Lesson 4:
The Metric Tensor

Mathematics of General Relativity: A Complete Course

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Profound Physics

In the previous lesson, we looked at the differences between vectors, covectors and their components - in other words, these objects with *upstairs* and *downstairs* indices like v^i or v_i. At the end, we also got to discuss the relation between these and the central concept related to this was the **metric tensor**.

In this lesson, we'll continue discussing the metric tensor much more deeply.

The metric tensor is probably the most important mathematical object you'll come across in tensor calculus and general relativity. One of the central tools in general relativity are the Einstein field equations, and the solutions to these equations are the components of a metric tensor. A huge part of general relativity is revolved around finding metric tensors - so, this is stuff you definitely should learn!

This lesson - the one you are reading now - presents the metric tensor as a tool used to **calculate distances** and other quantities in any coordinate system or geometry. Therefore, we'll get to some very useful and practical stuff in this lesson!

My current plan is to also include a "bonus lesson" that goes into much more depth about how the metric tensor is used in the context of **working with tensor indices** and manipulating these. We already saw (briefly) how this is done in the previous lesson.

Essentially, these two lessons will cover the "two sides" of the metric tensor - how it is used to calculate distances in practice, but also how it is used to manipulate tensor equations. Both topics are hugely important in general relativity, and we will see plenty of general relativity -related examples in these two lessons.

Lesson Contents:

1. Why Is The Metric Tensor Important?

To get started, it's worth discussing why we even care about the metric tensor so much. The metric tensor is pretty much the most important mathematical object in tensor calculus and general relativity - but why?

Well, the reason for this won't be clear yet once we get to differential geometry, but the metric essentially defines the geometry of any given space. It tells us how **lengths** and **angles** are calculated in the given geometry and allows us to easily define things like *curvature*.

An example of this would be the surface of a sphere - this is a geometry that can be described by a metric tensor. By knowing the metric for the surface of a sphere, we can calculate, for example, distances along the surface. After this lesson, you should be able to understand why this is and how it works.

In the context of general relativity, this is important because **gravity** is caused by the geometry of spacetime, and we can also describe the geometry of spacetime by a metric.

Therefore, the metric tensor essentially holds all the information about gravity in general relativity - we can use it to, for example, calculate how light bends around a star or predict where the event horizon of a black hole is located at.

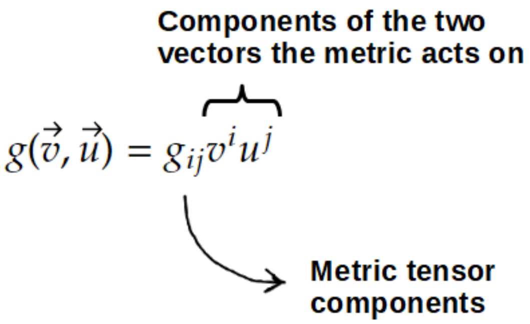
This idea of the metric tensor defining the geometry of a given space is completely new to us so far. In the previous lesson, we only looked at how the metric tensor relates vectors and covectors and allows us to "raise" or "lower" their indices.

This is the *other* reason the metric is so important - it allows us to practical calculations using index notation very efficiently. In fact, this single feature of the metric is the basis for pretty much all tensor index manipulation techniques that we'll come to talk about much more in the upcoming lessons.

2. The Metric Tensor & Dot Product

In the previous lesson, we looked at the metric tensor as an abstract function that takes in two vectors to produce a scalar. We also saw how leaving one of its "slots" empty takes a vector to a covector, but we won't need that feature much in this lesson.

We denote the metric tensor as g and in an "abstract" sense, it has two slots for input arguments. These input arguments take in two vectors, so $g = g(\vec{v}, \vec{u})$. The result of the metric acting on two vectors like this is a **scalar** calculated as:



This formula was a result of the metric tensor components being defined as $g(\vec{e}_i, \vec{e}_j) = g_{ij}$ - in other words, the metric acting on the basis vectors give back its components with respect to those basis vectors.

Now, this is all stuff we talked about in the previous lesson. In the context of this lesson, the most important thing for us are the components of the metric, g_{ij}.

These are completely analogous to vector components - they allow us to represent the metric in a given coordinate system, just like vector components do. They are also not unique in that the metric components depend on which coordinate system we choose, and they change under coordinate transformations just like the components of any vector.

A vector can be represented in terms of its components as a linear combination over the basis vectors, for example $\vec{v} = v^i \vec{e}_i$. Analogously, the metric (or any tensor) can be represented as a linear combination that looks like this; $g = g_{ij}\tilde{e}^i \otimes \tilde{e}^j$.

We'll talk much more about this in the lesson on tensors. The main point here is that the metric components are very similar to ordinary vector components - the metric components are not the same as the metric itself, strictly speaking, but just a representation of it in a given coordinate system. However, because of the tools of index notation, we are able to fully describe the metric by its components - just like with vectors!

One of the main differences here is that the metric components have **two indices** instead of just one like vectors do. This means that if we are in, say, a 3D space, the metric will have 3x3=9 components.

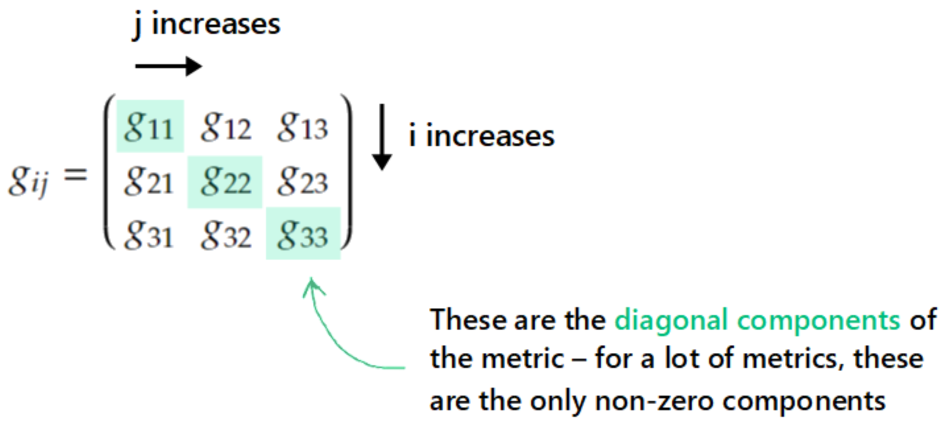
A convenient way to represent these components is by a *matrix*:

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

Again, as a reminder; this is just a matrix representation of the <u>components