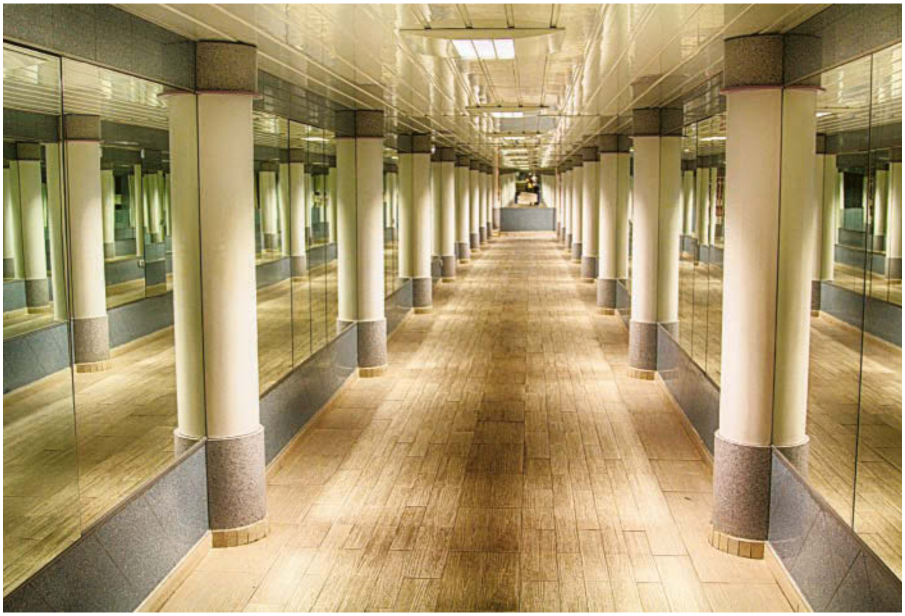
Mathematical modelling

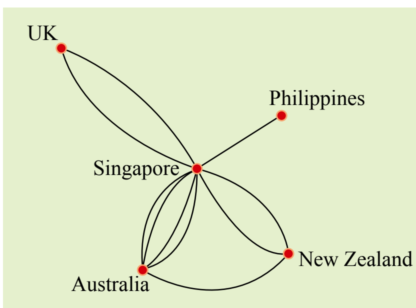
Mathematical modelling can be applied to many different situations and problems, leading to predictions and solutions. A variety of mathematical content areas and techniques may be required to create the model. Once the model has been created and applied, the results can be interpreted to give predictions and information about the real world.

These key concepts are reinforced in the different question types included in this book: **Problem-solving**, **Communication and proof**, and **Modelling**.

1 Matrices and transformations

As for everything else, so for a mathematical theory – beauty can be perceived but not explained.
Arthur Cayley (1821–1895)





▲ **Figure 1.1** Direct flights between countries by one airline.

?

Figure 1.1 shows some of the direct flights between countries by one airline. How many direct flights are there from:

- Singapore to Australia
- Australia to New Zealand
- the UK to the Philippines?

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

1.1 Matrices

You can represent the number of direct flights between each pair of countries (shown in Figure 1.1) as an array of numbers like this:

	A	N	P	S	U
A	0	1	0	4	0
N	1	0	0	2	0
P	0	0	0	1	0
S	4	2	1	0	2
U	0	0	0	2	0

The array is called a **matrix** (the plural is **matrices**) and is usually written inside curved brackets.

$$\begin{pmatrix} 0 & 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 4 & 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

It is usual to represent matrices by capital letters, often in bold print.

A matrix consists of rows and columns, and the entries in the various cells are known as **elements**.

The matrix $\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 4 & 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$ representing the flights between the counties has 25 elements, arranged in five rows and five columns. **M** is described as a 5×5 matrix, and this is the **order** of the matrix. You state the number of rows first then the number of columns. So, for example, the matrix

$\mathbf{M} = \begin{pmatrix} 3 & -1 & 4 \\ 2 & 0 & 5 \end{pmatrix}$ is a 2×3 matrix and $\mathbf{N} = \begin{pmatrix} 4 & -4 \\ 3 & 4 \\ 0 & -2 \end{pmatrix}$ is a 3×2 matrix.

Special matrices

Some matrices are described by special names that relate to the number of rows and columns or the nature of the elements.

Matrices such as $\begin{pmatrix} 4 & 2 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 3 & 5 & 1 \\ 2 & 0 & -4 \\ 1 & 7 & 3 \end{pmatrix}$ that have the same number of rows as columns are called **square matrices**.

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the 2×2 **identity matrix** or **unit matrix**, and similarly $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is called the 3×3 identity matrix. Identity matrices must be square, and are usually denoted by I.

The matrix $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is called the 2×2 **zero matrix**. Zero matrices can be of any order.

Two matrices are said to be **equal** if, and only if, they have the same order and each element in one matrix is equal to the corresponding element in the other matrix. So, for example, the matrices **A** and **D** below are equal, but **B** and **C** are not equal to any of the other matrices.

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

Working with matrices

Matrices can be added or subtracted if they are of the same order.

$$\begin{pmatrix} 2 & 4 & 0 \\ -1 & 3 & 5 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 4 \\ 2 & 0 & -5 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 4 \\ 1 & 3 & 0 \end{pmatrix}$$

Add the elements in corresponding positions.

$$\begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} 7 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -5 & 0 \\ 5 & -1 \end{pmatrix}$$

Subtract the elements in corresponding positions.

But $\begin{pmatrix} 2 & 4 & 0 \\ -1 & 3 & 5 \end{pmatrix} + \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix}$ cannot be evaluated because the matrices are not of the same order. These matrices are **non-conformable** for addition.

You can also multiply a matrix by a **scalar** number:

$$2\begin{pmatrix} 3 & -4 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 6 & -8 \\ 0 & 12 \end{pmatrix}$$

Multiply each of the elements by 2.

Technology note

You can use a calculator to add and subtract matrices of the same order and to multiply a matrix by a number. If you have a calculator that can handle matrices, find out:

- the method for inputting matrices
- how to add and subtract matrices
- how to multiply a matrix by a number for matrices of varying sizes.

Associativity and commutativity

When working with numbers the properties of **associativity** and **commutativity** are often used.

Associativity

Addition of numbers is **associative**.

$$(3 + 5) + 8 = 3 + (5 + 8)$$

When you add numbers, it does not matter how the numbers are grouped, the answer will be the same.

Commutativity

Addition of numbers is **commutative**.

$$4 + 5 = 5 + 4$$

When you add numbers, the order of the numbers can be reversed and the answer will still be the same.

?

- Give examples to show that subtraction of numbers is not commutative or associative.
- Are matrix addition and matrix subtraction associative and/or commutative?

Exercise 1A

1 Write down the order of these matrices.

(i) $\begin{pmatrix} 2 & 4 \\ 6 & 0 \\ -3 & 7 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 8 & 4 \\ -2 & -3 & 1 \\ 5 & 3 & -2 \end{pmatrix}$ (iii) $\begin{pmatrix} 7 & -3 \end{pmatrix}$

(iv) $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ (v) $\begin{pmatrix} 2 & -6 & 4 & 9 \\ 5 & 10 & 11 & -4 \end{pmatrix}$ (vi) $\begin{pmatrix} 8 & 5 \\ -2 & 0 \\ 3 & -9 \end{pmatrix}$

2 For the matrices

$$\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 0 & 4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 7 & -3 \\ 1 & 4 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 3 & 5 & -9 \\ 2 & 1 & 4 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 0 & -4 & 5 \\ 2 & 1 & 8 \end{pmatrix}$$

$$\mathbf{E} = \begin{pmatrix} -3 & 5 \\ -2 & 7 \end{pmatrix} \quad \mathbf{F} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

find, where possible

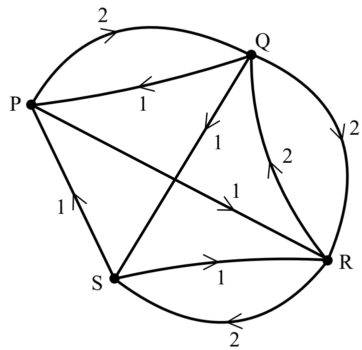
(i) $\mathbf{A} - \mathbf{E}$ (ii) $\mathbf{C} + \mathbf{D}$ (iii) $\mathbf{E} + \mathbf{A} - \mathbf{B}$

(iv) $\mathbf{F} + \mathbf{D}$ (v) $\mathbf{D} - \mathbf{C}$ (vi) $4\mathbf{F}$

(vii) $3\mathbf{C} + 2\mathbf{D}$ (viii) $\mathbf{B} + 2\mathbf{F}$ (ix) $\mathbf{E} - (2\mathbf{B} - \mathbf{A})$

3 The diagram below shows the number of direct ferry crossings on one day offered by a ferry company between cities P, Q, R and S.

The same information is also given in the partly completed matrix **X**.



$$\mathbf{X} = \begin{array}{c c} & \text{To} \\ & \begin{array}{cccc} \text{P} & \text{Q} & \text{R} & \text{S} \end{array} \\ \text{From} \begin{array}{c} \text{P} \\ \text{Q} \\ \text{R} \\ \text{S} \end{array} & \begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & & & \\ & & & \\ & & & \end{pmatrix} \end{array}$$

(i) Copy and complete the matrix **X**.

A second ferry company also offers ferry crossings between these four cities. The following matrix represents the total number of direct ferry crossings offered by the two ferry companies.

$$\begin{pmatrix} 0 & 2 & 3 & 2 \\ 2 & 0 & 2 & 1 \\ 2 & 2 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{pmatrix}$$

(ii) Find the matrix **Y** representing the ferry crossings offered by the second ferry company.

(iii) Draw a diagram similar to the one above, showing the ferry crossings offered by the second ferry company.

4 Find the values of w, x, y and z such that

$$\begin{pmatrix} 3 & w \\ -1 & 4 \end{pmatrix} + x\begin{pmatrix} 2 & -1 \\ y & z \end{pmatrix} = \begin{pmatrix} -9 & 8 \\ 11 & -8 \end{pmatrix}.$$

5 Find the possible values of p and q such that

$$\begin{pmatrix} p^2 & -3 \\ 2 & 9 \end{pmatrix} - \begin{pmatrix} 5p & -2 \\ -7 & q^2 \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ 9 & 4 \end{pmatrix}.$$

M **6** Four local football teams took part in a competition in which every team plays each of the others twice, once at home and once away. The results matrix after half of the games had been played is:

	Win	Draw	Lose	Goals for	Goals against
Stars	2	1	0	6	3
Cougars	0	0	3	2	8
Town	2	0	1	4	3
United	1	1	1	5	3

(i) The results of the next three matches are as follows:

Stars	2	Cougars	0
Town	3	United	3
Stars	2	Town	4

Find the results matrix for these three matches and hence find the complete results matrix for all the matches so far.

(ii) Here is the complete results matrix for the whole competition.

$$\begin{pmatrix} 4 & 1 & 1 & 12 & 8 \\ 1 & 1 & 4 & 5 & 12 \\ 3 & 1 & 2 & 12 & 10 \\ 1 & 3 & 2 & 10 & 9 \end{pmatrix}$$

Find the results matrix for the last three matches (Stars vs United, Cougars vs Town and Cougars vs United) and deduce the result of each of these three matches.

M **7** A mail-order clothing company stocks a jacket in three different sizes and four different colours.

The matrix $\mathbf{P} = \begin{pmatrix} 17 & 8 & 10 & 15 \\ 6 & 12 & 19 & 3 \\ 24 & 10 & 11 & 6 \end{pmatrix}$ represents the number of jackets in stock at the start of one week.

The matrix $\mathbf{Q} = \begin{pmatrix} 2 & 5 & 3 & 0 \\ 1 & 3 & 4 & 6 \\ 5 & 0 & 2 & 3 \end{pmatrix}$ represents the number of orders for jackets received during the week.

(i) Find the matrix $\mathbf{P} - \mathbf{Q}$.

What does this matrix represent? What does the negative element in the matrix mean?

A delivery of jackets is received from the manufacturers during the week.

The matrix $\mathbf{R} = \begin{pmatrix} 5 & 10 & 10 & 5 \\ 10 & 10 & 5 & 15 \\ 0 & 0 & 5 & 5 \end{pmatrix}$ shows the number of jackets received.

(ii) Find the matrix that represents the number of jackets in stock at the end of the week after all the orders have been dispatched.

(iii) Assuming that this week is typical, find the matrix that represents sales of jackets over a six-week period. How realistic is this assumption?

1.2 Multiplication of matrices

When you multiply two matrices you do not just multiply corresponding terms. Instead you follow a slightly more complicated procedure. The following example will help you to understand the rationale for the way it is done.

There are four ways of scoring points in rugby: a try (five points), a conversion (two points), a penalty (three points) and a drop goal (three points). In a match, Tonga scored three tries, one conversion, two penalties and one drop goal.

So their score was

$3 \times 5 + 1 \times 2 + 2 \times 3 + 1 \times 3 = 26.$

You can write this information using matrices. The tries, conversions, penalties and drop goals that Tonga scored are written as the 1×4 row matrix $(3 \quad 1 \quad 2 \quad 1)$ and the points for the different methods of scoring as the 4×1 column matrix $\begin{pmatrix} 5 \\ 2 \\ 3 \\ 3 \end{pmatrix}$.

These are combined to give the 1×1 matrix
$(3 \times 5 + 1 \times 2 + 2 \times 3 + 1 \times 3) = (26)$.

Combining matrices in this way is called **matrix multiplication** and this example is written as $(3 \quad 1 \quad 2 \quad 1) \times \begin{pmatrix} 5 \\ 2 \\ 3 \\ 3 \end{pmatrix} = (26)$.

The use of matrices can be extended to include the points scored by the other team, Japan. They scored two tries, two conversions, four penalties and one drop goal. This information can be written together with Tonga's scores as a 2×4 matrix, with one row for Tonga and the other for Japan. The multiplication is then written as

$$\begin{pmatrix} 3 & 1 & 2 & 1 \\ 2 & 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 26 \\ 29 \end{pmatrix}.$$

So Japan scored 29 points and won the match.

This example shows you two important points about matrix multiplication. Look at the orders of the matrices involved.

The two 'middle' numbers, in this case 4, must be the same for it to be possible to multiply two matrices. If two matrices can be multiplied, they are conformable for multiplication.

$$2 \times 4 \quad \times \quad 4 \times 1$$

The two 'outside' numbers give you the order of the product matrix, in this case 2×1.

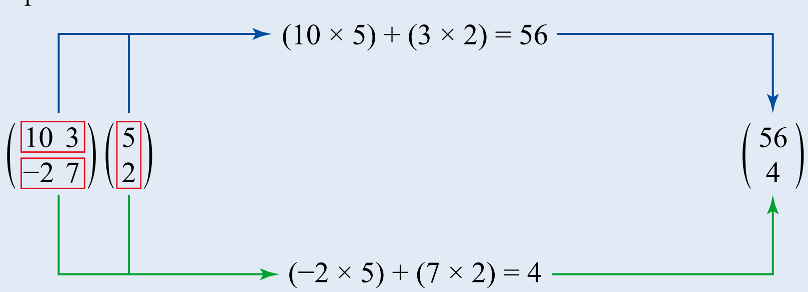
You can see from the previous example that multiplying matrices involves multiplying each element in a row of the left-hand matrix by each element in a column of the right-hand matrix and then adding these products.

Example 1.1

Find $\begin{pmatrix} 10 & 3 \\ -2 & 7 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

Solution

The product will have order 2×1.



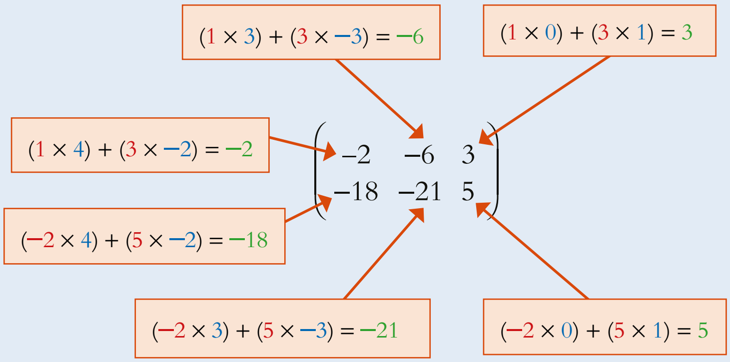
▲ **Figure 1.2**

Example 1.2

Find $\begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 4 & 3 & 0 \\ -2 & -3 & 1 \end{pmatrix}$.

Solution

The order of this product is 2 × 3.



So $\begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 4 & 3 & 0 \\ -2 & -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -6 & 3 \\ -18 & -21 & 5 \end{pmatrix}$

?

› If $\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 \\ -2 & 4 & 1 \\ 0 & 3 & 7 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 8 & -1 \\ -2 & 3 \\ 4 & 0 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 & 0 \\ 3 & -4 \end{pmatrix}$, which of the products **AB**, **BA**, **AC**, **CA**, **BC** and **CB** exist?

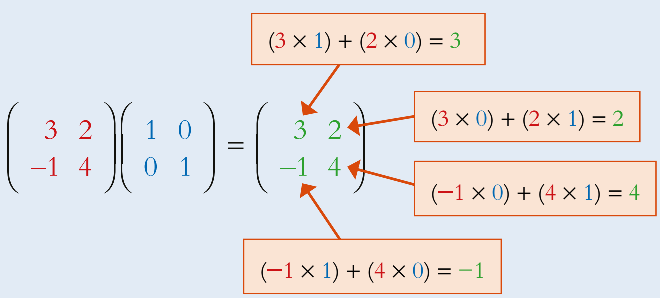
Example 1.3

Find $\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

What do you notice?

Solution

The order of this product is 2 × 2.



Multiplying a matrix by the identity matrix has no effect.

1

Properties of matrix multiplication

In this section you will look at whether matrix multiplication is:

- commutative
- associative.

On page 4 you saw that for numbers, addition is both associative and commutative. Multiplication is also both associative and commutative. For example:

$$(3 \times 4) \times 5 = 3 \times (4 \times 5)$$

and

$$3 \times 4 = 4 \times 3$$

ACTIVITY 1.1

Using $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -4 & 0 \\ -2 & 1 \end{pmatrix}$ find the products **AB** and **BA** and hence comment on whether or not matrix multiplication is commutative. Find a different pair of matrices, **C** and **D**, such that **CD** = **DC**.

Technology note

You could use the matrix function on your calculator.

ACTIVITY 1.2

Using $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -4 & 0 \\ -2 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, find the matrix products:

(i) **AB**

(ii) **BC**

(iii) (**AB**)**C**

(iv) **A**(**BC**)

Does your answer suggest that matrix multiplication is associative? Is this true for all 2 × 2 matrices? How can you prove your answer?

Exercise 1B

In this exercise, do not use a calculator unless asked to. A calculator can be used for checking answers.

1 Write down the orders of these matrices.

(i) (a) $\mathbf{A} = \begin{pmatrix} 3 & 4 & -1 \\ 0 & 2 & 3 \\ 1 & 5 & 0 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 2 & 3 & 6 \end{pmatrix}$

(c) $\mathbf{C} = \begin{pmatrix} 4 & 9 & 2 \\ 1 & -3 & 0 \end{pmatrix}$ (d) $\mathbf{D} = \begin{pmatrix} 0 & 2 & 4 & 2 \\ 0 & -3 & -8 & 1 \end{pmatrix}$

(e) $\mathbf{E} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ (f) $\mathbf{F} = \begin{pmatrix} 2 & 5 & 0 & -4 & 1 \\ -3 & 9 & -3 & 2 & 2 \\ 1 & 0 & 0 & 10 & 4 \end{pmatrix}$

(ii) Which of the following matrix products can be found? For those that can, state the order of the matrix product.

(a) **AE** (b) **AF** (c) **FA** (d) **CA** (e) **DC**

2 Calculate these products.

(i) $\begin{pmatrix} 3 & 0 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} 7 & 2 \\ 4 & -3 \end{pmatrix}$

(ii) $\begin{pmatrix} 2 & -3 & 5 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 5 & 8 \\ -3 & 1 \end{pmatrix}$

(iii) $\begin{pmatrix} 2 & 5 & -1 & 0 \\ 3 & 6 & 4 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ -9 \\ 11 \\ -2 \end{pmatrix}$

Check your answers using the matrix function on a calculator if possible.

CP **3** Using the matrices $\mathbf{A} = \begin{pmatrix} 5 & 9 \\ -2 & 7 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -3 & 5 \\ 2 & -9 \end{pmatrix}$, confirm that matrix multiplication is not commutative.

4 For the matrices

$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} -3 & 7 \\ 2 & 5 \end{pmatrix}$ $\mathbf{C} = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 7 & 1 \end{pmatrix}$

$\mathbf{D} = \begin{pmatrix} 3 & 4 \\ 7 & 0 \\ 1 & -2 \end{pmatrix}$ $\mathbf{E} = \begin{pmatrix} 4 & 7 \\ 3 & -2 \\ 1 & 5 \end{pmatrix}$ $\mathbf{F} = \begin{pmatrix} 3 & 7 & -5 \\ 2 & 6 & 0 \\ -1 & 4 & 8 \end{pmatrix}$

calculate, where possible, the following:

(i) **AB** (ii) **BA** (iii) **CD** (iv) **DC** (v) **EF** (vi) **FE**

5 Using the matrix function on a calculator, find $\mathbf{M}^4$ for the matrix

$$\mathbf{M} = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -1 & 4 & 3 \end{pmatrix}.$$

> **Note**
>
> $\mathbf{M}^4$ means $\mathbf{M} \times \mathbf{M} \times \mathbf{M} \times \mathbf{M}$

PS **6** $\mathbf{A} = \begin{pmatrix} x & 3 \\ 0 & -1 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 2x & 0 \\ 4 & -3 \end{pmatrix}$:

(i) Find the matrix product **AB** in terms of x.

(ii) If $\mathbf{AB} = \begin{pmatrix} 10x & -9 \\ -4 & 3 \end{pmatrix}$, find the possible values of x.

(iii) Find the possible matrix products **BA**.

7 (i) For the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, find

(a) $\mathbf{A}^2$

(b) $\mathbf{A}^3$

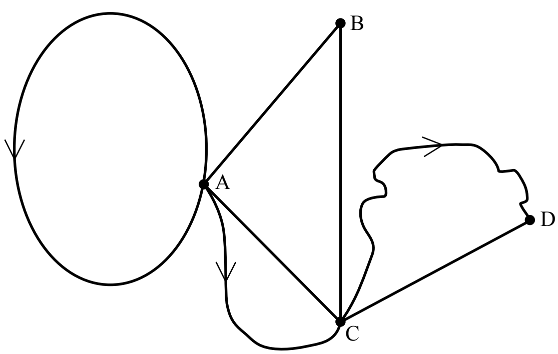
(c) $\mathbf{A}^4$

(ii) Suggest a general form for the matrix $\mathbf{A}^n$ in terms of n.

(iii) Verify your answer by finding $\mathbf{A}^{10}$ on your calculator and confirming it gives the same answer as (ii).

8 The map below shows the bus routes in a holiday area. Lines represent routes that run each way between the resorts. Arrows indicated one-way scenic routes.

M is the partly completed 4 × 4 matrix that shows the number of direct routes between the various resorts.



$$\mathbf{M} = \begin{array}{c} \\ \text{From} \\ \end{array} \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \text{To} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 1 & 1 & 2 & 0 \\ & & & \\ & & & \\ & & & \end{pmatrix} \end{array}$$

(i) Copy and complete the matrix **M**.

(ii) Calculate $\mathbf{M}^2$ and explain what information it contains.

(iii) What information would $\mathbf{M}^3$ contain?

9 $\mathbf{A} = \begin{pmatrix} 4 & x & 0 \\ 2 & -3 & 1 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 2 & -5 \\ 4 & x \\ x & 7 \end{pmatrix}$:

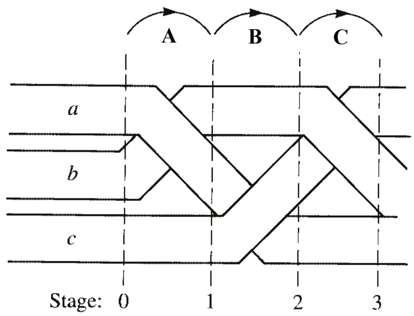
(i) Find the product **AB** in terms of x.

A symmetric matrix is one in which the entries are symmetrical about the leading diagonal, for example $\begin{pmatrix} 2 & 5 \\ 5 & 0 \end{pmatrix}$ and $\begin{pmatrix} 3 & 4 & -6 \\ 4 & 2 & 5 \\ -6 & 5 & 1 \end{pmatrix}$.

(ii) Given that the matrix **AB** is symmetric, find the possible values of x.

(iii) Write down the possible matrices **AB**.

PS 10 The diagram on the right shows the start of the plaiting process for producing a leather bracelet from three leather strands a, b and c.



The process has only two steps, repeated alternately:

Step 1: cross the top strand over the middle strand

Step 2: cross the middle strand under the bottom strand.

At the start of the plaiting process,

Stage 0, the order of the strands is given by $\mathbf{S}_0 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

(i) Show that pre-multiplying $\mathbf{S_0}$ by the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

gives $\mathbf{S_1}$, the matrix that represents the order of the strands at Stage 1.

(ii) Find the 3×3 matrix **B** that represents the transition from Stage 1 to Stage 2.

(iii) Find matrix $\mathbf{M} = \mathbf{BA}$ and show that $\mathbf{MS_0}$ gives $\mathbf{S_2}$, the matrix that represents the order of the strands at Stage 2.

(iv) Find $\mathbf{M^2}$ and hence find the order of the strands at Stage 4.

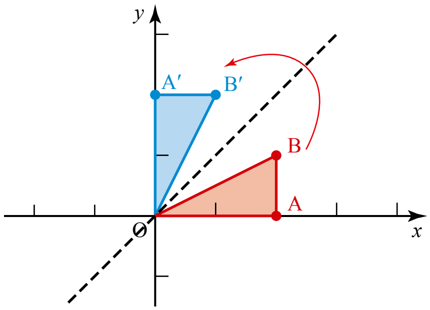
(v) Calculate $\mathbf{M^3}$. What does this tell you?

1.3 Transformations

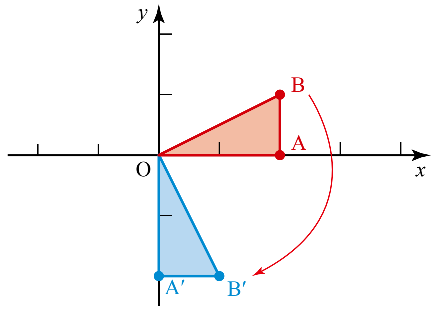
You are already familiar with several different types of transformation, including reflections, rotations and enlargements.

- The original point, or shape, is called the **object**.
- The new point, or shape, after the transformation, is called the **image**.
- A transformation is a **mapping** of an object onto its image.

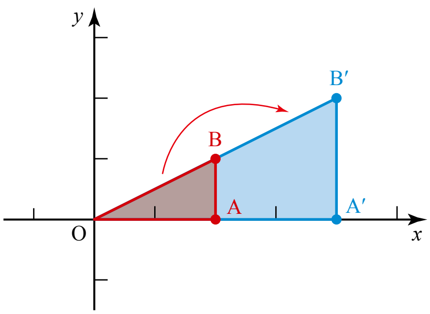
Some examples of transformations are illustrated in Figures 1.3 to 1.5 (note that the vertices of the image are denoted by the same letters with a dash, e.g. A′, B′).



▲ **Figure 1.3 Reflection in the line $y = x$**



▲ **Figure 1.4** Rotation through 90° clockwise, centre O



▲ **Figure 1.5** Enlargement centre O, scale factor 2

In this section, you will also meet the idea of

- a **stretch** parallel to the x-axis or y-axis
- a **shear**.

A transformation maps an object according to a rule and can be represented by a matrix (see next section). The effect of a transformation on an object can be found by looking at the effect it has on the **position vector** of the point $\begin{pmatrix} x \\ y \end{pmatrix}$, i.e. the vector from the origin to the point (x, y). So, for example, to find the effect of a transformation on the point $(2, 3)$ you would look at the effect that the transformation matrix has on the position vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Vectors that have length or **magnitude** of 1 are called **unit vectors**.

In two dimensions, two unit vectors that are of particular interest are

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ – a unit vector in the direction of the x-axis

$\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ – a unit vector in the direction of the y-axis.

The equivalent unit vectors in three dimensions are

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ – a unit vector in the direction of the x-axis

$\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ – a unit vector in the direction of the y-axis

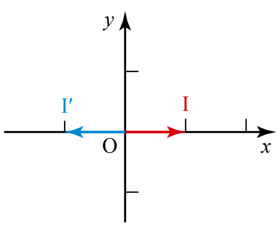
$\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ – a unit vector in the direction of the z-axis.

Finding the transformation represented by a given matrix

Start by looking at the effect of multiplying the unit vectors $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

The image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ under this transformation is given by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$



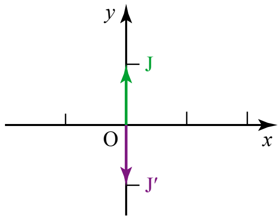
▲ **Figure 1.6**

Note

The letter I is often used for the point $(1, 0)$.

The image of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ under the transformation is given by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$



▲ **Figure 1.7**

Note

The letter J is often used for the point $(0, 1)$.

You can see from this that the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ represents a rotation, centre the origin, through 180°.

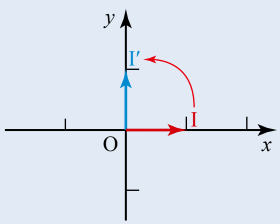
Example 1.4

Describe the transformations represented by the following matrices.

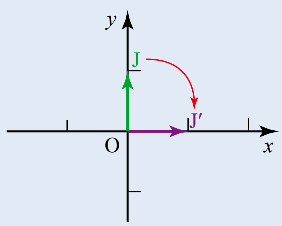
(i) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Solution

(i) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



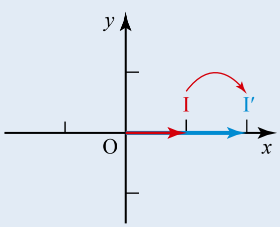
▲ **Figure 1.8**



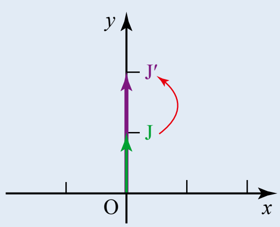
▲ **Figure 1.9**

The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents a reflection in the line $y = x$.

(ii) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$



▲ **Figure 1.10**



▲ **Figure 1.11**

The matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ represents an enlargement, centre the origin, scale factor 2.

You can see that the images of $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ are the two columns of the transformation matrix.

Finding the matrix that represents a given transformation

The connection between the images of the unit vectors **i** and **j** and the matrix representing the transformation provides a quick method for finding the matrix representing a transformation.

You may find it easier to see what the transformation is when you use a shape, like the unit square, rather than points or lines.

It is common to use the unit square with coordinates O (0, 0), I (1, 0), P (1, 1) and J (0, 1).

You can think about the images of the points I and J, and from this you can write down the images of the unit vectors **i** and **j**.

This is done in the next example.

Example 1.5

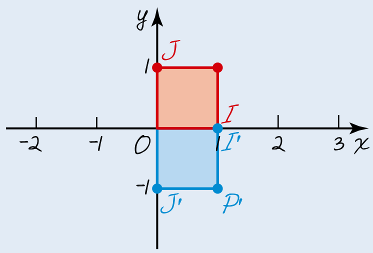
By drawing a diagram to show the image of the unit square, find the matrices that represent each of the following transformations:

(i) a reflection in the x-axis

(ii) an enlargement of scale factor 3, centre the origin.

Solution

(i)



▲ **Figure 1.12**

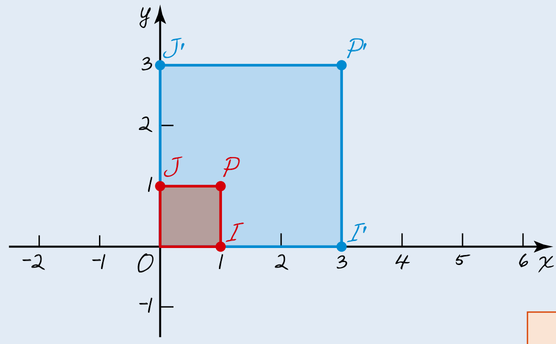
You can see from Figure 1.12 that I (1, 0) is mapped to itself and J (0, 1) is mapped to J′ (0, −1).

So the image of **I** is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

and the image of **J** is $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

So the matrix that represents a reflection in the x-axis is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(ii)



▲ Figure 1.13

So the image of **I** is $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

You can see from Figure 1.13 that I (1, 0) is mapped to I′ (3, 0), and J (0, 1) is mapped to J′ (0, 3).

and the image of **J** is $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

So the matrix that represents an enlargement, centre the origin, scale factor 3 is $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

?

- For a general transformation represented by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, what are the images of the unit vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$?
- What is the image of the origin (0, 0)?

ACTIVITY 1.3

Using the image of the unit square, find the matrix which represents a rotation of 45° anticlockwise about the origin.
Use your answer to write down the matrices that represent the following transformations:

(i) a rotation of 45° clockwise about the origin

(ii) a rotation of 135° anticlockwise about the origin.

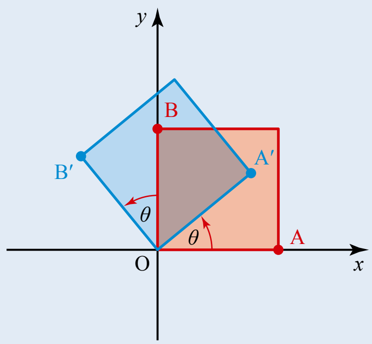
Example 1.6

(i) Find the matrix that represents a rotation through angle θ anticlockwise about the origin.

(ii) Use your answer to find the matrix that represents a rotation of 60° anticlockwise about the origin.

Solution

(i) Figure 1.14 shows a rotation of angle θ anticlockwise about the origin.



▲ **Figure 1.14**

Call the coordinates of the point A′ (p, q). Since the lines OA and OB are perpendicular, the coordinates of B′ will be $(-q, p)$.

From the right-angled triangle with OA′ as the hypotenuse, $\cos\theta = \frac{p}{1}$ and so $p = \cos\theta$.

Similarly, from the right-angled triangle with OB′ as the hypotenuse, $\sin\theta = \frac{q}{1}$ so $q = \sin\theta$.

So, the image point A′ (p, q) has position vector $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and the image point B′ $(-q, p)$ has position vector $\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$.

Therefore, the matrix that represents a rotation of angle θ anticlockwise about the origin is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

(ii) The matrix that represents an anticlockwise rotation of 60° about the origin is $\begin{pmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$.

?

› What matrix would represent a rotation through an angle θ clockwise about the origin?

ACTIVITY 1.4

Investigate the effect of the matrices:

(i) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$

Describe the general transformation represented by the matrices $\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$.

Technology note

You could use geometrical software to try different values of m and n.

Activity 1.4 illustrates two important general results:

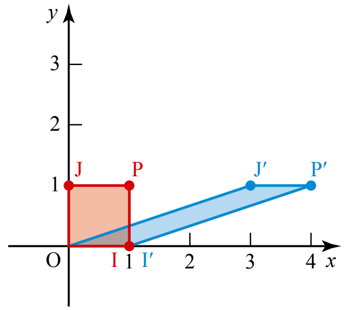
- The matrix $\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$ represents a stretch of scale factor m parallel to the x-axis.
- The matrix $\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$ represents a stretch of scale factor n parallel to the y-axis.

Shears

Figure 1.15 shows the unit square and its image under the transformation represented by the matrix $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ on the unit square. The matrix $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ transforms the unit vector $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and transforms the unit vector $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to the vector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

The point with position vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is transformed to the point with position vector $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

As $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$



▲ **Figure 1.15**

This transformation is called a **shear**. Notice that the points on the x-axis stay the same, and the points J and P move parallel to the x-axis to the right.

This shear can be described fully by saying that the x-axis is fixed, and giving the image of one point not on the x-axis, e.g. (0, 1) is mapped to (3, 1).

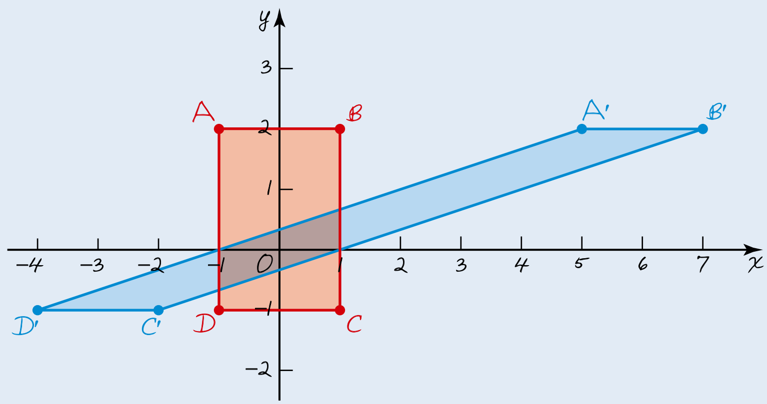
Generally, a shear with the x-axis fixed has the form $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ and a shear with the y-axis fixed has the form $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$.

Example 1.7

Find the image of the rectangle with vertices A (–1, 2), B (1, 2), C (1, –1) and D (–1, –1) under the shear $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ and show the rectangle and its image on a diagram.

Solution

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 1 & -1 \\ 2 & 2 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 7 & -2 & -4 \\ 2 & 2 & -1 & -1 \end{pmatrix}$$



▲ **Figure 1.16**

→

1

The effect of this shear is to transform the sides of the rectangle parallel to the y-axis into sloping lines. Notice that the gradient of the side A′D′ is $\frac{1}{3}$, which is the reciprocal of the top right-hand element of the matrix $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$.

Note

Notice that under the shear transformation, points above the x-axis move to the right and points below the x-axis move to the left.

ACTIVITY 1.5

For each of the points A, B, C and D in Example 1.7, find

$$\frac{\text{distance between the point and its image}}{\text{distance of original point from } x\text{-axis}}.$$

What do you notice?

In the activity above, you should have found that dividing the distance between the point and its image by the distance of the original point from the x-axis (which is fixed), gives the answer 3 for all points, which is the number in the top right of the matrix. This is called the **shear factor** for the shear.

Technology note

If you have access to geometrical software, investigate how shears are defined.

!

There are different conventions about the sign of a shear factor, and for this reason shear factors are not used to define a shear in this book. It is possible to show the effect of matrix transformations using some geometrical computer software packages. You might find that some packages use different approaches towards shears and define them in different ways.

Example 1.8

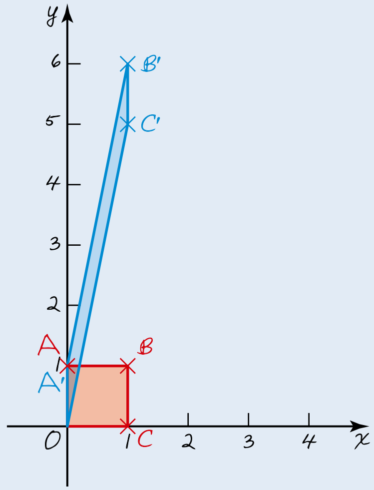
In a shear, S, the y-axis is fixed, and the image of the point $(1, 0)$ is the point $(1, 5)$.

(i) Draw a diagram showing the image of the unit square under the transformation S.

(ii) Find the matrix that represents the shear S.

Solution

(i)



▲ **Figure 1.17**

(ii) Under S $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ← Since the y-axis is fixed.

So the matrix representing S is $\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$.

Notice that this matrix is of the form $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ for shears with the y-axis fixed.

Summary of transformations in two dimensions

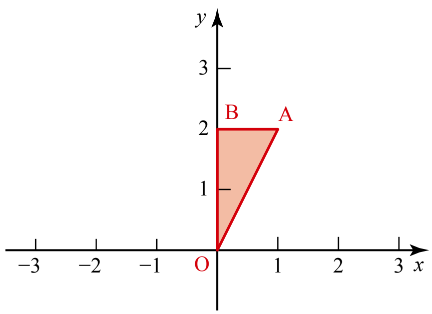
Reflection in the x-axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	Reflection in the y-axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Reflection in the line $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	Reflection in the line $y = -x$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
Rotation anticlockwise about the origin through angle θ	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$	Enlargement, centre the origin, scale factor k	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
Stretch parallel to the x-axis, scale factor k	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$	Stretch parallel to the y-axis, scale factor k	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
Shear, x-axis fixed, with (0, 1) mapped to (k, 1)	$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$	Shear, y-axis fixed, with (1, 0) mapped to (1, k)	$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$

Note

All these transformations are examples of linear transformations. In a linear transformation, straight lines are mapped to straight lines, and the origin is mapped to itself.

Exercise 1C

1 The diagram shows a triangle with vertices at O, A (1, 2) and B (0, 2).



For each of the transformations below

(a) draw a diagram to show the effect of the transformation on triangle OAB

(b) give the coordinates of A′ and B′, the images of points A and B

(c) find expressions for x' and y', the coordinates of P′, the image of a general point P (x, y)

(d) find the matrix that represents the transformation.

(i) Enlargement, centre the origin, scale factor 3

(ii) Reflection in the x-axis

(iii) Reflection in the line $x + y = 0$

(iv) Rotation 90° clockwise about O

(v) Two-way stretch, scale factor 3 horizontally and scale factor $\frac{1}{2}$ vertically.

2 Describe the geometrical transformations represented by these matrices.

(i) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ (iii) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

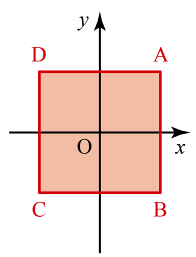
(iv) $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ (v) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

3 Each of the following matrices represents a rotation about the origin. Find the angle and direction of rotation in each case.

(i) $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ (ii) $\begin{pmatrix} 0.574 & -0.819 \\ 0.819 & 0.574 \end{pmatrix}$

(iii) $\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ (iv) $\begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$

4 The diagram below shows a square with vertices at the points A (1, 1), B (1, –1), C (–1, –1) and D (–1,1).



(i) Draw a diagram to show the image of this square under the transformation matrix $\mathbf{M} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$.

(ii) Describe fully the transformation represented by the matrix **M**. State the fixed line and the image of the point A.

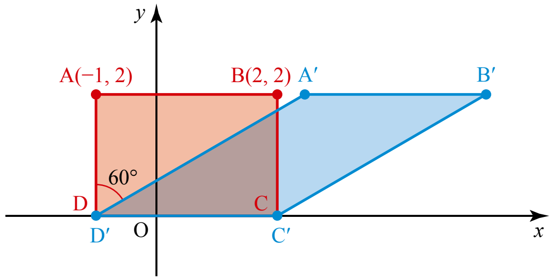
5 (i) Find the image of the unit square under the transformations represented by the matrices

(a) $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}$.

(ii) Use your answers to part (i) to fully describe the transformations represented by each of the matrices **A** and **B**.

6 The diagram below shows a shear that maps the rectangle ABCD to the parallelogram A′B′C′D′.

The angle A′DA is 60°.



(i) Find the coordinates of A′.

(ii) Find the matrix that represents the shear.

7 The unit square OABC has its vertices at (0, 0), (1, 0), (1, 1) and (0, 1).

OABC is mapped to OA′B′C′ by the transformation defined by the matrix $\begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix}$.

Find the coordinates of A′, B′ and C′ and show that the area of the shape has not been changed by the transformation.

8 The transformation represented by the matrix $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is applied to the triangle ABC with vertices A (−1, 1), B (1, −1) and C (−1, −1).

(i) Draw a diagram showing the triangle ABC and its image A′B′C′.

(ii) Find the gradient of the line A′C′ and explain how this relates to the matrix **M**.

PS **9** A transformation maps P to P′ as follows:

» Each point is mapped on to the line $y = x$.

» The line joining a point to its image is parallel to the y-axis.

Find the coordinates of the image of the point (x, y) and hence show that this transformation can be represented by means of a matrix.

What is that matrix?

PS **10** A square has corners with coordinates A (1, 0), B (1, 1), C (0, 1) and O (0, 0). It is to be transformed into another quadrilateral in the first quadrant of the coordinate grid.

Find a matrix that would transform the square into:

(i) a rectangle with one vertex at the origin, the sides lie along the axes and one side of length is 5 units

(ii) a rhombus with one vertex at the origin, two angles of 45° and side lengths of $\sqrt{2}$ units; one of the sides lies along an axis

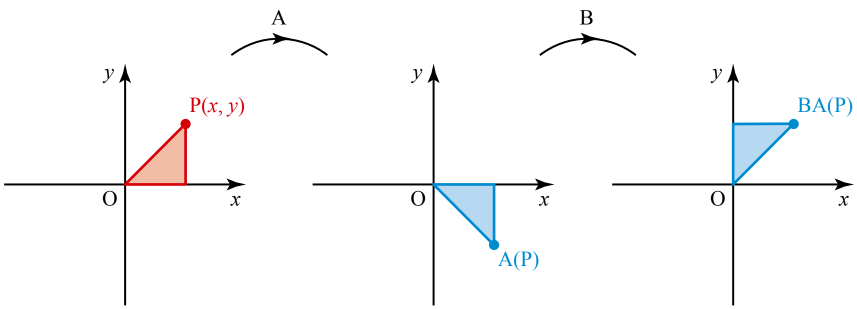
(iii) a parallelogram with one vertex at the origin and two angles of 30°; one of the longest sides lies along an axis and has length 7 units; the shortest sides have length 3 units.

Is there more than one possibility for any of these matrices? If so, write down alternative matrices that satisfy the same description.

1.4 Successive transformations

Figure 1.18 shows the effect of two successive transformations on a triangle. The transformation A represents a reflection in the x-axis. A maps the point P to the point A(P).

The transformation B represents a rotation of 90° anticlockwise about O. When you apply B to the image formed by A, the point A(P) is mapped to the point B(A(P)). This is abbreviated to BA(P).



▲ **Figure 1.18**

Note

Notice that a transformation written as BA means 'carry out A, then carry out B'.

This process is sometimes called **composition of transformations**.

?

Look at Figure 1.18 and compare the original triangle with the final image after both transformations.

- Describe the single transformation represented by BA.
- Write down the matrices which represent the transformations A and B. Calculate the matrix product **BA** and comment on your answer.

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

Note

A transformation is often denoted by a capital letter. The matrix representing this transformation is usually denoted by the same letter, in bold.

Technology note

If you have access to geometrical software, you could investigate this using several different matrices for **T** and **S**.

In general, the matrix for a composite transformation is found by multiplying the matrices of the individual transformations in reverse order. So, for two transformations the matrix representing the first transformation is on the right and the matrix for the second transformation is on the left. For n transformations $T_1, T_2, \ldots, T_{n-1}, T_n$, the matrix product would be $\mathbf{T}_n\mathbf{T}_{n-1} \ldots \mathbf{T}_2\mathbf{T}_1$.

You will prove this result for two transformations in Activity 1.6.

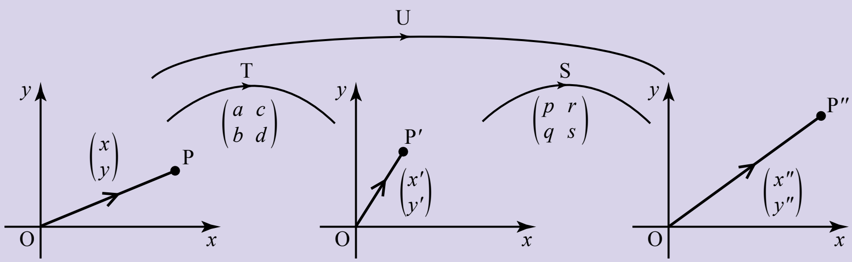
ACTIVITY 1.6

The transformations T and S are represented by the matrices

$\mathbf{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{S} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$.

T is applied to the point P with position vector $\mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix}$. The image of P is P′.

S is then applied to the point P′. The image of P′ is P″. This is illustrated in Figure 1.19.



▲ **Figure 1.19**

(i) Find the position vector $\begin{pmatrix} x' \\ y' \end{pmatrix}$ of P′ by calculating the matrix product $\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix}$.

(ii) Find the position vector $\begin{pmatrix} x'' \\ y'' \end{pmatrix}$ of P″ by calculating the matrix product $\mathbf{S}\begin{pmatrix} x' \\ y' \end{pmatrix}$.

(iii) Find the matrix product $\mathbf{U} = \mathbf{ST}$ and show that $\mathbf{U}\begin{pmatrix} x \\ y \end{pmatrix}$ is the same as $\begin{pmatrix} x'' \\ y'' \end{pmatrix}$.

› How can you use the idea of successive transformations to explain the associativity of matrix multiplication $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$?

Proving results in trigonometry

If you carry out a rotation about the origin through angle θ, followed by a rotation about the origin through angle ϕ, then this is equivalent to a single rotation about the origin through angle $\theta + \phi$. Using matrices to represent these transformations allows you to prove the formulae for $\sin(\theta + \phi)$ and $\cos(\theta + \phi)$. This is done in Activity 1.7.

Note

Assume that a rotation is anticlockwise unless otherwise stated

ACTIVITY 1.7

(i) Write down the matrix **A** representing a rotation about the origin through angle θ, and the matrix **B** representing a rotation about the origin through angle ϕ.

(ii) Find the matrix **BA**, representing a rotation about the origin through angle θ, followed by a rotation about the origin through angle ϕ.

(iii) Write down the matrix **C** representing a rotation about the origin through angle $\theta + \phi$.

(iv) By equating **C** to **BA**, write down expressions for $\sin(\theta + \phi)$ and $\cos(\theta + \phi)$.

(v) Explain why **BA** = **AB** in this case.

Example 1.9

(i) Write down the matrix **A** that represents an anticlockwise rotation of 135° about the origin.

(ii) Write down the matrices **B** and **C** that represent rotations of 45° and 90° respectively about the origin. Find the matrix **BC** and verify that **A** = **BC**.

(iii) Calculate the matrix $\mathbf{B}^3$ and comment on your answer.

Solution

(i) $$\mathbf{A} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

(ii) $$\mathbf{B} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\mathbf{BC} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \mathbf{A}$$

→

(iii) $\mathbf{B}^3 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.

This verifies that three successive anticlockwise rotations of 45° about the origin is equivalent to a single anticlockwise rotation of 135° about the origin.

Exercise 1D

1 $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(i) Describe the transformations that are represented by matrices **A**, **B**, **C** and **D**.

(ii) Find the following matrix products and describe the single transformation represented in each case:

(a) **BC** (b) **CB** (c) **DC** (d) $\mathbf{A}^2$ (e) **BCB** (f) $\mathbf{DC}^2\mathbf{D}$

(iii) Write down two other matrix products, using the matrices **A**, **B**, **C** and **D**, which would produce the same single transformation as $\mathbf{DC}^2\mathbf{D}$.

2 The matrix **X** represents a reflection in the *x*-axis.

The matrix **Y** represents a reflection in the *y*-axis.

(i) Write down the matrices **X** and **Y**.

(ii) Find the matrix **XY** and describe the transformation it represents.

(iii) Find the matrix **YX**.

(iv) Explain geometrically why **XY** = **YX** in this case.

PS 3 The matrix **P** represents a rotation of 180° about the origin.

The matrix **Q** represents a reflection in the line $y = x$.

(i) Write down the matrices **P** and **Q**.

(ii) Find the matrix **PQ** and describe the transformation it represents.

(iii) Find the matrix **QP**.

(iv) Explain geometrically why **PQ** = **QP** in this case.

4 The transformations R and S are represented by the matrices

$\mathbf{R} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ and $\mathbf{S} = \begin{pmatrix} 3 & 0 \\ -2 & 4 \end{pmatrix}$.

(i) Find the matrix which represents the transformation RS.

(ii) Find the image of the point (3, -2) under the transformation RS.

PS 5 The transformation represented by $\mathbf{C} = \begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix}$ is equivalent to a single transformation B followed by a single transformation A. Give geometrical descriptions of a pair of possible transformations B and A and state the matrices that represent them.

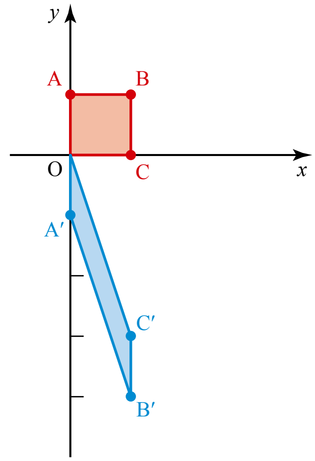
Comment on the order in which the transformations are performed.

6 The diagram on the right shows the image of the unit square OABC under the combined transformation with matrix **PQ**.

(i) Write down the matrix **PQ**.

Matrix **P** represents a reflection.

(ii) State the matrices **P** and **Q** and define fully the two transformations represented by these matrices. When describing matrix **Q** you should refer to the image of the point B.



PS **7** Find the matrix **X** that represents rotation of 135° about the origin followed by a reflection in the y-axis.

Explain why matrix **X** cannot represent a rotation about the origin.

Note

Assume that a rotation is anticlockwise unless otherwise stated

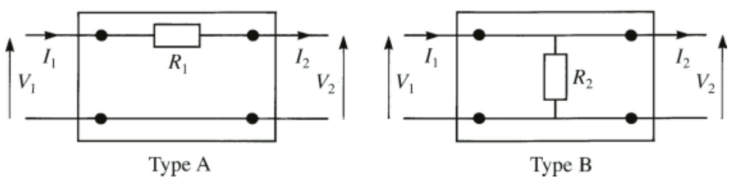
PS **8** (i) Write down the matrix **P** that represents a stretch of scale factor 2 parallel to the y-axis.

(ii) The matrix $\mathbf{Q} = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$. Write down the two single transformations that are represented by the matrix **Q**.

(iii) Find the matrix **PQ**. Write a list of the three transformations that are represented by the matrix **PQ**. In how many different orders could the three transformations occur?

(iv) Find the matrix **R** for which the matrix product **RPQ** would transform an object to its original position.

PS **9** There are two basic types of four-terminal electrical networks, as shown in the diagrams below.



In Type A the output voltage V_2 and current I_2 are related to the input voltage V_1 and current I_1 by the simultaneous equations:

$$V_2 = V_1 - I_1 R_1$$
$$I_2 = I_1$$

The simultaneous equations can be written as $\begin{pmatrix} V_2 \\ I_2 \end{pmatrix} = \mathbf{A}\begin{pmatrix} V_1 \\ I_1 \end{pmatrix}$.

(i) Find the matrix **A**.

In Type B the corresponding simultaneous equations are:

$$V_2 = V_1$$

$$I_2 = I_1 - \frac{V_1}{R_2}$$

(ii) Write down the matrix **B** that represents the effect of a Type B network.

(iii) Find the matrix that represents the effect of Type A followed by Type B.

(iv) Is the effect of Type B followed by Type A the same as the effect of Type A followed by Type B?

10 The matrix **B** represents a rotation of 45° anticlockwise about the origin.

$\mathbf{B} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ where a and b are positive real numbers

Given that $\mathbf{D}^2 = \mathbf{B}$, find exact values for a and b. Write down the transformation represented by the matrix **D**. What do the exact values a and b represent?

In questions **11** and **12** you will need to use the matrix that represents a reflection in the line $y = mx$. This can be written as $\frac{1}{1+m^2}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$.

11 (i) Find the matrix **P** that represents reflection in the line $y = \frac{1}{\sqrt{3}}x$, and the matrix **Q** that represents reflection in the line $y = \sqrt{3}x$.

(ii) Use matrix multiplication to find the single transformation equivalent to reflection in the line $y = \frac{1}{\sqrt{3}}x$ followed by reflection in the line $y = \sqrt{3}x$.

Describe this transformation fully.

(iii) Use matrix multiplication to find the single transformation equivalent to reflection in the line $y = \sqrt{3}x$ followed by reflection in the line $y = \frac{1}{\sqrt{3}}x$.

Describe this transformation fully.

CP **12** The matrix **R** represents a reflection in the line $y = mx$.

Show that $\mathbf{R}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and explain geometrically why this is the case.

1.5 Invariance

Invariant points

- In a reflection, are there any points that map to themselves?
- In a rotation, are there any points that map to themselves?

Points that map to themselves under a transformation are called **invariant points**. The origin is always an invariant point under a transformation that can be represented by a matrix, as the following statement is always true:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

More generally, a point (x, y) is invariant if it satisfies the matrix equation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}$$

For example, the point $(-2, 2)$ is invariant under the transformation represented by the matrix $\begin{pmatrix} 6 & 5 \\ 2 & 3 \end{pmatrix}$: $\begin{pmatrix} 6 & 5 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} -2 \\ 2 \end{pmatrix}=\begin{pmatrix} -2 \\ 2 \end{pmatrix}$

Example 1.10

M is the matrix $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$.

(i) Show that $(5, 5)$ is an invariant point under the transformation represented by **M**.

(ii) What can you say about the invariant points under this transformation?

Solution

(i) $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5 \\ 5 \end{pmatrix}=\begin{pmatrix} 5 \\ 5 \end{pmatrix}$ so $(5, 5)$ is an invariant point under the transformation represented by **M**.

(ii) Suppose the point $\begin{pmatrix} x \\ y \end{pmatrix}$ maps to itself. Then

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 2x-y \\ x \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}$$

$\Leftrightarrow 2x - y = x$ and $x = y$.

Both equations simplify to $y = x$.

So the invariant points of the transformation are all the points on the line $y = x$.

These points all have the form (λ, λ). The point (5,5) is just one of the points on this line.

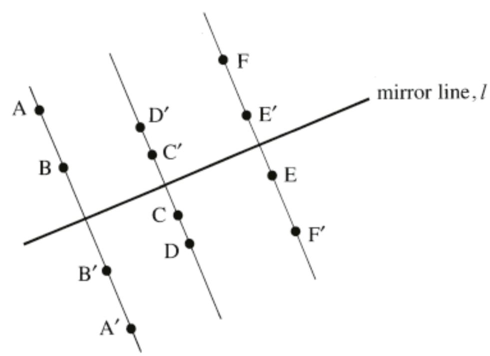
The simultaneous equations in Example 1.10 were equivalent and so all the invariant points were on a straight line. Generally, any matrix equation set up to find the invariant points will lead to two equations of the form $ax + by = 0$, which can also be expressed in the form $y = -\frac{ax}{b}$. These equations may be equivalent, in which case this is a line of invariant points. If the two equations are not equivalent, the origin is the only point that satisfies both equations, and so this is the only invariant point.

Invariant lines

A line AB is known as an **invariant line** under a transformation if the image of every point on AB is also on AB. It is important to note that it is not necessary for each of the points to map to itself; it can map to itself or to some other point on the line AB.

Sometimes it is easy to spot which lines are invariant. For example, in Figure 1.20 the position of the points A–F and their images A′–F′ show that the transformation is a reflection in the line l. So every point on l maps onto itself and l is a **line of invariant points**.

Look at the lines perpendicular to the mirror line in Figure 1.20, for example the line ABB′A′. Any point on one of these lines maps onto another point on the same line. Such a line is invariant but it is not a line of invariant points.



▲ **Figure 1.20**

Example 1.11

Find the invariant lines of the transformation given by the matrix $\mathbf{M} = \begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix}$.

Solution

Suppose the invariant line has the form $y = mx + c$

Let the original point be (x, y) and the image point be (x', y').

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow x' = 5x + y \text{ and } y' = 2x + 4y$$

$$\Leftrightarrow \begin{cases} x' = 5x + mx + c = (5 + m)x + c \\ y' = 2x + 4(mx + c) = (2 + 4m)x + 4c \end{cases}$$

Using $y = mx + c$.

As the line is invariant, (x', y') also lies on the line, so $y' = mx' + c$.

Therefore,

$(2 + 4m)x + 4c = m[(5 + m)x + c] + c$

$\Leftrightarrow 0 = (m^2 + m - 2)x + (m - 3)c$

For the left-hand side to equal zero, both $m^2 + m - 2 = 0$ and $(m - 3)c = 0$.

$(m - 1)(m + 2) = 0 \Leftrightarrow m = 1$ or $m = -2$

and

$(m - 3)c = 0 \Leftrightarrow m = 3$ or $c = 0$

$m = 3$ is not a viable solution as $m^2 + m - 2 \neq 0$.

So, there are two possible solutions for the invariant line:

$m = 1, c = 0 \Leftrightarrow y = x$

or

$m = -2, c = 0 \Leftrightarrow y = -2x$

Figure 1.21 shows the effect of this transformation, together with its invariant lines.

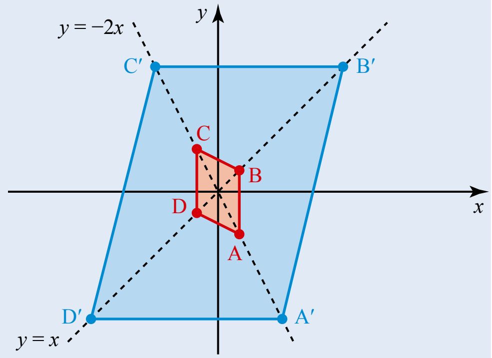


Figure 1.21

Exercise 1E

1 Find the invariant points under the transformations represented by the following matrices.

(i) $\begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$ (iii) $\begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix}$ (iv) $\begin{pmatrix} 7 & -4 \\ 3 & -1 \end{pmatrix}$

2 What lines, if any, are invariant under the following transformations?

(i) Enlargement, centre the origin

(ii) Rotation through 180° about the origin

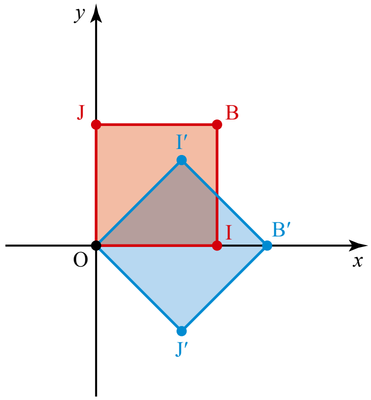
(iii) Rotation through 90° about the origin

(iv) Reflection in the line $y = x$

(v) Reflection in the line $y = -x$

(vi) Shear, x-axis fixed

3 The diagram below shows the effect on the unit square of a transformation represented by $\mathbf{A} = \begin{pmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{pmatrix}$.



(i) Find three points that are invariant under this transformation.

(ii) Given that this transformation is a reflection, write down the equation of the mirror line.

(iii) Using your answer to part (ii), write down the equation of an invariant line, other than the mirror line, under this reflection.

(iv) Justify your answer to part (iii) algebraically.

4 For the matrix $\mathbf{M} = \begin{pmatrix} 4 & 11 \\ 11 & 4 \end{pmatrix}$

(i) show that the origin is the only invariant point

(ii) find the invariant lines of the transformation represented by **M**.

5 (i) Find the invariant lines of the transformation given by the matrix $\begin{pmatrix} 3 & 4 \\ 9 & -2 \end{pmatrix}$.

(ii) Draw a diagram to show the effect of the transformation on the unit square, and show the invariant lines on your diagram.

6 For the matrix $\mathbf{M} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$

(i) find the line of invariant points of the transformation given by **M**

(ii) find the invariant lines of the transformation

(iii) draw a diagram to show the effect of the transformation on the unit square.

CP **7** The matrix $\begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{pmatrix}$ represents a reflection in the line $y = mx$.

Prove that the line $y = mx$ is a line of invariant points.

CP **8** The transformation T maps $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Show that invariant points other than the origin exist if $ad - bc = a + d - 1$.

PS **9** T is a translation of the plane by the vector $\begin{pmatrix} a \\ b \end{pmatrix}$. The point (x, y) is mapped to the point (x', y').

(i) Write down equations for x' and y' in terms of x and y.

(ii) Verify that $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ produces the same equations as those obtained in part (i).

The point (X, Y) is the image of the point (x, y) under the combined transformation TM where

$$\begin{pmatrix} X \\ Y \\ 1 \end{pmatrix} = \begin{pmatrix} -0.6 & 0.8 & a \\ 0.8 & 0.6 & b \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

(iii) (a) Show that if $a = -4$ and $b = 2$ then $(0, 5)$ is an invariant point of TM.

(b) Show that if $a = 2$ and $b = 1$ then TM has no invariant point.

(c) Find a relationship between a and b that must be satisfied if TM is to have any invariant points.

KEY POINTS

1 A matrix is a rectangular array of numbers or letters.

2 The shape of a matrix is described by its order. A matrix with r rows and c columns has order $r \times c$.

3 A matrix with the same number of rows and columns is called a square matrix.

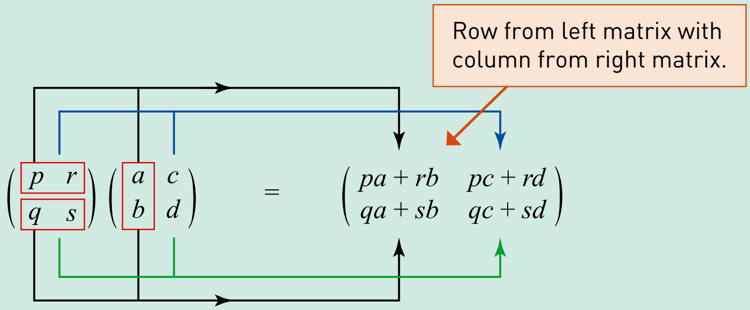
4 The matrix $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is known as the 2×2 zero matrix. Zero matrices can be of any order.

5 A matrix of the form $\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is known as an identity matrix. All identity matrices are square, with 1s on the leading diagonal and zeros elsewhere.

6 Matrices can be added or subtracted if they have the same order.

7 Two matrices **A** and **B** can be multiplied to give matrix **AB** if their orders are of the form $p \times q$ and $q \times r$ respectively. The resulting matrix will have the order $p \times r$.

8 Matrix multiplication



$$\begin{pmatrix} p & r \\ q & s \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} pa+rb & pc+rd \\ qa+sb & qc+sd \end{pmatrix}$$

9 Matrix addition and multiplication are associative:

$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$

10 Matrix addition is commutative but matrix multiplication is generally not commutative:

$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

$\mathbf{AB} \neq \mathbf{BA}$

11 The matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represents the transformation that maps the point with position vector $\begin{pmatrix} x \\ y \end{pmatrix}$ to the point with position vector $\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$.

12 A list of the matrices representing common transformations, including rotations, reflections, enlargements, stretches and shears, is given on page 24.

13 Under the transformation represented by **M**, the image of $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the first column of **M** and the image of $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the second column of **M**.

14 The composite of the transformation represented by M followed by that represented by N is represented by the matrix product **NM**.

15 If (x, y) is an invariant point under a transformation represented by the matrix **M**, then $\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$.

16 A line AB is known as an invariant line under a transformation if the image of every point on AB is also on AB.

Note

Work on matrices is developed further in Chapter 6 ‘Matrices and their inverses’.

LEARNING OUTCOMES

Now that you have finished this chapter, you should be able to

- understand what is meant by the terms
  - order of a matrix
  - square matrix
  - zero matrix
  - equal matrices
- carry out the matrix operations
  - addition
  - subtraction
  - multiplication by a scalar
- understand when matrices are conformable for multiplication and be able to carry out matrix multiplication
- use a calculator to carry out matrix operations
- understand the use of matrices to represent the geometric transformations in the x–y plane
  - rotation about the origin
  - reflection in lines through the origin
  - enlargement with centre the origin
  - stretch parallel to the coordinate axes
  - shear with the axes as fixed lines
- recognise that the matrix product **AB** represents the transformation that results from the transformation represented by **B** followed by the transformation represented by **A**
- find the matrix that represents a given transformation or sequence of transformations
- understand the meaning of 'invariant' in the context of transformations represented by matrices
  - as applied to points
  - as applied to lines
- solve simple problems involving invariant points and invariant lines, for example
  - locate the invariant points of the transformation
  - find the invariant lines of the transformation
  - show lines of a given gradient are invariant for a certain transformation.

2 Series and induction

Great things are not done by impulse, but by a series of small things brought together.
Vincent Van Gogh (1853–1890)

▲ **Figure 2.1** Phases of the Moon.

› How would you describe the sequence of pictures of the Moon shown in Figure 2.1?

2.1 Sequences and series

A **sequence** is an ordered set of objects with an underlying rule.

For example:

2, 5, 8, 11, 14

A **series** is the sum of the terms of a numerical sequence:

2 + 5 + 8 + 11 + 14

> ❯ How would you describe this sequence?

Notation

There are a number of different notations which are commonly used in writing down sequences and series:

- The terms of a sequence are often written as $a_1, a_2, a_3, \ldots$ or $u_1, u_2, u_3, \ldots$
- The general term of a sequence may be written as a_r or u_r. (Sometimes the letters k or i are used instead of r.)
- The last term is usually written as a_n or u_n.
- The sum S_n of the first n terms of a sequence can be written using the symbol Σ (the Greek capital S, sigma).

  $$S_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{r=1}^{n} a_r$$

  The numbers above and below the Σ are the limits of the sum. They show that the sum includes all the a_r from a_1 to a_n. The limits may be omitted if they are obvious, so that you would just write Σa_r or you might write $\sum_r a_r$ (meaning the sum of a_r for all values of r).

When discussing sequences you may find the following vocabulary helpful:

- In an **increasing sequence**, each term is greater than the previous term.
- In a **decreasing sequence**, each term is smaller than the previous term.
- In an **oscillating sequence**, the terms lie above and below a middle number.
- The terms of a **convergent sequence** get closer and closer to a limiting value.

Defining sequences

One way to define a sequence is by thinking about the relationship between one term and the next.

The sequence 2, 5, 8, 11, 14, … can be written as

$u_1 = 2$ ← You need to say where the sequence starts.

$u_{r+1} = u_r + 3$ ← You find each term by adding 3 to the previous term.

This is called an **inductive** definition or **term-to-term** definition.

An alternative way to define a sequence is to describe the relationship between the term and its position.

In this case,

$$u_r = 3r - 1.$$

You can see that, for example, substituting $r = 2$ into this definition gives $u_2 = (3 \times 2) - 1 = 5$, which is the second term of the sequence.

This is called a **deductive** definition or **position-to-term** definition.

The series of positive integers

One of the simplest of all sequences is the sequence of the integers:

1, 2, 3, 4, 5, 6, ...

As simple as it is, it may not be immediately obvious how to calculate the sum of the first few integers, for example the sum of the first 100 integers.

$$\sum_{r=1}^{100} r = 1 + 2 + \ldots + 100$$

One way of reaching a total is illustrated below.

$S_{100} = 1 + 2 + 3 + \ldots + 98 + 99 + 100$ ← Call the sum S_{100}

Rewrite S_{100} in reverse:

$$S_{100} = 100 + 99 + 98 + \ldots + 3 + 2 + 1$$

Adding these two lines together, by matching up each term with the one below it, produces pairings of 101 each time, while giving you $2S_{100}$ on the left-hand side.

$$\begin{array}{rcl} S_{100} & = & 1 + 2 + 3 + \ldots + 98 + 99 + 100 \\ S_{100} & = & 100 + 99 + 98 + \ldots + 3 + 2 + 1 \\ \hline 2S_{100} & = & 101 + 101 + 101 + \ldots + 101 + 101 + 101 \end{array}$$

There are 100 terms on the right-hand side (since you were originally adding 100 terms together), so simplify the right-hand side:

$$2S_{100} = 100 \times 101$$

and solve for S_{100}:

$$2S_{100} = 10100$$

$$S_{100} = 5050$$

The sum of the first 100 integers is 5050.

You can use this method to find a general result for the sum of the first n integers (call this S_n).

$$S_n = 1 + 2 + 3 + \ldots + (n-2) + (n-1) + n$$
$$S_n = n + (n-1) + (n-2) + \ldots + 3 + 2 + 1$$
$$2S_n = (n+1) + (n+1) + (n+1) + \ldots + (n+1) + (n+1) + (n+1)$$
$$2S_n = n(n+1)$$
$$S_n = \frac{1}{2}n(n+1)$$

Technology note

You could use a spreadsheet to verify this result for different values of n.

This result is an important one and you will often need to use it.

Note

A common confusion occurs with the sigma notation when there is no r term present.

For example,

$$\sum_{r=1}^{5} 3$$

This means 'The sum of 3, with r changing from 1 to 5'.

means

$$3 + 3 + 3 + 3 + 3 = 15$$

since there are five terms in the sum (it's just that there is no r term to change anything each time).

In general:

$$\sum_{r=1}^{n} 1 = 1 + 1 + \ldots + 1 + 1$$

with n repetitions of the number 1.

So,

$$\sum_{r=1}^{n} 1 = n$$

This apparently obvious result is important and you will often need to use it.

You can use the results $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ and $\sum_{r=1}^{n} 1 = n$ to find the sum of other series.

Example 2.1

For the series 2 + 5 + 8 + ... + 500:

(i) Find a formula for the rth term, u_r.

(ii) How many terms are in this series?

(iii) Find the sum of the series using the reverse/add method.

(iv) Express the sum using sigma notation, and use this to confirm your answer to part (iii).

Solution

(i) The terms increase by 3 each time and start at 2. So $u_r = 3r - 1$.

(ii) Let the number of terms be n. The last term (the nth term) is 500.

$$u_n = 3n - 1$$
$$3n - 1 = 500$$
$$3n = 501$$
$$n = 167$$

There are 167 terms in this series.

(iii)
$$S = 2 + 5 + \ldots + 497 + 500$$
$$S = 500 + 497 + \ldots + 5 + 2$$
$$2S = 167 \times 502$$
$$S = 41\,917$$

(iv)
$$S = \sum_{r=1}^{167}(3r - 1)$$
$$S = \sum_{r=1}^{167} 3r - \sum_{r=1}^{167} 1$$
$$S = 3\sum_{r=1}^{167} r - \sum_{r=1}^{167} 1$$
$$S = 3 \times \frac{1}{2} \times 167 \times 168 - 167$$
$$S = 41917$$

Using the results $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ and $\sum_{r=1}^{n} 1 = n$

Example 2.2

Calculate the sum of the integers from 100 to 200 inclusive.

Solution

$$\sum_{r=100}^{200} r = \sum_{1}^{200} r - \sum_{1}^{99} r$$

Start with all the integers from 1 to 200, and subtract the integers from 1 to 99, leaving those from 100 to 200.

$$= \frac{1}{2} \times 200 \times 201 - \frac{1}{2} \times 99 \times 100$$
$$= 20\,100 - 4950$$
$$= 15\,150$$

Exercise 2A

1 For each of the following definitions, write down the first five terms of the sequence and describe the sequence.

(i) $u_r = 5r + 1$

(ii) $v_r = 3 - 6r$

(iii) $p_r = 2^{r+2}$

(iv) $q_r = 10 + 2 \times (-1)^r$

(v) $a_{r+1} = 2a_r + 1, a_1 = 2$

(vi) $u_r = \frac{5}{r}$

2 For the sequence 1, 5, 9, 13, 17, …,

(i) write down the next four terms of the sequence

(ii) write down an inductive rule for the sequence, in the form $u_1 = \ldots, u_{r+1} = \ldots$

(iii) write down a deductive rule for the general term of the sequence, in the form $u_r = \ldots$.

3 For each of the following sequences,

(a) write down the next four terms of the sequence

(b) write down an inductive rule for the sequence

(c) write down a deductive rule for the general term of the sequence

(d) find the 20th term of the sequence.

(i) 10, 8, 6, 4, 2, …

(ii) 1, 2, 4, 8, 16, …

(iii) 50, 250, 1250, 6250, …

4 Find the sum of the series $\sum_{1}^{5} u_r$ for each of the following.

(i) $u_r = 2 + r$

(ii) $u_r = 3 - 11r$

(iii) $u_r = 3^r$

(iv) $u_r = 7.5 \times (-1)^r$

5 For $S = 50 + 44 + 38 + 32 + \ldots + 14$,

(i) express S in the form $\sum_{r=1}^{n} u_r$

where n is an integer, and u_r is an algebraic expression for the rth term of the series

(ii) hence, or otherwise, calculate the value of S.

6 Given $u_r = 6r + 2$, calculate $\sum_{r=11}^{30} u_r$.

7 The general term of a sequence is given by $u_r = (-1)^r \times 5$.

(i) Write down the first six terms of the sequence and describe it.

(ii) Find the sum of the series $\sum_{r=1}^{n} u_r$

(a) when n is even

(b) when n is odd.

(iii) Find an algebraic expression for the sum to n terms, whatever the value of n.

8 A sequence is given by

$b_{r+2} = b_r + 2,\ b_1 = 0,\ b_2 = 100.$

(i) Write down the first six terms of the sequence and describe it.

(ii) Find the smallest odd value of r for which $b_r \geqslant 200$.

(iii) Find the largest even value of r for which $b_r \leqslant 200$.

PS **9** A sawmill receives an order requesting many logs of various specific lengths, that must come from the same particular tree. The log lengths must start at 5 cm long and increase by 2 cm each time, up to a length of 53 cm.

The saw blade destroys 1 cm (in length) of wood (turning it to sawdust) at every cut. What is the minimum height of tree required to fulfil this order?

PS **10** Find the sum of the integers from n to its square (inclusive). Express your answer in a fully factorised form.

PS **11** Write down the first five terms of the following sequence:

$$c_{r+1} = \begin{cases} 3c_r + 1 & \text{if } c_r \text{ is odd} \\ \dfrac{c_r}{2} & \text{if } c_r \text{ is even} \end{cases} \qquad c_1 = 10$$

If you have access to the internet, you can find out more about this sequence by a web search for the Collatz conjecture.

Try some other starting values (e.g. $c_1 = 6$ or 13) and make a conjecture about the behaviour of this sequence for any starting value.

2.2 Using standard results

In the previous section you used two important results:

$$\sum_{r=1}^{n} 1 = n$$

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1)$$

The sum of the integers.

There are similar results for the sum of the first n squares, and the first n cubes.

The sum of the squares: $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

The sum of the cubes: $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

These are important results. You will prove they are true later in the chapter.

These results can be used to sum other series, as shown in the following examples.

Technology note

You could use a spreadsheet to verify these results for different values of n.

Example 2.3

(i) Write out the first three terms of the sequence $u_r = r^2 + 2r - 1$.

(ii) Find $\sum_{r=1}^{n} u_r$.

(iii) Use your answers from part (i) to check that your answer to part (ii) works for $n = 3$.

Solution

(i) 2, 7, 14

(ii)
$$\begin{aligned}\sum_{r=1}^{n} u_r &= \sum_{r=1}^{n}(r^2 + 2r - 1)\\ &= \sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1\\ &= \frac{1}{6}n(n+1)(2n+1) + 2 \times \frac{1}{2}n(n+1) - n\\ &= \frac{1}{6}n\left[(n+1)(2n+1) + 6(n+1) - 6\right]\\ &= \frac{1}{6}n\left(2n^2 + 3n + 1 + 6n + 6 - 6\right)\\ &= \frac{1}{6}n\left(2n^2 + 9n + 1\right)\end{aligned}$$

(iii) $n = 3$

$$\begin{aligned}\frac{1}{6}n\left(2n^2 + 9n + 1\right) &= \frac{1}{6} \times 3 \times (18 + 27 + 1)\\ &= \frac{1}{2} \times 46\\ &= 23\end{aligned}$$

$2 + 7 + 14 = 23$

It is a good idea to check your results like this, if you can.

Example 2.4

(i) Write the sum of this series using Σ notation.

$(1 \times 3) + (2 \times 4) + (3 \times 5) + \ldots + n(n+2)$

(ii) Hence find an expression for the sum in terms of n.

Solution

(i) $\displaystyle\sum_{r=1}^{n} r(r+2)$

(ii)
$$\begin{aligned}\sum_{r=1}^{n} r(r+2) &= \sum_{r=1}^{n}(r^2+2r)\\ &= \sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r\\ &= \tfrac{1}{6}n(n+1)(2n+1) + 2\times\tfrac{1}{2}n(n+1)\\ &= \tfrac{1}{6}n(n+1)[2n+1+6]\\ &= \tfrac{1}{6}n(n+1)(2n+7)\end{aligned}$$

Exercise 2B

1 (i) Write out the first three terms of the sequence $u_r = 2r - 1$.

(ii) Find an expression for $\displaystyle\sum_{r=1}^{n}(2r-1)$.

(iii) Use part (i) to check part (ii).

2 (i) Write out the first three terms of the sequence $u_r = r(3r+1)$.

(ii) Find an expression for $\displaystyle\sum_{r=1}^{n} r(3r+1)$.

(iii) Use part (i) to check part (ii).

3 (i) Write out the first three terms of the sequence $u_r = (r+1)r^2$.

(ii) Find an expression for $\displaystyle\sum_{r=1}^{n}(r+1)r^2$.

(iii) Use part (i) to check part (ii).

4 Find $\displaystyle\sum_{r=1}^{n}(4r^3 - 6r^2 + 4r - 1)$.

5 Find $(1 \times 2) + (2 \times 3) + (3 \times 4) + \ldots + n(n+1)$.

6 Find $(1 \times 2 \times 3) + (2 \times 3 \times 4) + (3 \times 4 \times 5) + \ldots + n(n+1)(n+2)$.

7 Find the sum of integers above n, up to and including $2n$, giving your answer in a fully factorised form.

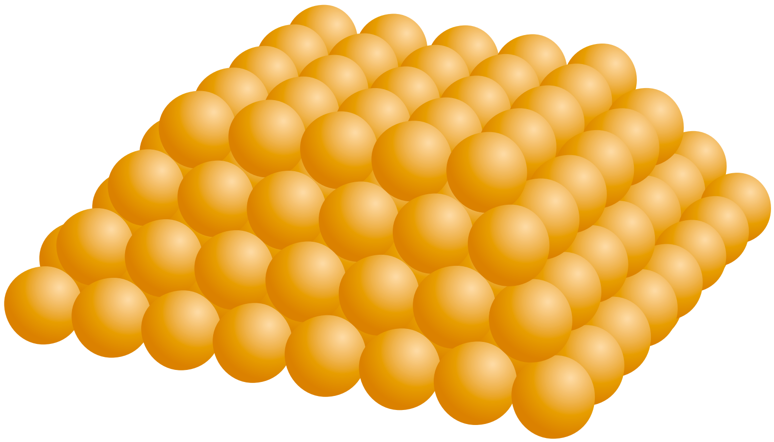
8 Find the sum of the cubes of the integers larger than n, up to and including $3n$, giving your answer in a fully factorised form.

M **9** On a particularly artistic fruit stall, a pile of oranges is arranged to form a truncated square pyramid. Each layer is a square, with the lengths of the side of successive layers reducing by one orange (as shown below).

The bottom layer measures $2n \times 2n$ oranges, and there are n layers.

(i) Prove that the number of oranges used is $\frac{1}{6}n(2n+1)(7n+1)$.

(ii) How many complete layers can the person setting up the stall use for this arrangement, given their stock of 1000 oranges? How many oranges are left over?



M **10** You have \$20 000 to invest for one year. You put it in the following bank account:

'Flexible Saver': 1.5% interest APR

- Interest calculated monthly (i.e. $\frac{1.5}{12}$% of balance each month).
- Interest paid annually, into a separate account.
- No limits on withdrawals or balance.

Your bank then informs you of a new savings account, which you are allowed to open as well as the Flexible Saver.

'Regular Saver': 5% interest APR

- Interest calculated monthly (i.e. $\frac{5}{12}$% each month).
- Interest paid annually, into a separate account.
- Maximum \$1000 balance increase per month.

(i) Assuming you initially have your money in the Flexible Saver, but transfer as much as you can into a Regular Saver each month, calculate how much extra money you will earn, compared to what would happen if you just left it in the Flexible Saver all year.

(ii) Generalise your result – given an investment of I (in thousands of dollars), and a time of n months – what interest will you earn? (Assume $n < I$, or you'll run out of funds to transfer.)

M **11** It is given that

$$S_n = \sum_{r=1}^{n} u_r = 2n^2 + n.$$

Write down the values of S_1, S_2, S_3, S_4. Express u_r in terms of r, justifying your answer.

Find

$$\sum_{r=n+1}^{2n} u_r.$$

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2.3 The method of differences

Sometimes it is possible to find the sum of a series by subtracting it from a related series, with most of the terms cancelling out. This is called the method of differences and is shown in the following example.

Example 2.5

Calculate the value of the series: $5 + 10 + 20 + 40 + \ldots + 2560 + 5120$

Solution

Each term is double the previous one.

> In fact, the sequence is $u_r = 5 \times 2^{r-1}$ but you won't need that here.

Call the sum S.

$$S = 5 + 10 + 20 + \ldots + 2560 + 5120$$

Double it:

$$2S = 10 + 20 + 40 + \ldots + 5120 + 10\,240$$

Subtract the first line from the second and notice that most terms cancel. In fact, only two remain.

$$2S - S = 10\,240 - 5$$

$$S = 10\,235$$

> This is the sum you needed.

This example worked because of the doubling of the terms.

Calculating the sums of much more complicated series can also use this technique, if each term can be expressed as the difference of two (or more) terms. Look at the following examples carefully to see the idea, paying particular attention to the way the series are laid out to help find the cancelling terms.

Example 2.6

(i) Show that $\frac{1}{r} - \frac{1}{r+1} = \frac{1}{r(r+1)}$.

(ii) Hence find $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{30 \times 31}$.

Solution

(i) $\text{LHS} = \dfrac{1}{r} - \dfrac{1}{r+1} = \dfrac{(r+1)-r}{r(r+1)}$

$$= \frac{1}{r(r+1)}$$

$$= \text{RHS} \qquad \text{as required}$$

(ii) $\dfrac{1}{1\times 2} + \dfrac{1}{2\times 3} + \dfrac{1}{3\times 4} + \ldots + \dfrac{1}{30\times 31} = \displaystyle\sum_{r=1}^{30} \frac{1}{r(r+1)}$

$$= \sum_{r=1}^{30}\left(\frac{1}{r} - \frac{1}{r+1}\right)$$

Using the result from part (i)

start writing out the sum, but it is helpful to lay it out like this to see which parts cancel.

$$= 1 - \frac{1}{2}$$
$$+ \frac{1}{2} - \frac{1}{3}$$
$$+ \frac{1}{3} - \frac{1}{4}$$
$$+ \ldots\ldots$$
$$+ \frac{1}{29} - \frac{1}{30}$$
$$+ \frac{1}{30} - \frac{1}{31}$$

The terms in the red loops cancel out – so all the terms in the green box vanish.

$$= 1 - \frac{1}{31}$$

$$= \frac{30}{31}$$

Notice that the result in the example can easily be generalised for a sequence of any length. If the sequence has n terms, then the terms would still cancel in pairs, leaving the first term, 1, and the last term, $-\dfrac{1}{n+1}$.

The sum of the terms would therefore be

$$1 - \frac{1}{n+1} = \frac{n+1-1}{n+1} = \frac{n}{n+1}.$$

> ? What happens to this series when n becomes very large?

The cancelling of nearly all the terms is similar to the way in which the interior sections of a collapsible telescope disappear when it is compressed, so a sum like this is sometimes described as a **telescoping sum**.

The next example uses a telescoping sum to prove a familiar result.

When a series converges you can use the sum to n terms to deduce the **sum to infinity** by considering what happens to the series as n approaches infinity.

Example 2.7

(i) Show that $(2r+1)^2 - (2r-1)^2 = 8r$.

(ii) Hence find $\sum_{r=1}^{n} 8r$.

(iii) Deduce that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$.

Solution

(i)
$$(2r+1)^2 - (2r-1)^2 = (4r^2 + 4r + 1) - (4r^2 - 4r + 1)$$
$$= 8r$$

as required.

(ii)
$$\sum_{r=1}^{n} 8r = \sum_{r=1}^{n}\left[(2r+1)^2 - (2r-1)^2\right]$$
$$\begin{aligned} = {} & 3^2 - 1^2 \\ & + 5^2 - 3^2 \\ & + 7^2 - 5^2 \\ & + \ldots \\ & + (2(n-1)+1)^2 - (2(n-1)-1)^2 \\ & + (2n+1)^2 - (2n-1)^2 \end{aligned}$$

The only terms remaining are the 2nd and the 2nd from last.

$$= (2n+1)^2 - 1^2$$
$$= 4n^2 + 4n + 1 - 1$$
$$= 4n^2 + 4n$$

(iii) Since $\sum_{r=1}^{n} 8r = 4n^2 + 4n$

so $\sum_{r=1}^{n} r = \frac{1}{2}n^2 + \frac{1}{2}n$

$$= \frac{1}{2}n(n+1)$$

as required.

This result was also proved on page 43 using a different method.

Example 2.8

(i) Show that $\frac{2}{r} - \frac{3}{r+1} + \frac{1}{r+2} = \frac{r+4}{r(r+1)(r+2)}$.

(ii) Hence find $\sum_{r=1}^{n} \frac{r+4}{r(r+1)(r+2)}$.

(iii) Deduce the value of the sum to infinity of the series.

Solution

(i) $$\frac{2}{r} - \frac{3}{r+1} + \frac{1}{r+2} = \frac{2(r+1)(r+2) - 3r(r+2) + r(r+1)}{r(r+1)(r+2)}$$

$$= \frac{2r^2 + 6r + 4 - 3r^2 - 6r + r^2 + r}{r(r+1)(r+2)}$$

$$= \frac{r+4}{r(r+1)(r+2)}$$

(ii) $$\sum_{r=1}^{n} \frac{r+4}{r(r+1)(r+2)} = \sum_{r=1}^{n} \left(\frac{2}{r} - \frac{3}{r+1} + \frac{1}{r+2} \right)$$

$$\begin{aligned}
= 2 - \frac{3}{2} &+ \frac{1}{3} \\
+ \frac{2}{2} &- \frac{3}{3} + \frac{1}{4} \\
&+ \frac{2}{3} - \frac{3}{4} + \frac{1}{5} \\
&\quad + \ldots - \ldots + \ldots \\
&\quad\quad + \ldots - \ldots + \ldots \\
&\quad\quad + \frac{2}{n-1} - \frac{3}{n-1} + \frac{1}{n} \\
&\quad\quad\quad + \frac{2}{n-1} - \frac{3}{n} + \frac{1}{n+1} \\
&\quad\quad\quad\quad + \frac{2}{n} - \frac{3}{n+1} + \frac{1}{n+2}
\end{aligned}$$

The terms in the red loops cancel out – so all the terms in the green box vanish.

Most of the terms cancel, leaving

$$\sum_{r=1}^{n} \frac{r+4}{r(r+1)(r+2)} = 2 - \frac{3}{2} + \frac{2}{2} + \frac{1}{n+1} - \frac{3}{n+1} + \frac{1}{n+2}$$

$$= \frac{3}{2} - \frac{2}{n+1} + \frac{1}{n+2}$$

Note

The terms which do not cancel form a symmetrical pattern, three at the start and three at the end.

(iii) As $n \to \infty$

$$\frac{2}{n+1} \to 0 \text{ and } \frac{1}{n+2} \to 0$$

So the sum to infinity is $\frac{3}{2}$.

?

- Show that the final expression in the previous example can be simplified to give $\frac{n(3n+7)}{2(n+1)(n+2)}$.
- Show that this expression gives the same sum to infinity as found in part (iii).

Exercise 2C

CP **1** This question is about the series $1 + 3 + 5 + \ldots + (2n - 1)$.

You can write this as $\sum_{r=1}^{n}(2r - 1)$.

(i) Show that $r^2 - (r - 1)^2 = 2r - 1$.

(ii) Write out the first three terms and the last three terms of $\sum_{r=1}^{n}\left(r^2 - (r - 1)^2\right)$.

(iii) Hence find $\sum_{r=1}^{n}(2r - 1)$.

(iv) Show that using the standard formulae to find $\sum_{r=1}^{n}(2r - 1)$ gives the same result as in (iii).

CP **2** This question is about the series $\frac{2}{1 \times 3} + \frac{2}{3 \times 5} + \frac{2}{5 \times 7} + \ldots + \frac{2}{19 \times 21}$.

(i) Show that the general term of the series is $\frac{2}{(2r - 1)(2r + 1)}$, and find the values of r for the first term and the last term of the series.

(ii) Show that $\frac{1}{2r - 1} - \frac{1}{2r + 1} = \frac{2}{(2r - 1)(2r + 1)}$.

(iii) Hence find $\frac{2}{1 \times 3} + \frac{2}{3 \times 5} + \frac{2}{5 \times 7} + \ldots + \frac{2}{19 \times 21}$.

CP **3** (i) Show that $(r + 1)^2(r + 2) - r^2(r + 1) = (r + 1)(3r + 2)$.

(ii) Hence find $(2 \times 5) + (3 \times 8) + (4 \times 11) + \ldots + (n + 1)(3n + 2)$.

(iii) Show that you can obtain the same result by using the standard formulae to find the sum of this series.

(iv) Using trial and improvement, find the smallest value of n for which the sum is greater than one million.

CP **4** (i) Show that $\frac{1}{r^2} - \frac{1}{(r + 1)^2} = \frac{2r + 1}{r^2(r + 1)^2}$.

(ii) Hence find $\sum_{r=1}^{n}\frac{2r + 1}{r^2(r + 1)^2}$.

CP **5** (i) Show that $\frac{1}{2r} - \frac{1}{2(r + 2)} = \frac{1}{r(r + 2)}$.

(ii) Hence find $\sum_{r=1}^{n}\frac{1}{r(r + 2)}$.

(iii) Find the value of this sum for $n = 100$, $n = 1000$ and $n = 10\,000$ and comment on your answer.

CP **6** (i) Show that $-\frac{1}{r + 2} + \frac{3}{r + 3} - \frac{2}{r + 4} = \frac{r}{(r + 2)(r + 3)(r + 4)}$.

(ii) Hence find $\sum_{r=1}^{12}\frac{r}{(r + 2)(r + 3)(r + 4)}$.

CP 7 (i) Show that $\dfrac{1}{2r} - \dfrac{1}{r+1} + \dfrac{1}{2(r+2)} = \dfrac{1}{r(r+1)(r+2)}$.

(ii) Hence find $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$.

(iii) Find the value of this sum for $n = 100$ and $n = 1000$, and comment on your answer.

In questions **8** and **9** you will prove the standard results for $\sum r^2$ and $\sum r^3$.

CP 8 (i) Show that $(2r+1)^3 - (2r-1)^3 = 24r^2 + 2$.

(ii) Hence find $\displaystyle\sum_{r=1}^{n}\left(24r^2 + 2\right)$.

(iii) Deduce that $\displaystyle\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

CP 9 (i) Show that $(2r+1)^4 - (2r-1)^4 = 64r^3 + 16r$.

(ii) Hence find $\displaystyle\sum_{r=1}^{n}\left(64r^3 + 16r\right)$.

(iii) Deduce that $\displaystyle\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$.

(You may use the standard result for $\sum r$.)

CP 10 (i) Show that $\dfrac{2}{r^2-1}$ can be written as $\dfrac{1}{r-1} - \dfrac{1}{r+1}$.

(ii) Hence find the values of A and B in the identity

$$\frac{1}{r^2-1} = \frac{A}{r-1} + \frac{B}{r+1}$$

(iii) Find $\displaystyle\sum_{r=2}^{n} \frac{1}{r^2-1}$.

(iv) Find the sum to infinity of the series.

11 Given that

$$u_k = \frac{1}{\sqrt{(2k-1)}} - \frac{1}{\sqrt{(2k+1)}},$$

express $\displaystyle\sum_{k=13}^{n} u_k$ in terms of n.

Deduce the value of $\displaystyle\sum_{k=13}^{\infty} u_k$.

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12 The sequence $a_1, a_2, a_3, \ldots$ is such that, for all positive integers n,

$$a_n = \frac{n+5}{\sqrt{(n^2-n+1)}} - \frac{n+6}{\sqrt{(n^2+n+1)}}.$$

The sum $\displaystyle\sum_{n=1}^{N} a_n$ is denoted by S_N.

Find

(i) the value of S_{30} correct to 3 decimal places,

(ii) the least value of N for which $S_N > 4.9$.

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13 Let $f(r) = r(r+1)(r+2)$. Show that

$$f(r) - f(r-1) = 3r(r+1).$$

Hence show that $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$.

Using the standard result for $\sum_{r=1}^{n} r$, deduce that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

Find the sum of the series

$$1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2 + \ldots + 2(n-1)^2 + n^2,$$

where n is odd.

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2.4 Proof by induction

The oldest person to have ever lived, with documentary evidence, is believed to be a French woman called Jeanne Calment who died aged 122, in 1997.

Aisha is an old woman who claims to have broken the record. A reporter asked her, 'How do you know you're 122 years old?'

She replied, 'Because I was 121 last year.'

? Is this a valid argument?

The sort of argument that Aisha was trying to use is called inductive reasoning. If all the elements are present it can be used in proof by induction. This is the subject of the rest of this chapter. It is a very beautiful form of proof but it is also very delicate; if you miss out any of the steps in the argument, as Aisha did, you invalidate your whole proof.

ACTIVITY 2.1

Work out the first four terms of this pattern:

$$\frac{1}{1\times 2} =$$

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} =$$

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} =$$

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \frac{1}{4\times 5} =$$

Activity 2.1 illustrates one common way of solving problems in mathematics. Looking at a number of particular cases may show a pattern, which can be used to form a **conjecture** (i.e. a theory about a possible general result).

Conjectures are often written algebraically.

The conjecture can then be tested in further particular cases.

In this case, the sum of the first n terms of the sequence can be written as

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{n(n+1)}.$$

The activity shows that the conjecture

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

is true for $n = 1, 2, 3$ and 4.

Try some more terms, say, the next two.

If you find a **counter-example** at any point (a case where the conjecture is not true) then the conjecture is definitely disproved. If, on the other hand, the further cases agree with the conjecture then you may feel that you are on the right lines, but you can never be mathematically certain that trying another particular case might not reveal a counter-example: the conjecture is supported by more evidence but not proved.

The ultimate goal is to prove this conjecture is true for *all* positive integers. But it is not possible to prove this conjecture by deduction from known results. A different approach is needed: **mathematical induction**.

In Activity 2.1 you established that the conjecture is true for particular cases of n ($n = 1, 2, 3, 4, 5$ and 6).

Now, assume that the conjecture is true for a particular integer, $n = k$ say, so that

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$$

and use this assumption to check what happens for the next integer, $n = k + 1$.

If the conjecture is true then you should get

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{(k+1)}{(k+1)+1}$$

$$=\frac{k+1}{k+2}$$

Replacing k with $k+1$ in the result $\frac{k}{k+1}$

This is your target result. It is what you need to establish.

Look at the left-hand side (LHS). You can see that the first k terms are part of the assumption.

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$ (the LHS)

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$ Using the assumption

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$ getting a common denominator

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$ expanding the top bracket

$$=\frac{(k+1)^2}{(k+1)(k+2)}$$ factorising the top quadratic

cancelling the $(k+1)$ factor – since $k \neq -1$

$$=\frac{k+1}{k+2}$$ which is the required result.

These steps show that *if* the conjecture is true for $n = k$, *then* it is true for $n = k + 1$.

Since you have already proved it is true for $n = 1$, you can deduce that it is therefore true for $n = 2$ (by taking $k = 2$).

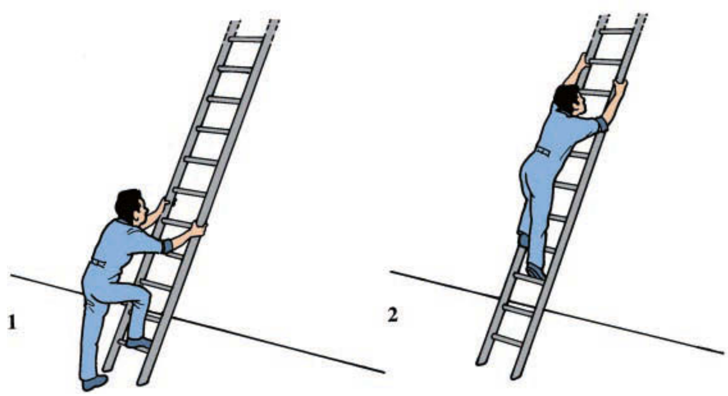
You can continue in this way (e.g. take $n = 2$ and deduce it is true for $n = 3$) as far as you want to go. Since you can reach *any* positive integer n you have now proved the conjecture is true for *every positive integer.*

This method of **proof by mathematical induction** (often shortened to **proof by induction**) is a bit like the process of climbing a ladder:

If you can

1 get on the ladder (the bottom rung), and
2 get from one rung to the next,

then you can climb as far up the ladder as you like.



▲ **Figure 2.2**

The corresponding steps in the previous proof are

1 showing the conjecture is true for $n = 1$, and
2 showing that *if* it is true for a particular value ($n = k$ say), *then* it is true for the next one ($n = k + 1$).

(Notice the *if*… *then*… structure to this step.)

You should conclude any argument by mathematical induction with a statement of what you have shown.

Steps in mathematical induction

To prove something by mathematical induction you need to state a conjecture to begin with. Then there are five elements needed to try to prove the conjecture is true.

- Proving that it is true for a starting value (e.g. $n = 1$).
- Finding the target expression: using the result for $n = k$ to find the equivalent result for $n = k + 1$.
- Proving that: *if* it is true for $n = k$, *then* it is true for $n = k + 1$.
- Arguing that since it is true for $n = 1$, it is also true for $n = 1 + 1 = 2$, and so for $n = 2 + 1 = 3$ and for all subsequent values of n.
- Concluding the argument by writing down the result and stating that it has been proved.

This can be done before or after finding the target expression, but you may find it easier to find the target expression first so that you know what you are working towards.

To find the target expression you replace k with $k + 1$ in the result for $n = k$.

This ensures the argument is properly rounded off. You will often use the word 'therefore'.

Example 2.9

(The sum of the squares of the first *n* integers)

Prove that, for all positive integers n:

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$$

Note

You have already had the opportunity to prove this result using the method of differences, in question 8 of Exercise 2C.

Solution

When $n = 1$, LHS $= 1^2 = 1$ RHS $= \frac{1}{6} \times 1 \times 2 \times 3 = 1$

So it is true for $n = 1$.

Assume the result is true for $n = k$, so

$$1^2 + 2^2 + \dots + k^2 = \frac{1}{6}k(k + 1)(2k + 1)$$

Target expression:

$$1^2 + 2^2 + 3^2 + \dots + k^2 + (k + 1)^2 = \frac{1}{6}(k + 1)[(k + 1) + 1)][(2(k + 1) + 1]$$

$$= \frac{1}{6}(k + 1)(k + 2)(2k + 3)$$

You want to prove that the result is true for $n = k + 1$ (if the assumption is true).

Look at the LHS of the result you want to prove:

$$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2$$

Use the assumed result for $n = k$, to replace the first k terms.

$$= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$$

The $(k + 1)$th. term.

The first k terms.

$$= \frac{1}{6}(k+1)\left[k(2k+1) + 6(k+1)\right]$$

Take out a factor $\frac{1}{6}(k+1)$. You can see from the target expression that this will be helpful.

$$= \frac{1}{6}(k+1)(2k^2 + 7k + 6)$$

$$= \frac{1}{6}(k+1)(k+2)(2k+3)$$

This is the same as the target expression, as required.

If the result is true for $n = k$, then it is true for $n = k + 1$.

Since it is true for $n = k$, it is true for all positive integer values of n.

Therefore the result that $1 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ is true.

ACTIVITY 2.2

Nita is investigating the sum of the first n even numbers.
She writes

$$2 + 4 + 6 + \ldots + 2n = \left(n + \frac{1}{2}\right)^2.$$

(i) Prove that *if* this result is true when $n = k$, *then* it is true when $n = k + 1$. Explain why Nita's conjecture is *not* true for all positive integers n.

(ii) Suggest a different conjecture for the sum of the first n even numbers, that is true for $n = 1$ but not for other values of n. At what point does an attempt to use proof by induction on this result break down?

Exercise 2D

CP

1 (i) Show that the result $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ is true for the case $n = 1$.

(ii) Assume that $1 + 3 + 5 + \ldots + (2k - 1) = k^2$ and use this to prove that:

$$1 + 3 + \ldots + (2k-1) + (2k+1) = (k+1)^2.$$

(iii) Explain how parts (i) and (ii) together prove the sum of the first n odd integers is n^2.

2 (i) Show that the result $1 + 5 + 9 + \ldots + (4n-3) = n(2n-1)$ is true for the case $n = 1$.

(ii) Assume that $1 + 5 + 9 + \ldots + (4k-3) = k(2k-1)$ and use this to prove that:

$$1 + 5 + \ldots + (4k-3) + \left(4(k+1)-3\right) = (k+1)\left(2(k+1)-1\right).$$

(iii) Explain how parts (i) and (ii) together prove that:

$$1 + 5 + 9 + \ldots + (4n-3) = n(2n-1)$$

Prove the following results by induction.

You have already seen two proofs of this result, on pages 43 and 52.

3 $1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n+1)$

(the sum of the first n integers)

You have already had the opportunity to prove this result using the method of differences, in question 9 of Exercise 2C.

4 $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

(the sum of the first n cubes)

5 $2^1 + 2^2 + 2^3 + 2^4 + \ldots + 2^n = 2\left(2^n - 1\right)$

6 $\sum_{r=0}^{n} x^r = \frac{1 - x^{n+1}}{1 - x} \quad (x \neq 1)$

7 $(1 \times 2 \times 3) + (2 \times 3 \times 4) + \ldots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$

8 $\sum_{r=1}^{n} (3r+1) = \frac{1}{2}n(3n+5)$

9 $\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \ldots + \frac{1}{4n^2 - 1} = \frac{n}{2n+1}$

10 $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\ldots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$ for $n \geqslant 2$

11 $1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n+1)! - 1$

12 (i) Prove by induction that

$$\sum_{r=1}^{n} \left(5r^4 + r^2\right) = \frac{1}{2}n^2(n+1)^2(2n+1).$$

(ii) Using the result in part (i), and the formula for $\sum_{r=1}^{n} r^2$, show that

$$\sum_{r=1}^{n} r^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2 + 3n - 1).$$

2.5 Other proofs by induction

So far, you have used induction to prove results involving the sums of series. It can also be used in other situations.

You have seen that induction can be used to prove a given result for the sum of a series in which the terms have been given using a deductive definition. In the next example you will see how induction can be used to prove a given result for the general term of a sequence, when the terms of a sequence have been given inductively.

Example 2.10

A sequence is defined by $u_{n+1} = 4u_n - 3, u_1 = 2$.

Prove that $u_n = 4^{n-1} + 1$.

Solution

For $n = 1, u_1 = 4^0 + 1 = 1 + 1 = 2$, so the result is true for $n = 1$.

Assume that the result is true for $n = k$, so that $u_k = 4^{k-1} + 1$.

Target expression: $u_{k+1} = 4^k + 1$.

For $n = k + 1$, $u_{k+1} = 4u_k - 3$

$= 4\left(4^{k-1} + 1\right) - 3$

$= 4 \times 4^{k-1} + 4 - 3$

$= 4^k + 1$

If the result is true for $n = k$, then it is true for $n = k + 1$.

Since it is true for $n = 1$, it is true for all positive integer values of n. Therefore the result $u_n = 4^{n-1} + 1$ is true.

Example 2.11

Prove that $n! > 2^n$ for all positive integer n greater than 4.

Solution

When $n = 4$: $4! = 24$ and $2^4 = 16$, so $4! > 2^4$ and the result is shown to be true for $n = 4$.

In this proof you need to start with $n = 4$, not $n = 1$.

Assume that the result is true for $n = k$, so $k! > 2^k$

Now prove for $n = k + 1$, $(k + 1)! > 2^{k+1}$

$(k + 1)! = (k + 1) \times k!$

$(k + 1) \times k! > (k + 1) \times 2^k$ — You assumed that when $n = k$, $k! > 2^k$

$> 2 \times 2^k$ — Since $k > 4$ then $k + 1 > 2$

$> 2^{k+1}$

So when $n = k + 1$ then $(k + 1) \times k! > 2^{k+1}$ as required.

If the result is true for $n = k$, then it is true for $n = k + 1$.

Since it is true for $n = 4$, it is true for all positive integer values of n greater than 4.

Therefore the result $n! > 2^n$ is true.

Example 2.12

Prove that, for every positive integer n,

$$\frac{d^n}{dx^n}(xe^{ax}) = na^{n-1}e^{ax} + a^n xe^{ax} \text{ (where } a \text{ is a constant).}$$

Solution

Substituting $n = 1$ into $\frac{d^n}{dx^n}(xe^{ax}) = na^{n-1}e^{ax} + a^n xe^{ax}$ gives

$$\frac{d}{dx}(xe^{ax}) = 1 \times a^0 e^{ax} + a^1 xe^{ax}$$

$$= e^{ax} + axe^{ax}$$

And when $n = 1$: $\frac{d}{dx}(xe^{ax}) = 1 \times e^{ax} + x \times ae^{ax}$

Using the product rule for differentiation.

$= e^{ax} + axe^{ax}$, so the result is shown to be true when $n = 1$.

Assume that the result is true for $n = k$, so

$$\frac{d^k}{dx^k}(xe^{ax}) = ka^{k-1}e^{ax} + a^k xe^{ax}$$

To find $\frac{d^{k+1}}{dx^{k+1}}(xe^{ax})$ you need to differentiate $\frac{d^k}{dx^k}(xe^{ax})$

Now prove for $n = k + 1$, $\frac{d^{k+1}}{dx^{k+1}}(xe^{ax}) = \frac{d}{dx}(ka^{k-1}e^{ax} + a^k xe^{ax})$

Using the product rule for differentiation.

$$\frac{d}{dx}(ka^{k-1}e^{ax} + a^k\ xe^{ax}) = ka^{k-1} \times ae^{ax} + a^k \times 1 \times e^{ax} + a^k x \times ae^{ax}$$

$$= ka^k e^{ax} + a^k e^{ax} + a^{k+1}xe^{ax}$$

$$= (k+1)a^k e^{ax} + a^{k+1}xe^{ax}$$

So when $n = k + 1$ then $\frac{d^{(k+1)}}{dx^{(k+1)}}(xe^{ax}) = (k+1)a^k e^{ax} + a^{k+1}xe^{ax}$ as required.

If the result is true for $n = k$, then it is true for $n = k + 1$.

Since it is true for $n = 1$, it is true for all positive integer values of n.

Therefore the result $\frac{d^n}{dx^n}(xe^{ax}) = na^{n-1}e^{ax} + a^n xe^{ax}$ is true.

You can sometimes use induction to prove results involving powers of matrices.

Example 2.13

Given $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$, prove by induction that

$$\mathbf{A}^n = \frac{1}{4}\begin{pmatrix} 3 \times 5^n + 1 & 5^n - 1 \\ 3 \times 5^n - 3 & 5^n + 3 \end{pmatrix}.$$

Solution

Let $n = 1$ LHS $= \mathbf{A}^1 = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

$$\text{RHS} = \frac{1}{4}\begin{pmatrix} 3 \times 5 + 1 & 5 - 1 \\ 3 \times 5 - 3 & 5 + 3 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 16 & 4 \\ 12 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

$=$ LHS as required

Assume true for $n = k$, i.e.

$$\mathbf{A}^k = \frac{1}{4}\begin{pmatrix} 3 \times 5^k + 1 & 5^k - 1 \\ 3 \times 5^k - 3 & 5^k + 3 \end{pmatrix}$$

Target expression:

$$\mathbf{A}^{k+1} = \frac{1}{4}\begin{pmatrix} 3 \times 5^{k+1} + 1 & 5^{k+1} - 1 \\ 3 \times 5^{k+1} - 3 & 5^{k+1} + 3 \end{pmatrix}$$

You want to prove it is true for $n = k + 1$.

$$\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A}$$

$$= \frac{1}{4}\begin{pmatrix} 3 \times 5^k + 1 & 5^k - 1 \\ 3 \times 5^k - 3 & 5^k + 3 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 12 \times 5^k + 4 + 3 \times 5^k - 3 & 3 \times 5^k + 1 + 2 \times 5^k - 2 \\ 12 \times 5^k - 12 + 3 \times 5^k + 9 & 3 \times 5^k - 3 + 2 \times 5^k + 6 \end{pmatrix}$$

Multiplying matrices.

$$= \frac{1}{4}\begin{pmatrix} 15 \times 5^k + 1 & 5 \times 5^k - 1 \\ 15 \times 5^k - 3 & 5 \times 5^k + 3 \end{pmatrix}$$

Using $15 = 3 \times 5$.

$$= \frac{1}{4}\begin{pmatrix} 3 \times 5^{k+1} + 1 & 5^{k+1} - 1 \\ 3 \times 5^{k+1} - 3 & 5^{k+1} + 3 \end{pmatrix}$$

This is the target matrix.

as required.

If it is true for $n = k$, then it is true for $n = k + 1$

Since it is true for $n = 1$, it is true for all $n \geqslant 1$.

Therefore the result $\mathbf{A}^n = \frac{1}{4}\begin{pmatrix} 3 \times 5^n + 1 & 5^n - 1 \\ 3 \times 5^n - 3 & 5^n + 3 \end{pmatrix}$ is true.

Exercise 2E

1 A sequence is defined by $u_{n+1} = 3u_n + 2$, $u_1 = 2$.
Prove by induction that $u_n = 3^n - 1$.

2 A sequence is defined by $u_{n+1} = 2u_n - 1$, $u_1 = 2$.
Prove by induction that $u_n = 2^{n-1} + 1$.

3 Given that $\mathbf{M} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, prove by induction that $\mathbf{M}^n = \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix}$.

4 A sequence is defined by $u_{n+1} = 4u_n - 6$, $u_1 = 3$.
Prove by induction that $u_n = 4^{n-1} + 2$.

5 (i) Given that $\mathbf{M} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, prove by induction that $\mathbf{M}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

(ii) Describe the transformations represented by **M** and by $\mathbf{M}^n$.

6 A sequence is defined by $u_{n+1} = \dfrac{u_n}{u_n + 1}$, $u_1 = 1$.

(i) Find the values of u_2, u_3 and u_4.

(ii) Suggest a general formula for u_n, and prove your conjecture by induction.

7 You are given the matrix $\mathbf{A} = \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}$.

(i) Calculate $\mathbf{A}^2$ and $\mathbf{A}^3$.

(ii) Show that the formula $\mathbf{A}^n = \begin{pmatrix} 1-2n & -4n \\ n & 1+2n \end{pmatrix}$ is consistent with the given value of **A** and your calculations for $n = 2$ and $n = 3$.

(iii) Prove by induction that the formula for $\mathbf{A}^n$ is correct when n is a positive integer.

8 You are given the matrix $\mathbf{M} = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix}$.

(i) Calculate $\mathbf{M}^2$, $\mathbf{M}^3$ and $\mathbf{M}^4$.

(ii) Write down separate conjectures for formulae for $\mathbf{M}^n$, for even n (i.e. $\mathbf{M}^{2m}$) and for odd n (i.e. $\mathbf{M}^{2m+1}$)

(iii) Prove each conjecture by induction, and hence write down what $\mathbf{M}^n$ is for any $n \geqslant 1$.

9 Let $F_n = 2^{(2^n)} + 1$.

(i) Calculate F_0, F_1, F_2, F_3, and F_4.

(ii) Prove, by induction, that $F_0 \times F_1 \times F_2 \times \ldots \times F_{n-1} = F_n - 2$.

(iii) Use part (ii) to prove that F_i and F_j are coprime (for $i \neq j$).

(iv) Use part (iii) to prove there are infinitely many prime numbers.

Note

The F_n numbers are called Fermat Numbers. The first five are prime: the Fermat Primes. Nobody (yet) knows if any other Fermat Numbers are prime.

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

10 It is given that $u_r = r \times r!$ for $r = 1, 2, 3, \ldots,$.
Let $S_n = u_1 + u_2 + u_3 + \cdots + u_n$. Write down the values of

$$2! - S_1, \quad 3! - S_2, \quad 4! - S_3, \quad 5! - S_4.$$

Conjecture a formula for S_n.

Prove, by mathematical induction, a formula for S_n, for all positive integers n.

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11 It is given that $y = (1+x)^2 \ln(1+x)$. Find $\frac{d^3y}{dx^3}$.

Prove by mathematical induction that, for every integer $n \geqslant 3$

$$\frac{d^n y}{dx^n} = (-1)^{n-1} = \frac{2(n-3)!}{(1+x)^{n-2}}.$$

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12 Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n}{dx^n}(e^x \sin x) = 2^{\frac{1}{2}n} e^x \sin(x + \frac{1}{4}n\pi).$$

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KEY POINTS

1 The terms of a sequence are often written as $a_1, a_2, a_3, \ldots$ or $u_1, u_2, u_3, \ldots$
The general term of a sequence may be written as a_r or u_r (sometimes the letters k or i are used instead of r). The last term is usually written as a_n or u_n.

2 A series is the sum of the terms of a sequence. The sum S_n of the first n terms of a sequence can be written using the symbol Σ (the Greek capital S, sigma).

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{r=1}^{n} a_r$$

The numbers above and below the Σ are the limits of the sum. They show that the sum includes all the terms a_r from a_1 to a_n.

3 Some series can be expressed as combinations of these standard results:

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1) \quad \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1) \quad \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

4 Some series can be summed by using the method of differences.
If the terms of the series can be written as the difference of terms of another series, then many terms may cancel out. This is called a telescoping sum.

5 To prove by induction that a statement involving an integer n is true for all $n \geqslant n_0$, you need to:

- prove that the result is true for an initial value of n_0, typically $n = 1$
- find the target expression:
  use the result for $n = k$ to find the equivalent result for $n = k + 1$.
- prove that:
  if it is true for $n = k$, *then* it is true for $n = k + 1$.
- argue that since it is true for $n = 1$, it is also true for $n = 1 + 1 = 2$, and so for $n = 2 + 1 = 3$ and for all subsequent values of n.
- conclude the argument with a precise statement about what has been proved.

LEARNING OUTCOMES

Now that you have finished this chapter, you should be able to

- define what is meant by a sequence and a series
- find the sum of a series using standard formulae for $\sum r$, $\sum r^2$ and $\sum r^3$
- find the sum of a series using the method of differences
- use the sum to n terms, to find the sum to infinity of a convergent series
- use proof by induction to prove given results for the sum of a series
- use proof by induction to prove given results for the nth term of a sequence.

3 Roots of polynomials

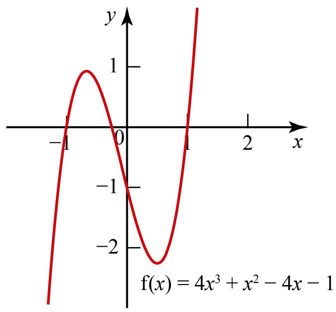
In mathematics it is new ways of looking at old things that seem to be the most prolific sources of far-reaching discoveries.
Eric Temple Bell (1883–1960)

A **polynomial** is an expression like $4x^3 + x^2 - 4x - 1$. Its terms are all positive integer powers of a variable (in this case x) like x^2, or multiples of them like $4x^3$. There are no square roots, reciprocals, etc.

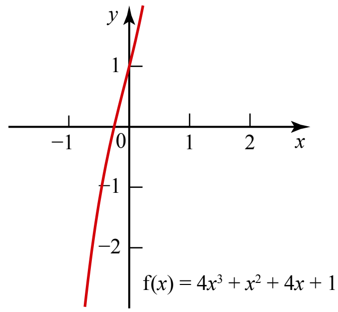
The **order** (or degree) of a polynomial is the highest power of the variable. So the order of $4x^3 + x^2 - 4x - 1$ is 3; this is why it is called a **cubic**.

You often need to solve polynomial equations, and it is usually helpful to think about the associated graph.

The following diagrams show the graphs of two cubic polynomial functions. The first example (in Figure 3.1) has three real roots (where the graph of the polynomial crosses the x-axis). The second example (in Figure 3.2) has only one real root. In this case there are also two complex roots.



▲ **Figure 3.1**



▲ **Figure 3.2**

In general a polynomial equation of order n has n roots. However, some of these may be complex rather than real numbers and sometimes they coincide so that two or more distinct roots become one repeated root.

3.1 Polynomials

The following two statements are true for all polynomials:

- A polynomial equation of order n has at most n real roots.
- The graph of a polynomial function of order n has at most $n - 1$ turning points.

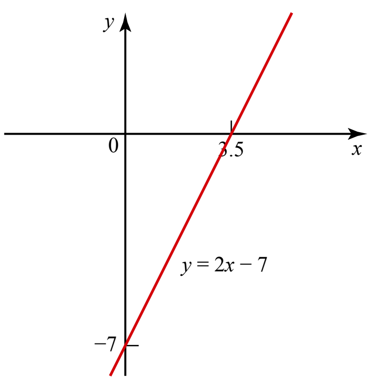
?

- How would you solve the polynomial equation $4x^3 + x^2 - 4x - 1 = 0$?
- What about $4x^3 + x^2 + 4x + 1 = 0$?

Here are some examples that illustrate these results.

Order 1 (a linear equation)

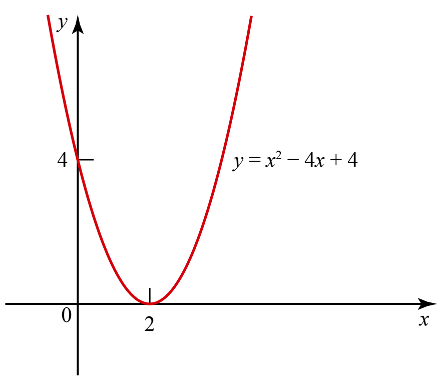
Example: $2x - 7 = 0$



▲ **Figure 3.3** The graph is a straight line with no turning points. There is one real root at $x = 3.5$.

Order 2 (a quadratic equation)

Example: $x^2 - 4x + 4 = 0$

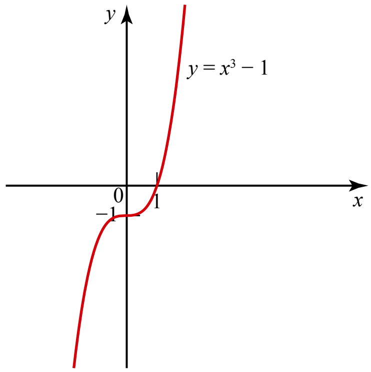


▲ **Figure 3.4** The curve has one turning point. There is one repeated root at $x = 2$.

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

Order 3 (a cubic equation)

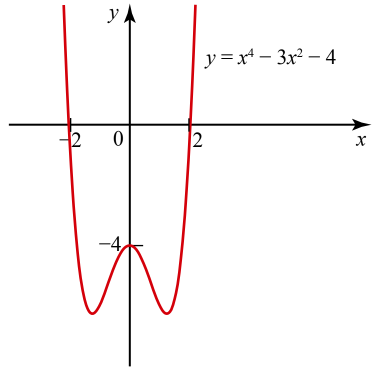
Example: $x^3 - 1 = 0$



▲ **Figure 3.5** The two turning points of this curve coincide to give a point of inflection at (0, −1). There is one real root at $x = 1$ and two complex roots at $x = \frac{-1 \pm \sqrt{3}i}{2}$.

Order 4 (a quartic equation)

Example: $x^4 - 3x^2 - 4 = 0$



▲ **Figure 3.6** This curve has three turning points. There are two real roots at $x = -2$ and $x = 2$ and two complex roots at $x = \pm i$.

You learned about complex roots of polynomial eqautions in *Pure Mathematics 3*

The same patterns continue for higher order polynomials.

The rest of this chapter explores some properties of polynomials, and ways to use these properties to avoid the difficulties of actually finding the roots of polynomials directly.

It is important that you recognise that the roots of polynomials may be complex. For this reason, in the work that follows, z is used as the variable (or unknown) instead of x to emphasise that the results apply regardless of whether the roots are complex or real.

3.2 Quadratic equations

ACTIVITY 3.1

Solve each of the following quadratic equations (by factorising or otherwise).
Also write down the *sum* and *product* of the two roots.
What do you notice?

Equation	Two roots	Sum of roots	Product of roots
(i) $z^2 - 3z + 2 = 0$			
(ii) $z^2 + z - 6 = 0$			
(iii) $z^2 - 6z + 8 = 0$			
(iv) $z^2 - 3z - 10 = 0$			
(v) $2z^2 - 3z + 1 = 0$			
(vi) $z^2 - 4z + 5 = 0$			

Technology note

You could use the equation solver on a calculator.

? What is the connection between the sums and products of the roots, and the coefficients in the original equation?

The roots of polynomial equations are usually denoted by Greek letters such as α and β.

α (alpha) and β (beta) are the first two letters of the Greek alphabet.

Always be careful to distinguish between:
a – the coefficient of z^2 and,
α – one of the roots of the quadratic.

If you know the roots are α and β, you can write the equation

$$az^2 + bz + c = 0$$

in factorised form as

$$a(z - \alpha)(z - \beta) = 0.$$

Assuming $a \neq 0$

This gives the identity,

$$az^2 + bz + c \equiv a(z - \alpha)(z - \beta).$$

$$az^2 + bz + c \equiv a\left(z^2 - \alpha z - \beta z + \alpha\beta\right)$$

$$\equiv az^2 - a(\alpha + \beta)z + a\alpha\beta$$

Multiplying out

$$b = -a(\alpha + \beta) \Rightarrow \alpha + \beta = -\frac{b}{a}$$

Equating coefficients of z

$$c = a\alpha\beta \Rightarrow \alpha\beta = \frac{c}{a}$$

Equating constant terms

So the sum of the roots is

$$\alpha + \beta = -\frac{b}{a}$$

and the product of the roots is

$$\alpha\beta = \frac{c}{a}.$$

From these results you can obtain information about the roots without actually solving the equation.

> What happens if you try to find the values of α and β by solving the equations $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$ as a pair of simultaneous equations?

ACTIVITY 3.2

The quadratic formula gives the roots of the quadratic equation $az^2 + bz + c = 0$ as

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Use these expressions to prove that $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

Example 3.1

Find a quadratic equation with roots 5 and –3.

Solution

The sum of the roots is $5 + (-3) = 2$ $\Rightarrow -\frac{b}{a} = 2$

The product of the roots is $5 \times (-3) = -15$ $\Rightarrow \frac{c}{a} = -15$

Taking a to be 1 gives

$b = -2$ and $c = -15$

You could choose any value for a but choosing 1 in this case gives the simplest form of the equation.

A quadratic equation with roots 5 and –3 is $z^2 - 2z - 15 = 0$.

Forming new equations

Using these properties of the roots sometimes allows you to form a new equation with roots that are related to the roots of the original equation. The next example illustrates this.

Example 3.2

The roots of the equation $2z^2 + 3z + 5 = 0$ are α and β.

(i) Find the values of $\alpha + \beta$ and $\alpha\beta$.

(ii) Find the quadratic equation with roots 2α and 2β.

Solution

(i) $\alpha + \beta = -\frac{3}{2}$ and

$\alpha\beta = \frac{5}{2}$

> These lines come from looking at the original quadratic, and quoting the facts $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.
> It might be confusing to introduce a, b and c here, since you need different values for them later in the question.

(ii) The sum of the new roots $= 2\alpha + 2\beta$

$$= 2(\alpha + \beta)$$

$$= 2 \times -\frac{3}{2}$$

$$= -3$$

The product of the new roots $= 2\alpha \times 2\beta$

$$= 4\alpha\beta$$

$$= 4 \times \frac{5}{2}$$

$$= 10$$

Let a, b and c be the coefficients in the new quadratic equation, then

$-\frac{b}{a} = -3$ and $\frac{c}{a} = 10$.

Taking $a = 1$ gives $b = 3$ and $c = 10$.

So a quadratic equation with the required roots is $z^2 + 3z + 10 = 0$.

Example 3.3

The roots of the equation $3z^2 - 4z - 1 = 0$ are α and β.
Find the quadratic equation with roots $\alpha + 1$ and $\beta + 1$.

Solution

$\alpha + \beta = \frac{4}{3}$ and

$\alpha\beta = -\frac{1}{3}$

→

$$\begin{aligned}\text{The sum of the new roots} &= \alpha + 1 + \beta + 1 \\ &= \alpha + \beta + 2 \\ &= \frac{4}{3} + 2 \\ &= \frac{10}{3}\end{aligned}$$

$$\begin{aligned}\text{The product of the new roots} &= (\alpha + 1)(\beta + 1) \\ &= \alpha\beta + (\alpha + \beta) + 1 \\ &= -\frac{1}{3} + \frac{4}{3} + 1 \\ &= 2\end{aligned}$$

So $-\frac{b}{a} = \frac{10}{3}$ and $\frac{c}{a} = 2$.

Choose $a = 3$, then $b = -10$ and $c = 6$.

Choosing $a = 1$ would give a value for b which is not an integer. It is easier here to use $a = 3$.

So a quadratic equation with the required roots is $3z^2 - 10z + 6 = 0$.

Symmetric functions of the roots of a quadratic equation

If α and β are the roots of a quadratic equation then functions like:

$$f(\alpha, \beta) = \alpha + \beta,$$

$$f(\alpha, \beta) = \alpha^2 + \beta^2,$$

and $f(\alpha, \beta) = \frac{1}{\alpha} + \frac{1}{\beta}$

are called **symmetric functions of the roots** because when you interchange α and β the function remains the same, i.e. $f(\alpha, \beta) = f(\beta, \alpha)$

Functions like $f(\alpha, \beta) = \alpha - \beta$ or $f(\alpha, \beta) = \alpha^2 - \beta^2$ are not symmetric functions, because the function changes when α and β are interchanged. (They would become $f(\beta, \alpha) = \beta - \alpha$ and $f(\beta, \alpha) = \beta^2 - \alpha^2$ respectively).

All symmetric functions of the roots of a quadratic equation can be expressed in terms of $\alpha + \beta$ and $\alpha\beta$.

Example 3.4

The quadratic equation $ax^2 + bx + c = 0$, where a, b and c are constants, has roots α and β.

Prove that $\alpha^2 + \beta^2 = \frac{b^2 - 2ac}{a^2}$.

$\sum \alpha = \alpha + \beta = -\frac{b}{a}$

Solution

You do not know what $\alpha^2 + \beta^2$ equals, but you do know $(\alpha + \beta)^2$, so start by expanding the brackets.

$$(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$$

$$\Rightarrow \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

Notice that $\alpha^2 + \beta^2$ is a symmetric function of the roots.

Substituting in $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$ gives:

$$\alpha^2 + \beta^2 = \left(-\frac{b}{a}\right)^2 - 2 \times \frac{c}{a}$$

$$= \frac{b^2}{a^2} - \frac{2c}{a}$$

$$= \frac{b^2 - 2ac}{a^2}$$

In the last example, you saw that the sum of the squares of the roots is:

$$\alpha^2 + \beta^2 = \frac{b^2}{a^2} - \frac{2c}{a}$$

ACTIVITY 3.3

Solve the quadratic equations from Examples 3.2 and 3.3 (perhaps using the equation solver on your calculator, or a computer algebra system):

(i) $2z^2 + 3z + 5 = 0$ $\quad z^2 + 3z + 10 = 0$

(ii) $3z^2 - 4z - 1 = 0$ $\quad 3z^2 - 10z + 6 = 0$

Verify that the relationships between the roots are correct.

Exercise 3A

1 Write down the sum and product of the roots of each of these quadratic equations.

(i) $2z^2 + 7z + 6 = 0$ (ii) $5z^2 - z - 1 = 0$

(iii) $7z^2 + 2 = 0$ (iv) $5z^2 + 24z = 0$

(v) $z(z + 8) = 4 - 3z$ (vi) $3z^2 + 8z - 6 = 0$

2 Write down quadratic equations (in expanded form, with integer coefficients) with the following roots:

(i) 7, 3 (ii) 4, –1

(iii) –5, –4.5 (iv) 5, 0

(v) 3 (repeated) (vi) $3 - 2\text{i}, 3 + 2\text{i}$

3 The roots of $2z^2 + 5z - 9 = 0$ are α and β.

Find quadratic equations with these roots.

(i) 3α and 3β (ii) $-\alpha$ and $-\beta$

(iii) $\alpha - 2$ and $\beta - 2$ (iv) $1 - 2\alpha$ and $1 - 2\beta$

4 The roots of a quadratic equation $z^2 - 4z - 2 = 0$ are α and β.

Find the quadratic equation with roots α^2 and β^2.

PS **5** A quadratic equation has roots α and β. Express the following symmetric functions in terms of $\alpha+\beta$ and $\alpha\beta$.

(i) $\alpha^2+\beta^2$

(ii) $\frac{1}{\alpha}+\frac{1}{\beta}$

(iii) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

(iv) $\alpha^2\beta+\beta^2\alpha$

(v) $\alpha^3+\beta^3$

CP **6** Using the fact that $\alpha+\beta=-\frac{b}{a}$, and $\alpha\beta=\frac{c}{a}$, what can you say about the roots, α and β, of $az^2+bz+c=0$ in the following cases:

(i) a, b, c are all positive and $b^2-4ac>0$

(ii) $b=0$

(iii) $c=0$

(iv) a and c have opposite signs

CP **7** One root of $az^2+bz+c=0$ is twice the other. Prove that $2b^2=9ac$.

CP **8** The quadratic equation $x^2+px+q=0$, where p and q are constants, has roots α and β. Prove that $\alpha^2+\beta^2=p^2-2q$.

CP **9** The quadratic equation $ax^2+bx+c=0$, where a, b and c are constants, has roots α and β. Prove that $\frac{1}{\alpha}+\frac{1}{\beta}=-\frac{b}{c}$.

PS **10** The roots of $az^2+bz+c=0$ are, α and β. Find quadratic equations with the following roots:

(i) $k\alpha$ and $k\beta$

(ii) $k+\alpha$ and $k+\beta$

> You may wish to introduce different letters (say p, q and r instead of a, b and c) for the coefficients of your target equation.

PS **11** (i) A quadratic equation with *real* coefficients $ax^2+bx+c=0$ has complex roots z_1 and z_2. Explain how the relationships between roots and coefficients show that z_1 and z_2 must be complex conjugates.

(ii) Find a quadratic equation with *complex* coefficients that has roots $2+3\text{i}$ and $3-\text{i}$.

3.3 Cubic equations

There are corresponding properties for the roots of higher order polynomials.

To see how to generalise the properties you can begin with the cubics in a similar manner to the discussion of the quadratics. As before, it is conventional to use Greek letters to represent the three roots: α, β and γ (gamma, the third letter of the Greek alphabet).

You can write the general cubic as

$$az^3 + bz^2 + cz + d = 0$$

or in factorised form as

$$a(z - \alpha)(z - \beta)(z - \gamma) = 0.$$

This gives the identity

$$az^3 + bz^2 + cz + d \equiv a(z - \alpha)(z - \beta)(z - \gamma).$$

Check this for yourself.

Multiplying out the right-hand side gives

$$az^3 + bz^2 + cz + d \equiv az^3 - a(\alpha + \beta + \gamma)z^2 + a(\alpha\beta + \beta\gamma + \gamma\alpha)z - a\alpha\beta\gamma.$$

Comparing coefficients of z^2:

$$b = -a(\alpha + \beta + \gamma) \Rightarrow \alpha + \beta + \gamma = -\frac{b}{a}$$

Sum of the roots: $\sum\alpha$

Comparing coefficients of z:

$$c = a(\alpha\beta + \beta\gamma + \gamma\alpha) \Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

Sum of products of pairs of roots: $\sum\alpha\beta$

Comparing constant terms:

$$d = -a\alpha\beta\gamma \Rightarrow \alpha\beta\gamma = -\frac{d}{a}$$

Product of the three roots: $\sum\alpha\beta\gamma$

Note: Notation

It often becomes tedious writing out the sums of various combinations of roots, so shorthand notation is often used:

$\sum\alpha = \alpha + \beta + \gamma$	the sum of individual roots (however many there are)
$\sum\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha$	the sum of the products of pairs of roots
$\sum\alpha\beta\gamma = \alpha\beta\gamma$	the sum of the products of triples of roots (in this case only one)

Provided you know the degree of the equation (e.g. cubic, quartic, etc,) it will be quite clear what this means. Functions like these are called symmetric functions of the roots, since exchanging any two of α, β, γ will not change the value of the function.

Using this notation you can shorten tediously long expressions. For example, for a cubic with roots α, β and γ,

$$\alpha^2\beta + \alpha\beta^2 + \beta^2\gamma + \beta\gamma^2 + \gamma^2\alpha + \gamma\alpha^2 = \sum\alpha^2\beta.$$

This becomes particularly useful when you deal with quartics in the next section.

Example 3.5

The roots of the equation $2z^3 - 9z^2 - 27z + 54 = 0$ form a geometric progression (i.e. they may be written as $\frac{a}{r}$, a, ar).

Solve the equation.

Solution

Remember: $\sum\alpha\beta\gamma = \alpha\beta\gamma$

$$\alpha\beta\gamma = -\frac{d}{a} \quad \Rightarrow \quad \frac{a}{r} \times a \times ar = -\frac{54}{2}$$

$$\Rightarrow \quad a^3 = -27$$

$$\Rightarrow \quad a = -3$$

$$\sum\alpha = -\frac{b}{a} \quad \Rightarrow \quad \frac{a}{r} + a + ar = \frac{9}{2}$$

$$\Rightarrow \quad -3\left(\frac{1}{r} + 1 + r\right) = \frac{9}{2}$$

$$\Rightarrow \quad 2\left(\frac{1}{r} + 1 + r\right) = -3$$

$$\Rightarrow \quad 2 + 2r + 2r^2 = -3r$$

$$\Rightarrow \quad 2r^2 + 5r + 2 = 0$$

$$\Rightarrow (2r + 1)(r + 2) = 0$$

$$\Rightarrow \quad r = -2 \text{ or } r = -\frac{1}{2}$$

Either value of r gives three roots: $\frac{3}{2}, -3, 6$.

You can also form symmetric functions roots of cubic equations. For example, $f(\alpha, \beta, \gamma) = \alpha^2 + \beta^2 + \gamma^2$ is a symmetric function as when you interchange any two of α, β and γ the function remains the same. You can express symmetric functions in terms of $\sum\alpha$, $\sum\alpha\beta$ and $\alpha\beta\gamma$.

Example 3.6

The roots of the cubic equation $x^3 - 4x^2 + x + 6 = 0$, are α, β and γ. Find the values of:

(i) $\alpha^2 + \beta^2 + \gamma^2$

(ii) $\alpha^3 + \beta^3 + \gamma^3$.

Notice these are symmetric functions of the roots.

Solution

(i) $$(\alpha+\beta+\gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma$$

$$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$

$$= \left(\sum\alpha\right)^2 - 2\sum\alpha\beta$$

$$= 4^2 - 2\times 1$$

$$= 14$$

(ii) α is a root of $x^3 - 4x^2 + x + 6 = 0 \Rightarrow \alpha^3 - 4\alpha^2 + \alpha + 6 = 0$

So $\quad \alpha^3 = 4\alpha^2 - \alpha - 6 \quad$ ①

Likewise, $\quad \beta^3 = 4\beta^2 - \beta - 6 \quad$ ②

and $\quad \gamma^3 = 4\gamma^2 - \gamma - 6 \quad$ ③

Adding equations ①, ② and ③ gives:

$$\begin{aligned}\alpha^3 + \beta^3 + \gamma^3 &= 4(\alpha^2 + \beta^2 + \gamma^2) - (\alpha + \beta + \gamma) - (6 + 6 + 6)\\ &= 4(\alpha^2 + \beta^2 + \gamma^2) - \sum\alpha - 3\times 6\\ &= 4\times 14 - 4 - 18\\ &= 34\end{aligned}$$

Forming new equations: the substitution method

In the next example you are asked to form a new cubic equation with roots related to the roots of the original equation. Using the same approach as in the quadratic example is possible, but this gets increasingly complicated as the order of the equation increases. A substitution method is often a quicker alternative. The following example shows both methods for comparison.

Example 3.7

The roots of the cubic equation $2z^3 + 5z^2 - 3z - 2 = 0$ are α, β, γ.

Find the cubic equation with roots $2\alpha + 1, 2\beta + 1, 2\gamma + 1$.

Solution 1

$\sum\alpha = \alpha + \beta + \gamma = -\frac{5}{2}$

$\sum\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = -\frac{3}{2}$

$\sum\alpha\beta\gamma = \alpha\beta\gamma = \frac{2}{2} = 1$

> $\sum\alpha = -\frac{b}{a}$
>
> $\sum\alpha\beta = \frac{c}{a}$
>
> $\sum\alpha\beta\gamma = \alpha\beta\gamma = -\frac{d}{a}$

For the new equation:

$$\begin{aligned}\text{Sum of roots } &= 2\alpha + 1 + 2\beta + 1 + 2\gamma + 1\\ &= 2(\alpha + \beta + \gamma) + 3\\ &= -5 + 3 = -2\end{aligned}$$

Product of the roots in pairs

$= (2\alpha + 1)(2\beta + 1) + (2\beta + 1)(2\gamma + 1) + (2\gamma + 1)(2\alpha + 1)$

$= [4\alpha\beta + 2(\alpha + \beta) + 1] + [4\beta\gamma + 2(\beta + \gamma) + 1] + [4\gamma\alpha + 2(\gamma + \alpha) + 1]$

$= 4(\alpha\beta + \beta\gamma + \gamma\alpha) + 4(\alpha + \beta + \gamma) + 3$

$= 4 \times -\frac{3}{2} + 4 \times -\frac{5}{2} + 3$

$= -13$

➜

Product of roots $= (2\alpha + 1)(2\beta + 1))(2\gamma + 1)$

$= 8\alpha\beta\gamma + 4(\alpha\beta + \beta\gamma + \gamma\alpha) + 2(\alpha + \beta + \gamma) + 1$

$= 8 \times 1 + 4 \times -\frac{3}{2} + 2 \times -\frac{5}{2} + 1$

$= -2$

Check this for yourself.

In the new equation, $-\frac{b}{a} = -2$, $\frac{c}{a} = -13$, $-\frac{d}{a} = -2$.

These are all integers, so choose $a = 1$ and this gives the simplest integer coefficients.

The new equation is $z^3 + 2z^2 - 13z + 2 = 0$.

Solution 2 (substitution method)

This method involves a new variable $w = 2z + 1$. You write z in terms of w, and substitute into the original equation:

This is a transformation of z in the same way as the new roots are a transformation of the original z roots.

$z = \frac{w-1}{2}$ α, β, γ are the roots of $2z^3 + 5z^2 - 3z - 2 = 0$

$\Leftrightarrow 2\alpha + 1, 2\beta + 1, 2\gamma + 1$ are the roots of

$$2\left(\frac{w-1}{2}\right)^3 + 5\left(\frac{w-1}{2}\right)^2 - 3\left(\frac{w-1}{2}\right) - 2 = 0$$

$$\Leftrightarrow \frac{2}{8}(w-1)^3 + \frac{5}{4}(w-1)^2 - \frac{3}{2}(w-1) - 2 = 0$$

$$\Leftrightarrow (w-1)^3 + 5(w-1)^2 - 6(w-1) - 8 = 0$$

$$\Leftrightarrow w^3 - 3w^2 + 3w - 1 + 5w^2 - 10w + 5 - 6w + 6 - 8 = 0$$

$$\Leftrightarrow w^3 + 2w^2 - 13w + 2 = 0$$

The substitution method can sometimes be much more efficient, although you need to take care with the expansion of the cubic brackets.

Technology note

If you have access to graphing software, use it to draw the graphs of $y = 2x^3 + 5x^2 - 3x - 2$ and $y = x^3 + 2x^2 - 13x + 2$. How do these graphs relate to Example 3.7? What transformations map the first graph on to the second one?

Example 3.8

The roots of the cubic equation $2z^3 - 5z^2 + z + 2 = 0$ are α, β, γ.

Find the cubic equation with roots $\alpha^2, \beta^2, \gamma^2$.

This equation has roots $\alpha^2, \beta^2, \gamma^2$, but it is not a cubic. You need to rearrange the equation to remove the square roots.

Solution

Let $w = z^2 \Rightarrow z = \sqrt{w}$

Substituting $z = \sqrt{w}$ into $2z^3 - 5z^2 + z + 2 = 0$ gives

$$2\left(\sqrt{w}\right)^3 - 5\left(\sqrt{w}\right)^2 + \sqrt{w} + 2 = 0$$

$$\Rightarrow \quad 2w\sqrt{w} - 5w + \sqrt{w} + 2 = 0$$

Gather terms containing $\sqrt{w}$: $\sqrt{w}(2w+1) = 5w-2$

Square both sides: $w(2w+1)^2 = (5w-2)^2$

$$\Rightarrow w(4w^2 + 4w + 1) = 25w^2 - 20w + 4$$

$$\Rightarrow \quad 4w^3 + 4w^2 + w = 25w^2 - 20w + 4$$

$$\Rightarrow \quad 4w^3 - 21w^2 + 21w - 4 = 0$$

Exercise 3B

1 The roots of the cubic equation $2z^3 + 3z^2 - z + 7 = 0$ are α, β, γ.

Find the following:

(i) $\sum\alpha$ (ii) $\sum\alpha\beta$ (iii) $\sum\alpha\beta\gamma$

2 Find cubic equations (with integer coefficients) with the following roots:

(i) 1, 2, 4
(ii) 2, –2, 3
(iii) 0, –2, –1.5
(iv) 2 (repeated), 2.5
(v) –2, –3, 5
(vi) 1, 2 + i, 2 – i

3 The roots of each of these equations are in arithmetic progression (i.e. they may be written as $a - d, a, a + d$).

Solve each equation.

(i) $z^3 - 15z^2 + 66z - 80 = 0$
(ii) $9z^3 - 18z^2 - 4z + 8 = 0$
(iii) $z^3 - 6z^2 + 16 = 0$
(iv) $54z^3 - 189z^2 + 207z - 70 = 0$

4 The roots of the equation $z^3 + z^2 + 2z - 3 = 0$ are α, β, γ.

(i) The substitution $w = z + 3$ is made. Write z in terms of w.

(ii) Substitute your answer to part (i) for z in the equation $z^3 + z^2 + 2z - 3 = 0$

(iii) Give your answer to part (ii) as a cubic equation in w with integer coefficients.

(iv) Write down the roots of your equation in part (iii), in terms of α, β and γ.

5 The roots of the equation $z^3 - 2z^2 + z - 3 = 0$ are α, β, γ. Use the substitution $w = 2z$ to find a cubic equation in w with roots:

(i) $2\alpha, 2\beta, 2\gamma$

(ii) $\alpha^2, \beta^2, \gamma^2$

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

6 The roots of the equation $2z^3 + 4z^2 - 3z + 1 = 0$ are α, β, γ.
Find cubic equations with these roots:

(i) $2 - \alpha, 2 - \beta, 2 - \gamma$
(ii) $3\alpha - 2, 3\beta - 2, 3\gamma - 2$
(iii) $\alpha^2, \beta^2, \gamma^2$

7 The roots of the equation $2z^3 - 12z^2 + kz - 15 = 0$ are in arithmetic progression.

Solve the equation and find k.

PS 8 Solve $32z^3 - 14z + 3 = 0$ given that one root is twice another.

PS 9 The equation $z^3 + pz^2 + 2pz + q = 0$ has roots $\alpha, 2\alpha, 4\alpha$.

Find all possible values of p, q, α.

CP 10 The cubic equation $ax^3 + bx^2 + cx + d = 0$, where a, b, c and d are constants, has roots α, β and γ. Prove that $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = -\frac{c}{d}$.

CP 11 The cubic equation $ax^3 + bx^2 + cx + d = 0$, where a, b, c and d are constants, has roots α, β and γ. Prove that $\alpha^2 + \beta^2 + \gamma^2 = \frac{b^2}{a^2} - \frac{2c}{a}$.

CP 12 The roots of $z^3 + pz^2 + qz + r = 0$ are $\alpha, -\alpha, \beta$ and $r \neq 0$.

Show that $r = pq$, and find all three roots in terms of p and q.

CP 13 The cubic equation $8x^3 + px^2 + qx + r = 0$ has roots α and $\frac{1}{2\alpha}$ and β.

(i) Express p, q and r in terms of α and β.

(ii) Show that $2r^2 - pr + 4q = 16$.

(iii) Given that $p = 6$ and $q = -23$, find the two possible values of r and, in each case, solve the equation $8x^3 + 6x^2 - 23x + r = 0$.

CP 14 Show that one root of $az^3 + bz^2 + cz + d = 0$ is the reciprocal of another root if and only if $a^2 - d^2 = ac - bd$.

Verify that this condition is satisfied for the equation $21z^3 - 16z^2 - 95z + 42 = 0$ and hence solve the equation.

CP 15 Find a formula connecting a, b, c and d that is a necessary and sufficient condition for the roots of the equation $az^3 + bz^2 + cz + d = 0$ to be in geometric progression.

Show that this condition is satisfied for the equation

$8z^3 - 52z^2 + 78z - 27 = 0$ and hence solve the equation.

16 The cubic equation $x^3 + px^2 + qx + r = 0$, where p, q and r are integers, has roots α, β and γ, such that

$$\alpha + \beta + \gamma = 15,$$

$$\alpha^2 + \beta^2 + \gamma^2 = 83.$$

Write down the value of p and find the value of q.

Given that α, β and γ are all real and that $\alpha\beta + \alpha\gamma = 36$, find α and hence find the value of r.

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17 The cubic equation $x^3 - px - q = 0$, where p and q are constants, has roots α, β and γ. Show that

(i) $\alpha^2 + \beta^2 + \gamma^2 = 2p$,

(ii) $\alpha^3 + \beta^3 + \gamma^3 = 3q$,

(iii) $6(\alpha^5 + \beta^5 + \gamma^5) = 5(\alpha^3 + \beta^3 + \gamma^3)(\alpha^2 + \beta^2 + \gamma^2)$.

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18 The equation $x^3 + px + q = 0$ has a repeated root. Prove that $4p^3 + 27q^2 = 0$.

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3.4 Quartic equations

Quartic equations have four roots, denoted by the first four Greek letters: α, β, γ and δ (delta).

?

› By looking back at the two formulae for quadratics and the three formulae for cubics, predict the four formulae that relate the roots α, β, γ and δ to the coefficients a, b, c and d of the quartic equation $az^4 + bz^3 + cz^2 + dz + e = 0$.

You may wish to check/derive these results yourself before looking at the derivation on the next page.

Historical note

The formulae used to relate the coefficients of polynomials with sums and products of their roots are called Vieta's Formulae after François Viète (a Frenchman who commonly used a Latin version of his name: Franciscus Vieta). He was a lawyer by trade but made important progress (while doing mathematics in his spare time) on algebraic notation and helped pave the way for the more logical system of notation you use today.

Derivation of formulae

As before, the quartic equation

$$az^4 + bz^3 + cz^2 + dz + e = 0$$

can be written is factorised form as

$$a(z - \alpha)(z - \beta)(z - \gamma)(z - \delta) = 0.$$

This gives the identity

$$az^4 + bz^3 + cz^2 + dz + e \equiv a(z - \alpha)(z - \beta)(z - \gamma)(z - \delta).$$

Multiplying out the right-hand side gives

$$az^4 + bz^3 + cz^2 + dz + e \equiv az^4 - a(\alpha + \beta + \gamma + \delta)z^3 + a(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\alpha)z^2 - a(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)z + a\alpha\beta\gamma\delta.$$

Check this for yourself.

Equating coefficients shows that

$$\sum\alpha = \alpha + \beta + \gamma + \delta = -\frac{b}{a}$$

The sum of the individual roots.

$$\sum\alpha\beta = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$$

The sum of the products of roots in pairs.

$$\sum\alpha\beta\gamma = \alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -\frac{d}{a}$$

The sum of the products of roots in threes.

$$\sum\alpha\beta\gamma\delta = \alpha\beta\gamma\delta = \frac{e}{a}$$

There are four roots so there is only one way to form a product of them all.

Example 3.9

The roots of the quartic equation $4z^4 + pz^3 + qz^2 - z + 3 = 0$ are $\alpha, -\alpha, \alpha + \lambda, \alpha - \lambda$ where α and γ are real numbers.

(i) Express p and q in terms of α and λ.

(ii) Show that $\alpha = -\frac{1}{2}$, and find the values of p and q.

(iii) Give the roots of the quartic equation.

Solution

(i) $\sum\alpha = \alpha - \alpha + \alpha + \lambda + \alpha - \lambda = -\frac{p}{4}$

$$\Rightarrow 2\alpha = -\frac{p}{4}$$

$$\Rightarrow p = -8\alpha$$

Use the sum of the individual roots to find an expression for p.

$$\sum\alpha\beta = -\alpha^2 + \alpha(\alpha + \lambda) + \alpha(\alpha - \lambda) - \alpha(\alpha + \lambda) - \alpha(\alpha - \lambda) + (\alpha + \lambda)(\alpha - \lambda) = \frac{q}{4}$$

$$\Rightarrow -\lambda^2 = \frac{q}{4}$$

$$\Rightarrow q = -4\lambda^2$$

Use the sum of the product of the roots in pairs to find an expression for q.

(ii)

$$\sum\alpha\beta\gamma = -\alpha^2(\alpha+\lambda) - \alpha(\alpha+\lambda)(\alpha-\lambda) + \alpha(\alpha+\lambda)(\alpha-\lambda) - \alpha^2(\alpha-\lambda) = \frac{1}{4}$$

$$\Rightarrow -2\alpha^3 = \frac{1}{4}$$

$$\Rightarrow \alpha = -\frac{1}{2}$$

$$p = -8\alpha = -8 \times -\frac{1}{2} = 4$$

Use the sum of the product of the roots in threes to find α (λ cancels out) and hence find p, using your answer to part (i).

$$\alpha\beta\gamma\delta = -\alpha^2(\alpha+\lambda)(\alpha-\lambda) = \frac{3}{4}$$

$$\Rightarrow -\alpha^2(\alpha^2 - \lambda^2) = \frac{3}{4}$$

$$\Rightarrow -\frac{1}{4}\left(\frac{1}{4} - \lambda^2\right) = \frac{3}{4}$$

$$\Rightarrow \frac{1}{4} - \lambda^2 = -3$$

$$\Rightarrow \lambda^2 = \frac{13}{4}$$

$$q = -4\lambda^2 = -4 \times \frac{13}{4} = -13$$

Use the sum of the product of the roots and the value for α to find λ, and hence find q, using your answer to part (i).

Substitute the values for α and λ to give the roots.

(iii) The roots of the equation are $\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} + \frac{1}{2}\sqrt{13}, -\frac{1}{2} - \frac{1}{2}\sqrt{13}$.

Exercise 3C

1 The roots of $2z^4 + 3z^3 + 6z^2 - 5z + 4 = 0$ are α, β, γ and δ.

Write down the following:

(i) $\sum\alpha$

(ii) $\sum\alpha\beta$

(iii) $\sum\alpha\beta\gamma$

(iv) $\sum\alpha\beta\gamma\delta$

2 Find quartic equations (with integer coefficients) with the roots.

(i) $1, -1, 2, 4$

(ii) $0, 1.5, -2.5, -4$

(iii) 1.5 (repeated), -3 (repeated)

(iv) $1, -3, 1 + \mathrm{i}, 1 - \mathrm{i}$.

3 The roots of the quartic equation $2z^4 + 4z^3 - 3z^2 - z + 6 = 0$ are α, β, γ and δ.

Find quartic equations with these roots:

(i) $2\alpha, 2\beta, 2\gamma, 2\delta$

(ii) $\alpha - 1, \beta - 1, \gamma - 1, \delta - 1$.

4 The roots of the quartic equation $x^4 + 4x^3 - 8x + 4 = 0$ are α, β, γ and δ.

(i) By making a suitable substitution, find a quartic equation with roots $\alpha + 1$, $\beta + 1$, $\gamma + 1$ and $\delta + 1$.

(ii) Solve the equation found in part (i), and hence find the values of α, β, γ and δ.

PS **5** The quartic equation $x^4 + px^3 - 12x + q = 0$, where p and q are real, has roots $\alpha, 3\alpha, \beta, -\beta$.

(i) By considering the coefficients of x^2 and x, find α and β, where $\beta > 0$.

(ii) Show that $p = 4$ and find the value of q.

(iii) By making the substitution $y = x - k$, for a suitable value of k, find a **cubic** equation in y, with integer coefficients, which has roots $-2\alpha, \beta - 3\alpha, -\beta - 3\alpha$.

CP **6** (i) Make conjectures about the five properties of the roots $\alpha, \beta, \gamma, \delta$ and ε (epsilon) of the general quintic $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$.

(ii) Prove your conjectures.

Note

For question 6, you should try the algebra by hand, thinking about keeping good presentation habits for long algebraic expansions. You may want to check any long expansions using CAS (computer algebra software). You then might also like to consider whether a 'proof' is still valid if it relies on a computer system to prove it – look up the history of *The Four Colour Theorem* to explore this idea further.

7 The roots of the quartic equation $x^4 + 4x^3 + 2x^2 - 4x + 1 = 0$ are α, β, γ and δ. Find the values of

(i) $\alpha + \beta + \gamma + \delta$

(ii) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

(iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} + \dfrac{1}{\delta}$

(iv) $\dfrac{\alpha}{\beta\gamma\delta} + \dfrac{\beta}{\alpha\gamma\delta} + \dfrac{\gamma}{\alpha\beta\delta} + \dfrac{\delta}{\alpha\beta\gamma}$.

Using the substitution $y = x + 1$, find a quartic equation in y. Solve this quartic equation and hence find the roots of the equation $x^4 + 4x^3 + 2x^2 - 4x + 1 = 0$.

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8 The roots of the equation $x^4 - 3x^2 + 5x - 2 = 0$ are $\alpha, \beta, \gamma, \delta$, and $\alpha^n + \beta^n + \gamma^n + \delta^n$ is denoted by S_n.

Show that

$$S_{n+4} - 3S_{n+2} + 5S_{n+1} - 2S_n = 0.$$

Find the values of

(i) S_2 and S_4

(ii) S_3 and S_5.

Hence find the value of

$$\alpha^2(\beta^3+\gamma^3+\delta^3)+\beta^2(\gamma^3+\delta^3+\alpha^3)+\gamma^2(\delta^3+\alpha^3+\beta^3)+\delta^2(\alpha^3+\beta^3+\gamma^3).$$

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KEY POINTS

1 If α and β are the roots of the quadratic equation $az^2 + bz + c = 0$, then

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

2 If α, β and γ are the roots of the cubic equation $az^3 + bz^2 + cz + d = 0$, then

$$\sum \alpha = \alpha + \beta + \gamma = -\frac{b}{a},$$

$$\sum \alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} \text{ and,}$$

$$\alpha\beta\gamma = -\frac{d}{a}.$$

3 If α, β, γ and δ are the roots of the quaartic equation $az^4 + bz^3 + cz^2 + dz + e = 0$, then

$$\sum \alpha = \alpha + \beta + \gamma + \delta = -\frac{b}{a},$$

$$\sum \alpha\beta = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a},$$

$$\sum \alpha\beta\gamma = \alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -\frac{d}{a} \text{ and}$$

$$\alpha\beta\gamma\delta = \frac{e}{a}.$$

4 All of these formulae may be summarised using the shorthand sigma notation for elementary symmetric functions as follows:

$$\sum \alpha = -\frac{b}{a}$$

$$\sum \alpha\beta = \frac{c}{a}$$

$$\sum \alpha\beta\gamma = -\frac{d}{a}$$

$$\sum \alpha\beta\gamma\delta = \frac{e}{a}$$

(using the convention that polynomials of degree n are labelled $az^n + bz^{n-1} + \ldots = 0$ and have roots $\alpha, \beta, \gamma, \ldots$)

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

LEARNING OUTCOMES

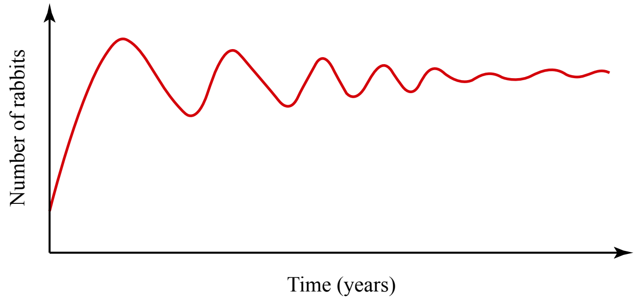
Now that you have finished this chapter, you should be able to

- recall the relationships between the roots and coefficients of quadratic, cubic and quartic equations
- form new equations whose roots are related to the roots of a given equation by a linear transformation
- evaluate symmetric functions of the roots
- understand that complex roots of polynomial equations with real coefficients occur in conjugate pairs.

4 Rational functions and graphs

The purpose of visualisation is insight, not pictures.
Ben Shneiderman (1947–)

The graph in Figure 4.1 shows how the population of rabbits on a small island changes over time after a small group is introduced to the island.



▲ **Figure 4.1**

› What can you conclude from the graph?

4.1 Graphs of rational functions

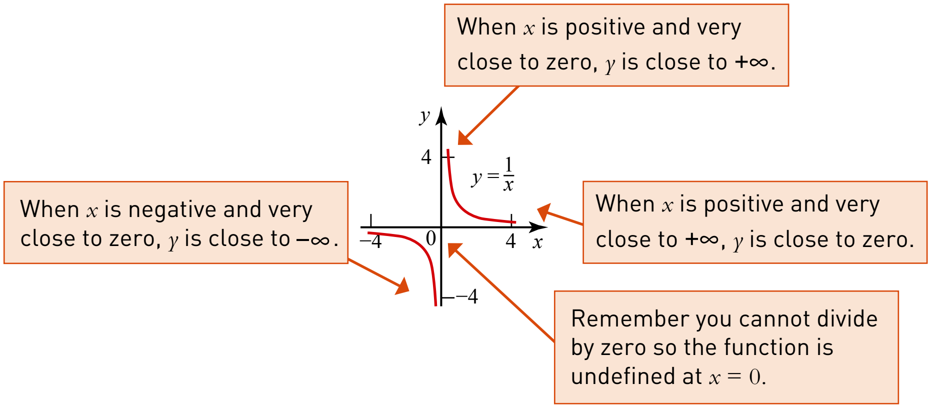
A **rational number** is defined as a number that can be expressed as $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

In a similar way, a **rational function** is defined as a function that can be expressed in the form $y = \frac{\mathrm{f}(x)}{\mathrm{g}(x)}$, where $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are polynomials, and $\mathrm{g}(x) \neq 0$.

This chapter will only look at polynomials of degree 2 (quadratics) or less.

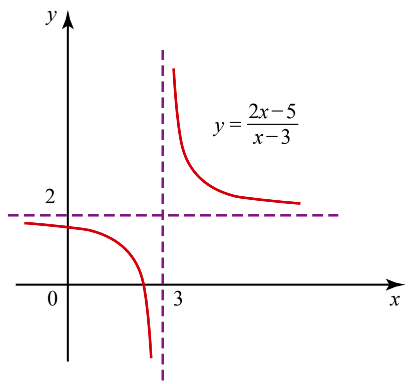
In this chapter you will learn how to sketch graphs of rational functions.

Think about the graph of $y = \frac{1}{x}$ (see Figure 4.2).



▲ **Figure 4.2**

Translating $y = \frac{1}{x}$ three units to the right and two units up gives the graph of $y = \frac{1}{x-3} + 2$ which can be written as $y = \frac{2x-5}{x-3}$ (see Figure 4.3).



▲ **Figure 4.3**

Imagine yourself moving along the curve $y = \frac{2x-5}{x-3}$ from the left. As your x coordinate gets close to 3, your y coordinate tends to $-\infty$, and you get closer and closer to the vertical line $x = 3$, shown dashed.

If you move along the curve again, letting your x coordinate increase without limit, you get closer and closer to the horizontal line $y = 2$, also shown dashed.

These dashed lines are examples of **asymptotes**. An asymptote is a straight line that a curve approaches tangentially as x and/or y tends to infinity. The line $x = 3$ is a vertical asymptote; the line $y = 2$ is a horizontal asymptote. It is usual for asymptotes to be shown by dashed lines in books. In your own work you may find it helpful to use a different colour for asymptotes.

Finding vertical asymptotes

To find any vertical asymptotes look for the values of x for which the function is undefined.

The curve $y = \frac{f(x)}{g(x)}$ is undefined when $g(x) = 0$.

Remember, you cannot divide by 0.

So there is a vertical asymptote at $x = a$ where $g(a) = 0$.

Look back at Figure 4.2 and see *Step 2* in the next section.

The signs of f(x) and g(x) when x is close to a, let you determine whether y tends to positive or negative infinity, as x tends to a from the left or from the right.

$y = \frac{2x-5}{x-3}$ is undefined when $x = 3$, so $x = 3$ is a vertical asymptote (see Figure 4.3).

?

› What are the vertical asymptotes of the graphs of the following rational functions?

(i) $y = \frac{1}{x+2}$ (ii) $y = \frac{x-2}{(x-1)(x+2)}$ (iii) $y = \frac{2}{(2x-1)(x^2+1)}$

Finding horizontal asymptotes

To find any horizontal asymptotes look at:

- the value of y as $x \to \infty$
- the value of y as $x \to -\infty$.

Say 'x tends to infinity' this means as x becomes very large and positive.

When x is close to negative infinity.

For the curve $y = \frac{2x-5}{x-3}$, when x is numerically very large (either positive or negative) the -5 in the numerator and the -3 in the denominator become negligible compared to the value of x.

So as $x \to \pm\infty$, $\quad y = \frac{2x-5}{x-3} \to \frac{2x}{x} = 2$.

Hence the line $y = 2$ is a horizontal asymptote (see Figure 4.3).

Think about the value of y when x is:

- large and positive, e.g. +1000 and +10 000
- 'large' and negative, e.g. −1000 and −10 000.

?

› What are the horizontal asymptotes of the graphs of the following rational functions?

(i) $y = \frac{1}{x+2}$ (ii) $y = \frac{x}{x+2}$ (iii) $y = \frac{1-2x}{x+2}$

Technology note

If you have graphing software, you can use it to sketch graphs and check that you have found the asymptotes correctly.

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

4.2 How to sketch a graph of a rational function

The five steps below show how to draw a sketch graph of $y = \dfrac{(x+2)}{(x-2)(x+1)}$.

Step 1: Find where the graph cuts the axes

When $x = 0$:

$$y = \frac{(x+2)}{(x-2)(x+1)} = \frac{2}{-2\times 1} = -1$$

$\Rightarrow$ the y intercept is at $(0,-1)$

When $y = 0$:

$$\frac{(x+2)}{(x-2)(x+1)} = 0$$

$\Rightarrow x + 2 = 0$

$\Rightarrow x = -2$

$\Rightarrow$ the x intercept is at $(-2,0)$

Step 2: Find the vertical asymptotes and examine the behaviour of the graph either side of them

$y = \dfrac{(x+2)}{(x-2)(x+1)}$ has vertical asymptotes when $(x-2)(x+1) = 0$

$\Rightarrow$ the vertical asymptotes are at $x = -1$ and $x = 2$

On either side of the asymptote, y will either be large and positive (tending to $+\infty$) or large and negative (tending to $-\infty$). To find out which, you need to examine the sign of y on either side of the asymptote.

Behaviour of the graph $y = \dfrac{(x+2)}{(x-2)(x+1)}$ either side of the asymptote $x = 2$:

When x is slightly less than 2 then

$$y \text{ is } \frac{(+ve\ number)}{(-ve\ number\ close\ to\ zero)(+ve\ number)}.$$

So y is large and negative.

For example, when $x = 1.999$

$$y = \frac{(1.999+2)}{(1.999-2)(1.999+1)} = \frac{3.999}{-0.001\times 2.999} = -1333.44...$$

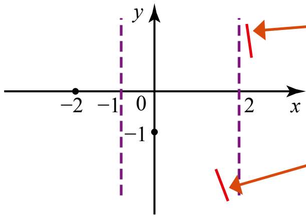
When x is slightly more than 2 then

$$y \text{ is } \frac{(+ve\ number)}{(+ve\ number\ close\ to\ zero)(+ve\ number)}.$$

So y is large and positive.

For example, when $x = 2.001$

$$y = \frac{(2.001+2)}{(2.001-2)(2.001+1)} = \frac{4.001}{0.001 \times 3.001} = 1333.22\ldots$$



$y \to +\infty$ as $x \to 2$ from the right.

$y \to -\infty$ as $x \to 2$ from the left.

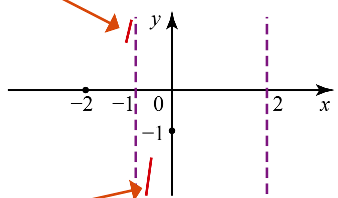
▲ **Figure 4.4**

Behaviour of the graph $y = \dfrac{(x+2)}{(x-2)(x+1)}$ either side of the asymptote $x = -1$:

When x is slightly less than −1 then:

$$y \text{ is } \frac{(+ve\ number)}{(-ve\ number)(-ve\ number\ close\ to\ zero)}$$

so y is large and positive.



When x is slightly more than −1 then:

$$y \text{ is } \frac{(+ve\ number)}{(-ve\ number)(+ve\ number\ close\ to\ zero)}$$

so y is large and negative.

▲ **Figure 4.5**

Step 3: Find the horizontal asymptotes and examine the behaviour of the graph either side of them

Examine the behaviour as x tends to infinity.

As $x \to \pm\infty$, $y = \dfrac{(x+2)}{(x-2)(x+1)} \to \dfrac{x}{x^2} = \dfrac{1}{x} \to 0$

This means that the line $y = 0$ is a horizontal asymptote.

When x is very large (either positive or negative) the 2 in the numerator and the −2 and the 1 in the denominator become negligible compared to the vaxlues of x, so you can ignore them.

› What are the signs of $(x + 2)$, $(x - 2)$ and $(x + 1)$ for:

(i) large, positive values of x

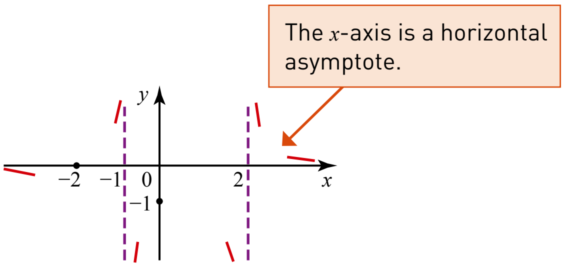
(ii) large, negative values of x?

› What is the sign of $y = \dfrac{(x+2)}{(x-2)(x+1)}$ for:

(iii) large, positive values of x

(iv) large, negative values of x?

From the discussion point above, you now know that $y \to 0$ from above as $x \to \infty$, and $y \to 0$ from below as $x \to -\infty$. This additional information is shown in Figure 4.6.



▲ **Figure 4.6**

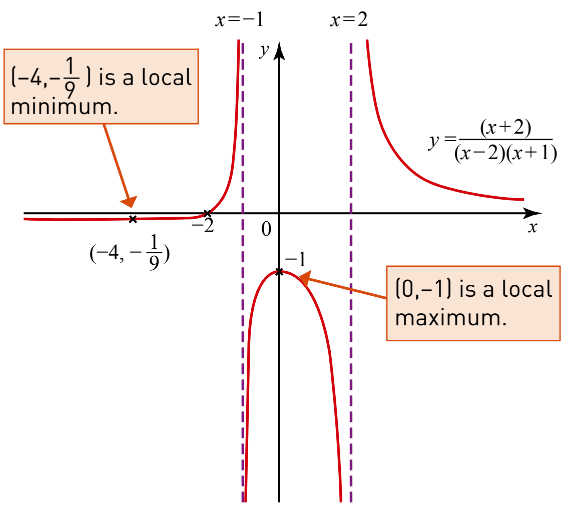
Step 4: Identify any stationary points

At a stationary point, $\dfrac{dy}{dx} = 0$.

› Show that $y = \dfrac{(x+2)}{(x-2)(x+1)}$ has stationary points at $(-4, -\frac{1}{9})$ and $(0, -1)$.

Step 5: Complete the sketch

The sketch is completed in Figure 4.7.



▲ **Figure 4.7**

Notice that, in this case, you can conclude that there is a local minimum at $(-4,-\frac{1}{9})$ and a local maximum at $(0,-1)$ without the need for further differentiation.

So the range of the function $f(x) = \dfrac{(x+2)}{(x-2)(x+1)}$ is $f(x) \geq -\frac{1}{9}$ and $f(x) \leqslant -1$.

You can use the sketch to solve inequalities.

For example: $\dfrac{(x+2)}{(x-2)(x+1)} \leq 0$ when $x \leqslant -2$ or $-1 < x < 2$.

For these values of x, the curve is below the x-axis.

Using the discriminant to find the range of a function

You can use the discriminant to find the range of the function instead of using differentiation. This method is often more straightforward than differentiation.

Look at Figure 4.7, this graph meets the horizontal line $y = k$ where $k = \dfrac{x+2}{(x-2)(x+1)}$. The range of the function $f(x) = \dfrac{x+2}{(x-2)(x+1)}$ will be the values of k for which this equation has real roots.

$$k = \frac{x+2}{(x-2)(x+1)}$$

$$\Rightarrow \quad k(x-2)(x+1) = x+2$$

$$\Rightarrow \quad kx^2 - kx - 2k = x+2$$

$$\Rightarrow \quad kx^2 - (k+1)x - 2k - 2 = 0$$

This equation has real roots when the discriminant, $b^2 - 4ac$, is positive or zero.

So $(k+1)^2 - 4k(-2k-2) \geqslant 0$

$$k^2 + 2k + 1 + 8k^2 + 8k \geqslant 0$$

$$9k^2 + 10k + 1 \geqslant 0$$

$$(k+1)(9k+1) \geqslant 0$$

$$\Rightarrow k \leqslant -1 \text{ or } k \geqslant -\frac{1}{9}$$

So the function $f(x) = \dfrac{x+2}{(x-2)(x+1)}$ cannot take any values between -1 and $-\frac{1}{9}$, and the range of the function is $f(x) \leqslant -1$ and $f(x) \geqslant -\frac{1}{9}$ as found before.

Using symmetry

Recognising symmetry can help you to draw a sketch.

- If $f(x) = f(-x)$ the graph of $y = f(x)$ is symmetrical about the y-axis.
- If $f(x) = -f(-x)$ the graph of $y = f(x)$ has rotational symmetry of order 2 about the origin.

4

Note

A function is an **even function** if its graph has the y-axis as a line of symmetry.

So when $f(x) = f(-x)$ the function is **even**.

A function is an **odd function** if its graph has rotational symmetry of order 2 about the origin.

So when $f(x) = -f(-x)$ the function is **odd**.

Example 4.1

(i) Sketch the graph of $y = f(x)$, where $f(x) = \dfrac{x^2+1}{x^2+2}$.

(ii) State the range of $f(x)$.

(iii) The equation $f(x) = k$ has no real solutions.

Find the values of k.

Solution

(i) *Step 1:*

When $x = 0$, $y = \frac{1}{2}$, so the graph passes through $\left(0, \frac{1}{2}\right)$.

No (real) value of x makes $x^2 + 1 = 0$, so the graph does not cut the x-axis.

Step 2:

No (real) value of x makes $x^2 + 2 = 0$, so there are no vertical asymptotes.

Step 3:

As $x \to \pm\infty$, $y = \dfrac{x^2+1}{x^2+2} \to \dfrac{x^2}{x^2} = 1$

So $y = 1$ is a horizontal asymptote.

The denominator is larger than the numerator for all values of x, so $y < 1$ for all x.

So $y \to 1$ from below as $x \to \pm\infty$.

Step 4:

Differentiate to find the stationary points.

$$y = \frac{x^2+1}{x^2+2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2+2)\times 2x - (x^2+1)\times 2x}{(x^2+2)^2}$$

$$= \frac{2x^3+4x-2x^3-2x}{(x^2+2)^2}$$

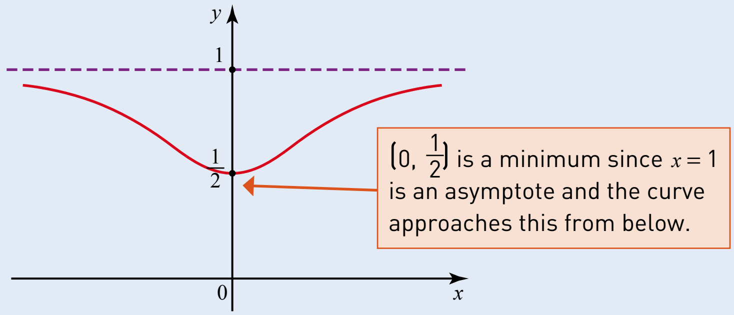
$$= \frac{2x}{(x^2+2)^2}$$

At a stationary point, $\frac{dy}{dx} = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$.

When $x = 0$ then $y = \frac{1}{2}$, so there is a stationary point at $\left(0, \frac{1}{2}\right)$.

Step 5:

f(x) contains only even powers of x, so $f(x) = f(-x)$ and the graph is symmetrical about the y-axis (see Figure 4.8).



▲ **Figure 4.8**

(ii) From the sketch the range is $\frac{1}{2} \leqslant f(x) < 1$.

(iii) Solutions of the equation $f(x) = k$ occur where the horizontal line $y = k$ meets the curve $y = f(x)$.

From the sketch of $y = f(x)$, you can see that when $k < \frac{1}{2}$ or $k \geqslant 1$, the line $y = k$ will not meet the curve and so there are no solutions to the equation $f(x) = k$.

Show that you get the same answer when you use the discriminant method to find the range.

Exercise 4A

Follow the steps below for each of questions **1** to **12**.

Step 1: Find the coordinates of the point(s) where the graph cuts the axes.
Step 2: Find the vertical asymptote(s).
Step 3: State the behaviour of the graph as $x \to \pm\infty$.
Step 4: Sketch the graph.
Step 5: Find the range.

1 $y = \frac{2}{x-3}$

2 $y = \frac{2}{(x-3)^2}$

3 $y = \frac{1}{x^2+1}$

4 $y = \frac{x}{x^2-4}$

5 $y = \frac{2-x}{x+3}$

6 $y = \frac{x-5}{(x+2)(x-3)}$

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

7 $y = \dfrac{3-x}{(2-x)(4-x)}$

8 $y = \dfrac{x}{x^2+3}$

9 $y = \dfrac{x-3}{(x-4)^2}$

10 $y = \dfrac{(2x-3)(5x+2)}{(x+1)(x-4)}$

11 $y = \dfrac{x^2-6x+9}{x^2+1}$

12 $y = \dfrac{x^2-5x-6}{(x+1)(x-4)}$

PS **13** (i) Sketch the graph of $y = \dfrac{4-x^2}{4+x^2}$.

(ii) The equation $\dfrac{4-x^2}{4+x^2} = k$ has no real solutions.

Find the possible values of k.

PS **14** (i) Sketch the graph of $y = \dfrac{1}{(x+1)(3-x)}$.

(ii) Write down the equation of the line of symmetry of the graph and hence find the coordinates of the local minimum point.

(iii) For what values of k does the equation $\dfrac{1}{(x+1)(3-x)} = k$ have

(a) two real distinct solutions

(b) one real solution

(c) no real solutions?

PS **15** Solve these inequalities, by first sketching one or more appropriate curve(s).

(i) $\dfrac{x+2}{x-1} \geqslant 0$

(ii) $\dfrac{2x+3}{x-2} \leqslant 1$

(iii) $\dfrac{x-5}{x+1} \leqslant \dfrac{1}{x-3}$

(iv) $\dfrac{x+3}{2x-1} \geqslant 2$

(v) $\dfrac{2x-1}{x+3} \leqslant \dfrac{1}{2}$

(vi) $\dfrac{1}{x+6} \leqslant \dfrac{2}{2-3x}$

4.3 Oblique asymptotes

In general, when the numerator of any rational function is of lower degree than the denominator (e.g. $y = \dfrac{(x+2)}{(x-2)(x+1)}$), then $y = 0$ is a horizontal asymptote.

When the numerator has the same degree as the denominator (e.g. $y = \dfrac{2x-5}{x-3}$), then as $x \to \pm\infty$, y tends to a fixed rational number. So there is a horizontal asymptote of the form $y = c$.

When the degree of the numerator is one greater than that of the denominator (e.g. $y = \frac{2x^2 - 4x - 1}{x - 3}$), then as $x \to \pm\infty$ then y tends to an expression in the form $ax + b$. So the asymptote is a sloping line, you say there is an **oblique asymptote**.

To find the equation of the oblique asymptote of $y = \frac{2x^2 - 4x - 1}{x - 3}$ you need to rewrite the equation using long division.

$$
\begin{array}{r}
2x \quad + 2 \\
(x-3)\overline{)2x^2 - 4x - 1} \\
-\ \underline{2x^2 - 6x} \quad\quad \\
2x - 1 \\
-\ \underline{2x - 6} \\
5
\end{array}
$$

So $$y = \frac{2x^2 - 4x - 1}{x - 3} = 2x + 2 + \frac{5}{x - 3}$$

As x increases then $\frac{5}{x-3} \to 0$ and $y \to 2x + 2$, so the equation of the oblique asymptote is $y = 2x + 2$.

Notice you didn't need to complete the division.

In order to sketch the graph of $y = \frac{2x^2 - 4x - 1}{x - 3}$ you need to consider any vertical asymptotes and then examine the behaviour of the function on either side of the asymptotes.

The vertical asymptote is at $x = 3$.

You cannot divide by 0, so $x = 3$ is an asymptote.

Examine the behaviour of $y = \frac{2x^2 - 4x - 1}{x - 3}$ on either side of the vertical asymptote.

- When x is slightly less than 3 then $y \to -\infty$.
- When x is slightly more than 3 then $y \to +\infty$.

When x is slightly less than 3 then $y = \frac{+ve}{-ve} = -ve$

When x is slightly more than 3 then $y = \frac{+ve}{+ve} = +ve$

Examine the behaviour on either side of the oblique asymptote:

- When x is slightly more than 3 then $y = 2x + 2 + \frac{5}{x-3} > 2x + 2$ so the curve lies above the asymptote.
- When x is slightly less than 3 then $y = 2x + 2 + \frac{5}{x-3} < 2x + 2$ so the curve lies below the asymptote.

For $x > 3$, $\frac{5}{x-3}$ is positive so $y > 2x + 2$.

For $x < 3$, $\frac{5}{x-3}$ is negative so $y < 2x + 2$.

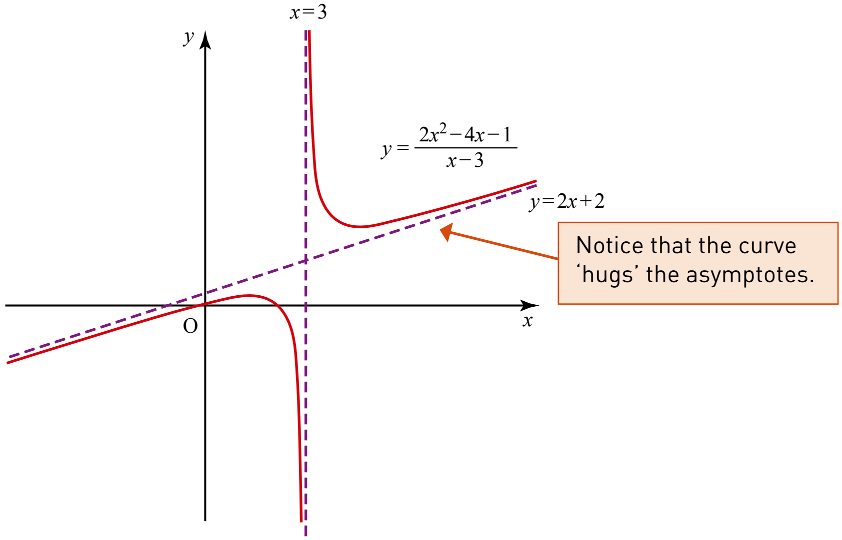


Figure 4.9

Note

Rewriting the equation as $y = 2x + 2 + \frac{5}{x-3}$ makes it easier to differentiate and find the turning points.

$$y = 2x + 2 + \frac{5}{x-3}$$

$$\Rightarrow \frac{dy}{dx} = 2 - \frac{5}{(x-3)^2}$$

$$\frac{dy}{dx} = 0 \Rightarrow 2 - \frac{5}{(x-3)^2} = 0$$

So $\frac{5}{(x-3)^2} = 2 \Rightarrow (x-3)^2 = \frac{5}{2}$

$$\Rightarrow \quad (x-3) = \pm\sqrt{\frac{5}{2}}$$

$$\Rightarrow \quad x = 3 \pm \sqrt{\frac{5}{2}}$$

So there are turning points at $x = 3 \pm \sqrt{\frac{5}{2}}$.

The turning points are at $x = 1.42$ and $x = 4.58$ (to 3 significant figures).

Exercise 4B

For each of the curves given in questions **1** to **3**:

(a) find the coordinates of any point(s) where the graph cuts the axes

(b) find the equations of the asymptotes

(c) find the coordinates of the turning points

(d) sketch the graph.

1 (i) $y = 2x - 1 + \frac{2}{x-4}$ (ii) $y = 2x - 1 + \frac{2}{4-x}$

2 (i) $y = \frac{x^2 - 4x + 6}{x-1}$ (ii) $y = \frac{x^2 - 4x + 6}{1-x}$

3 (i) $y = \frac{(2x-1)(x-4)}{(x-3)}$ (ii) $y = \frac{(2x-1)(x-4)}{(3-x)}$

4 A curve C has equation $y = \frac{2x^2 + x - 1}{x-1}$. Find the equations of the asymptotes of C.

Show that there is no point on C for which $1 < y < 9$.

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5 The curve C has equation $y = \frac{x^2 + px + 1}{x-2}$, where p is a constant. Given that C has two asymptotes, find the equation of each asymptote.

Find the set of values of p for which C has two distinct turning points.

Sketch C in the case $p = -1$. Your sketch should indicate the coordinates of any intersections with the axes, but need not show the coordinates of any turning points.

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6 The curve C has equation $y = \frac{2x^2 + kx}{x+1}$, where k is a constant. Find the set of values of k for which C has no stationary points.

For the case $k = 4$, find the equations of the asymptotes of C and sketch C, indicating the coordinates of the points where C intersects the coordinate axes.

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7 The curve C has equation

$$y = \frac{px^2 + 4x + 1}{x+1},$$

where p is a positive constant and $p \neq 3$.

(i) Obtain the equations of the asymptotes of C.

(ii) Find the value of p for which the x-axis is a tangent to C, and sketch C in this case.

(iii) For the case $p = 1$, show that C has no turning points, and sketch C, giving the exact coordinates of the points of intersection of C with the x-axis.

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Answers to exercises are available at www.hoddereducation.com/cambridgeextras

8 The curve C has equation

$$y = \lambda x + \frac{x}{x-2},$$

where λ is a non-zero constant. Find the equations of the asymptotes of C.

Show that C has no turning points if $\lambda < 0$.

Sketch C in the case $\lambda = -1$, stating the coordinates of the intersections with the axes.

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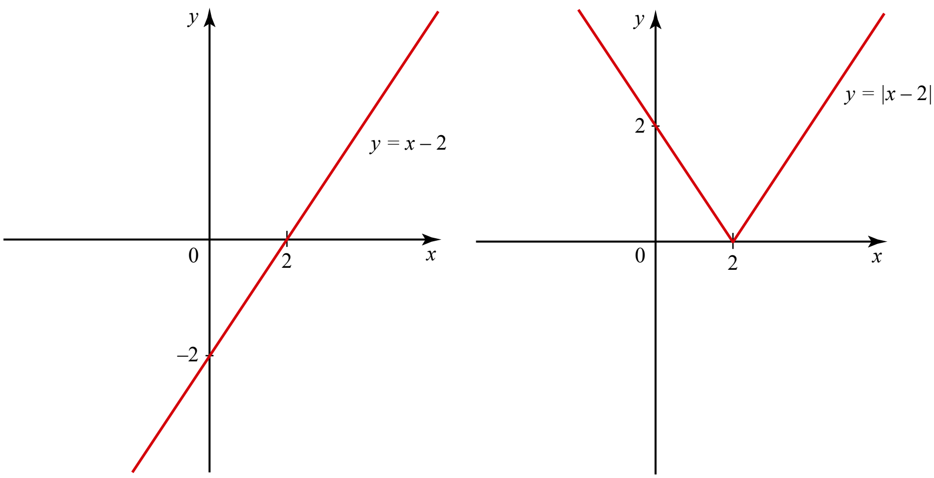
4.4 Sketching curves related to $y = f(x)$

The curve $y = |f(x)|$

The function $|f(x)|$ is the **modulus** of $f(x)$. $|f(x)|$ always takes the positive numerical value of $f(x)$. For example, when $f(x) = -2$, then $|f(x)| = 2$.



▲ **Figure 4.10** **(i)** $y = x - 2$ **(ii)** $y = |x - 2|$

The graph of $y = |f(x)|$ can be obtained from the graph of $y = f(x)$ by replacing values where $f(x)$ is negative by the equivalent positive values. This is the same as reflecting that part of the curve in the x-axis.

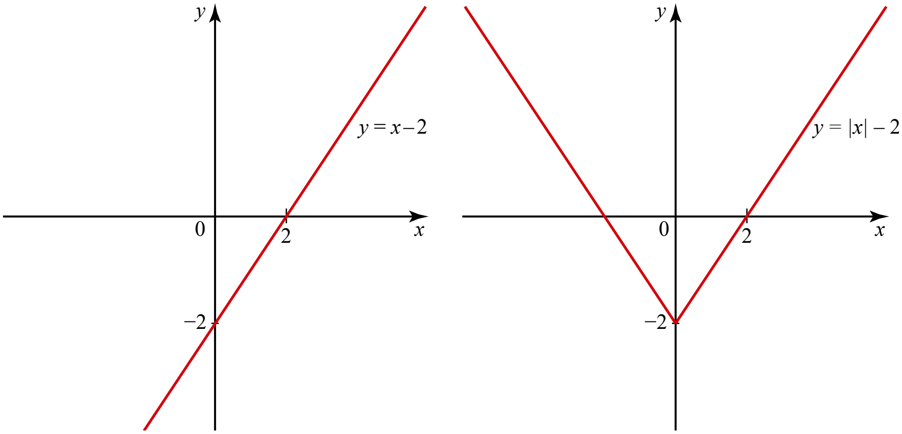
To draw the curve $y = |f(x)|$ you should:

- draw $y = f(x)$
- reflect the part(s) of the curve where $y < 0$ in the x-axis.

The curve $y = |f(x)|$

Remember that $|x|$ always takes the positive numerical value of x.

When $x = 2$ then $f(|2|) = f(2)$, and when $x = -2$ then $f(|-2|) = f(2)$

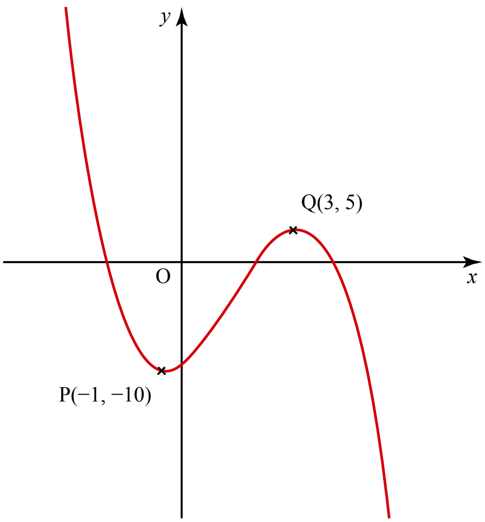


▲ **Figure 4.11** **(i)** $y = x - 2$ **(ii)** $y = |x| - 2$

To draw the curve $y = f(|x|)$ you should:

- draw $y = f(x)$ for $x > 0$
- reflect the part of the curve where $x > 0$ in the y axis.

Example 4.2



▲ **Figure 4.12**

The diagram shows the curve $y = f(x)$. The curve has stationary points at P and Q.

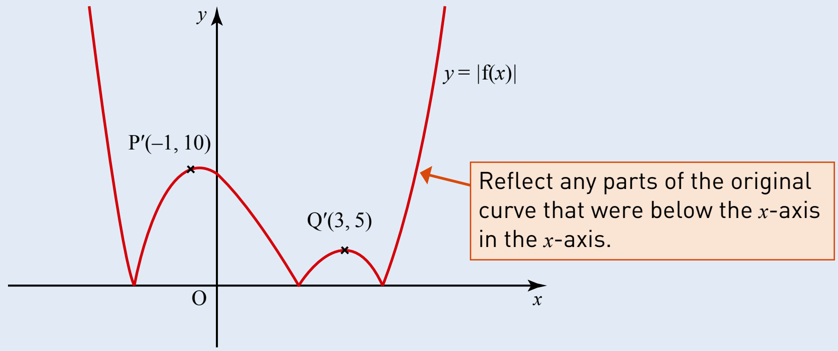
Sketch the curves

(i) $y = |f(x)|$ (ii) $y = f(|x|)$.

→

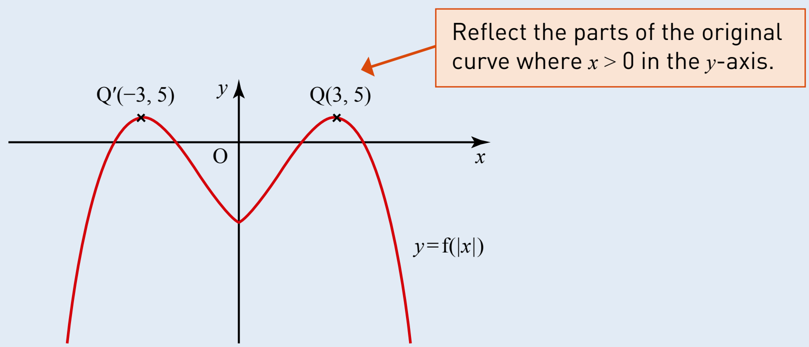
Solution

(i)



▲ **Figure 4.13**

(ii)



▲ **Figure 4.14**

The curve $y = \frac{1}{f(x)}$

You can use the following points to help you sketch $y = \frac{1}{f(x)}$ given the graph of $y = f(x)$.

- When $f(x) = 1$, then $\frac{1}{f(x)} = 1$.

  Also when $f(x) = -1$, then $\frac{1}{f(x)} = -1$.

  So the curves $y = f(x)$ and $y = \frac{1}{f(x)}$ intersect when $y = 1$ and $y = -1$.

- $f(x)$ and $\frac{1}{f(x)}$ have the same sign.

- $\frac{1}{f(x)}$ is undefined when $f(x) = 0$.

  So $y = \frac{1}{f(x)}$ has a discontinuity at any point where $f(x) = 0$ (i.e. the x intercept). This is usually an asymptote.
  For example, if $x = 3$ is a root of $y = f(x)$ then $x = 3$ is an asymptote of $\frac{1}{f(x)}$.

The points $(x, 1)$ and $(x, -1)$ are fixed points; they are the same on both curves.

When $f(x)$ is above the x-axis then so is $\frac{1}{f(x)}$.

Similarly, when $f(x)$ is below the x-axis then so is $\frac{1}{f(x)}$.

Roots become asymptotes.

Asymptotes become discontinuities. You should show these 'apparent roots' with a small open circle on the x-axis.

» If $f(x)$ has a vertical asymptote at $x = a$, then $f(x) \to \pm\infty$ as $x \to a$.

As $f(x) \to \infty$, $\frac{1}{f(x)} \to 0$ so $\frac{1}{f(x)}$ approaches the x-axis as $x \to a$.

However, since $f(a)$ is not defined, $\frac{1}{f(a)}$ is not defined either, so $y = \frac{1}{f(x)}$ has a discontinuity at $x = a$.

For example, if $x = 3$ is an asymptote of $y = f(x)$ then $x = 3$ is a discontinuity of $\frac{1}{f(x)}$.

The gradient of $\frac{1}{f(x)}$ at a given point has the opposite sign to $f(x)$.

» $\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{-f'(x)}{[f(x)]^2}$

Hence:

- when $f(x)$ is increasing, $\frac{1}{f(x)}$ is decreasing
- when $f(x)$ is decreasing, $\frac{1}{f(x)}$ is increasing
- when $f'(x) = 0$ then $\frac{d}{dx}\left(\frac{1}{f(x)}\right) = 0$.

So where $f(x)$ is a maximum, $\frac{1}{f(x)}$ is a minimum...

... and where $f(x)$ is a minimum, $\frac{1}{f(x)}$ is a maximum

?

› Prove that $\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{[f(x)]^2}$.

Example 4.3

Given

$$f(x) = \cos x$$

and $$g(x) = \tan x,\ x \neq \pm 90,\ \pm 270,$$

sketch the graphs of

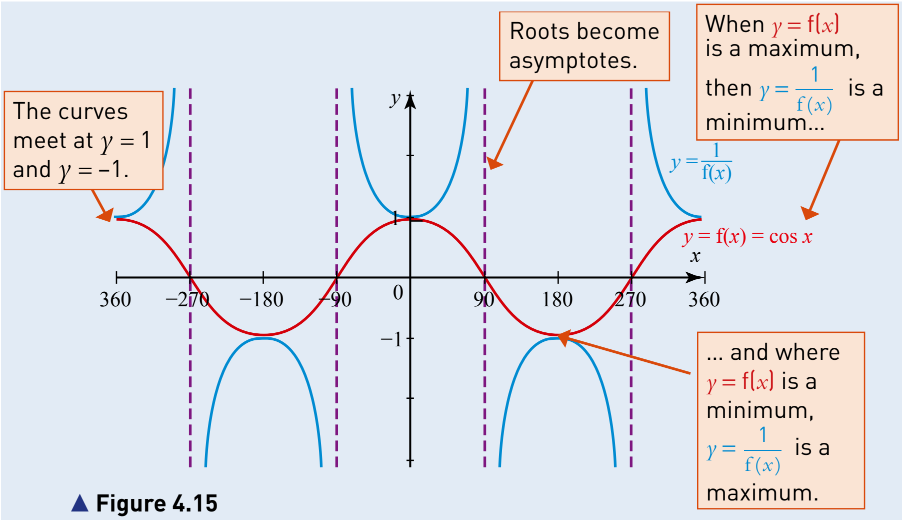
(i) $y = \frac{1}{f(x)}$ (ii) $y = \frac{1}{g(x)}$

for $-360° \leqslant x \leqslant 360°$.

Solution

(i) Start by drawing the graph of $y = \cos x$. This is the curve shown in red in Figure 4.15.

→

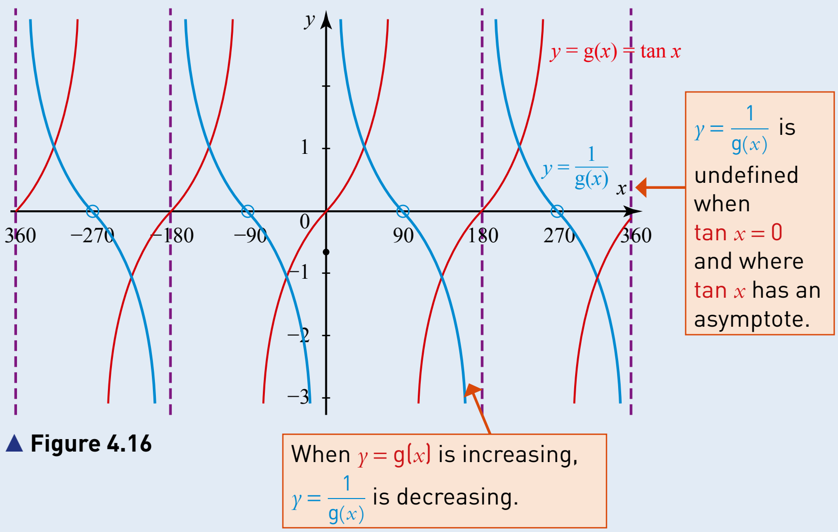


▲ **Figure 4.15**

(ii) Start by drawing the graph of $y = \tan x$. This is the curve shown in red in Figure 4.16.

Note the asymptotes for $y = \tan\theta$ have been omitted for clarity.

Notice $y = f(x)$ and $y = \frac{1}{f(x)}$ meet at the points where $y = 1$ and when $y = -1$



▲ **Figure 4.16**

Note

You met the graphs of the reciprocal trigonometric functions in *Pure Mathematics 2 and 3*.

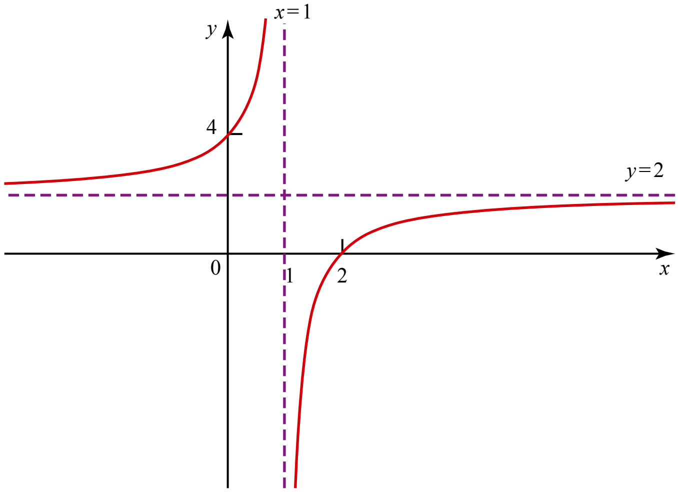
In part (ii) of Example 4.3, the points where $x = \pm 90, \pm 270$ were excluded because the function $g(x)$ was not defined for $x = \pm 90, \pm 270$ and so $\frac{1}{g(x)}$ is also not defined for these values this is shown by the small open circles on the x-axis.

Note when you draw $y = \cot x$, then there will be roots at $x = \pm 90, \pm 270$ ... (where $\cos x = 0$) and asymptotes where $\sin x = 0$ since $\cot x = \frac{\cos x}{\sin x}$.

Example 4.4

The diagram shows the curve $y = f(x)$. The lines $x = 1$ and $y = 2$ are asymptotes to the curve and the curve intercepts the axes at (0, 4) and (2, 0).

Sketch the curve $y = \frac{1}{f(x)}$.



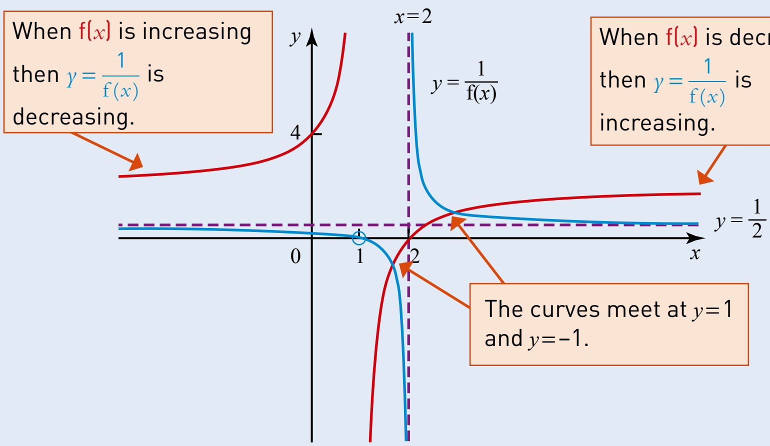
▲ **Figure 4.17**

Solution

From the graph you can see that:

- $x = 2$ is a root of $y = f(x) \Rightarrow x = 2$ is an asymptote of $y = \frac{1}{f(x)}$.
- $y = 2$ is an asymptote of $y = f(x) \Rightarrow y = \frac{1}{2}$ is an asymptote of $y = \frac{1}{f(x)}$.
- $x = 1$ is an asymptote of $y = f(x) \Rightarrow x = 1$ is a discontinuity of $y = \frac{1}{f(x)}$.

Use an open circle to show that $x = 1$ is not part of the curve.



▲ **Figure 4.18**

?

› Let $f(x) = \frac{x-4}{x-1}, x \neq 1$ and $g(x) = \frac{x-1}{x-4}, x \neq 4$.

Are the graphs of $y = \frac{1}{f(x)}$ and $y = g(x)$ the same? Explain your answer fully.

The curve $y^2 = f(x)$

You can think of $y^2 = f(x)$ as two curves:

$$y = \sqrt{f(x)} \text{ and } y = -\sqrt{f(x)}$$

$y = -\sqrt{f(x)}$ is a reflection of $y = \sqrt{f(x)}$ in the x-axis.

You can use the following points to help you sketch $y^2 = f(x)$ given the graph of $y = f(x)$.

- $y^2 = f(x)$ is symmetrical about the x-axis.
- $y^2 = f(x)$ is undefined where $f(x) < 0$.
  So any parts of $y = f(x)$ that are below the x-axis will not be part of $y^2 = f(x)$.

  Since the square root of a negative number is not real.

- $y = f(x)$ and $y^2 = f(x)$ intersect where $y = 0$ or $y = 1$.

  $0^2 = 0$ and $1^2 = 1$.

- $y^2 = f(x) \Rightarrow 2y\frac{dy}{dx} = f'(x) \Rightarrow \frac{dy}{dx} = \frac{f'(x)}{2y}$.
  So when $y > 0$, the gradients of $y = f(x)$ and $y^2 = f(x)$ have the same sign for a given value of x.
  Also $y = f(x)$ and $y^2 = f(x)$ have stationary points located at the same x values.

  So when $y = f(x)$ is increasing so is $y^2 = f(x)$...

  ...and when $y = f(x)$ is decreasing so is $y^2 = f(x)$.

- When $y = f(x)$ has a root then, provided $f'(x) \neq 0$, $y^2 = f(x)$ passes vertically through the x-axis.

  Provided the root is not at a stationary point.

Example 4.5

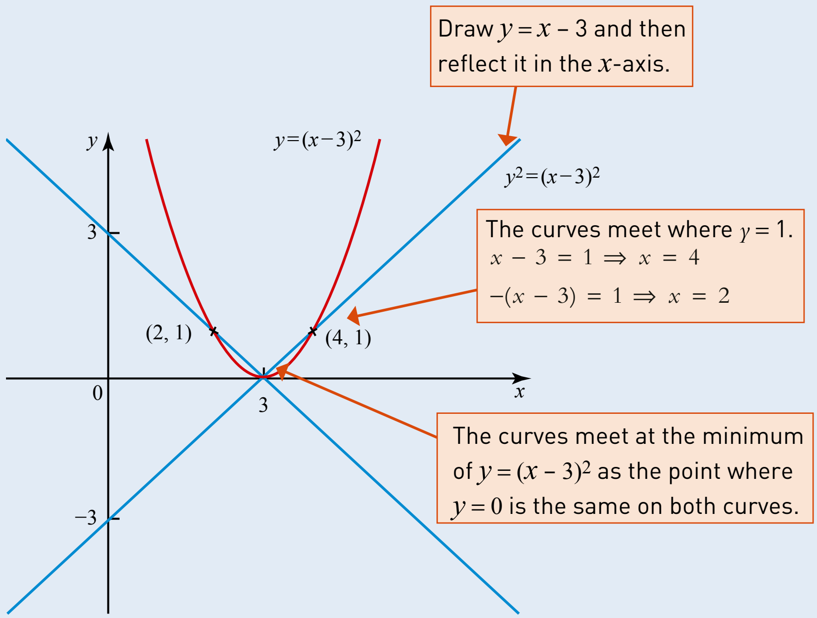
(i) Given $f(x) = (x-3)^2$, sketch $y = f(x)$ and $y^2 = f(x)$ on the same axes.

(ii) Given $f(x) = \cos x$ for $-2\pi \leqslant x \leqslant 2\pi$, sketch $y = f(x)$ and $y^2 = f(x)$ on the same axes.

Solution

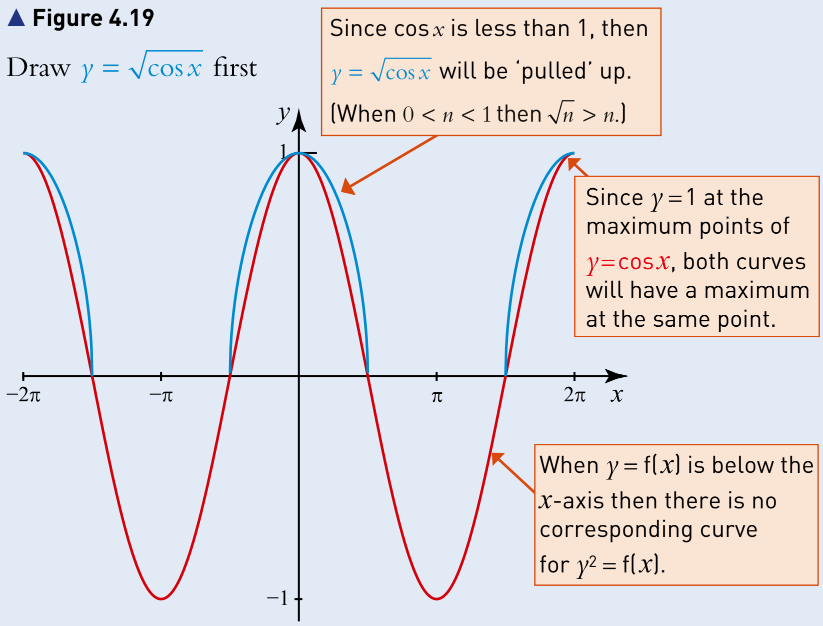
(i) $y = (x-3)^2$ is a translation of $y = x^2$ by the vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

$y^2 = (x-3)^2 \Rightarrow y = \pm(x-3)$



▲ **Figure 4.19**

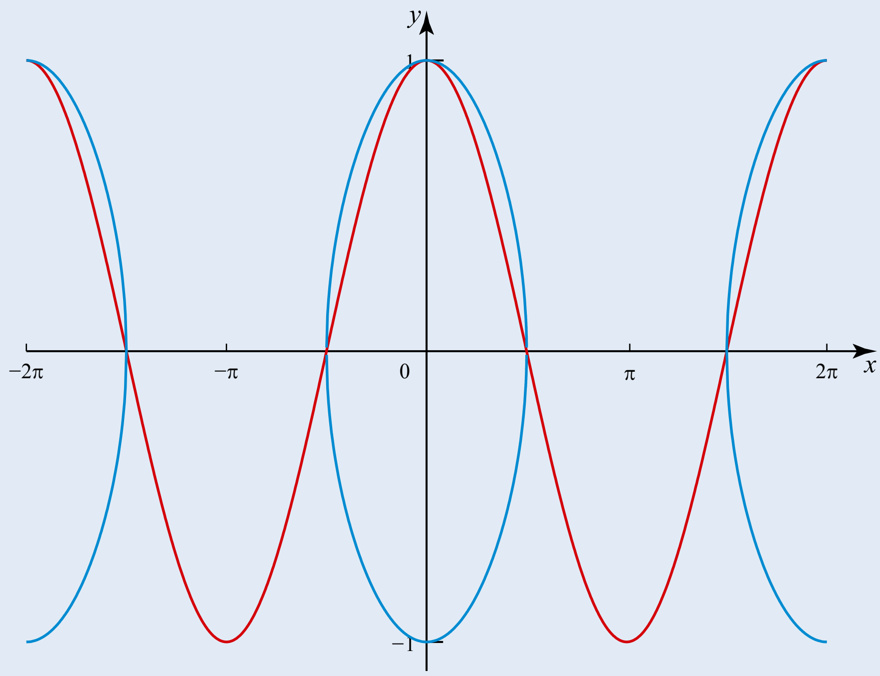
(ii) Draw $y = \sqrt{\cos x}$ first



▲ **Figure 4.20**

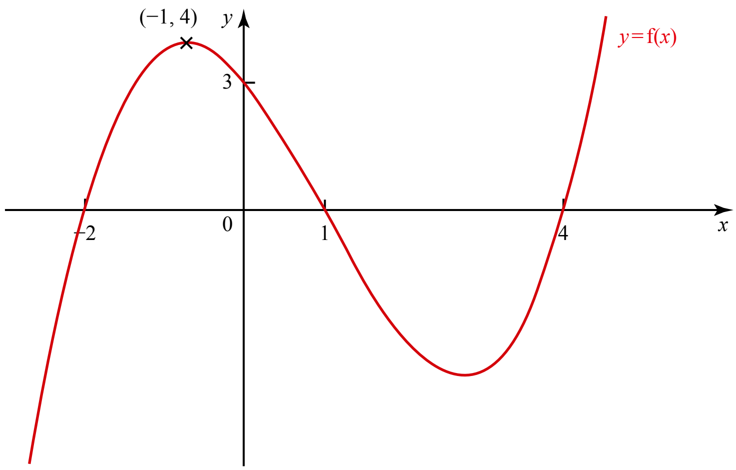
Reflect the curve $y = \sqrt{\cos x}$ in the x-axis to give $y = \pm\sqrt{\cos x}$.

→



▲ **Figure 4.21**

Example 4.6



▲ **Figure 4.22**

The diagram shows the curve $y = f(x)$.

The curve has a maximum point at $(-1, 4)$. It crosses the x-axis at $(-2, 0)$, $(1, 0)$ and $(3, 0)$ and the y-axis at $(0, 3)$.

Sketch the curve $y^2 = f(x)$, showing the coordinates of any turning points and where the curve crosses the axes.

Square root the y coordinate to find the corresponding point on $y = \sqrt{f(x)}$.

Solution

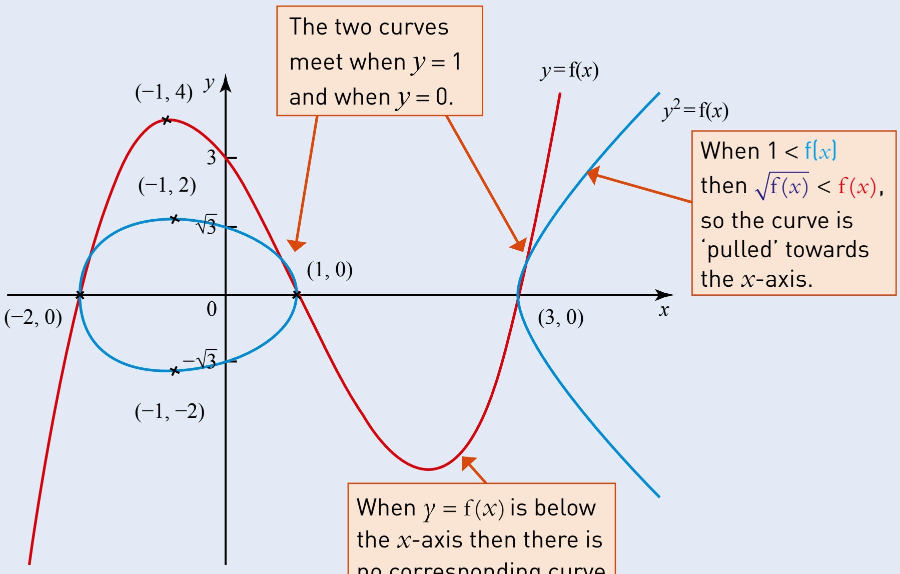
Think about the curve $y = \sqrt{f(x)}$ first.

$y = f(x)$ has a maximum at $(-1, 4)$ so $y = \sqrt{f(x)}$ has a maximum at $(-1, 2)$.

$y = f(x)$ has a y intercept at $(0, 3)$ so $y = \sqrt{f(x)}$ has a y intercept at $(0, \sqrt{3})$

The x intercepts remain the same.

Draw $y = \sqrt{f(x)}$ first and then reflect the curve in the x-axis to complete the sketch.



▲ **Figure 4.23**

Exercise 4C

For each of questions **1** to **5**:

Sketch the following curves, showing the coordinates of any turning points and where the curve crosses the axes. Label any asymptotes with their equations.

(i) $y = f(x)$

(ii) $y = |f(x)|$

(iii) $y = f(|x|)$

(iv) $y = \dfrac{1}{f(x)}$

(v) $y^2 = f(x)$

1 $f(x) = x^2 - 4$

2 $f(x) = \sin x$, for $-2\pi \leqslant x \leqslant 2\pi$.

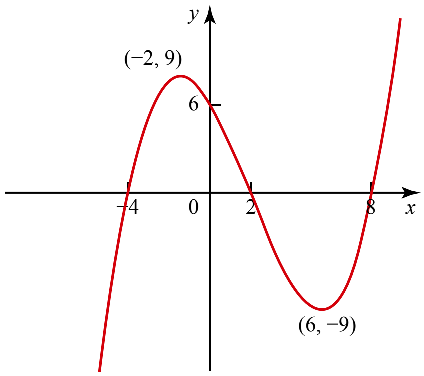
3 $f(x) = e^x$

4 $f(x) = (2 - x)(x + 1)(2x - 1)$

5 $f(x) = \dfrac{x + 3}{x + 1}$, $x \neq -1$

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

6



The diagram shows the curve $y = \mathrm{f}(x)$.

The curve has turning points at $(-2, 9)$ and $(6, -9)$. It crosses the x-axis at $(-4, 0)$, $(2, 0)$ and $(8, 0)$ and the y-axis at $(0, 6)$.

Sketch the following curves, showing the coordinates of any turning points and where the curve crosses the axes. Include the equations of any asymptotes on your diagrams.

(i) $y = |\mathrm{f}(x)|$ (ii) $y = \mathrm{f}(|x|)$ (iii) $y = \dfrac{1}{\mathrm{f}(x)}$ (iv) $y^2 = \mathrm{f}(x)$

7 $\mathrm{f}(x) = \dfrac{(x+3)}{(x-2)(x+1)}$, $x \neq -1$ and $x \neq 2$

(i) (a) Sketch the curve $y = \mathrm{f}(x)$.

(b) Solve the inequality $\dfrac{(x+3)}{(x-2)(x+1)} < 0$.

(ii) (a) Sketch the curve $y = \mathrm{f}(|x|)$.

(b) Solve the inequality $\dfrac{(|x|+3)}{(|x|-2)(|x|+1)} < 0$.

PS **8** $\mathrm{f}(x) = \dfrac{4ax}{x^2 + a^2}$.

(i) For the curve with equation $y = \mathrm{f}(x)$, find

(a) the equation of the asymptote

(b) the range of values that y can take.

(ii) For the curve with equation $y^2 = \mathrm{f}(x)$, write down

(a) the equation of the line of symmetry

(b) the maximum and minimum values of y

(c) the set of values of x for which the curve is defined.

9 $\mathrm{f}(x) = \dfrac{9-2x}{x+3}$, $x \neq -3$

(i) Sketch the curve with equation $y = \mathrm{f}(x)$.

(ii) State the values of x for which $y = \dfrac{1}{\mathrm{f}(x)}$ is undefined.

Sketch the curve with equation $y = \dfrac{1}{\mathrm{f}(x)}$.

State the coordinates of any points where each curve crosses the axes, and give the equations of any asymptotes.

PS **10** The curve C has equation $y = \frac{p(x)}{x+a}$, $x \neq -a$ where $p(x)$ is a polynomial of degree 2 and a is an integer.

The asymptotes of the curve are $x = 1$ and $y = 2x + 5$, and the curve passes through the point (2, 12).

(i) Express the equation of the curve C in the form $y = \frac{p(x)}{x+a}$.

(ii) Find the range of values that y can take.

(iii) Sketch the curve with equation $y^2 = \frac{p(x)}{x+a}$ where $p(x)$ and a are as found in part (i).

KEY POINTS

1 A rational function is defined as a function that can be expressed in the form $y = \frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials, and $g(x) \neq 0$.

2 To sketch the graph of $y = f(x)$ follow these steps.

Step 1 Find the intercepts (where the graph cuts the axes).

Step 2 Examine the behaviour of the graph near the vertical asymptotes; these are the lines $x = a$ if $g(a) = 0$ and $f(a) \neq 0$.

Step 3 Examine the behaviour as $x \to \pm\infty$.

Step 4 Find the coordinates of any stationary points.

Step 5 Sketch the graph.

Remember:

- when the order of $g(x)$ is less than the order of $f(x)$ then $y = 0$ is a horizontal asymptote
- when the order of $g(x)$ equals the order of $f(x)$ then $y = c$, for a constant c, is a horizontal asymptote
- when the order of $g(x)$ is greater than the order of $f(x)$ then there is an oblique asymptote. (Use long division to find it.)

3 You can use the discriminant to find the range of a rational function.

4 The graph of $y = f(x)$ can be used to help you solve inequalities and equations.

5 To sketch the graph of $y = |f(x)|$ given the graph of $y = f(x)$ reflect the part of the curve where $y < 0$ in the x-axis.

6 To sketch the graph of $y = f(|x|)$ given the graph of $y = f(x)$ reflect the part of the curve where $x > 0$ in the y-axis.

7 To sketch the graph of $y = \frac{1}{f(x)}$ given the graph of $y = f(x)$ remember:

- the curves $y = f(x)$ and $y = \frac{1}{f(x)}$ intersect at $y = 1$ and $y = -1$
- $f(x)$ and $\frac{1}{f(x)}$ have the same sign
- any roots of $y = f(x)$ become asymptotes of $y = \frac{1}{f(x)}$
- as $f(x) \to \pm\infty$ then $\frac{1}{f(x)} \to 0$
- there are no x intercepts (roots) for $y = \frac{1}{f(x)}$; any apparent 'roots' (from vertical asymptotes of $y = f(x)$) are discontinuities and should be shown by a small open circle
- when $f(x)$ is increasing then $\frac{1}{f(x)}$ is decreasing
- when $f(x)$ is decreasing then $\frac{1}{f(x)}$ is increasing
- where $f(x)$ has a maximum then $\frac{1}{f(x)}$ is a minimum
- where $f(x)$ has a minimum then $\frac{1}{f(x)}$ has a maximum.

8 When using the graph of $y = f(x)$ to sketch the graph of $y^2 = f(x)$ remember:

- $y^2 = f(x)$ can be thought of as two curves $y = \pm\sqrt{f(x)}$; sketch $y = \sqrt{f(x)}$ first and reflect the result in the x-axis to complete the sketch
- $y^2 = f(x)$ has a line of symmetry in the x-axis
- all y values above the x-axis are replaced by their positive square roots
  - if $y > 1$, then the points get 'pulled down' towards the x-axis ($\sqrt{4} = 2$)
  - if $0 < y < 1$, then the points get 'pulled up' from the x-axis ($\sqrt{\frac{1}{4}} = \frac{1}{2}$)
  - if $y = 1$ the point is invariant (stays the same) ($\sqrt{1} = 1$)
  - at a root (when $y = 0$) the point is invariant (stays the same) and the new curve passes vertically through this point
- any vertical asymptotes stay the same
- any horizontal asymptotes, $x = k$, above the x-axis become the horizontal asymptotes $x = \sqrt{k}$
- any turning point, e.g. (2, 5), above the x-axis remain the same type (i.e. maxima remain as maxima) and the square root of the y coordinate is taken, e.g. $(2, \sqrt{5})$.

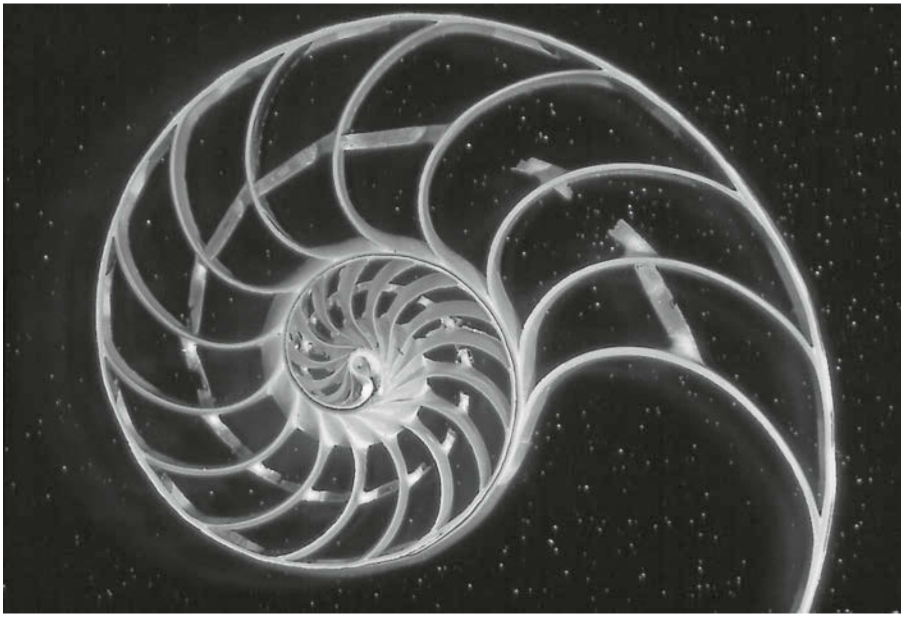
LEARNING OUTCOMES

Now that you have finished this chapter, you should be able to

- sketch the graph of a rational function, where the numerator and the denominator are polynomials of degree two or less, including finding
  - equations of vertical and horizontal asymptotes
  - the equation of an oblique asymptote (should it exist)
  - intersections with the x- and y-axes
  - turning points
  - the set of values taken by the function (range)
- use differentiation or the discriminant to determine the range of a function
- use the graph of $y = f(x)$ to sketch the graph of
  - $y = |f(x)|$
  - $y = f(|x|)$
  - $y = \dfrac{1}{f(x)}$
  - $y^2 = f(x)$
- use sketch graphs to help you solve inequalities and equations.

5 Polar coordinates

Let no-one ignorant of geometry enter my door.
Inscription over the entrance to the Academy of Plato, c.430–349BC



5.1 Polar coordinates

This nautilus shell forms a shape called an **equiangular spiral**.

> › How could you describe this mathematically?

You will be familiar with using Cartesian coordinates (x, y) to show the position of a point in a plane.

Figure 5.1, on the next page, shows an alternative way to describe the position of a point P by giving:

- its distance from a fixed point O, known as the **pole**;
- the angle θ between OP and a line called the **initial line**.

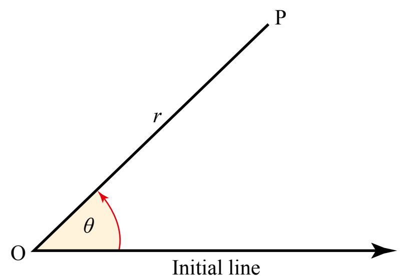
The numbers (r, θ) are called the polar coordinates of P.

The length r is the distance of the point P from the origin.

The angle θ is usually measured in radians, in an anticlockwise direction from the initial line, which is drawn horizontally to the right.

> Is it possible to provide more than one set of polar coordinates (r, θ) to define a given point P?
>
> If so, in how many ways can a point be defined?

At the point O, $r = 0$ and θ is undefined. Each pair of polar coordinates (r, θ) gives a unique point in the plane.



▲ **Figure 5.1**

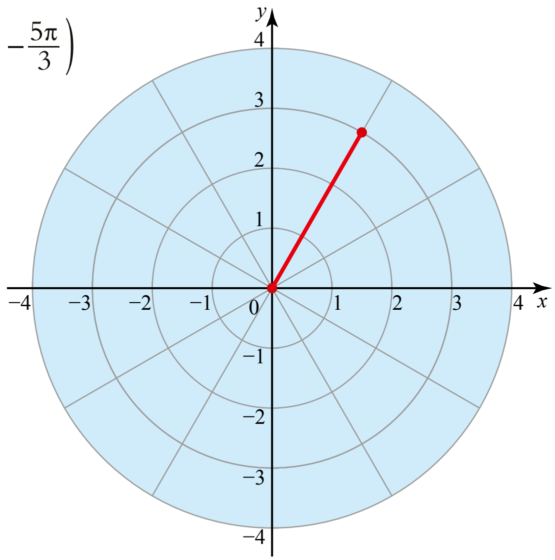
You may have noticed that adding or subtracting any integer multiple of 2π to the angle θ does not change the point P.

For example, the point in Figure 5.2 could be expressed as

$\left(3, \frac{\pi}{3}\right), \left(3, \frac{7\pi}{3}\right), \left(3, \frac{13\pi}{3}\right), \left(3, -\frac{5\pi}{3}\right)$

and so on.

This means that each point P can be written in an infinite number of ways.



▲ **Figure 5.2**

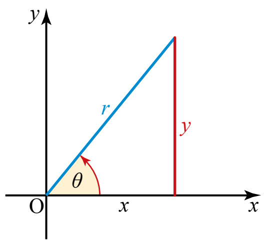
ACTIVITY 5.1

Check by drawing a diagram that the polar coordinates $\left(5, \frac{\pi}{6}\right)$, $\left(5, \frac{13\pi}{6}\right)$ and $\left(5, -\frac{11\pi}{6}\right)$ all describe the same point.

Give the polar coordinates for the point $\left(6, \frac{3\pi}{4}\right)$ in three other ways.

If you need to specify the polar coordinates of a point uniquely, you use the **principal polar coordinates**, where $r > 0$ and $-\pi < \theta \leqslant \pi$. This is similar to the convention used when writing a complex number in modulus argument form.

Converting between polar and Cartesian coordinates



▲ **Figure 5.3**

It is easy to convert between polar coordinates (r, θ) and Cartesian coordinates (x, y).

From Figure 5.3 you can see:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x}$$

You need to be careful to choose the right quadrant when finding θ, as the equation $\tan\theta = \frac{y}{x}$ always gives two values of θ that differ by π. Always draw a sketch to make sure you know which angle is correct.

Example 5.1

(i) Find the Cartesian coordinates of the following points:

(a) $\left(4, \frac{2\pi}{3}\right)$ (b) $\left(12, -\frac{\pi}{6}\right)$

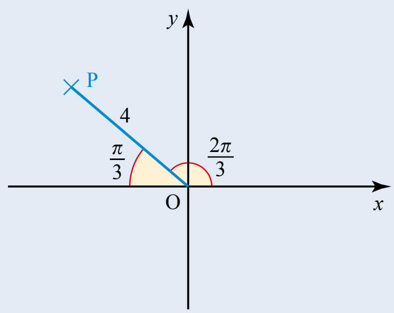
(ii) Find the polar coordinates of the following points:

(a) $\left(-\sqrt{3}, 1\right)$ (b) $(4, -4)$

Solution

First draw a diagram to represent the coordinates of the point:

(i) (a)



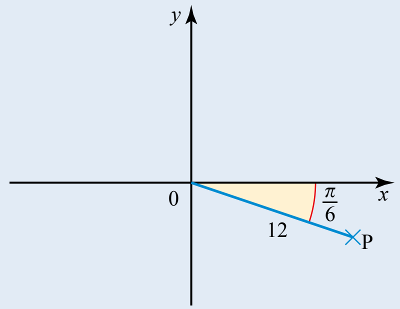
▲ **Figure 5.4**

$$4\cos\frac{\pi}{3} = 2 \text{ so } x = -2$$

$$4\sin\frac{\pi}{3} = 2\sqrt{3} \text{ so } y = 2\sqrt{3}$$

$\left(4, \frac{2\pi}{3}\right)$ has Cartesian coordinates $\left(-2, 2\sqrt{3}\right)$.

(b)



Since θ is negative, P is below the x-axis.

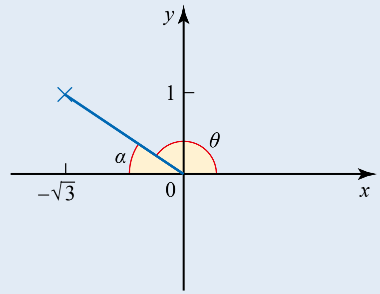
▲ **Figure 5.5**

$12\cos\frac{\pi}{6} = 6\sqrt{3}$ so $x = 6\sqrt{3}$

$12\sin\frac{\pi}{6} = 6$ so $y = -6$

$\left(12, \frac{-\pi}{6}\right)$ has Cartesian coordinates $\left(6\sqrt{3}, -6\right)$.

(ii) (a)



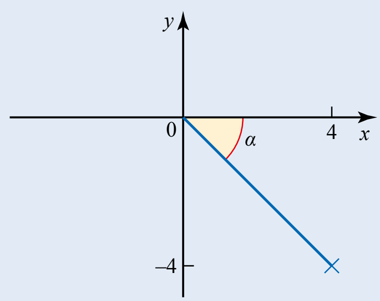
▲ **Figure 5.6**

$r = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$

$\tan\alpha = \frac{1}{\sqrt{3}}$ so $\alpha = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$

$\left(-\sqrt{3}, 1\right)$ has polar coordinates $\left(2, \frac{5\pi}{6}\right)$.

(b)



▲ **Figure 5.7**

$r = \sqrt{4^2 + 4^2} = 4\sqrt{2}$

$\tan\alpha = \frac{4}{4}$ so $\alpha = \frac{\pi}{4}$ so $\theta = -\frac{\pi}{4}$.

$(4, -4)$ has polar coordinates $\left(4\sqrt{2}, -\frac{\pi}{4}\right)$.

Converting equations between polar and Cartesian forms

You will be familiar with using Cartesian equations such as $y = 2x^2 + 5$ to represent the relationship between the coordinates (x, y) of points on a curve. Curves can also be represented using the relationship between polar coordinates (r, θ) of points on the curve. The **polar equation** $r = f(\theta)$ is sometimes simpler than the Cartesian equation, especially if the curve has rotational symmetry. Polar equations have many important applications, for example in the study of orbits.

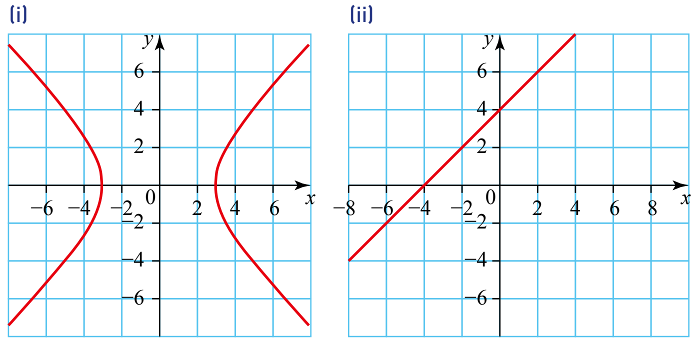
Coordinates of points can be converted from polar to Cartesian form and vice versa. Similarly the Cartesian equation of a curve can be converted from one form to the other.

Use $x = r\cos\theta$ and $y = r\sin\theta$, to convert from Cartesian form, and use $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{y}{x}$ to convert from polar form.

Equations that appear quite simple in one form may be quite complicated in the other.

Example 5.2

Find the polar equations of the following curves.



▲ **Figure 5.8** (i) $x^2 - y^2 = 9$ (ii) $y = x + 4$

Solution

(i) Substitute for x and y.

$$(r\cos\theta)^2 - (r\sin\theta)^2 = 9$$

$$r^2(\cos^2\theta - \sin^2\theta) = 9$$

The bracket is the double angle formula for $\cos 2\theta$

$$r^2 \cos 2\theta = 9$$

$$r^2 = 9\sec 2\theta$$

(ii) Substitute for y and x

$r\sin\theta = r\cos\theta + 4$

$r(\sin\theta - \cos\theta) = 4$

Rewriting

$r\left(\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right)\right) = 4$

$r = 2\sqrt{2}\operatorname{cosec}\left(\theta - \frac{\pi}{4}\right)$

> $\sin\theta - \cos\theta = R\sin(\theta - \alpha)$
> $= R(\sin\theta\cos\alpha - \cos\alpha\sin\alpha)$
> So $1 = R\cos\alpha$ and $1 = R\sin\alpha$
> Squaring and adding $\Rightarrow R^2 = 2$ so $R = \sqrt{2}$.
> Dividing $\Rightarrow \tan\alpha = 1$ so $\alpha = \frac{\pi}{4}$.
> So $\sin\theta - \cos\theta = \sqrt{2}\sin(\theta - \alpha)$.

Example 5.3

Find the Cartesian equations of the curves.

(i) $r = 5$ (ii) $r^2 = 4\sin\theta$

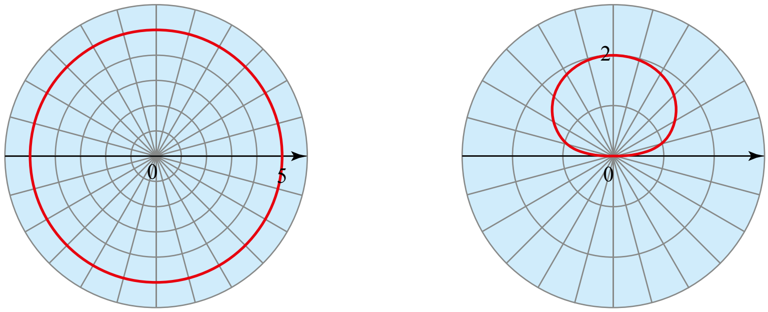


Figure 5.9

Solution

(i) Substitute for r.

$\sqrt{x^2 + y^2} = 5$

$x^2 + y^2 = 25$

(ii) Multiply both sides by r.

$r^3 = 4r\sin\theta$

$\left(\sqrt{x^2 + y^2}\right)^3 = 4y$

$\left(x^2 + y^2\right)^3 = 16y^2$

Exercise 5A

1 Find the Cartesian coordinates of the following points:

(i) $\left(8, -\frac{\pi}{2}\right)$ (ii) $\left(8, -\frac{3\pi}{4}\right)$ (iii) $\left(8, \frac{\pi}{3}\right)$ (iv) $\left(8, \frac{5\pi}{6}\right)$

2 Find the principal polar coordinates of the following points, giving answers as exact values or to three significant figures as appropriate:

(i) $(5, -12)$ (ii) $(-5, 0)$ (iii) $\left(-\sqrt{3}, -1\right)$ (iv) $(3, 4)$

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

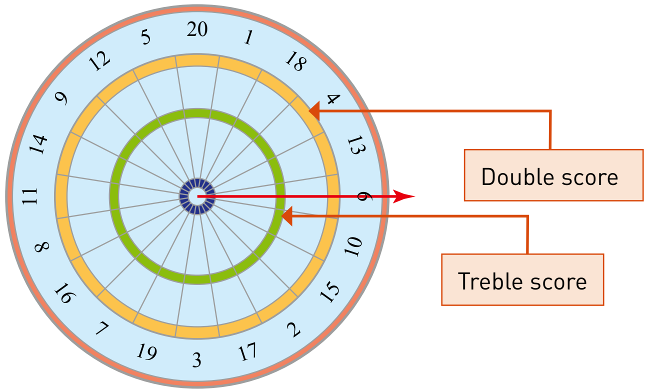
3 Plot the points with polar coordinates A $\left(5, \frac{5\pi}{6}\right)$, B $\left(3, -\frac{3\pi}{4}\right)$, C $\left(5, -\frac{\pi}{6}\right)$ and D $\left(3, \frac{\pi}{4}\right)$.

Write down the name of the quadrilateral ABCD. Explain your answer.

4 Plot the points with polar coordinates A $\left(3, \frac{\pi}{5}\right)$, B $\left(2, \frac{7\pi}{10}\right)$, C $\left(3, -\frac{4\pi}{5}\right)$ and D $\left(4, -\frac{3\pi}{10}\right)$.

Write down the name of the quadrilateral ABCD. Explain your answer.

M **5**



The diagram shows a dartboard made up of six concentric circles. The radii of the six circles are 6, 16, 99, 107, 162 and 170 mm respectively.

The smallest circle at the centre is called the inner bull and the next circle is called the outer bull. If a dart lands in either of these two regions it scores 50 or 25 points respectively.

The areas that get a double score or treble score are labelled. If a dart lands in one of these two rings it doubles or trebles the sector number.

The initial line passes through the middle of the sector labelled 6 and angles θ are measured in degrees from this line.

(i) Find the score in the region for which 16 mm $< r <$ 99 mm and $27° < \theta < 45°$.

(ii) Give conditions for r and θ that define the boundary between sectors 10 and 15.

(iii) Give conditions for r and θ for which the score is:

(a) treble 14 (b) 17 (c) double 18.

PS **6** One vertex of an equilateral triangle has polar coordinates A$\left(4, \frac{\pi}{4}\right)$.

Find the polar coordinates of all the other possible vertices B and C of the triangle, when:

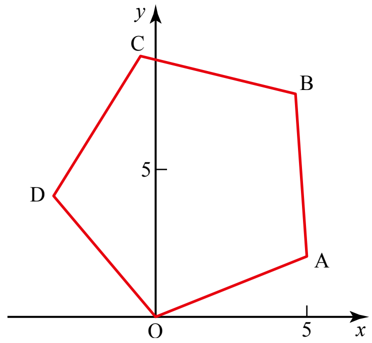
(i) the origin O is at the centre of the triangle

(ii) B is the origin

(iii) O is the midpoint of one of the sides of the triangle.

PS

7 The diagram shows a regular pentagon OABCD in which A has Cartesian coordinates (5, 2).



(i) Show that OB = 8.71, correct to 2 decimal places.

(ii) Find the polar coordinates of the vertices A, B, C and D, giving angles in radians.

(iii) Hence find the Cartesian coordinates of the vertices B, C and D.

In parts (ii) and (iii) give your answers correct to 2 decimal places.

8 Convert these Cartesian equations into polar form.

(i) $y = 2$ (ii) $y = \sqrt{3}x$ (iii) $x^2 + y^2 = 3$

(iv) $xy = 9$ (v) $y^2 = x$ (vi) $(x - y)^2 = 5$

9 Convert these polar equations into Cartesian form.

(i) $r\cos\theta = 5$ (ii) $r = 5$ (iii) $r = 4\operatorname{cosec}\theta$

(iv) $r = 4\cos\theta$ (v) $r^2 = \sin 2\theta$ (vi) $r = 2 - 2\cos\theta$

10 Prove that $r = a\sec\theta$ and $r = b\operatorname{cosec}\theta$, where a and b are non-zero constants, are the polar equations of two straight lines. Find their Cartesian equations.

Note

For AS & A Level Further Mathematics you need to be familiar with using radians to represent an angle. You also need to know the definitions of the reciprocal trigonometric functions:

$$\sec\theta \equiv \frac{1}{\cos\theta}$$

$$\operatorname{cosec}\theta \equiv \frac{1}{\sin\theta}$$

and the double angle formulae:

$$\sin 2\theta \equiv 2\sin\theta\cos\theta$$

$$\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$$

5.2 Sketching curves with polar equations

The polar equation of a curve is usually written in the form $r = f(\theta)$. The curve can be plotted or sketched without converting to Cartesian form.

Example 5.4

Sketch the graphs of the following.

(i) $r = 3$ (ii) $\theta = \frac{\pi}{4}$

Solution

(i) The distance from the origin is constant so this is a circle centre the origin and radius 3. (Notice this is $x^2 + y^2 = 9$.)

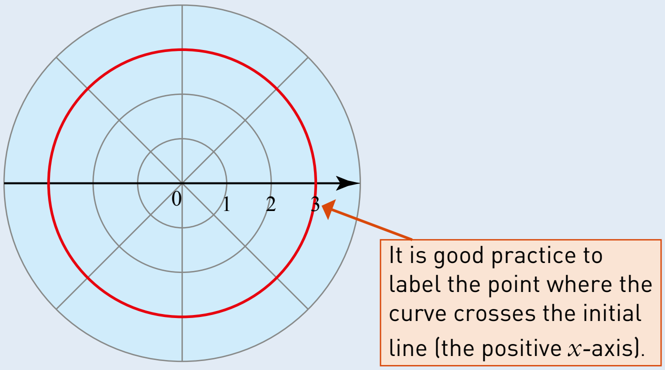


Figure 5.10

(ii) θ is constant so the graph is a half-line from the pole at an angle of $\frac{\pi}{4}$ to the initial line. (It is not the same as the line $y = x$, which extends in both directions from the pole.)

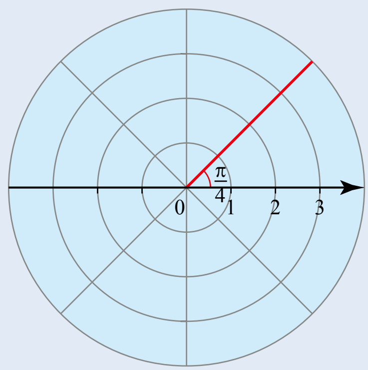


Figure 5.11

ACTIVITY 5.2

Sketch the curves with polar equations:

$r = 7$

$\theta = \frac{\pi}{3}$

Example 5.5

A curve is defined by the polar equation $r = 5\cos^2\theta$.

(i) Create a table of values for $0 \leqslant \theta \leqslant \pi$.

(ii) Plot the curve for $0 \leqslant \theta \leqslant 2\pi$.

(iii) State the minimum and maximum values of r and the values of θ for which they occur.

Solution

(i) Create a table of values for $0 \leqslant \theta \leqslant \frac{\pi}{2}$. This table has values of θ that increase in intervals of $\frac{\pi}{12}$, which gives a convenient number of points.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
r	5	4.67	3.75	2.5	1.25	0.34	0

For values of θ from $\frac{\pi}{2}$ to π, the values for r are repeated as the value of $\cos\theta$ is squared.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
r	5	4.67	3.75	2.5	1.25	0.34	0	0.34	1.25	2.5	3.75	4.67	5

(ii) Plotting these points gives this graph. Use the symmetry of the cosine function to reduce the number of calculations needed. The values of r for negative values of θ are the same as those for the corresponding positive values. The initial line is a line of symmetry.

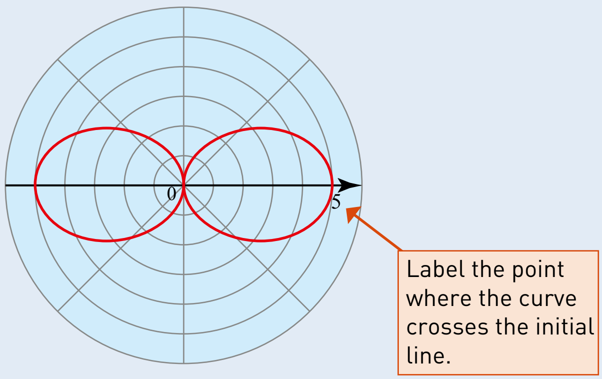


Figure 5.12

(iii) The maximum value for $r = 5\cos^2\theta$ is 5 and occurs when $\cos\theta = \pm 1$; i.e. when $\theta = 0$ or π.

The minimum value of $r = 5\cos^2\theta$ is 0 and occurs when $\cos\theta = 0$; i.e. when $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

Example 5.6

Find the minimum and maximum values of r for the curve $r = 3 + 2\cos 4\theta$ and the values of θ for which they occur.

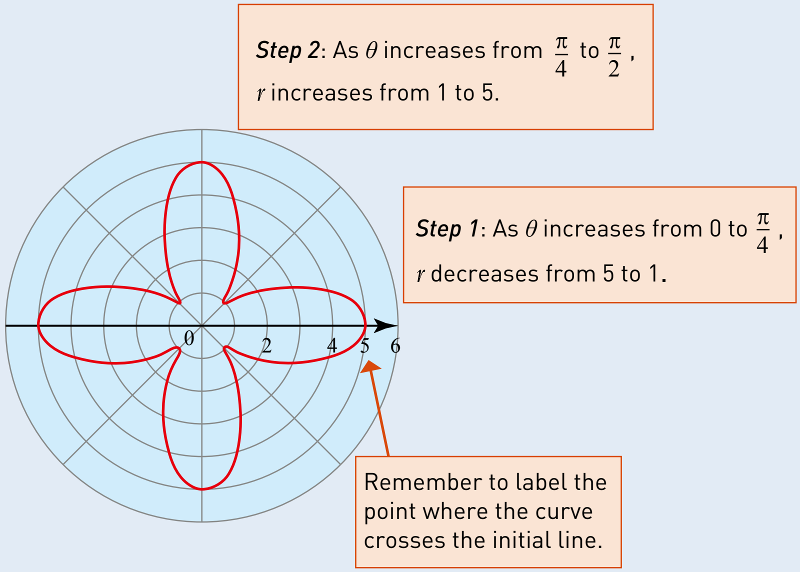
Hence sketch the graph.

Solution

Minimum value is $3 - 2 = 1$ and occurs when $\cos 4\theta = -1$, giving $\theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$.

Maximum value is $3 + 2 = 5$ and occurs when $\cos 4\theta = 1$, giving $\theta = 0, \pm\frac{\pi}{2}$.

The function is periodic with period $\frac{\pi}{2}$ so the graph is repeated in each quadrant and the graph has rotational symmetry order 4.



▲ **Figure 5.13**

Example 5.7

A curve is given by the equation $r = \theta(\pi - \theta)$.

(i) Plot the curve for $0 \leqslant \theta \leqslant \pi$, where r is positive.

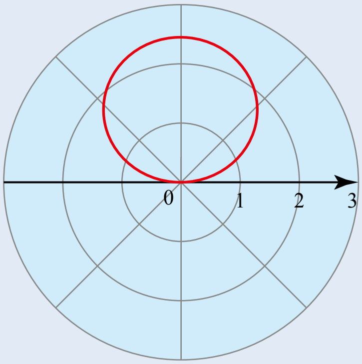
(ii) Create a table of values for the parts of the graph $\pi \leqslant \theta \leqslant \frac{3\pi}{2}$ and $-\frac{\pi}{2} \leqslant \theta \leqslant 0$.

(iii) Sketch the curve for $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{2}$.

Solution

(i) $0 \leqslant \theta \leqslant \pi$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	0	$\frac{5\pi^2}{36}$	$\frac{2\pi^2}{9}$	$\frac{\pi^2}{4}$	$\frac{2\pi^2}{9}$	$\frac{5\pi^2}{36}$	0



▲ **Figure 5.14**

(ii) $\pi \leqslant \theta \leqslant \frac{3\pi}{2}$

θ	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
r	0	$-\frac{7\pi^2}{36}$	$-\frac{4\pi^2}{9}$	$-\frac{3\pi^2}{4}$

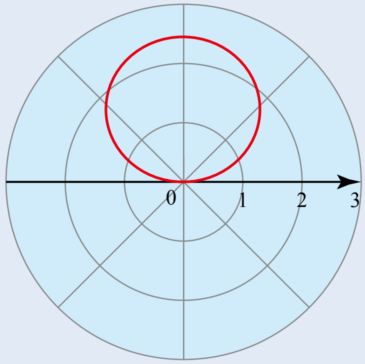
$-\frac{\pi}{2} \leqslant \theta \leqslant 0$

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0
r	$-\frac{3\pi^2}{4}$	$-\frac{4\pi^2}{9}$	$-\frac{7\pi^2}{36}$	0

(iii) There are three conventions for dealing with negative values for r.

→

Convention 1: The curve is only taken to exist for positive values of r, using the convention that $r \geqslant 0$ and no curve would be drawn outside the range $0 \leqslant \theta \leqslant \pi$.

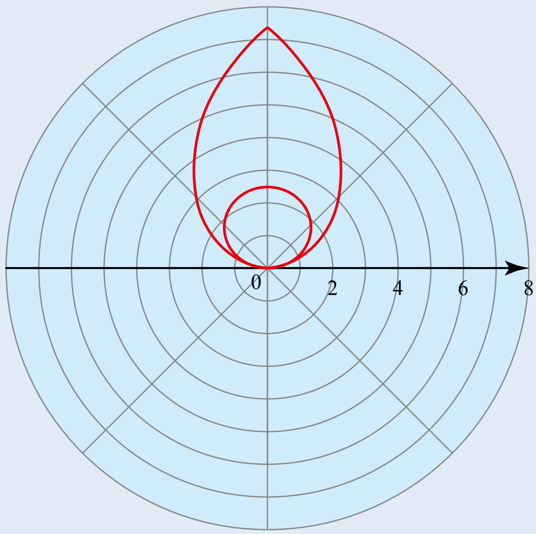


▲ **Figure 5.16**

Think of this as r being measured in the opposite direction from the origin.

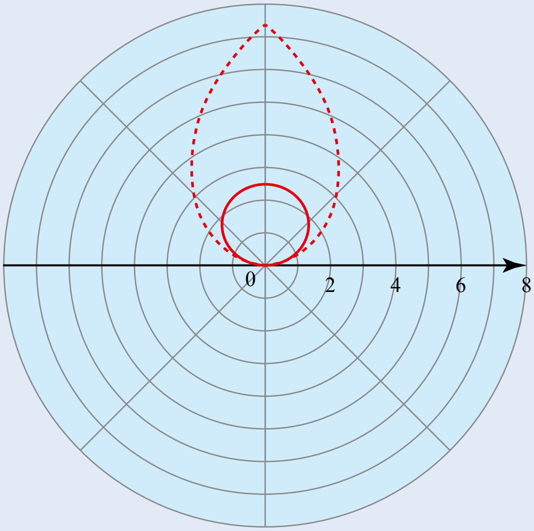
Convention 2: The point $\left(-\frac{7\pi^2}{36}, \frac{7\pi}{6}\right)$ is plotted at $\left(\frac{7\pi^2}{36}, \frac{\pi}{6}\right)$

The curve then appears above the initial line, as shown.



▲ **Figure 5.15**

Convention 3: The curve is drawn with the negative values shown using a broken line.



▲ **Figure 5.17**

Summary: Sometimes an equation gives negative values of r, for example $r = \pi\theta - \theta^2$.

There are three conventions for dealing with this.

1. The curve is only taken to exist for positive values of r.
2. Think about the distance as being measured in the opposite direction.
3. The curve is drawn with the negative values using a broken line.

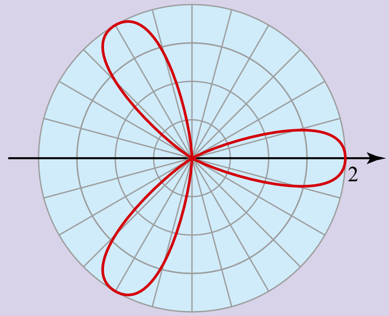
In this course, Convention 1 is used.

ACTIVITY 5.3

If you have graphing software, find out how to use it to draw a curve from its polar equation by drawing the curve used in Example 5.7, $r = \theta(\pi - \theta)$. Which convention does your software use?
Extension question:
Investigate the curve for other values of θ.

ACTIVITY 5.4

This is the curve $r = 2\cos 3\theta$.



▲ **Figure 5.18**

(i) How do the constants 2 and 3 in the equation relate to the shape of the curve?

(ii) Copy the sketch of the curve. Use arrows to show how a point would move around the curve as the value of θ varies from 0 to π.

(iii) If you have graphing software, use it to investigate the curve $r = k\sin n\theta$ where k and n are positive and n is an integer. Using the convention for this course, only include the parts of the graph for which r is positive.

The type of curve shown in Activity 5.4 is called a **rhodonea** (rose curve).

Exercise 5B

Technology note

If you have graphing software, you may wish to use it to check your graphs in this exercise. Remember that the scales used on the axes can affect how the shape of the graph appears.

1 Sketch the curve given by the equations:

(i) $r = 5$ (ii) $\theta = -\frac{3\pi}{4}$

(iii) $r = 3\cos\theta$ (iv) $r = 2\sin\theta$

(v) $r = 3\theta$ for the interval $0 \leqslant \theta \leqslant 2\pi$

2 Make a table of values of $r = 8\sin\theta$ for θ from 0 to π in intervals of $\frac{\pi}{12}$, giving answers to two decimal places where appropriate.

Explain what happens when $\pi \leqslant \theta \leqslant 2\pi$.

By plotting the points, confirm that the curve $r = 8\sin\theta$ represents a circle that is symmetrical about the y-axis.

Write down the Cartesian equation of the circle.

3 Sketch the curves with equations $r = 3\cos 2\theta$ and $r = 3\cos 3\theta$ for the values of θ from 0 to 2π. Only include the parts of the graph for which $r \geqslant 0$.

State the number of petals on each curve. Generalise this result.

Extension question: Investigate how the result for the number of petals is different when negative values of r are also used.

4 The curve $r = \frac{4\theta}{\pi}$ for $0 \leqslant \theta \leqslant 4\pi$ is called the **spiral of Archimedes**.

Draw the curve.

5 A curve with polar equation $r = a(1 + \cos\theta)$ is called a **cardioid**.

(i) Draw the curve when $a = 8$. How do you think the curve got its name?

(ii) Sketch the curve with polar equation $r = a(1 - \cos\theta)$ when $a = 8$. How does the shape of your graph compare to that in part (i)?

CP **6** (i) If available, use graphing software to draw the curves $r = k + 3\cos\theta$ for $k = 2, 3, 4, 5, 6, 7$. For each graph note the minimum and maximum values of r.

(ii) Investigate the shape of the curve $r = a + b\cos\theta$ for other values of a and b and use this to define the shape of the curve when:

(a) $a \geqslant 2b$ (b) $2b > a > b$ (c) $a = b$

(iii) In the case $a < b$, find the range of values of θ for which $r \geqslant 0$.

(iv) Investigate how the shape of the curve differs for polar curves of the form $r = a + b\sin\theta$.

7 A **lemniscate** has the equation $r^2 = a^2\cos 2\theta$ or $r^2 = a^2\sin 2\theta$. Taking $r > 0$, sketch these curves for $0 \leqslant \theta \leqslant 2\pi$ explaining clearly what happens in each interval of $\frac{\pi}{4}$ radians.

CP **8** The straight line L passes through the point A with polar coordinates (p, α) and is perpendicular to OA.

(i) Prove that the polar equation of L is $r\cos(\theta - \alpha) = p$.

(ii) Use the identity $\cos(\theta - \alpha) \equiv \cos\theta\cos\alpha + \sin\theta\sin\alpha$ to find the Cartesian equation of L.

9 The curves C_1 and C_2 have polar equations

$$C_1 : r = \frac{1}{\sqrt{2}}, \quad \text{for } 0 \leqslant \theta \leqslant 2\pi,$$

$$C_2 : r = \sqrt{\sin\left(\tfrac{1}{2}\theta\right)}, \quad \text{for } 0 \leqslant \theta \leqslant \pi.$$

Find the polar coordinates of the point of intersection of C_1 and C_2. Sketch C_1 and C_2 on the same diagram.

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5.3 Finding the area enclosed by a polar curve

> **Note**
>
> You need to be able to integrate polynomial functions and trigonometric functions of the form $a\sin bx$ and $a\cos bx$.

Look at the region in Figure 5.19 bounded by the lines OU and OV and the curve UV. To find the area of this region, start by dividing it up into smaller regions OPQ. Let OU and OV have angles $\theta = \alpha$ and $\theta = \beta$ respectively. If the curve has equation $r = f(\theta)$, P and Q have coordinates (r, θ) and $(r + \delta r, \theta + \delta\theta)$.

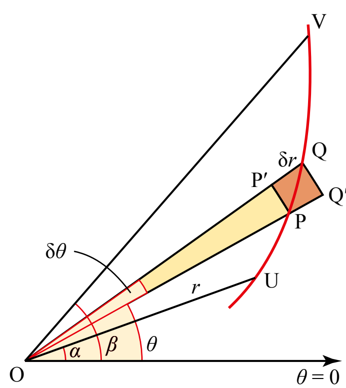


Figure 5.19

Let the area of OUV be A and the area of OPQ be δA.

The area δA lies between the circular sectors OPP′ and OQQ′, so:

$$\tfrac{1}{2}r^2\delta\theta < \delta A < \tfrac{1}{2}(r+\delta r)^2\,\delta\theta$$

Remember that the area of a sector of a circle is given by $\frac{1}{2}r^2\theta$, where θ is in radians.

therefore

$$\tfrac{1}{2}r^2 < \frac{\delta A}{\delta\theta} < \tfrac{1}{2}(r+\delta r)^2$$

As $\delta\theta \to 0$, $\delta r \to 0$ and so $\frac{1}{2}(r+\delta r)^2 \to \frac{1}{2}r^2$. Therefore $\frac{\delta A}{\delta\theta}$ must also tend to $\frac{1}{2}r^2$ as $\delta\theta \to 0$.

But as $\delta\theta \to 0$, $\frac{\delta A}{\delta\theta} \to \frac{\mathrm{d}A}{\mathrm{d}\theta}$

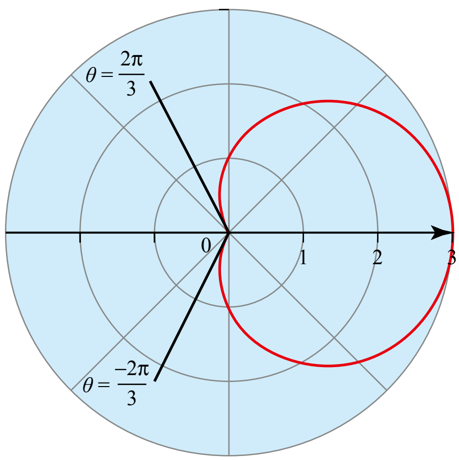
Therefore $\frac{\mathrm{d}A}{\mathrm{d}\theta} = \frac{1}{2}r^2$.

Integrating both sides with respect to θ shows the result for the area of a region bounded by a polar curve and two straight lines $\theta = \alpha$ and $\theta = \beta$ is:

$$A = \int_\alpha^\beta \tfrac{1}{2}r^2\,d\theta$$

Example 5.8

The diagram below shows the graph of $r = 1 + 2\cos\theta$. Find the area enclosed.



▲ **Figure 5.20**

Solution

The area is formed with values of θ from $-\frac{2\pi}{3}$ to $\frac{2\pi}{3}$ (the values of θ for which $r = 0$).

The shape is symmetrical, so the area is given by $2\left(\frac{1}{2}\int_0^{\frac{2\pi}{3}}(1+2\cos\theta)^2\,\mathrm{d}\theta\right)$

$$\int_0^{\frac{2\pi}{3}} \left(1+4\cos\theta+4\cos^2\theta\right) d\theta =$$

$$\int_0^{\frac{2\pi}{3}} \left(1+4\cos\theta+2(\cos 2\theta+1)\right) d\theta =$$

$$\int_0^{\frac{2\pi}{3}} (3+4\cos\theta+2\cos 2\theta)\, d\theta =$$

$$[3\theta+4\sin\theta+\sin 2\theta]_0^{\frac{2\pi}{3}} =$$

$$\left(3\times\tfrac{2\pi}{3}+4\sin\tfrac{2\pi}{3}+\sin\tfrac{4\pi}{3}\right)-0 =$$

$$\left(2\pi+2\sqrt{3}-\frac{\sqrt{3}}{2}\right)-0 = 2\pi+\frac{3\sqrt{3}}{2}$$

Exercise 5C

CP

1 Check that the integral $\int \frac{1}{2} r^2\, d\theta$ correctly gives the area of the circle $r = 10\cos\theta$ when it is evaluated from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

2 A curve has equation $r = 5\cos 4\theta$.

(i) Sketch the curve for the interval $0 \leqslant \theta \leqslant 2\pi$.

(ii) Find the area of one loop of the curve.

3 A curve has equation $r = 3 + 3\sin\theta$.

(i) Sketch the curve for the interval 0 to 2π.

(ii) Find the area enclosed by the curve.

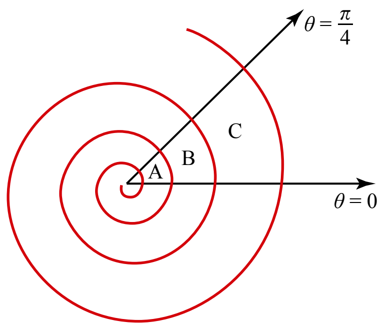
4 Find the area bounded by the spiral $r = \frac{4\theta}{\pi}$ from $\theta = 0$ to $\theta = 2\pi$ and the initial line.

5 Find the exact areas of the two portions into which the line $\theta = \frac{\pi}{2}$ divides the upper half of the cardioid $r = 8(1 + \cos\theta)$.

6 Sketch the lemniscate $r^2 = a^2\cos 2\theta$ and find the area of one of its loops.

CP

7 The diagram shows the **equiangular spiral** $r = ae^{k\theta}$ where a and k are positive constants and e is the exponential constant 2.71828…



Prove that the areas A, B and C formed by the lines $\theta = 0$ and $\theta = \frac{\pi}{4}$ and the spiral form a geometric sequence and find its common ratio.

8 Find the area enclosed between the curves $r = 3 - 3\cos\theta$ and $r = 4\cos\theta$. Give your final answer to three significant figures.

9 The curve C has polar equation $r = a(1 + \sin\theta)$, where a is a positive constant and $0 \leqslant \theta < 2\pi$. Draw a sketch of C.

Find the exact value of the area of the region enclosed by C and the half-lines $\theta = \frac{1}{3}\pi$ and $\theta = \frac{2}{3}\pi$.

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10 The curve C has polar equation $r = 2 + 2\cos\theta$, $0 \leqslant \theta < \pi$. Sketch the graph of C.

Find the area of the region R enclosed by C and the initial line.

The half-line $\theta = \frac{1}{5}\pi$ divides R into two parts. Find the area of each part, correct to 3 decimal places.

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KEY POINTS

1 To convert from polar coordinates to Cartesian coordinates $x = r\cos\theta$, $y = r\sin\theta$.

2 To convert from Cartesian coordinates to polar coordinates $r = \sqrt{x^2 + y^2}$, $\theta = \arctan\frac{y}{x}$ ($\pm\pi$ if necessary).

3 The principal polar coordinates (r, θ) are those for which $r > 0$ and $-\pi < \theta \leqslant \pi$.

4 The area of a sector is $\int_\alpha^\beta \frac{1}{2} r^2 \, d\theta$.

LEARNING OUTCOMES

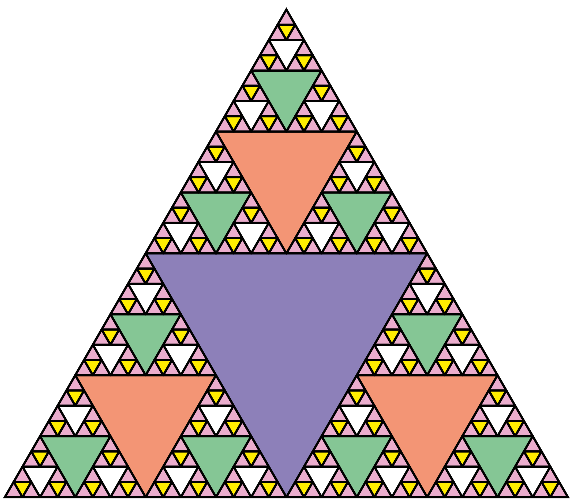
Now that you have finished this chapter, you should be able to

- understand the relations between Cartesian and polar coordinates and be able to
  - convert from Cartesian to polar coordinates
  - convert from polar to Cartesian coordinates
- sketch simple polar curves showing significant features
  - the value where the curve intersects the initial line
  - symmetry
  - the form of the curve at the pole
  - the greatest and least values of r
- find the area enclosed by a polar curve using the formula $A = \frac{1}{2}\int r^2 \, \mathrm{d}\theta$.

6 Matrices and their inverses

The grand thing is to be able to reason backwards.
Arthur Conan Doyle (1859–1930)

The diagram in Figure 6.1 is called a Sierpinsky triangle. The pattern can be continued with smaller and smaller triangles.



▲ **Figure 6.1** Sierpinsky triangle.

?

- What is the same about each of the triangles in the diagram?
- How many of the yellow triangles are needed to cover the large purple triangle?

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

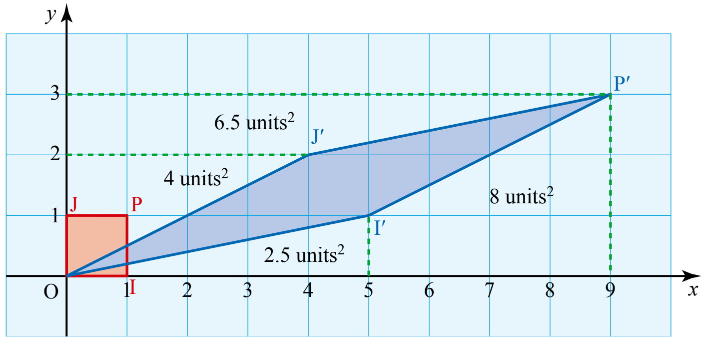
6

Note

You need to have covered the work on matrices and transformations from Chapter 1.

6.1 The determinant of a 2×2 matrix

Figure 6.2 shows the unit square, labelled OIPJ, and the parallelogram OI′P′J′ formed when OIPJ is transformed using the matrix $\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$.



▲ **Figure 6.2**

What effect does the transformation have on the area of OIPJ?

The area of OIPJ is 1 unit^2.

To find the area of OI′P′J′, a rectangle has been drawn surrounding it. The area of the rectangle is $9 \times 3 = 27$ units^2. The part of the rectangle that is not inside OI′P′J′ has been divided up into two triangles and two trapezia and their areas are shown on the diagram.

So, area OI′P′J′ $= 27 - 2.5 - 8 - 6.5 - 4 = 6$ units^2.

The interesting question is whether you could predict this answer from the numbers in the matrix $\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$.

You can see that $5 \times 2 - 4 \times 1 = 6$. Is this just a coincidence?

To answer that question you need to transform the unit square by the general 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and see whether the area of the transformed figure is $(ad - bc)$ units^2. The answer is, 'Yes', and the proof is left for you to do in the activity below.

You are advised to use the same method as the example above but replace the numbers by the appropriate letters.

ACTIVITY 6.1

The unit square is transformed by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Prove that the resulting shape is a parallelogram with area $(ad - bc)$ units^2.

It is now evident that the quantity $(ad - bc)$ is the area scale factor associated with the transformation matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. It is called the **determinant** of the matrix.

Example 6.1

A shape S has area 8 cm^2. S is mapped to a shape T under the transformation represented by the matrix $\mathbf{M} = \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}$.

Find the area of shape T.

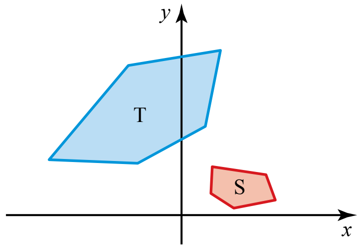


Figure 6.3

Note

In Example 6.1, it does not matter what shape S looks like; for any shape S with area 8 cm², the area of the image T will always be 48 cm².

Solution

$$\det\begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix} = (1 \times 0) - (-2 \times 3) = 0 + 6 = 6$$

The area scale factor of the transformation is 6 …

$$\text{Area of T} = 8 \times 6 = 48\,\text{cm}^2$$

… and so the area of the original shape is multiplied by 6.

Example 6.2

(i) Draw a diagram to show the image of the unit square OIPJ under the transformation represented by the matrix $\mathbf{M} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$.

(ii) Find det **M**.

(iii) Use your answer to part (ii) to find the area of the transformed shape.

Solution

(i) $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 5 & 3 \\ 0 & 4 & 5 & 1 \end{pmatrix}$

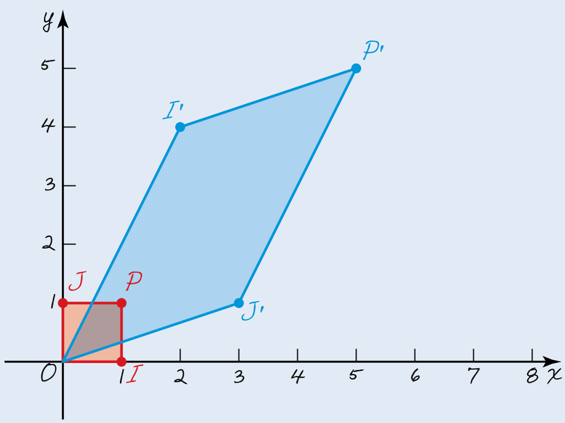


Figure 6.4

→

(ii) $\det\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} = (2 \times 1) - (3 \times 4) = 2 - 12 = -10$

(iii) The area of the transformed shape is 10 square units.

Notice that the determinant is negative. Since area cannot be negative, the area of the transformed shape is 10 square units.

The sign of the determinant does have significance. If you move anticlockwise around the original unit square you come to vertices O, I, P, J in that order. However, moving anticlockwise about the image reverses the order of the vertices, i.e. O, J′, P′, I′. This reversal in the order of the vertices produces the negative determinant.

> Which of the following transformations reverse the order of the vertices?
>
> (i) rotation
>
> (ii) reflection
>
> (iii) enlargement
>
> Check your answers by finding the determinants of matrices representing these transformations.

Example 6.3

Given that $\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, find

(i) det $\mathbf{P}$

(ii) det $\mathbf{Q}$

(iii) det $\mathbf{PQ}$.

What do you notice?

Solution

(i) $\det \mathbf{P} = 2 - 0 = 2$

(ii) $\det \mathbf{Q} = 4 - 1 = 3$

(iii) $\mathbf{PQ} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ $\det \mathbf{PQ} = 10 - 4 = 6$

The determinant of $\mathbf{PQ}$ is given by $\det \mathbf{P} \times \det \mathbf{Q}$.

Example 6.3 (iii) illustrates the general result that $\det \mathbf{MN} = \det \mathbf{M} \times \det \mathbf{N}$.

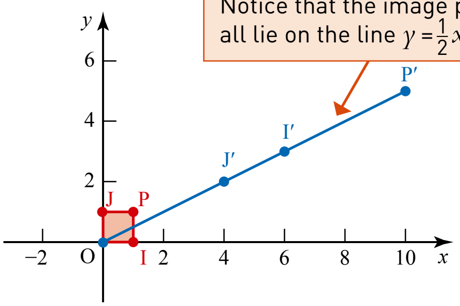
Remember that a transformation **MN** means 'apply **N**, then apply **M**'.

This result makes sense in terms of transformations. In Example 6.3, applying **Q** involves an area scale factor of 3, and applying **P** involves an area scale factor of 2. So applying **Q** followed by **P**, represented by the matrix **PQ**, involves an area scale factor of 6.

Matrices with determinant zero

Figure 6.5 shows the image of the unit square OIPJ under the transformation represented by the matrix $\mathbf{T} = \begin{pmatrix} 6 & 4 \\ 3 & 2 \end{pmatrix}$.

Notice that the image points all lie on the line $y = \frac{1}{2}x$.



▲ **Figure 6.5**

The determinant of $\mathbf{T} = (6 \times 2) - (4 \times 3) = 12 - 12 = 0$.

This means that the area scale factor of the transformation is zero, so any shape is transformed into a shape with area zero.

In this case, the image of a point (p, q) is given by

$$\begin{pmatrix} 6 & 4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 6p + 4q \\ 3p + 2q \end{pmatrix} = \begin{pmatrix} 2(3p + 2q) \\ 3p + 2q \end{pmatrix}.$$

You can see that for all the possible image points, the y coordinate is half the x coordinate, showing that all the image points lie on the line $y = \frac{1}{2}x$.

In this transformation, more than one point maps to the same image point.

For example, $(4, 0) \rightarrow (24, 12)$

$(0, 6) \rightarrow (24, 12)$

$(1, 4.5) \rightarrow (24, 12)$.

Exercise 6A

1 For each of the following matrices:

(a) draw a diagram to show the image of the unit square under the transformation represented by the matrix

(b) find the area of the image in part (a)

(c) find the determinant of the matrix.

(i) $\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 4 & 0 \\ -1 & 4 \end{pmatrix}$

(iii) $\begin{pmatrix} 4 & -8 \\ 1 & -2 \end{pmatrix}$ (iv) $\begin{pmatrix} 5 & -7 \\ -3 & 2 \end{pmatrix}$

2 The matrix $\begin{pmatrix} x-3 & -3 \\ 2 & x-5 \end{pmatrix}$ has determinant 9.

Find the possible values of x.

3 (i) Write down the matrices **A**, **B**, **C** and **D** which represent:

A – a reflection in the x-axis

B – a reflection in the y-axis

C – a reflection in the line $y = x$

D – a reflection in the line $y = -x$

(ii) Show that each of the matrices **A**, **B**, **C** and **D** has determinant of -1.

(iii) Draw diagrams for each of the transformations **A**, **B**, **C** and **D** to demonstrate that the images of the vertices labelled anticlockwise on the unit square OIPJ are reversed to a clockwise labelling.

4 A triangle has area $6\,\text{cm}^2$. The triangle is transformed by means of the matrix $\begin{pmatrix} 2 & 3 \\ -3 & 1 \end{pmatrix}$.

Find the area of the image of the triangle.

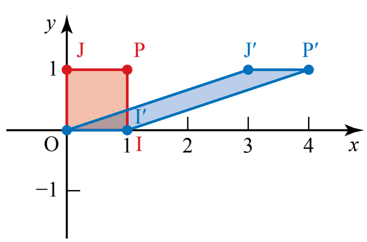
5 The two-way stretch with matrix $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ preserves the area (i.e. the area of the image is equal to the area of the original shape).

What is the relationship connecting a and d?

6 The diagram below shows the unit square transformed by a shear.

(i) Write down the matrix that represents this transformation.

(ii) Show that under this transformation the area of the image is always equal to the area of the object.



CP **7** $\mathbf{M} = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$.

(i) Find the determinants of **M** and **N**.

(ii) Find the matrix **MN** and show that det **MN** = det **M** × det **N**.

8 The plane is transformed by the matrix $\mathbf{M} = \begin{pmatrix} 4 & -6 \\ 2 & -3 \end{pmatrix}$.

(i) Draw a diagram to show the image of the unit square under the transformation represented by **M**.

(ii) Describe the effect of the transformation and explain this with reference to the determinant of **M**.

CP **9** The plane is transformed by the matrix $\mathbf{N} = \begin{pmatrix} 5 & -10 \\ -1 & 2 \end{pmatrix}$.

(i) Find the image of the point (p, q).

(ii) Hence show that the whole plane is mapped to a straight line and find the equation of this line.

(iii) Find the determinant of **N** and explain its significance.

PS **10** A matrix **T** maps all points on the line $x + 2y = 1$ to the point (1, 3).

(i) Find the matrix **T** and show that it has determinant of zero.

(ii) Show that **T** maps all points on the plane to the line $y = 3x$.

(iii) Find the coordinates of the point to which all points on the line $x + 2y = 3$ are mapped.

CP **11** The plane is transformed using the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $ad - bc = 0$.

Prove that the general point P(x, y) maps to P′ on the line $cx - ay = 0$.

12 The point P is mapped to P′ on the line $3y = x$ so that PP′ is parallel to the line $y = 3x$.

(i) Find the equation of the line parallel to $y = 3x$ passing through the point P with coordinates (s, t).

(ii) Find the coordinates of P′, the point where this line meets $3y = x$.

(iii) Find the matrix of the transformation that maps P to P′ and show that the determinant of this matrix is zero.

6.2 The inverse of a matrix

The identity matrix

Whenever you multiply a 2 × 2 matrix **M** by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the product is **M**.

It makes no difference whether you **pre-multiply**, for example,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 6 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 6 & 3 \end{pmatrix}$$

or **post-multiply**

$$\begin{pmatrix} 4 & -2 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 6 & 3 \end{pmatrix}.$$

ACTIVITY 6.2

(i) Write down the matrix **P** that represents a reflection in the x-axis.

(ii) Find the matrix $\mathbf{P}^2$.

(iii) Comment on your answer.

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is known as the 2×2 identity matrix.

Identity matrices are often denoted by the letter **I**.

For multiplication of matrices, **I** behaves in the same way as the number 1 when dealing with the multiplication of real numbers.

The transformation represented by the identity matrix maps every points to itself.

Similarly, the 3×3 identity matrix is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Example 6.4

(i) Write down the matrix **A** that represents a rotation of 90° anticlockwise about the origin.

(ii) Write down the matrix **B** that represents a rotation of 90° clockwise about the origin.

(iii) Find the product **AB** and comment on your answer.

Solution

(i) $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(ii) $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(iii) $\mathbf{AB} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

AB represents a rotation of 90° clockwise followed by a rotation of 90° anticlockwise. The result of this is to return to the starting point.

To undo the effect of a rotation through 90° anticlockwise about the origin, you need to carry out a rotation through 90° clockwise about the origin. These two transformations are inverses of each other.

Similarly, the matrices that represent these transformations are inverses of each other.

In Example 6.4, $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the inverse of $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and vice versa.

Finding the inverse of a 2 × 2 matrix

Finding the inverse of 3 × 3 matrices is covered in Section 6.3.

If the product of two square matrices, **M** and **N**, is the identity matrix **I**, then **N** is the inverse of **M**. You can write this as $\mathbf{N} = \mathbf{M}^{-1}$.

Generally, if $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ you need to find an inverse matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

ACTIVITY 6.3

Multiply $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

What do you notice?

Use your result to write down the inverse of the general matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

How does the determinant $|\mathbf{M}|$ relate to the matrix $\mathbf{M}^{-1}$?

You should have found in the activity that the inverse of the matrix

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is given by}$$

$$\mathbf{M}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

If the determinant is zero then the inverse matrix does not exist and the matrix is said to be **singular**. If det $\mathbf{M} \neq 0$ the matrix is said to be **non-singular**.

If a matrix is singular, then it maps all points on the plane to a straight line. So an infinite number of points are mapped to the same point on the straight line. It is therefore not possible to find the inverse of the transformation, because an inverse matrix would map a point on that straight line to just one other point, not to an infinite number of them.

A special case is the zero matrix, which maps all points to the origin.

Example 6.5

$\mathbf{A} = \begin{pmatrix} 11 & 3 \\ 6 & 2 \end{pmatrix}$

(i) Find $\mathbf{A}^{-1}$.

(ii) The point P is mapped to the point Q (5, 2) under the transformation represented by **A**. Find the coordinates of P.

Solution

(i) $\det \mathbf{A} = (11 \times 2) - (3 \times 6) = 4$

$$\mathbf{A}^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -3 \\ -6 & 11 \end{pmatrix}$$

(ii) $$\mathbf{A}^{-1}\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & -3 \\ -6 & 11 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 4 \\ -8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

A maps P to Q, so $\mathbf{A}^{-1}$ maps Q to P.

The coordinates of P are (1, –2).

As matrix multiplication is generally non-commutative, it is interesting to find out if $\mathbf{MM}^{-1} = \mathbf{M}^{-1}\mathbf{M}$. The next activity investigates this.

ACTIVITY 6.4

(i) In Example 6.5 you found that the inverse of $\mathbf{A} = \begin{pmatrix} 11 & 3 \\ 6 & 2 \end{pmatrix}$ is $\mathbf{A}^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -3 \\ -6 & 11 \end{pmatrix}$.

Show that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

(ii) If the matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, write down $\mathbf{M}^{-1}$ and show that

$\mathbf{MM}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$.

The result $\mathbf{MM}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$ is important as it means that the inverse of a matrix, if it exists, is unique. This is true for all square matrices, not just 2×2 matrices.

?

- How would you reverse the effect of a rotation followed by a reflection?
- How would you write down the inverse of a matrix product **MN** in terms of $\mathbf{M}^{-1}$ and $\mathbf{N}^{-1}$?

The inverse of a product of matrices

Suppose you want to find the inverse of the product **MN**, where **M** and **N** are non-singular matrices. This means that you need to find a matrix **X** such that

$$\mathbf{X}(\mathbf{MN}) = \mathbf{I}.$$

$$\mathbf{X}(\mathbf{MN}) = \mathbf{I} \Rightarrow \mathbf{XMNN}^{-1} = \mathbf{IN}^{-1}$$ ← Post multiply by $\mathbf{N}^{-1}$

$$\Rightarrow \mathbf{XM} = \mathbf{IN}^{-1}$$ ← Using $\mathbf{NN}^{-1} = \mathrm{I}$

$$\Rightarrow \mathbf{XMM}^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}$$ ← Post multiply by $\mathbf{M}^{-1}$

$$\Rightarrow \mathbf{X} = \mathbf{N}^{-1}\mathbf{M}^{-1}$$ ← Using $\mathbf{MM}^{-1} = \mathrm{I}$

So $(\mathbf{MN})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}$ for matrices **M** and **N** of the same order. This means that when working backwards, you must reverse the second transformation before reversing the first transformation.

Technology note

Investigate how to use a calculator to find the inverse of 2×2 and 3×3 matrices.

Check using a calculator that multiplying a matrix by its inverse gives the identity matrix.

Exercise 6B

1 For the matrix $\begin{pmatrix} 5 & -1 \\ -2 & 0 \end{pmatrix}$

(i) find the image of the point (3, 5)

(ii) find the inverse matrix

(iii) find the point that maps to the image (3, –2).

2 Determine whether the following matrices are singular or non-singular. For those that are non-singular, find the inverse.

(i) $\begin{pmatrix} 6 & 3 \\ -4 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$ (iii) $\begin{pmatrix} 11 & 3 \\ 3 & 11 \end{pmatrix}$

(iv) $\begin{pmatrix} 11 & 11 \\ 3 & 3 \end{pmatrix}$ (v) $\begin{pmatrix} 2 & -7 \\ 0 & 0 \end{pmatrix}$

(vi) $\begin{pmatrix} -2a & 4a \\ 4b & -8b \end{pmatrix}$ (vii) $\begin{pmatrix} -2 & 4a \\ 4b & -8 \end{pmatrix}$

3 $\mathbf{M} = \begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 8 & 5 \\ -2 & -1 \end{pmatrix}$.

Calculate the following:

(i) $\mathbf{M}^{-1}$ (ii) $\mathbf{N}^{-1}$ (iii) $\mathbf{MN}$

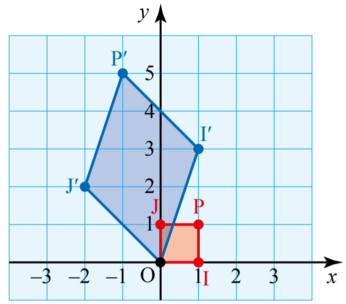
(iv) $\mathbf{NM}$ (v) $(\mathbf{MN})^{-1}$ (vi) $(\mathbf{NM})^{-1}$

(vii) $\mathbf{M}^{-1}\mathbf{N}^{-1}$ (viii) $\mathbf{N}^{-1}\mathbf{M}^{-1}$

4 The diagram shows the unit square OIPJ mapped to the image OI′P′J′ under a transformation represented by a matrix $\mathbf{M}$.

(i) Find the inverse of $\mathbf{M}$.

(ii) Use matrix multiplication to show that $\mathbf{M}^{-1}$ maps OI′P′J′ back to OIPJ.



5 The matrix $\begin{pmatrix} 1-k & 2 \\ -1 & 4-k \end{pmatrix}$ is singular.

Find the possible values of k.

PS **6** Given that $\mathbf{M} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$ and $\mathbf{MN} = \begin{pmatrix} 7 & 2 & -9 & 10 \\ 2 & -1 & -12 & 17 \end{pmatrix}$, find the matrix $\mathbf{N}$.

7 Triangle T has vertices at (1, 0), (0, 1) and (−2, 0).

It is transformed to triangle T′ by the matrix $\mathbf{M} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$.

(i) Find the coordinates of the vertices of T′.

Show the triangles T and T′ on a single diagram.

(ii) Find the ratio of the area of T′ to the area of T.

Comment on your answer in relation to the matrix $\mathbf{M}$.

(iii) Find $\mathbf{M}^{-1}$ and verify that this matrix maps the vertices of T′ to the vertices of T.

CP **8** $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a singular matrix.

(i) Show that $\mathbf{M}^2 = (a + d)\mathbf{M}$.

(ii) Find a formula that expresses $\mathbf{M}^n$ in terms of $\mathbf{M}$, where n is a positive integer.

Comment on your results.

9 Given that $\mathbf{PQR} = \mathbf{I}$, show algebraically that

(i) $\mathbf{Q} = \mathbf{P}^{-1}\mathbf{R}^{-1}$ (ii) $\mathbf{Q}^{-1} = \mathbf{RP}$.

Given that $\mathbf{P} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 12 & -3 \\ 2 & -1 \end{pmatrix}$

(iii) use part (i) to find the matrix $\mathbf{Q}$

(iv) calculate the matrix $\mathbf{Q}^{-1}$

(v) verify that your answer to part (ii) is correct by calculating $\mathbf{RP}$ and comparing it with your answer to part (iv).

6.3 Finding the inverse of a 3 × 3 matrix

The determinant of a 3 × 3 matrix is sometimes denoted |***a b c***|.

In this section you will find the determinant and inverse of 3 × 3 matrices using the calculator facility and also using a non-calculator method.

Finding the inverse of a 3 × 3 matrix using a calculator

ACTIVITY 6.5

Using a calculator, find the determinant and inverse of the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -2 & 1 \\ 0 & 1 & 2 \\ 4 & 0 & 1 \end{pmatrix}.$$

Still using a calculator, find out which of the following matrices are non-singular and find the inverse in each of these cases.

$$\mathbf{B} = \begin{pmatrix} 5 & 5 & 5 \\ 2 & 2 & 2 \\ 2 & 4 & -3 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 0 & 1 \\ 2 & 1 & 4 \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} 0 & 3 & -2 \\ 1 & -1 & 2 \\ 3 & 0 & 3 \end{pmatrix}$$

Finding the inverse of a 3 × 3 matrix without using a calculator

It is also possible to find the determinant and inverse of a 3 × 3 matrix without using a calculator. This is useful in cases where some of the elements of the matrix are algebraic rather than numerical.

If **M** is the 3 × 3 matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ then the determinant of **M** is defined by

$$\det \mathbf{M} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix},$$

which is sometimes referred to as the **expansion of the determinant by the first column**.

For example, to find the determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -2 & 1 \\ 0 & 1 & 2 \\ 4 & 0 & 1 \end{pmatrix} \text{ from Activity 6.5:}$$

$$\begin{aligned} \det \mathbf{A} &= 3\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} -2 & 1 \\ 0 & 1 \end{vmatrix} + 4\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} \\ &= 3(1-0) - 0(-2-0) + 4(-4-1) \\ &= 3 - 20 \\ &= -17 \end{aligned}$$

Notice that you do not really need to calculate $\begin{vmatrix} -2 & 1 \\ 0 & 1 \end{vmatrix}$ as it is going to be multiplied by zero. Keeping an eye open for helpful zeros can reduce the number of calculations needed.

This is the same answer as you will have obtained earlier using your calculator.

The 2×2 determinant $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ is called the minor of the element a_1. It is obtained by deleting the row and column containing a_1:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Other minors are defined in the same way, for example the minor of a_2 is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$$

You may have noticed that in the expansions of the determinant, the signs on the minors alternate as shown:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

A minor, together with its correct sign, is known as a **cofactor** and is denoted by the corresponding capital letter; for example, the cofactor of a_3 is A_3. This means that the expansion by the first column, say, can be written as

$$a_1 A_1 + a_2 A_2 + a_3 A_3.$$

Note

As an alternative to using the first column, you could use the **expansion of the determinant by the second column**:

$$\det \mathbf{M} = -b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix},$$

or the **expansion of the determinant by the third column**:

$$\det \mathbf{M} = c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

It is fairly easy to show that all three expressions above for det **M** simplify to:

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3$$

Example 6.6

Find the determinant of the matrix $\mathbf{M} = \begin{pmatrix} 3 & 0 & -4 \\ 7 & 2 & -1 \\ -2 & 1 & 3 \end{pmatrix}$.

To find the determinant you can also expand by rows. So, for example, expanding by the top row would give:

$$3\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} - 0\begin{vmatrix} 7 & -1 \\ -2 & 3 \end{vmatrix} + (-4)\begin{vmatrix} 7 & 2 \\ -2 & 1 \end{vmatrix}$$

which also gives the answer –23.

Solution

Expanding by the first column using the expression:

$$\det \mathbf{M} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

gives:

$$\det \mathbf{M} = 3\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} - 7\begin{vmatrix} 0 & -4 \\ 1 & 3 \end{vmatrix} + (-2)\begin{vmatrix} 0 & -4 \\ 2 & -1 \end{vmatrix}$$

$$= 3(6-(-1)) - 7(0-(-4)) - 2(0-(-8))$$

$$= 21 - 28 - 16$$

$$= -23$$

Notice that expanding by the top row would be quicker here as it has a zero element.

Earlier you saw that the determinant of a 2×2 matrix represents the area scale factor of the transformation represented by the matrix. In the case of a 3×3 matrix the determinant represents the volume scale factor. For example, the matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ has determinant 8; this matrix represents an enlargement of scale factor 2, centre the origin, so the volume scale factor of the transformation is $2 \times 2 \times 2 = 8$.

Recall that a minor, together with its correct sign, is known as a cofactor and is denoted by the corresponding capital letter; for example the cofactor of a_3 is A_3.

For the matrix $\mathbf{M} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, the matrix $\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$ is known as the **adjugate** or **adjoint** of $\mathbf{M}$, denoted **adj M**.

The adjugate of $\mathbf{M}$ is formed by

- replacing each element of $\mathbf{M}$ by its cofactor;
- then transposing the matrix (i.e. changing rows into columns and columns into rows).

The unique inverse of a 3×3 matrix can be calculated as follows:

$$\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}} \text{adj } \mathbf{M} = \frac{1}{\det \mathbf{M}} \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}, \ \det \mathbf{M} \neq 0$$

The steps involved in the method are shown in the following example.

Example 6.7

Find the inverse of the matrix $\mathbf{M}$ without using a calculator, where

$$\mathbf{M} = \begin{pmatrix} 2 & 3 & 4 \\ 2 & -5 & 2 \\ -3 & 6 & -3 \end{pmatrix}.$$

Solution

Step 1: Find the determinant Δ and check $\Delta \neq 0$.

Expanding by the first column

$$\Delta = 2\begin{vmatrix} -5 & 2 \\ 6 & -3 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 \\ 6 & -3 \end{vmatrix} + (-3)\begin{vmatrix} 3 & 4 \\ -5 & 2 \end{vmatrix}$$

$$= (2 \times 3) - (2 \times -33) - (3 \times 26) = -6$$

Therefore the inverse matrix exists.

Step 2: Evaluate the cofactors.

$$A_1 = \begin{vmatrix} -5 & 2 \\ 6 & -3 \end{vmatrix} = 3 \qquad A_2 = -\begin{vmatrix} 3 & 4 \\ 6 & -3 \end{vmatrix} = 33 \quad A_3 = \begin{vmatrix} 3 & 4 \\ -5 & 2 \end{vmatrix} = 26$$

$B_1 = -\begin{vmatrix} 2 & 2 \\ -3 & -3 \end{vmatrix} = 0 \quad B_2 = \begin{vmatrix} 2 & 4 \\ -3 & -3 \end{vmatrix} = 6 \quad B_3 = -\begin{vmatrix} 2 & 4 \\ 2 & 2 \end{vmatrix} = 4$

$C_1 = \begin{vmatrix} 2 & -5 \\ -3 & 6 \end{vmatrix} = -3 \quad C_2 = -\begin{vmatrix} 2 & 3 \\ -3 & 6 \end{vmatrix} = -21 \quad C_3 = \begin{vmatrix} 2 & 3 \\ 2 & -5 \end{vmatrix} = -16$

You can evaluate the determinant Δ using these cofactors to check your earlier arithmetic is correct:

2nd column:

$\Delta = 3B_1 - 5B_2 + 6B_3 = (3 \times 0) - (5 \times 6) + (6 \times 4) = -6$

3rd column:

$\Delta = 4C_1 + 2C_2 - 3C_3 = (4 \times -3) + (2 \times -21) - (3 \times -16) = -6$

Step 3: Form the matrix of cofactors and transpose it, then multiply by $\frac{1}{\Delta}$

$$\mathbf{M}^{-1} = \frac{1}{-6}\begin{pmatrix} 3 & 0 & -3 \\ 33 & 6 & -21 \\ 26 & 4 & -16 \end{pmatrix}^T$$

Multiply by $\frac{1}{\Delta}$.

The capital T indicates the matrix is to be transposed.

Matrix of cofactors.

$$= \frac{1}{-6}\begin{pmatrix} 3 & 33 & 26 \\ 0 & 6 & 4 \\ -3 & -21 & -16 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} -3 & -33 & -26 \\ 0 & -6 & -4 \\ 3 & 21 & 16 \end{pmatrix}$$

The final matrix could then be simplified and written as

$$\mathbf{M}^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{11}{2} & -\frac{13}{3} \\ 0 & -1 & -\frac{2}{3} \\ \frac{1}{2} & \frac{7}{2} & \frac{8}{3} \end{pmatrix}$$

Check: $\mathbf{MM}^{-1} = \begin{pmatrix} 2 & 3 & 4 \\ 2 & -5 & 2 \\ -3 & 6 & -3 \end{pmatrix} \frac{1}{6}\begin{pmatrix} -3 & -33 & -26 \\ 0 & -6 & -4 \\ 3 & 21 & 16 \end{pmatrix}$

$$= \frac{1}{6}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This adjugate method for finding the inverse of a 3×3 matrix is reasonably straightforward but it is important to check your arithmetic as you go along, as it is very easy to make mistakes. You can use your calculator to check that you have calculated the inverse correctly.

As shown in Example 6.7, you might also multiply the inverse by the original matrix and check that you obtain the 3×3 identity matrix.

Exercise 6C

1 Evaluate these determinants without using a calculator. Check your answers using your calculator.

(i) (a) $\begin{vmatrix} 1 & 1 & 3 \\ -1 & 0 & 2 \\ 3 & 1 & 4 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & -1 & 3 \\ 1 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix}$

(ii) (a) $\begin{vmatrix} 1 & -5 & -4 \\ 2 & 3 & 3 \\ -2 & 1 & 0 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 2 & -2 \\ -5 & 3 & 1 \\ -4 & 3 & 0 \end{vmatrix}$

(iii) (a) $\begin{vmatrix} 2 & 1 & 2 \\ 3 & 5 & 3 \\ 1 & -1 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 5 & 0 \\ 1 & 5 & 0 \\ 2 & 1 & -2 \end{vmatrix}$

What do you notice about the determinants?

2 Find the inverses of the following matrices, if they exist, without using a calculator.

(i) $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 5 \\ 0 & 1 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & 2 & 6 \\ 5 & 3 & 11 \\ 7 & 4 & 16 \end{pmatrix}$

(iii) $\begin{pmatrix} 5 & 5 & -5 \\ -9 & 3 & -5 \\ -4 & -6 & 8 \end{pmatrix}$ (iv) $\begin{pmatrix} 6 & 5 & 6 \\ -5 & 2 & -4 \\ -4 & -6 & -5 \end{pmatrix}$

3 Find the inverse of the matrix $\mathbf{M} = \begin{pmatrix} 1 & 3 & -2 \\ k & 0 & 4 \\ 2 & -1 & 4 \end{pmatrix}$ where $k \neq 0$.

For what value of k is the matrix $\mathbf{M}$ singular?

PS **4** (i) Investigate the relationship between the matrices

$$\mathbf{A} = \begin{pmatrix} 0 & 3 & 1 \\ 2 & 4 & 2 \\ -1 & 3 & 5 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 2 & 4 \\ 5 & -1 & 3 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 3 & 1 & 0 \\ 4 & 2 & 2 \\ 3 & 5 & -1 \end{pmatrix}$$

(ii) Find det $\mathbf{A}$, det $\mathbf{B}$ and det $\mathbf{C}$ and comment on your answer.

5 Show that $x = 1$ is one root of the equation $\begin{vmatrix} 2 & 2 & x \\ 1 & x & 1 \\ x & 1 & 4 \end{vmatrix} = 0$ and find the other roots.

6 Find the values of x for which the matrix $\begin{pmatrix} 3 & -1 & 1 \\ 2 & x & 4 \\ x & 1 & 3 \end{pmatrix}$ is singular.

7 Given that the matrix $\mathbf{M} = \begin{pmatrix} k & 2 & 1 \\ 0 & -k & 2 \\ 2k & 1 & 3 \end{pmatrix}$ has determinant greater than 5, find the range of possible values for k.

CP **8** (i) $\mathbf{P}$ and $\mathbf{Q}$ are non-singular matrices. Prove that $(\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}$.

(ii) Find the inverses of the matrices $\mathbf{P} = \begin{pmatrix} 0 & 3 & -1 \\ -2 & 2 & 2 \\ -3 & 0 & 1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 4 & -3 & 2 \end{pmatrix}$.

Using the result from part (i), find $(\mathbf{PQ})^{-1}$.

9 (i) Prove that $\begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, where k is a constant.

(ii) Explain in terms of volumes, why multiplying all the elements in the first column by a constant k multiplies the value of the determinant by k.

(iii) What would happen if you multiplied a different column by k?

10 Given that $\begin{vmatrix} 1 & 2 & 3 \\ 6 & 4 & 5 \\ 7 & 5 & 1 \end{vmatrix} = 43$, write down the values of the determinants:

(i) $\begin{vmatrix} 10 & 2 & 3 \\ 60 & 4 & 5 \\ 70 & 5 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} 4 & 10 & -21 \\ 24 & 20 & -35 \\ 28 & 25 & -7 \end{vmatrix}$

(iii) $\begin{vmatrix} x & 4 & 3y \\ 6x & 8 & 5y \\ 7x & 10 & y \end{vmatrix}$ (iv) $\begin{vmatrix} x^4 & \frac{1}{x} & 12y \\ 6x^4 & \frac{2}{x} & 20y \\ 7x^4 & \frac{5}{2x} & 4y \end{vmatrix}$

KEY POINTS

1 If $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the determinant of $\mathbf{M}$, written det $\mathbf{M}$ or $|\mathbf{M}|$, is given by $\det \mathbf{M} = ad - bc$.

2 The determinant of a 2×2 matrix represents the area scale factor of the transformation.

3 If $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\mathbf{M}^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

4 The determinant of a 3×3 matrix $\mathbf{M} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ is given by

$$\det \mathbf{M} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

5 For a 3×3 matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ the minor of an element is formed by crossing out the row and column containing that element and finding the determinant of the resulting 2×2 matrix.

6 A minor, together with its correct sign, given by the matrix $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$ is known as a cofactor and is denoted by the corresponding capital letter; for example the cofactor of a_3 is A_3.

7 The inverse of a 3×3 matrix $\mathbf{M} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ can be found using a calculator or using the formula

$$\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}} \operatorname{adj} \mathbf{M} = \frac{1}{\det \mathbf{M}}\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}, \ \Delta \neq 0.$$

The matrix $\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$ is the adjoint or adjugate matrix, denoted adj $\mathbf{M}$, formed by replacing each element of $\mathbf{M}$ by its cofactor and then transposing (i.e. changing rows into columns and columns into rows).

8 $(\mathbf{MN})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}$

9 A matrix is singular if the determinant is zero. If the determinant is non-zero the matrix is said to be non-singular.

10 If the determinant of a matrix is zero, all points are mapped to either a straight line (in two dimensions) or to a plane (three dimensions).

11 If $\mathbf{A}$ is a non-singular matrix, $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

LEARNING OUTCOMES

Now that you have finished this chapter, you should be able to

- find the determinant of 2×2 and 3×3 matrices using the notation det $\mathbf{M}$
- recall the meaning of the terms
  - singular
  - non-singular, as applied to square matrices
- recall how the area scale factor of a transformation is related to the determinant of the corresponding 2×2 matrix
- understand the significance of a zero determinant in terms of transformations
- recognise the identity matrix
- find the inverses
  - of non-singular 2×2 matrices
  - of non-singular 3×3 matrices
- understand that for non-singular matrices $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ and this can be extended to the product of more than two matrices
- understand the relationship between the transformations represented by $\mathbf{A}$ and $\mathbf{A}^{-1}$.

7 Vectors

Why is there space rather than no space? Why is space three-dimensional? Why is space big? We have a lot of room to move around in. How come it's not tiny? We have no consensus about these things. We're still exploring them.

Leonard Susskind (1940–)

› Are there any right angles in the building shown above?

Note

From the work on vectors in *Pure Mathematics 2*, you should be able to use the language of vectors, including the terms magnitude, direction and position vector. You should also be able to find the distance between two points represented by position vectors and be able to add and subtract vectors and multiply a vector by a scalar. You should know how to find the scalar product of two vectors and use it to find the angle between vectors. You should also understand the significance of the terms in the equation of a line in vector form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$.

7.1 The vector equation of a plane

- Which balances better, a three-legged stool or a four-legged stool? Why?
- What information do you need to specify a particular plane?

There are various ways of finding the equation of a plane and these are given in this book. Your choice of which one to use will depend on the information you are given.

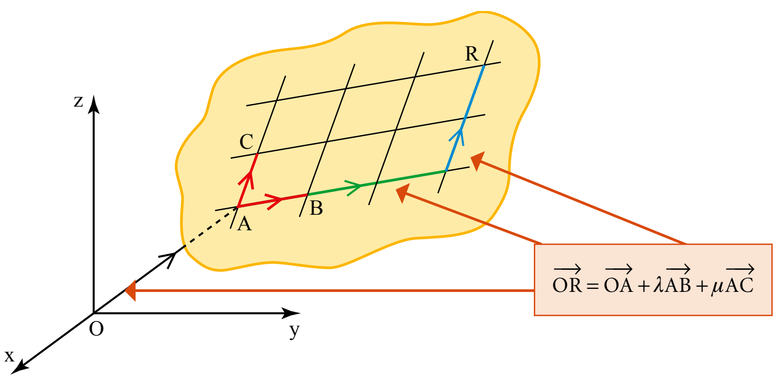
Finding the equation of a plane given three points on it

There are several methods used to find the equation of a plane through three points. The shortest method involves the use of vector products which can be found in Section 7.7, later in this chapter.

Vector form

To find the vector form of the equation of the plane through the points A, B and C (with position vectors $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$), think of starting at the origin, travelling along OA to join the plane at A, and then any distance in each of the directions $\overrightarrow{AB}$ and $\overrightarrow{AC}$ to reach a general point R with position vector **r**, where

$$\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}.$$



▲ **Figure 7.1**

This is a vector form of the equation of the plane. Since $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$, it may also be written as

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a}).$$

Example 7.1

Find the equation of the plane through A(4, 2, 0), B(3, 1, 1) and C(4, −1, 1).

Solution

$$\overrightarrow{OA} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$$

So the equation $\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}$ becomes

$$\mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}.$$

This is the vector form of the equation, written using components.

You can convert this equation into **Cartesian form** by writing it as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$$

and eliminating λ and μ. The three equations contained in this vector equation may be simplified to give

$$\lambda = -x + 4 \quad ①$$

$$\lambda + 3\mu = -y + 2 \quad ②$$

$$\lambda + \mu = z \quad ③$$

Substituting ① into ② gives

$$-x + 4 + 3\mu = -y + 2$$

$$3\mu = x - y - 2$$

$$\mu = \tfrac{1}{3}(x - y - 2)$$

Substituting this and ① into ③ gives

$$-x + 4 + \tfrac{1}{3}(x - y - 2) = z$$

$$-3x + 12 + x - y - 2 = 3z$$

$$2x + y + 3z = 10$$

and this is the Cartesian equation of the plane through A, B and C.

Note

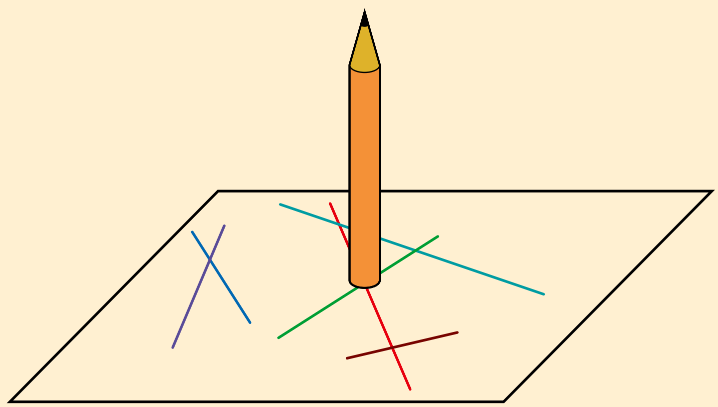
In contrast to the equation of a line, the equation of a plane is more neatly expressed in Cartesian form. The general Cartesian equation of a plane is often written as either

$$ax + by + cz = d \qquad \text{or} \qquad n_1x + n_2y + n_3z = d.$$

Finding the equation of a plane using the direction perpendicular to it

?

- Lay a sheet of paper on a flat horizontal table and mark several straight lines on it. Now take a pencil and stand it upright on the sheet of paper (see Figure 7.2).



▲ **Figure 7.2**

(i) What angle does the pencil make with any individual line?

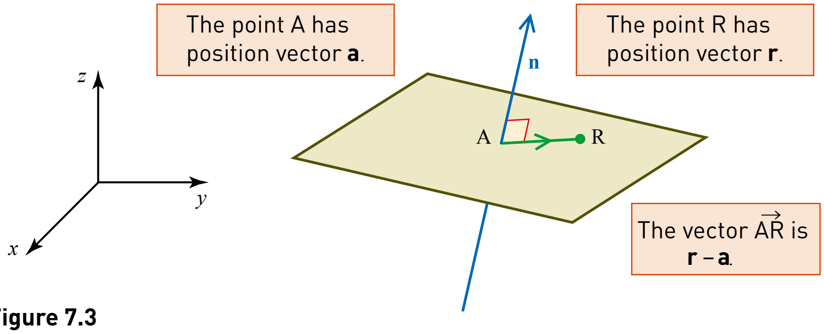
(ii) Would it make any difference if the table were tilted at an angle (apart from the fact that you could no longer balance the pencil)?

The discussion above shows you that there is a direction (that of the pencil) which is at right angles to every straight line in the plane. A line in that direction is said to be perpendicular to the plane or **normal** to the plane.

This allows you to find a different vector form of the equation of a plane, which you use when you know the position vector $\mathbf{a}$ of one point A in the plane and the direction $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ perpendicular to the plane.

What you want to find is an expression for the position vector **r** of a general point R in the plane (see Figure 7.3). Since AR is a line in the plane, it follows that AR is at right angles to the direction **n**.

$$\overrightarrow{AR}\,.\,\mathbf{n} = 0$$



▲ **Figure 7.3**

The vector $\overrightarrow{AR}$ is given by

$$\overrightarrow{AR} = \mathbf{r} - \mathbf{a}$$

and so $(\mathbf{r} - \mathbf{a})\,.\,\mathbf{n} = 0.$

For example, the plane through A $(2, 0, 0)$ perpendicular to $\mathbf{n} = (3\mathbf{i} - 4\mathbf{j} + \mathbf{k})$ can be written as $(\mathbf{r} - 2\mathbf{i})\,.\,(3\mathbf{i} - 4\mathbf{j} + \mathbf{k}) = 0$, which simplifies to $3x - 4y + z = 6$.

This can also be written as

$$\mathbf{r}\,.\,\mathbf{n} - \mathbf{a}\,.\,\mathbf{n} = 0$$

or

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} - \mathbf{a}\,.\,\mathbf{n} = 0$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \mathbf{a}\,.\,\mathbf{n}$$

$$\Rightarrow \quad n_1x + n_2y + n_3z = d$$

where $d = \mathbf{a}\,.\,\mathbf{n}$.

Notice that d is a constant scalar.

Example 7.2

Write down the equation of the plane through the point (2, 1, 3) given that the vector $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ is perpendicular to the plane.

Solution

In this case, the position vector **a** of the point (2, 1, 3) is given by $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$.

The vector perpendicular to the plane is

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$

The equation of the plane is

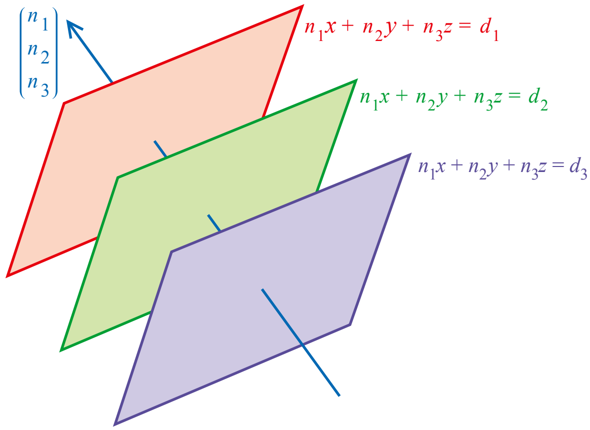
$$\begin{aligned} n_1x + n_2y + n_3z &= \mathbf{a}\,.\,\mathbf{n} \\ 4x + 5y + 6z &= 2 \times 4 + 1 \times 5 + 3 \times 6 \\ 4x + 5y + 6z &= 31 \end{aligned}$$

Look carefully at the equation of the plane in Example 7.2. You can see at once that the vector $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$, formed from the coefficients of x, y and z, is perpendicular to the plane.

The vector $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ is perpendicular to all planes of the form

$$n_1x + n_2y + n_3z = d$$

whatever the value of d (see Figure 7.4). Consequently, all planes of that form are parallel; the coefficients of x, y and z determine the direction of the plane, the value of d its location.



▲ **Figure 7.4**

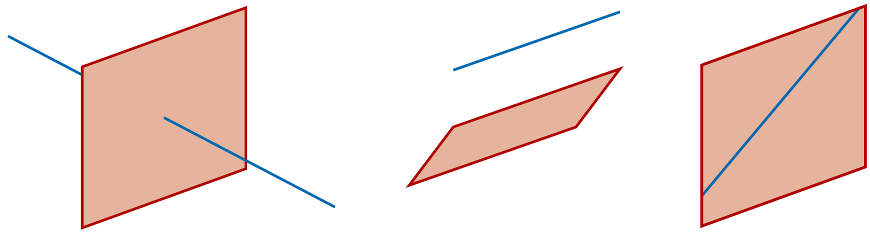
7.2 The intersection of a line and a plane

There are three possibilities for the intersection of a line and a plane.

1 The line and plane are not parallel and so they intersect in one point

2 The line and plane are parallel and so do not intersect

3 The line and plane are parallel and the line lies in the plane



▲ **Figure 7.5**

The point of intersection of a line and a plane is found by following the procedure in the next example.

Example 7.3

Find the point of intersection of the line $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ with the plane $5x + y - z = 1$.

Solution

The line is

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

and so for any point on the line

$$x = 2 + \lambda \qquad y = 3 + 2\lambda \qquad \text{and} \qquad z = 4 - \lambda.$$

Substituting these into the equation of the plane $5x + y - z = 1$ gives

$$\begin{aligned} 5(2 + \lambda) + (3 + 2\lambda) - (4 - \lambda) &= 1 \\ 8\lambda &= -8 \\ \lambda &= -1. \end{aligned}$$

Substituting $\lambda = -1$ in the equation of the line gives

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$$

so the point of intersection is (1, 1, 5).

As a check, substitute (1, 1, 5) into the equation of the plane:

$$\begin{aligned} 5x + y - z &= 5 + 1 - 5 \\ &= 1 \text{ as required.} \end{aligned}$$

When a line is parallel to a plane, its direction vector is perpendicular to the plane's normal vector.

Example 7.4

Show that the line $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ is parallel to the plane $2x + 4y + 5z = 8$.

Solution

The direction of the line is $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ and of the normal to the plane is $\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$.

If these two vectors are perpendicular, then the line and plane are parallel.

To prove that two vectors are perpendicular, you need to show that their scalar product is 0.

$$\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} . \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} = 3 \times 2 + 1 \times 4 + (-2) \times 5 = 0$$

So the line and plane are parallel as required.

To prove that a line lies in a plane, you need to show the line and the plane are parallel and that any point on the line also lies in the plane.

Example 7.5

Does the line $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ lie in the plane $2x + 4y + 5z = 8$?

Solution

You have already seen that this line and plane are parallel in Example 7.4.

Find a point on the line $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ by setting $t = 1$.

So the point $(5, 2, -2)$ lies on the line.

Now check that this point satisfies the equation of the plane, $2x + 4y + 5z = 8$.

$$2 \times 5 + 4 \times 2 + 5\,(-2) = 8 \checkmark$$

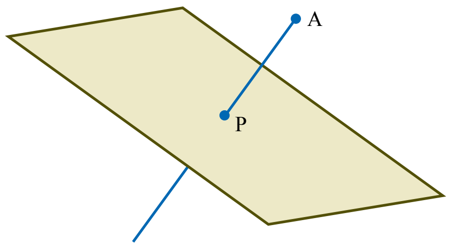
The line and the plane are parallel and the point $(5, 2, -2)$ lies both on the line and in the plane. Therefore the line must lie in the plane.

Note

The previous two examples showed you that the line $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ lies in the plane $2x + 4y + 5z = 8$. This line is parallel to all the planes in the form $2x + 4y + 5z = d$ but in the case when $d = 8$ it lies in the plane; for other values of d the line and the plane never meet.

7.3 The distance of a point from a plane

The shortest distance of a point, A, from a plane is the distance AP, where P is the point where the line through A perpendicular to the plane intersects the plane (see Figure 7.6). This is usually just called the distance of the point from the plane. The process of finding this distance is shown in the next example.



▲ **Figure 7.6**

Example 7.6

A is the point (7, 5, 3) and the plane π has the equation $3x + 2y + z = 6$. Find

(i) the equation of the line through A perpendicular to the plane π

(ii) the point of intersection, P, of this line with the plane

(iii) the distance AP.

Solution

(i) The direction perpendicular to the plane $3x + 2y + z = 6$ is $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ so the line through (7, 5, 3) perpendicular to the plane is given by

$$\mathbf{r} = \begin{pmatrix} 7 \\ 5 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

(ii) For any point on the line

$$x = 7 + 3\lambda \qquad y = 5 + 2\lambda \qquad \text{and} \qquad z = 3 + \lambda.$$

Substituting these expressions into the equation of the plane $3x + 2y + z = 6$ gives

$$3(7 + 3\lambda) + 2(5 + 2\lambda) + (3 + \lambda) = 6$$
$$14\lambda = -28$$
$$\lambda = -2.$$

So the point P has coordinates (1, 1, 1).

(iii) The vector $\overrightarrow{AP}$ is given by

$$\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}7\\5\\3\end{pmatrix} = \begin{pmatrix}-6\\-4\\-2\end{pmatrix}$$

and so the length AP is $\sqrt{(-6)^2 + (-4)^2 + (-2)^2} = \sqrt{56}$.

Note

In practice, you would not usually follow the procedure in Example 7.6 because there is a well-known formula for the distance of a point from a plane. You are invited to derive this in the following activity.

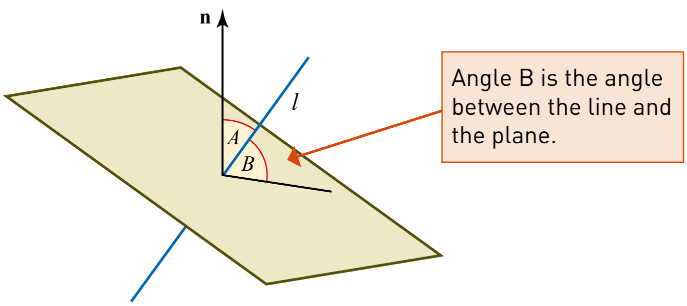
ACTIVITY 7.1

Generalise the work in Example 7.6 to show that the distance of the point (α, β, γ) from the plane $n_1x + n_2y + n_3z = d$ is given by

$$\frac{|n_1\alpha + n_2\beta + n_3\gamma - d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}.$$

7.4 The angle between a line and a plane

You can find the angle between a line and a plane by first finding the angle between the **normal** to the plane and the direction of the line. A normal to a plane is a line perpendicular to it.



▲ **Figure 7.7**

The angle between the normal, **n**, and the plane is 90°.

Angle A is the angle between the line l and the normal to the plane, so the angle between the line and the plane, angle B, is $90° - A$.

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

Example 7.7

Find the angle between the line $\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$ and the plane $2x + 3y + z = 4$.

Solution

The normal, **n**, to the plane is $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$. The direction, **d**, of the line is $\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$.

The angle between the normal to the plane and the direction of the line is given by:

$$\cos A = \frac{\mathbf{n} \,.\, \mathbf{d}}{|\mathbf{n}||\mathbf{d}|}$$

$\mathbf{n} \,.\, \mathbf{d} = 2 \times (-1) + 3 \times 2 + 1 \times 5 = 9$

$$\cos A = \frac{9}{\sqrt{14} \times \sqrt{30}}$$

$\Rightarrow \quad A = 63.95°$

$\Rightarrow \quad B = 26.05°$

Since $A + B = 90° = 9$

So the angle between the line and the plane is 26° to the nearest degree.

Exercise 7A

1 Determine whether the following planes and lines are parallel. If they are parallel, show whether the line lies in the plane.

(i) $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $3x + y - z = 8$

(ii) $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix} + t\begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$ and $x - 2y - 3z = 2$

(iii) $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 7 \end{pmatrix} + t\begin{pmatrix} -3 \\ 2 \\ -5 \end{pmatrix}$ and $2x - 3y + z = 5$

(iv) $\mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$ and $4x + 3y + z = -1$

(v) $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -5 \\ 4 \\ 7 \end{pmatrix}$ and $x + 2y - 6z = 0$

(vi) $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$ and $3x + 4y - z = 7$

2 The points L, M and N have coordinates (0, −1, 2), (2, 1, 0) and (5, 1, 1).

(i) Write down the vectors $\overrightarrow{LM}$ and $\overrightarrow{LN}$.

(ii) Show that $\overrightarrow{LM} \cdot \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} = \overrightarrow{LN} \cdot \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} = 0$.

(iii) Find the equation of the plane LMN.

3 (i) Show that the points A (1, 1, 1), B (3, 0, 0) and C (2, 0, 2) all lie in the plane $2x + 3y + z = 6$.

(ii) Show that $\overrightarrow{AB} \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \overrightarrow{AC} \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 0$

(iii) The point D has coordinates (7, 6, 2). D lies on a line perpendicular to the plane through one of the points A, B or C.
Through which of these points does the line pass?

CP **4** The lines l, $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and m, $\mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, lie in the same plane π.

(i) Find the coordinates of any two points on each of the lines.

(ii) Show that all the four points you found in part (i) lie on the plane $x - z = 2$.

(iii) Explain why you now have more than sufficient evidence to show that the plane π has equation $x - z = 2$.

(iv) Find the coordinates of the point where the lines l and m intersect.

5 Find the points of intersection of the following planes and lines.

(i) $x + 2y + 3z = 11$ and $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(ii) $2x + 3y - 4z = 1$ and $\mathbf{r} = \begin{pmatrix} -2 \\ -3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$

(iii) $3x - 2y - z = 14$ and $\mathbf{r} = \begin{pmatrix} 8 \\ 4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

(iv) $x + y + z = 0$ and $\mathbf{r} = \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

(v) $5x - 4y - 7z = 49$ and $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

6 In each of the following examples you are given a point A and a plane π. Find

(a) the equation of the line through A perpendicular to π

(b) the point of intersection, P, of this line with π

(c) the distance AP.

(i) A is (2, 2, 3); π is $x - y + 2z = 0$

(ii) A is (2, 3, 0); π is $2x + 5y + 3z = 0$

(iii) A is (3, 1, 3); π is $x = 0$

(iv) A is (2, 1, 0); π is $3x - 4y + z = 2$

(v) A is (0, 0, 0); π is $x + y + z = 6$

7 The points U and V have coordinates (4, 0, 7) and (6, 4, 13).
The line UV is perpendicular to a plane and the point U lies in the plane.

(i) Find the equation of the plane in Cartesian form.

(ii) The point W has coordinates (−1, 10, 2).
Show that $(\text{WV})^2 = (\text{WU})^2 + (\text{UV})^2$.

(iii) What information does this give you about the position of W?
Confirm this information by a different method.

8 (i) Find the equation of the line through (13, 5, 0) parallel to the line

$$\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}.$$

(ii) Where does this line meet the plane $3x + y - 2z = 2$?

(iii) How far is the point of intersection from (13, 5, 0)?

9 (i) Find the angle between the line $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + t(3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ and the plane $2x - 3y - z = 1$.

(ii) Find the angle between the line $\mathbf{r} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ and the plane $4x - 3z = -2$.

(iii) Find the angle between the line $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + t(3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ and the plane $7x - 2y + z = 1$.

10 A is the point (1, 2, 0), B is (0, 4, 1) and C is (9, −2, 1).

(i) Show that A, B and C lie in the plane $2x + 3y - 4z = 8$.

(ii) Write down the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and verify that they are at right angles to $\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$.

(iii) Find the angle BAC.

(iv) Find the area of triangle ABC $\left(\text{using area} = \frac{1}{2}bc \sin A\right)$.

11 P is the point (2, −1, 3), Q is (5, −5, 3) and R is (7, 2, −3). Find

(i) the lengths of PQ and QR

(ii) the angle PQR

(iii) the area of triangle PQR

(iv) the point S such that PQRS is a parallelogram.

CP

12 P is the point (2, 2, 4), Q is (0, 6, 8), X is (−2, −2, −3) and Y is (2, 6, 9).

(i) Write in vector form the equations of the lines PQ and XY.

(ii) Verify that the equation of the plane PQX is $2x + 5y - 4z = -2$.

(iii) Does the point Y lie in the plane PQX?

(iv) Does any point on PQ lie on XY? (That is, do the lines intersect?)

13 The line l has equation $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} - \mathbf{k} + t(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$. It is given that l lies in the plane with equation $2x + by + cz = 1$, where b and c are constants.

(i) Find the values of b and c.

(ii) The point P has position vector $2\mathbf{j} + 4\mathbf{k}$. Show that the perpendicular distance from P to l is $\sqrt{5}$.

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14 With respect to the origin O, the points A and B have position vectors given by

$$\overrightarrow{OA} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k}.$$

The line l has vector equation $\mathbf{r} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + s(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$.

(i) Prove that the line l does not intersect the line through A and B.

(ii) Find the equation of the plane containing l and the point A, giving your answer in the form $ax + by + cz = d$.

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15 The points A and B have position vectors, relative to the origin O, given by

$$\overrightarrow{OA} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix}.$$

The line l passes through A and is parallel to OB. The point N is the foot of the perpendicular from B to l.

(i) State a vector equation for the line l.

(ii) Find the position vector of N and show that BN = 3.

(iii) Find the equation of the plane containing A, B and N, giving your answer in the form $ax + by + cz = d$.

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16 The straight line l has equation $\mathbf{r} = \mathbf{i} + 6\mathbf{j} - 3\mathbf{k} + s(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$. The plane p has equation $(\mathbf{r} - 3\mathbf{i}) \cdot (2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) = 0$. The line l intersects the plane p at the point A.

(i) Find the position vector of A.

(ii) Find the acute angle between l and p.

(iii) Find a vector equation for the line which lies in p, passes through A and is perpendicular to l.

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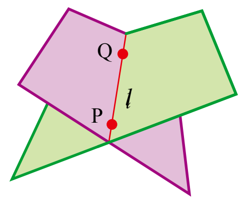
7.5 The intersection of two planes

If you look around you, you will find objects that can be used to represent planes – walls, floors, ceilings, doors, roofs, and so on. You will see that the intersection of two planes is a straight line.

Example 7.8

Find l, the line of intersection of the two planes

$$3x + 2y - 3z = -18 \quad \text{and} \quad x - 2y + z = 12.$$



▲ **Figure 7.8**

Solution 1

This solution depends on finding two points on l.

You can find one point by arbitrarily choosing to put $y = 0$ into the equations of the planes and solving simultaneously:

$$\left.\begin{aligned} 3x - 3z &= -18 \\ x + z &= 12 \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} x - z &= -6 \\ x + z &= 12 \end{aligned}\right\} \Leftrightarrow x = 3,\ z = 9.$$

So P with coordinates $(3, 0, 9)$ is a point on l.

(You could run into difficulties putting $y = 0$ as it is possible that the line has no points where $y = 0$. In this case your simultaneous equations for x and z would be inconsistent; you would then choose a value for x or z instead.)

In the same way, arbitrarily choosing to put $z = 1$ into the equations gives

$$\left.\begin{aligned} 3x + 2y &= -15 \\ x - 2y &= 11 \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} 4x &= -4 \\ 2y &= x - 11 \end{aligned}\right\} \Leftrightarrow x = -1,\ y = -6$$

so Q with coordinates $(-1, -6, 1)$ is a point on l.

$$\overrightarrow{PQ} = \begin{pmatrix} -1 \\ -6 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 9 \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \\ -8 \end{pmatrix} = -2\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

Removing factor –2 makes the arithmetic simpler.

Use $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ as the direction vector for l.

The vector equation for l is $\mathbf{r} = \begin{pmatrix} -1 \\ -6 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$.

Solution 2

In this solution the original two equations in x, y and z are solved, expressing each of x, y and z in terms of some parameter.

Put $x = \lambda$ into $\begin{cases} 3x + 2y - 3z = -18 \\ x - 2y + z = 12 \end{cases}$ and solve simultaneously for y and z:

$$\left.\begin{cases} 2y - 3z = -18 - 3\lambda \\ -2y + z = 12 - \lambda \end{cases}\right\} \Rightarrow -2z = -6 - 4\lambda \Rightarrow z = 2\lambda + 3$$

so that $2y = 3z - 18 - 3\lambda \Rightarrow 2y = 3(2\lambda + 3) - 18 - 3\lambda \Rightarrow 2y = 3\lambda - 9$
$\Rightarrow y = \frac{3}{2}\lambda = \frac{9}{2}$.

Thus the equations for l are

$$\begin{cases} x = \lambda \\ y = \frac{3}{2}\lambda - \frac{9}{2} \\ z = 2\lambda + 3 \end{cases} \quad \text{or} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{9}{2} \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ \frac{3}{2} \\ 2 \end{pmatrix}.$$

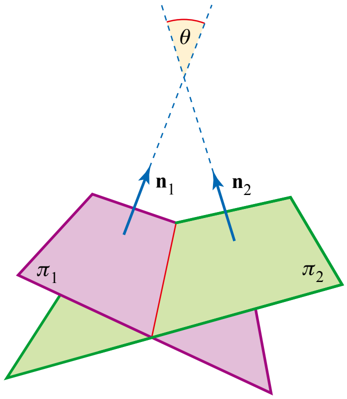
Note

This set of equations is different from but equivalent to the equations in Solution 1. The equivalence is most easily seen by substituting $2\mu - 1$ for λ, obtaining

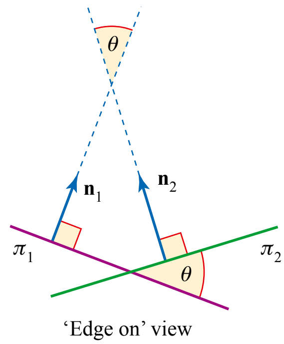
$$\begin{cases} x = 2\mu - 1 \\ y = \frac{3}{2}(2\mu - 1) - \frac{9}{2} = 3\mu - 6 \\ z = 2(2\mu - 1) + 3 = 4\mu + 1 \end{cases}$$

7.6 The angle between two planes

The angle between two planes can be found by using the scalar product. As Figures 7.9 and 7.10 make clear, the angle between planes π_1 and π_2 is the same as the angle between their normals, $\mathbf{n}_1$ and $\mathbf{n}_2$.



▲ **Figure 7.9**



▲ **Figure 7.10**

Example 7.9

Find the acute angle between the planes π_1: $2x + 3y + 5z = 8$ and π_2: $5x + y - 4z = 12$.

Solution

The planes have normals $\mathbf{n}_1 = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ and $\mathbf{n}_2 = \begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix}$, so $\mathbf{n}_1 \cdot \mathbf{n}_2 = 10 + 3 - 20 = -7$.

The angle between the normals is θ, where

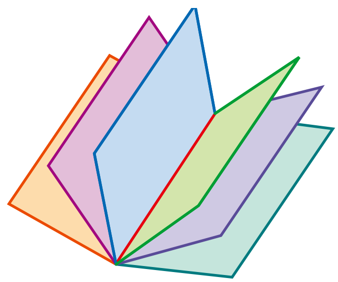
$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{-7}{\sqrt{38} \times \sqrt{42}}$$

$$\Rightarrow \qquad \theta = 100.1° \qquad \text{(to 1 decimal place)}$$

Therefore the acute angle between the planes is 79.9°.

Sheaf of planes

When several planes share a common line the arrangement is known as a **sheaf of planes** (Figure 7.11). The next example shows how you can find the equation of a plane that contains the line l common to two given planes, π_1 and π_2, without having to find the equation of l itself, or any points on l.



▲ **Figure 7.11**

Example 7.10

Find the equation of the plane that passes through the point $(1, 2, 3)$ and contains the common line of the planes π_1: $2x+2y+z+3=0$ and π_2: $2x+3y+z+13=0$.

Solution

The equation

$$p(2x+2y+z+3)+q(2x+3y+z+13)=0 \quad ①$$

can be rearranged in the form $n_1x+n_2y+n_3z=d$, where not all of a, b, c, d are zero provided p and q are not both zero. Therefore equation ① represents a plane. Further, any point (x, y, z) that satisfies both π_1 and π_2 will also satisfy equation ①. Thus equation ① represents a plane containing the common line of planes π_1 and π_2. Substituting $(1, 2, 3)$ into ① gives

$$12p+24q=0 \quad \Leftrightarrow \quad p=-2q.$$

The required equation is

$$-2q(2x+2y+z+3)+q(2x+3y+z+13)=0$$

$$\Leftrightarrow \quad -q(2x+y+z-7)=0$$

so that the required plane has equation $2x+y+z=7$.

?

› Planes π_1 and π_2 have equations $a_1x+b_1y+c_1z-d_1=0$ and $a_2x+b_2y+c_2z-d_2=0$ respectively. Plane π_3 has equation

$$p(a_1x+b_1y+c_1z-d_1)+q(a_2x+b_2y+c_2z-d_2)=0.$$

How is π_3 related to π_1 and π_2 if π_1 and π_2 are parallel?

Example 7.11

Find the equation of the common perpendicular to the lines

$$\mathbf{r}=\begin{pmatrix}1\\0\\-1\end{pmatrix}+\alpha\begin{pmatrix}1\\1\\0\end{pmatrix} \text{ and } \mathbf{r}=\begin{pmatrix}-2\\1\\4\end{pmatrix}+\beta\begin{pmatrix}-3\\0\\-1\end{pmatrix}.$$

When you are writing the vector equation of a line you can use any letter for the parameter. It does not have to be λ or μ. In these two equations α and β are used.

Solution

Let P be a general point on the first line $\overrightarrow{OP}=\begin{pmatrix}1+\alpha\\ \alpha\\ -1\end{pmatrix}$

Let Q be a general point on the second line $OQ=\begin{pmatrix}-2-3\beta\\ 1\\ 4+\beta\end{pmatrix}$

→

$$\overrightarrow{PQ} = \begin{pmatrix} -3-\alpha-3\beta \\ 1-\alpha \\ 5+\beta \end{pmatrix}$$ and is perpendicular to both lines

$$\overrightarrow{PQ} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3-\alpha-3\beta \\ 1-\alpha \\ 5+\beta \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = (-3-\alpha-3\beta) + (1-\alpha) = 0$$ giving

$$-2\alpha - 3\beta - 2 = 0$$

and $$\overrightarrow{PQ} \cdot \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3-\alpha-3\beta \\ 1-\alpha \\ 5+\beta \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} = -3(-3-\alpha-3\beta) + (5+\beta) = 0$$

giving $3\alpha + 10\beta + 14 = 0$.

Solving simultaneously gives $\alpha = 2$ and $\beta = -2$.

Substituting into the equations of the lines gives

$$\overrightarrow{OP} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$ and $$\overrightarrow{OQ} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

$$\overrightarrow{PQ} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$ so the equation of the line is $$\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

Exercise 7B

1 Find the vector equation of the line of intersection of each of these pairs of planes.

(i) $x + y - 6z = 4$, $5x - 2y - 3z = 13$

(ii) $5x - y + z = 8$, $x + 3y + z = -4$

(iii) $3x + 2y - 6z = 4$, $x + 5y - 7z = 2$

(iv) $5x + 2y - 3z = -2$, $3x - 3y - z = 2$

2 Find the acute angle between each pair of planes in question **1**.

3 Find the vector equation of the line that passes through the given point and that is parallel to the line of intersection of the two planes.

(i) $(-2, 3, 5)$, $4x - y + 3z = 5$, $3x - y + 2z = 7$

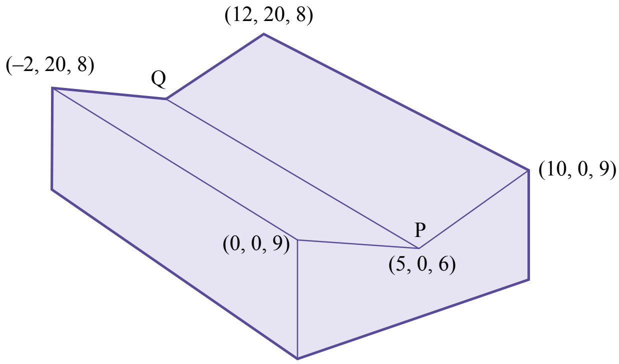
(ii) $(4, -3, 2)$, $2x + 3y + 2z = 6$, $4x - 3y + z = 11$

4 Find the equation of the plane that goes through $(3, 2, -2)$ and that contains the common line of $x + 7y - 2z = 3$ and $2x - 3y + 2z = 1$.

5 Find the equation of the plane that contains the point $(1, -2, 3)$ and that is perpendicular to the common line of $5x - 3y - 4z = 2$ and $2x + y + 5z = 7$.

6 Find the equation of the line that goes through $(4, -2, -7)$ and that is parallel to both $2x - 5y - 2z = 8$ and $x + 3y - 3z = 12$.

M **7** The diagram shows the coordinates of the corners of parts of the roof of a warehouse.



Find the equations of both roof sections, and the vector equation of the line PQ. Assuming that the z-axis is vertical, what angle does PQ make with the horizontal?

M **8** Test drilling in the Namibian desert has shown the existence of gold deposits at $(400, 0, -400)$, $(-50, 500, -250)$, $(-200, -100, -200)$, where the units are in metres, the x-axis points east, the y-axis points north, and the z-axis points up. Assume that these deposits are part of the same seam, contained in plane π.

(i) Find the equation of plane π.

(ii) Find the angle at which π is tilted to the horizontal.

The drilling positions $(400, 0, 3)$, $(-50, 500, 7)$, $(-200, -100, 5)$ are on the desert floor. Take the desert floor as a plane, Π.

(iii) Find the equation of Π.

(iv) Find the equation of the line where the plane containing the gold seam intersects the desert floor.

(v) How far south of the origin does the line found in part (iv) pass?

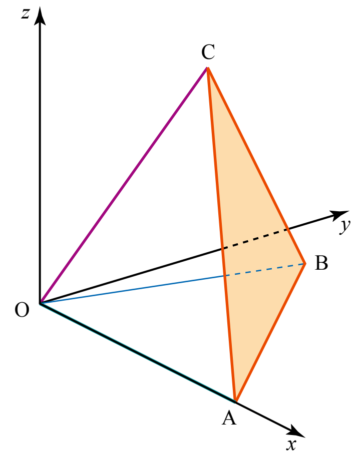
9 The plane p has equation $3x + 2y + 4z = 13$. A second plane q is perpendicular to p and has equation $ax + y + z = 4$, where a is a constant.

(i) Find the value of a.

(ii) The line with equation $\mathbf{r} = \mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ meets the plane p at the point A and the plane q at the point B. Find the length of AB.

Cambridge International AS & A Level Mathematics
9709 Paper 32 Q9 May/June 2010

10 The diagram shows a set of rectangular axes Ox, Oy and Oz, and three points A, B and C with position vectors $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.



(i) Find the equation of the plane ABC, giving your answer in the form $ax + by + cz = d$.

(ii) Calculate the acute angle between the planes ABC and OAB.

Cambridge International AS & A Level Mathematics 9709 Paper 3 Q9 May/June 2007

11 Two planes have equations $2x - y - 3z = 7$ and $x + 2y + 2z = 0$.

(i) Find the acute angle between the planes.

(ii) Find a vector equation for their line of intersection.

Cambridge International AS & A Level Mathematics 9709 Paper 3 Q7 October/November 2008

12 The plane p has equation $2x - 3y + 6z = 16$. The plane q is parallel to p and contains the point with position vector $\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$.

(i) Find the equation of q, giving your answer in the form $ax + by + cz = d$.

(ii) Calculate the perpendicular distance between p and q.

(iii) The line l is parallel to the plane p and also parallel to the plane with equation $x - 2y + 2z = 5$. Given that l passes through the origin, find a vector equation for l.

Cambridge International AS & A Level Mathematics 9709 Paper 32 Q10 October/November 2009

13 The lines l_1 and l_2 have equations $\mathbf{r} = 8\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j})$ and $\mathbf{r} = 5\mathbf{i} + 3\mathbf{j} - 14\mathbf{k} + \mu(-2\mathbf{j} - 3\mathbf{k})$ respectively. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find the position vector of the point P and the position vector of the point Q.

Cambridge International AS & A Level Further Mathematics 9231 Paper 11 Q11 (part question) May/June 2015

7.7 The vector product

The **vector product** is a different method for 'multiplying' two vectors. As the name suggests, in this case the result is a vector rather than a scalar. The vector product of **a** and **b** is a vector perpendicular to both **a** and **b** and it is written $\mathbf{a} \times \mathbf{b}$.

It is given by

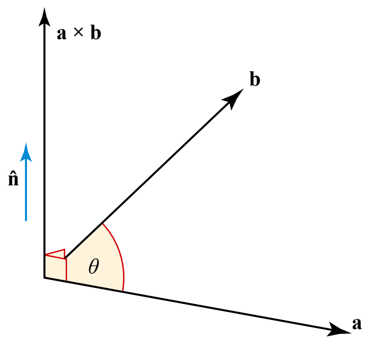
$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$$

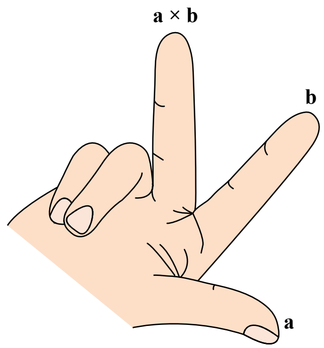
where θ is the angle between **a** and **b** and $\hat{\mathbf{n}}$ is a unit vector that is perpendicular to both **a** and **b**.

This is often described as having opposite senses.

There are two unit vectors perpendicular to both **a** and **b**, but they point in opposite directions.

The vector $\hat{\mathbf{n}}$ is chosen such that **a**, **b** and $\hat{\mathbf{n}}$ (in that order) form a **right-handed set** of vectors, as shown in Figure 7.12. If you point the thumb of your right hand in the direction of **a**, and your index finger in the direction of **b**, then your second finger coming up from your palm points in the direction $\mathbf{a} \times \mathbf{b}$ as shown below.





▲ **Figure 7.12**

In component form, the vector product is expressed as follows:

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

You will have the opportunity to prove this result in Exercise 7C.

Notice that the first component of $\mathbf{a} \times \mathbf{b}$ is the value of the 2×2 determinant $\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$ obtained by covering up the top row of $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$; the second component is the negative of the 2×2 determinant obtained by covering up the middle row; and the third component is the 2×2 determinant obtained by covering up the bottom row.

Notice that the value of the determinant $\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$ is the same as the value of $\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$. They are both $(a_2b_3 - a_3b_2)$. This means that the formula for the vector product can be expressed as a determinant:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Expanding this determinant by the first column gives:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

Note this sign.

Example 7.12

(i) Calculate $\mathbf{a} \times \mathbf{b}$ when $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

(ii) Hence find $\hat{\mathbf{n}}$, a unit vector that is perpendicular to both **a** and **b**.

Solution

(i) There are two possible methods:

Method 1

Using determinants:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 5 \\ 1 & -4 & 2 \end{vmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 5 \\ -4 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 2 \\ 1 & -4 \end{vmatrix}\mathbf{k}$$

$$= 24\mathbf{i} - \mathbf{j} - 14\mathbf{k}$$

Method 2

Using the result

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

gives

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \times \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \times 2 - 5 \times (-4) \\ 5 \times 1 - 3 \times 2 \\ 3 \times (-4) - 2 \times 1 \end{pmatrix} = \begin{pmatrix} 24 \\ -1 \\ -14 \end{pmatrix}$$

(ii) $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 24 \\ -1 \\ -14 \end{pmatrix}$ This vector is perpendicular to **a** and **b**.

$|\mathbf{a} \times \mathbf{b}| = \sqrt{24^2 + (-1)^2 + (-14)^2} = \sqrt{773}$

So a unit vector perpendicular to both **a** and **b** is $\hat{\mathbf{n}} = \frac{1}{\sqrt{773}}\begin{pmatrix} 24 \\ -1 \\ -14 \end{pmatrix}$.

?

› How can you use the scalar product to check that the answer to Example 7.12 is correct?

Technology note

Investigate whether a calculator will find the vector product of two vectors. If so, use a calculator to check the vector product calculated in Example 7.12.

Properties of the vector product

1 The vector product is anti-commutative

The vector products $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ have the same magnitude but are in opposite directions, so $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
This is known as the **anti-commutative property.**

2 The vector product of parallel vectors is zero

This is because the angle θ between two parallel vectors is 0° or 180°, so $\sin\theta = 0$.

In particular $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$.

3 The vector product is compatible with scalar multiplication

For scalars m and n,

$$(m\mathbf{a}) \times (n\mathbf{b}) = mn(\mathbf{a} \times \mathbf{b})$$

This is because the vector $m\mathbf{a}$ has magnitude $|m||\mathbf{a}|$; $m\mathbf{a}$ and $\mathbf{a}$ have the same direction if m is positive, but opposite directions if m is negative.

4 The vector product is distributive over vector addition

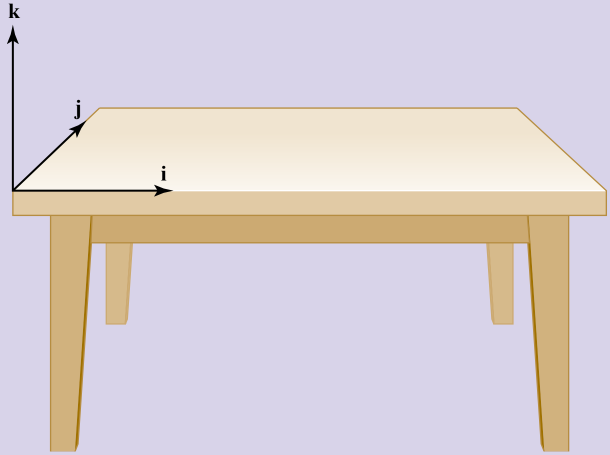
The result

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

enables you to change a product into the sum of two simpler products – in doing so the multiplication is 'distributed' over the two terms of the original sum.

ACTIVITY 7.2

In this activity you might find it helpful to take the edges of a rectangular table to represent the unit vectors **i**, **j** and **k** as shown in Figure 7.13.



▲ **Figure 7.13**

You could use pens to represent:

i, the unit vector pointing to the right along the x-axis

j, the unit vector pointing away from you along the y-axis

k, the unit vector pointing upwards along the z-axis.

The vector product of **a** and **b** is defined as

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$$

where θ is the angle between **a** and **b** and $\hat{\mathbf{n}}$ is a unit vector that is perpendicular to both **a** and **b** such that **a**, **b** and $\hat{\mathbf{n}}$ (in that order) form a right–handed set of vectors.

Using this definition, check the truth of each of the following results.

$\mathbf{i} \times \mathbf{i} = 0$ $\quad\quad$ $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ $\quad\quad$ $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$

Give a further six results for vector products of pairs of **i**, **j** and **k.**

Since the equation of a plane involves a vector that is perpendicular to the plane, the vector product is very useful in finding the equation of a plane.

Example 7.13

Find the Cartesian equation of the plane that contains the points A (3, 4, 2), B (2, 0, 5) and C (6, 7, 8).

Solution

Start by finding two vectors in the plane, for example $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix} \quad \text{and} \quad \overrightarrow{BC} = \begin{pmatrix} 6 \\ 7 \\ 8 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix}$$

You could find this result using your calculator.

You need to find a vector that is perpendicular to AB and BC.

Then $\overrightarrow{AB} \times \overrightarrow{BC} = \begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix} \times \begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix} = \begin{pmatrix} -33 \\ 15 \\ 9 \end{pmatrix}$ which can be written as $-3\begin{pmatrix} 11 \\ -5 \\ -3 \end{pmatrix}$.

So $\mathbf{n} = \begin{pmatrix} 11 \\ -5 \\ -3 \end{pmatrix}$ is a vector perpendicular to the plane containing A, B and C, and the equation of the plane is of the form $11x - 5y - 3z = d$.

Substituting the coordinates of one of the points, say A, allows you to find the value of the constant d:

$$(11 \times 3) - (5 \times 4) - (3 \times 2) = 7$$

Substituting for B and C provides a useful check of your answer.

The plane has equation $11x - 5y - 3z = 7$.

?

› Another way of finding the equation through three given points is to form three simultaneous equations and solve them.

Compare these two methods.

Example 7.14

A plane is given parametrically by the equation

$\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k} + \lambda(5\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + \mu(2\mathbf{i} + \mathbf{j} - 3\mathbf{k})$.

Find its Cartesian equation.

Solution

The Cartesian equation can be found from the vector equation $(\mathbf{r} - \mathbf{a})\cdot\mathbf{n} = \mathbf{0}$ where $\mathbf{n}$ is a normal vector.

Using the vector product to find $\mathbf{n} = (5\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \times (2\mathbf{i} + \mathbf{j} - 3\mathbf{k})$

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -2 & 1 \\ 2 & 1 & -3 \end{vmatrix}$$

$$\mathbf{n} = \begin{vmatrix} -2 & 1 \\ 1 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 5 & 1 \\ 2 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 5 & -2 \\ 2 & 1 \end{vmatrix}\mathbf{k} = 5\mathbf{i} + 17\mathbf{j} + 9\mathbf{k}$$

This could also be written.

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & 5 & 2 \\ \mathbf{j} & -2 & 1 \\ \mathbf{k} & 1 & -3 \end{vmatrix}$$

The determinant of the transpose of a matrix is equal to the determinant of the original matrix.

The equation of the plane is

$\mathbf{r}\cdot(5\mathbf{i} + 17\mathbf{j} + 9\mathbf{k}) = (3\mathbf{i} - \mathbf{j} + 5\mathbf{k})\cdot(5\mathbf{i} + 17\mathbf{j} + 9\mathbf{k})$

$5x + 17y + 9z = 43$

Exercise 7C

In this exercise you should calculate the vector products by hand. You could check your answers using the vector product facility on a calculator.

1 Calculate each of the following vector products:

(i) $\begin{pmatrix}3\\5\\2\end{pmatrix} \times \begin{pmatrix}2\\4\\-3\end{pmatrix}$ (ii) $\begin{pmatrix}7\\-4\\-5\end{pmatrix} \times \begin{pmatrix}-4\\5\\-3\end{pmatrix}$

(iii) $(5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) \times (\mathbf{i} + 5\mathbf{j} - 6\mathbf{k})$ (iv) $(3\mathbf{i} - 7\mathbf{k}) \times (2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k})$

2 Find a vector perpendicular to each of the following pairs of vectors:

(i) $\mathbf{a} = \begin{pmatrix}2\\0\\5\end{pmatrix}, \mathbf{b} = \begin{pmatrix}3\\-1\\-2\end{pmatrix}$ (ii) $\mathbf{a} = \begin{pmatrix}12\\3\\-2\end{pmatrix}, \mathbf{b} = \begin{pmatrix}7\\1\\4\end{pmatrix}$

(iii) $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}, \mathbf{b} = 3\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}$

(iv) $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}, \mathbf{b} = 8\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$

3 Three points A, B and C have coordinates (1, 4, −2), (2, 0, 1) and (5, 3, −2) respectively.

(i) Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(ii) Use the vector product to find a vector that is perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(iii) Hence find the equation of the plane containing points A, B and C.

4 Find a unit vector perpendicular to both $\mathbf{a} = \begin{pmatrix}1\\2\\7\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}3\\-1\\6\end{pmatrix}$.

5 Find the magnitude of $\begin{pmatrix}3\\1\\-4\end{pmatrix} \times \begin{pmatrix}1\\-1\\1\end{pmatrix}$.

6 Find the Cartesian equations of the planes containing the three points given:

(i) A(1, 4, 2), B(5, 1, 3) and C(1, 0, 0)

(ii) D(5, −3, 4), E(0, 1, 0) and F(6, 2, 5)

(iii) G(6, 2, −2), H(1, 4, 3) and L(−5, 7, 1)

(iv) M(4, 2, −1), N(8, 2, 4) and P(5, 8, −7)

7 Simplify the following:

(i) $4\mathbf{i} \times 2\mathbf{k}$ (ii) $2\mathbf{i} \times (5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k})$

(iii) $(6\mathbf{i} + \mathbf{j} - \mathbf{k}) \times 2\mathbf{k}$ (iv) $(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \times (\mathbf{i} - \mathbf{j} - 4\mathbf{k})$

CP **8** Prove algebraically that for two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix}a_2b_3 - a_3b_2\\a_3b_1 - a_1b_3\\a_1b_2 - a_2b_1\end{pmatrix}$$

CP

9 Two points A and B have position vectors $\mathbf{a} = 3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ respectively.

(i) Find the lengths of each of the sides of the triangle OAB, and hence find the area of the triangle.

(ii) Find $|\mathbf{a} \times \mathbf{b}|$.

(iii) How does the definition $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\hat{\mathbf{n}}$ explain the relationship between your answers to (i) and (ii)?

10 The plane Π_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$. Find a Cartesian equation of Π_1.

The plane Π_2 has equation $2x - y + z = 10$. Find the acute angle between Π_1 and Π_2.

Find an equation of the line of intersection of Π_1 and Π_2, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$.

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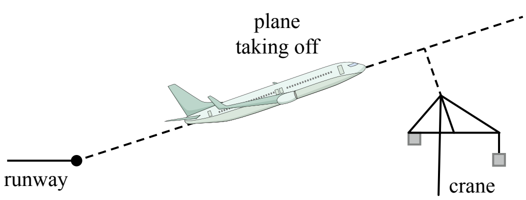
7.8 Finding distances

Sometimes you need to find the distance between points, lines and planes. In this section you will look at how to find:

- the distance from a point to a line, in two or three dimensions;
- the distance from a point to a plane;
- the distance between parallel or skew lines.

Finding the distance from a point to a line

Figure 7.14 shows building works at an airport that require the use of a crane near the end of the runway. How far is it from the top of the crane to the flight path of the aeroplane?

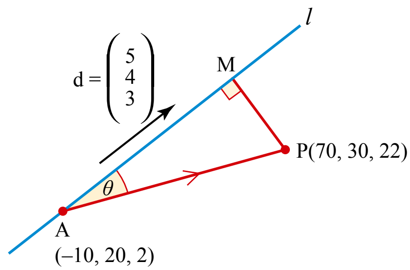


▲ **Figure 7.14**

To answer this question you need to know the flight path and the position of the top of the crane.

Working in metres, suppose the position of the top of the crane is at P (70, 30, 22)

and the aeroplanes take off along the line l: $\mathbf{r} = \begin{pmatrix} -10 \\ 20 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$ as illustrated in Figure 7.15.



▲ **Figure 7.15**

The shortest distance from P to the straight line l is measured along the line that is perpendicular to l. It is the distance PM in Figure 7.15. The vector product provides a convenient way of calculating such distances.

Since PM is perpendicular to l

$$\text{PM} = \text{AP} \sin \text{P}\hat{\text{A}}\text{M}$$

Compare this with the formula for the vector product of AP and AM:

$$\overrightarrow{\text{AP}} \times \overrightarrow{\text{AM}} = |\overrightarrow{\text{AP}}||\overrightarrow{\text{AM}}| \sin \text{P}\hat{\text{A}}\text{M}\,\hat{\mathbf{n}}$$

$$\text{so } |\overrightarrow{\text{AP}}| \sin \text{P}\hat{\text{A}}\text{M} = \frac{|\overrightarrow{\text{AP}} \times \overrightarrow{\text{AM}}|}{|\overrightarrow{\text{AM}}|}$$

AM is the direction vector **d** for the line l, so

$$\text{PM} = \frac{|\overrightarrow{\text{AP}} \times \mathbf{d}|}{|\mathbf{d}|}$$

Returning to calculating the distance from the top of the crane to the flight path:

$$\mathbf{p} = \begin{pmatrix} 70 \\ 30 \\ 22 \end{pmatrix}, \ \mathbf{a} = \begin{pmatrix} -10 \\ 20 \\ 2 \end{pmatrix} \text{ and } \mathbf{d} = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$$

so that

$$\overrightarrow{\text{AP}} = \mathbf{p} - \mathbf{a} = \begin{pmatrix} 70 \\ 30 \\ 22 \end{pmatrix} - \begin{pmatrix} -10 \\ 20 \\ 2 \end{pmatrix} = \begin{pmatrix} 80 \\ 10 \\ 20 \end{pmatrix} = 10\begin{pmatrix} 8 \\ 1 \\ 2 \end{pmatrix}$$

and

$$\overrightarrow{\text{AP}} \times \mathbf{d} = 10\begin{pmatrix} 8 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} = 10\begin{pmatrix} 1 \times 3 - 2 \times 4 \\ 2 \times 5 - 8 \times 3 \\ 8 \times 4 - 1 \times 5 \end{pmatrix} = 10\begin{pmatrix} -5 \\ -14 \\ 27 \end{pmatrix}$$

Therefore, $|\overrightarrow{AP} \times \mathbf{d}| = 10\sqrt{(-5)^2 + (-14)^2 + 27^2} = 50\sqrt{38}$ and $|\mathbf{d}| = \sqrt{5^2 + 4^2 + 3^2} = 5\sqrt{2}$.

So the shortest distance from P to l is:

$$PM = \frac{50\sqrt{38}}{5\sqrt{2}} = 10\sqrt{19} \approx 43.6 \text{ metres}$$

You might also need to find the point on l that is closest to P. The following example shows how you can do this for the scenario above.

Example 7.15

The line l has equation $\mathbf{r} = \begin{pmatrix} -10 \\ 20 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$ and the point P has coordinates (70, 30, 22).

The point M is the point on l that is closest to P.

(i) Express the position vector $\mathbf{m}$ of point M in terms of the parameter λ.

Hence find an expression for the vector $\overrightarrow{PM}$ in terms of the parameter λ.

(ii) By finding the scalar product of the vector $\overrightarrow{PM}$ with the direction vector $\mathbf{d}$, show that $\lambda = 10$ and hence find the coordinates of point M.

(iii) Verify that $PM = 10\sqrt{19}$ as found earlier.

Solution

(i) $$\mathbf{m} = \begin{pmatrix} -10 + 5\lambda \\ 20 + 4\lambda \\ 2 + 3\lambda \end{pmatrix}$$

$$\overrightarrow{PM} = \begin{pmatrix} -10 + 5\lambda - 70 \\ 20 + 4\lambda - 30 \\ 2 + 3\lambda - 22 \end{pmatrix} = \begin{pmatrix} -80 + 5\lambda \\ -10 + 4\lambda \\ -20 + 3\lambda \end{pmatrix}$$

(ii) $\overrightarrow{PM}.\mathbf{d} = 0$

Since M is the point on l closest to P, PM is perpendicular to l and so PM is perpendicular to $\mathbf{d}$.

$$\begin{pmatrix} -80 + 5\lambda \\ -10 + 4\lambda \\ -20 + 3\lambda \end{pmatrix} . \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} = 0$$

$$5(-80 + 5\lambda) + 4(-10 + 4\lambda) + 3(-20 + 3\lambda) = 0$$

$$-400 + 25\lambda - 40 + 16\lambda - 60 + 9\lambda = 0$$

$$50\lambda = 500$$

$$\lambda = 10$$

$$\mathbf{m} = \begin{pmatrix} -10 + 5\lambda \\ 20 + 4\lambda \\ 2 + 3\lambda \end{pmatrix} = \begin{pmatrix} 40 \\ 60 \\ 32 \end{pmatrix}$$

➜

(iii) $\overrightarrow{PM} = \begin{pmatrix} -30 \\ 30 \\ 10 \end{pmatrix}$

$|\overrightarrow{PM}| = 10\sqrt{(-3)^2 + 3^2 + 1^2} = 10\sqrt{19}$

As the vector product of vectors **a** and **b** is a vector perpendicular to both **a** and **b** the result $\frac{|\overrightarrow{AP} \times \mathbf{d}|}{|\mathbf{d}|}$ assumes that you are working in three dimensions.

When you are working in two dimensions, in which case the vectors have only two components, you can use the following result, which you can prove in Activity 7.3, which follows.

The distance between a point $P(x_1, y_1)$ and the line $ax + by + c = 0$ is:

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

ACTIVITY 7.3

In this activity, think of points $R(x, y)$ in two dimensional space as corresponding to the point $R'(x, y, 0)$ in three dimensional space.

Use the following steps to show that the distance between a point $P(x_1, y_1)$ and the line $ax + by + c = 0$ is:

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

(i) Write down the coordinates (x, y) of the point A where the line $ax + by + c = 0$ meets the y-axis. Write down the corresponding coordinates of A' in three dimensional space.

(ii) Write $ax + by + c = 0$ in the form $y = mx + c$. Find d, e, f so that $d\mathbf{i} + e\mathbf{j} + f\mathbf{k}$ is parallel to the line in three dimensional space, which corresponds to the line $ax + by + c = 0$ in two dimensional space.

(iii) Use the formula $\frac{|(\mathbf{p} - \mathbf{a}) \times \mathbf{d}|}{|\mathbf{d}|}$ to show that the distance between a point $P(x_1, y_1)$ and the line $ax + by + c = 0$ is:

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Example 7.16

Find the shortest distance from the point P(3, 5) to the line $5x - 3y + 4 = 0$.

Solution

In this case $x_1 = 3$, $y_1 = 5$ and $a = 5$, $b = -3$, $c = 4$ so the shortest distance from the point P to the line is

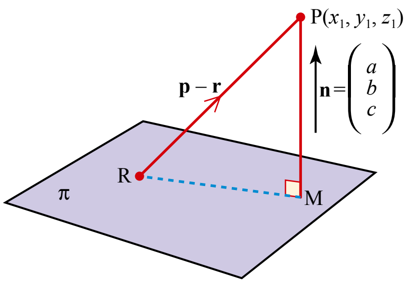
$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} = \frac{(5 \times 3) + (-3 \times 5) + 4}{\sqrt{5^2 + (-3)^2}} = \frac{4}{\sqrt{34}}$$

The distance from a point to a plane

The distance from the point $P(x_1, y_1, z_1)$ to the plane π with equation $ax + by + cz + d = 0$ is PM, where M is the foot of the perpendicular from P to the plane (see Figure 7.16).

Notice that since PM is normal to the plane, it is parallel to the vector $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

Take any point, other than M, on the plane and call it R, with position vector **r**.



▲ **Figure 7.16**

If the angle between the vectors $\mathbf{p} - \mathbf{r}$ and **n** is acute (as shown in Figure 7.16):

$$\text{PM} = \text{RP}\cos \text{R}\hat{\text{P}}\text{M} = \overrightarrow{\text{RP}}.\hat{\mathbf{n}}$$

$$= (\mathbf{p} - \mathbf{r}).\hat{\mathbf{n}}$$

Using the scalar product $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$.

If the angle between $\mathbf{p} - \mathbf{r}$ and **n** is obtuse, $\cos \text{R}\hat{\text{P}}\text{M}$ is negative and

$$\text{PM} = -(\mathbf{p} - \mathbf{r}).\hat{\mathbf{n}}$$

Now you want to choose coordinates for the point R that will keep your working simple. A suitable point is $\left(0, 0, -\frac{d}{c}\right)$. For this point, $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ -\frac{d}{c} \end{pmatrix}$.

$\left(0, 0, -\frac{d}{c}\right)$ lies on the plane $ax + by + cz = d$.

Then $(\mathbf{p} - \mathbf{r}).\mathbf{n} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 + \frac{d}{c} \end{pmatrix} . \begin{pmatrix} a \\ b \\ c \end{pmatrix} = ax_1 + by_1 + cz_1 + d$

and $\text{PM} = |(\mathbf{p} - \mathbf{r}).\hat{\mathbf{n}}|$

$$= \left|\frac{(\mathbf{p} - \mathbf{r}).\mathbf{n}}{|\mathbf{n}|}\right| = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Notice how this formula for the distance from a point to a plane in three dimensions resembles the distance from a point to a line in two dimensions.

Example 7.17

Find the shortest distance from the point (2, 4, −2) to the plane $6x - y - 3z + 1 = 0$.

Solution

The shortest distance from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is:

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

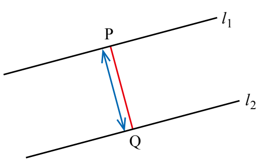
In this case, $x_1 = 2$, $y_1 = 4$, $z_1 = -2$ and $a = 6$, $b = -1$, $c = -3$, $d = 1$

so the shortest distance from the point to the plane is

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{(6 \times 2) + (-1 \times 4) + (-3 \times -2) + 1}{\sqrt{6^2 + (-1)^2 + (-3)^2}}$$

$$= \frac{15}{\sqrt{46}} \approx 2.21$$



▲ **Figure 7.17**

Finding the distance between two parallel lines

The distance between two parallel lines l_1 and l_2 is measured along a line PQ, which is perpendicular to both l_1 and l_2, as shown in Figure 7.17.

You can find this distance by simply choosing a point P on l_1, say, and then finding the shortest distance from P to the line l_2.

Example 7.18

Two straight lines in three dimensions are given by the equations:

$$l_1 : \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \text{ and } l_2 : \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 6 \\ -4 \end{pmatrix}$$

(i) Show that the two lines are parallel.

(ii) Find the shortest distance between the two lines.

Solution

(i) The direction vectors of the two lines are $\mathbf{d}_1 = \begin{pmatrix}1\\-3\\2\end{pmatrix}$ and $\mathbf{d}_2 = \begin{pmatrix}-2\\6\\-4\end{pmatrix}$.

Since $\mathbf{d}_2 = -2\mathbf{d}_1$ the two lines are parallel.

You could use any value for μ.

(ii) Choose a point P on l_2 by setting $\mu = 0$, which gives $\mathbf{p} = \begin{pmatrix}4\\2\\1\end{pmatrix}$.

To find the shortest distance of P from l_1, use $\mathbf{a} = \begin{pmatrix}2\\-3\\0\end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix}1\\-3\\2\end{pmatrix}$.

$$\overrightarrow{AP} = \begin{pmatrix}2\\5\\1\end{pmatrix} \text{ and so } \overrightarrow{AP} \times \mathbf{d} = \begin{pmatrix}2\\5\\1\end{pmatrix} \times \begin{pmatrix}1\\-3\\2\end{pmatrix} = \begin{pmatrix}13\\-3\\-11\end{pmatrix}.$$

$$|\overrightarrow{AP} \times \mathbf{d}| = \sqrt{13^2 + (-3)^2 + (-11)^2} = \sqrt{299}$$

$$|\mathbf{d}| = \sqrt{1^2 + (-3)^2 + 2^2} = \sqrt{14}$$

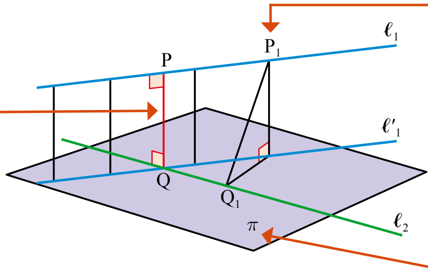
The shortest distance is $\dfrac{|\overrightarrow{AP} \times \mathbf{d}|}{|\mathbf{d}|} = \dfrac{\sqrt{299}}{\sqrt{14}} \approx 4.62.$

Finding the distance between skew lines

Two lines are **skew** if they do not intersect and are not parallel.

Figure 7.18 shows two skew lines l_1 and l_2. The shortest distance between the two lines is measured along a line that is perpendicular to both l_1 and l_2.

The common perpendicular of l_1 and l_2 is the perpendicular from l_1 that passes through the point Q, the point of intersection of l_2 and l'_1.



Drop perpendiculars from the points on l_1 to π to form l'_1, which is the projection of l_1 on π.

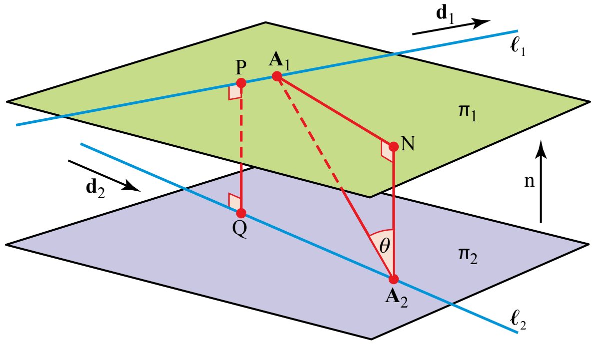
π is the plane containing l'_1 parallel to l_1.

▲ **Figure 7.18**

Figure 7.19 shows the lines l_1 and l_2 and two parallel planes. Then l_1 and l_2 have equations $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{d}_2$ respectively. A_1 and A_2 are points on the lines l_1 and l_2 with position vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ respectively.

π_1 contains l_1 and is parallel to l_2

π_2 contains l_2 and is parallel to l_1



▲ **Figure 7.19**

Then PQ, the common perpendicular of l_1 and l_2 has the same length as any other perpendicular between the planes such as A_2N. If angle $A_1A_2N = \theta$ then

$$PQ = A_2N = A_2A_1\cos\theta = \left|\overrightarrow{A_2A_1}.\hat{\mathbf{n}}\right|$$

where $\hat{\mathbf{n}}$ is a unit vector parallel to A_2N, i.e. perpendicular to both planes.

ACTIVITY 7.4

Explain why PQ is shorter than any other line, such as P_1Q_1 joining lines l_1 and l_2.

Notice that the modulus function is used to ensure a positive answer: the vector $\hat{\mathbf{n}}$ may be directed from π_1 to π_2 making $\overrightarrow{A_2A_1}.\hat{\mathbf{n}}$ negative.

Since π_1 and π_2 are parallel to l_1 and l_2, which are parallel to $\mathbf{d}_1$ and $\mathbf{d}_2$ respectively, you can take $\mathbf{d}_1 \times \mathbf{d}_2$ as $\mathbf{n}$ with $\hat{\mathbf{n}} = \dfrac{(\mathbf{d}_1 \times \mathbf{d}_2)}{|\mathbf{d}_1 \times \mathbf{d}_2|}$.

Then:

$$PQ = A_2N = \left|\overrightarrow{A_2A_1}.\hat{\mathbf{n}}\right| = \frac{\overrightarrow{A_2A_1}.(\mathbf{d}_1 \times \mathbf{d}_2)}{|\mathbf{d}_1 \times \mathbf{d}_2|} = \frac{(\mathbf{a}_1 - \mathbf{a}_2).(\mathbf{d}_1 \times \mathbf{d}_2)}{|\mathbf{d}_1 \times \mathbf{d}_2|}$$

So, the distance between two skew lines is given by:

$$\left|\frac{(\mathbf{d}_1 \times \mathbf{d}_2)}{|\mathbf{d}_1 \times \mathbf{d}_2|}.(\mathbf{a}_1 - \mathbf{a}_2)\right|$$

where $\mathbf{a}_1$ is the position vector of a point on the first line and $\mathbf{d}_1$ is parallel to the first line, and similarly $\mathbf{a}_2$ is the position vector of a point on the second line and $\mathbf{d}_2$ is parallel to the second line.

7

Example 7.19

Find the shortest distance between the lines l_1: $\mathbf{r} = \begin{pmatrix} 8 \\ 9 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and l_2: $\mathbf{r} = \begin{pmatrix} 6 \\ 0 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$.

Solution

Line l_1 contains the point $A_1(8, 9, -2)$ and is parallel to the vector $\mathbf{d}_1 = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

Line l_2 contains the point $A_2(6, 0, -2)$ and is parallel to the vector $\mathbf{d}_2 = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

$$\mathbf{a}_1 - \mathbf{a}_2 = \begin{pmatrix} 8 \\ 9 \\ -2 \end{pmatrix} - \begin{pmatrix} 6 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 9 \\ 0 \end{pmatrix}$$

and $\mathbf{d}_1 \times \mathbf{d}_2 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -4-3 \\ -3+2 \\ -1-2 \end{pmatrix} = \begin{pmatrix} -7 \\ -1 \\ -3 \end{pmatrix}$

Then $(\mathbf{d}_1 \times \mathbf{d}_2).(\mathbf{a}_1 - \mathbf{a}_2) = \begin{pmatrix} -7 \\ -1 \\ -3 \end{pmatrix}.\begin{pmatrix} 2 \\ 9 \\ 0 \end{pmatrix} = -14 - 9 = -23$

Also $|\mathbf{d}_1 \times \mathbf{d}_2| = \sqrt{(-7)^2 + (-1)^2 + (-3)^2} = \sqrt{59}$

Therefore the shortest distance between the skew lines is:

$$\left| \frac{(\mathbf{d}_1 \times \mathbf{d}_2)}{|\mathbf{d}_1 \times \mathbf{d}_2|}.(\mathbf{a}_1 - \mathbf{a}_2) \right| = \frac{23}{\sqrt{59}} \approx 2.99 \text{ units}$$

Exercise 7D

1 Calculate the distance from the point P to the line l:

(i) P(1, −2, 3) $\quad l: \frac{x-1}{2} = \frac{y-5}{2} = \frac{z+1}{-1}$

(ii) P(2, 3, −5) $\quad l: \mathbf{r} = \begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -7 \\ 6 \end{pmatrix}$

(iii) P(8, 9, 1) $\quad l: \frac{x-6}{12} = \frac{y-5}{-9} = \frac{z-11}{-8}$

2 Find the distance from the point P to the line l:

(i) P(8, 9) $\quad l: 3x + 4y + 5 = 0$

(ii) P(5, −4) $\quad l: 6x - 3y + 3 = 0$

(iii) P(4, −4) $\quad l: 8x + 15y + 11 = 0$

3 Find the distance from the point P to the plane π:

(i) P(5, 4, 0) $\quad \pi: 6x + 6y + 7z + 1 = 0$

(ii) P(7, 2, −2) $\quad \pi: 12x - 9y - 8z + 3 = 0$

(iii) P(−4, −5, 3) $\quad \pi: 8x + 5y - 3z - 4 = 0$

4 A line l_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$.

(i) Write down the equation of a line parallel to l_1 passing through the point (3, 1, 0).

(ii) Find the distance between these two lines.

5 (i) Show that the lines $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ are skew.

(ii) Find the shortest distance between these two lines.

6 Find the shortest distance between the lines l_1 and l_2.
In each case, state whether the lines are skew, parallel or intersect.

(i) $l_1: \dfrac{x-2}{1} = \dfrac{y-3}{2} = \dfrac{z-4}{2}$ and $l_2: \dfrac{x-2}{2} = \dfrac{y-9}{-2} = \dfrac{z-1}{1}$

(ii) $l_1: \dfrac{x-8}{4} = \dfrac{y+2}{3} = \dfrac{z-7}{5}$ and $l_2: \dfrac{x-2}{2} = \dfrac{y+6}{-6} = \dfrac{z-1}{-9}$

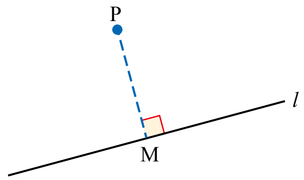
(iii) $l_1: \mathbf{r} = \begin{pmatrix} -5 \\ 6 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 6 \\ 3 \end{pmatrix}$ and $l_2: \mathbf{r} = \begin{pmatrix} 5 \\ 8 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}$

(iv) $l_1: \mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $l_2: \mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -2 \\ -4 \end{pmatrix}$

7 (i) Find the shortest distance from the point P(13, 4, 2) to the line l:

$$\mathbf{r} = \begin{pmatrix} 2 \\ -8 \\ -21 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}.$$

(ii) Find the coordinates of the point M, which is the foot of the perpendicular from P to the line l.



8 (i) Find the exact distance from the point A (2, 0, −5) to the plane $\pi: 4x - 5y + 2z + 4 = 0$.

(ii) Write down the equation of the line l through the point A that is perpendicular to the plane π.

(iii) Find the exact coordinates of the point M where the perpendicular from the point A meets the plane π.

M

9 In a school production some pieces of the stage set are held in place by steel cables. The location of points on the cables can be measured, in metres, from an origin O at the side of the stage.

Cable 1 passes through the points A(2, −3, 4) and B(1, −3, 5) whilst cable 2 passes through the points C(0, 3, −2) and D(2, 3, 5).

(i) Find the vector equations of the lines AB and CD and determine the shortest distance between these two cables.

One piece of the stage set, with corner at E(1, 6, −1), needs to be more firmly secured with an additional cable. It is decided that the additional cable should be attached to cable 2.

(ii) If the additional cable available is three metres long, determine whether it will be long enough to attach point E to cable 2.

10 The point P has coordinates (4, k, 5) where k is a constant.

The line L has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$.

The line M has equation $\mathbf{r} = \begin{pmatrix} 4 \\ k \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ 3 \\ -4 \end{pmatrix}$.

(i) Show that the shortest distance from the point P to the line L is $\frac{1}{3}\sqrt{5(k^2 + 12k + 117)}$.

(ii) Find, in terms of k, the shortest distance between the lines L and M.

(iii) Find the value of k for which the lines L and M intersect.

(iv) When $k = 12$, show that the distances in parts (i) and (ii) are equal. In this case, find the equation of the line that is perpendicular to, and intersects, both L and M.

11 The line l_1 passes through the points A (2, 3, −5) and B (8, 7, −13). The line l_2 passes through the points C (−2, 1, 8) and D (3, −1, 4).

Find the shortest distance between the lines l_1 and l_2.

The plane Π_1 passes through the points A, B and D. The plane Π_2 passes though the points A, C and D.

Find the acute angle between Π_1 and Π_2, giving your answer in degrees.

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12 The position vectors of the points A, B, C, D are

$$2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}, \quad -2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}, \quad \mathbf{i} + 4\mathbf{j} + \mathbf{k}, \quad \mathbf{i} + 5\mathbf{j} + m\mathbf{k},$$

respectively, where m is an integer. It is given that the shortest distance between the line through A and B and the line through C and D is 3. Show that the only possible value of m is 2.

Find the shortest distance of D from the line through A and C.

Show that the acute angle between the planes ACD and BCD is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

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KEY POINTS

1 The Cartesian equation of a plane perpendicular to the vector $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ is $n_1x + n_2y + n_3z = d$.

2 The vector equation of the plane through the points A, B and C is $\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}$.

3 The equation of the plane through the point with position vector $\mathbf{a}$, and perpendicular to $\mathbf{n}$, is given by $(\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$.

4 The distance of the point (α, β, γ) from the plane $n_1x + n_2y + n_3z = d$ is

$$\frac{|n_1\alpha + n_2\beta + n_3\gamma - d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}.$$

If the plane is written $ax + by + cz = d$, the formula for the distance is

$$\frac{|a\alpha + b\beta + c\gamma - d|}{\sqrt{a^2 + b^2 + c^2}}$$

5 The angle between a line and a plane is found by first considering the angle between the line and a normal to the plane.

6 The vector product $\mathbf{a} \times \mathbf{b}$ of $\mathbf{a}$ and $\mathbf{b}$ is a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$

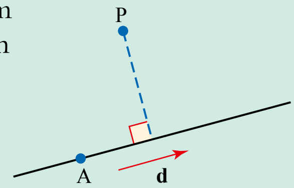
$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\hat{\mathbf{n}}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ and $\hat{\mathbf{n}}$ is a unit vector that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}$, $\mathbf{b}$ and $\hat{\mathbf{n}}$ (in that order) form a right-handed set of vectors.

7 $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$

8 In three dimensions, the shortest distance from a point P with position vector **p** to a line with direction vector **d** and passing through the point A, with position vector **a**, is given by:

$$\frac{|\overrightarrow{AP} \times \mathbf{d}|}{|\mathbf{d}|}$$

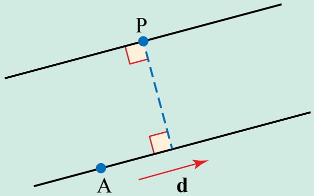


9 The shortest distance from a point $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is given by:

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

10 In three dimensions there are three possibilities for the arrangement of the lines. They are either parallel, intersecting or skew.

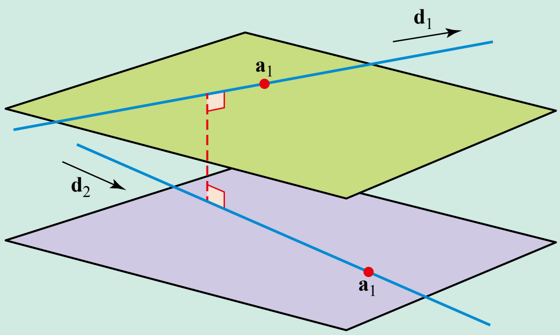
11 The shortest distance between two parallel lines can be found by choosing any point on one of the lines and finding the shortest distance from that point to the second line.



12 The distance between two skew lines is given by:

$$\left|\frac{(\mathbf{d}_1 \times \mathbf{d}_2)}{|\mathbf{d}_1 \times \mathbf{d}_2|} . (\mathbf{a}_1 - \mathbf{a}_2)\right|$$

where $\mathbf{a}_1$ is the position vector of a point on the first line and $\mathbf{d}_1$ is parallel to the first line, similarly for the second line.



LEARNING OUTCOMES

Now that you have finished this chapter, you should be able to

- use the equation of the plane in the form
  - $ax + by + cz = d$
  - $\mathbf{r}.\mathbf{n} = p$
  - $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$
- convert the equations of planes from one form to anther as necessary in problem solving
- recall that the vector product $\mathbf{a} \times \mathbf{b}$ can be expressed as:
  - $|\mathbf{a}||\mathbf{b}|\sin\theta\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is a unit vector
  - $(a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$
  - $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
- use
  - equations of lines
  - equations of planes
  - scalar product of vectors
  - vector product

  to solve problems, including
  - determining whether a line lies in a plane
  - determining whether a line is parallel to a plane or intersects a plane
  - finding the point of intersection of a line and a plane when it exists
  - finding the foot of the perpendicular from a point to a plane
  - finding the angle between a line and a plane
  - finding the angle between two planes
  - finding an equation for the line of intersection of two planes
  - calculating the shortest distance between two skew lines
  - finding an equation for the common perpendicular to two skew lines.

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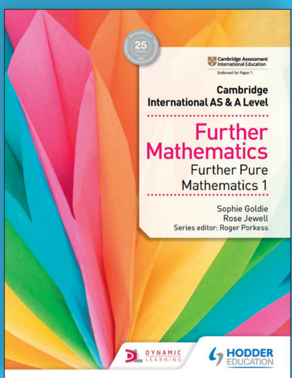

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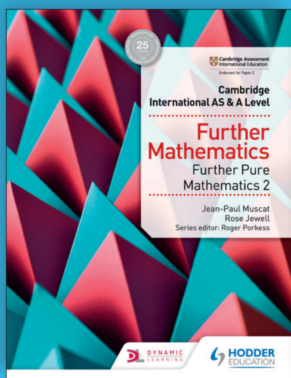

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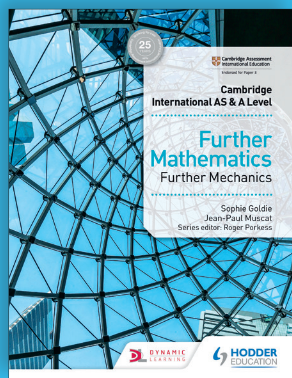

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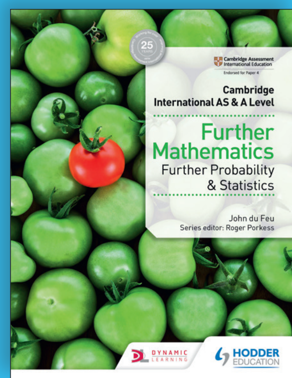

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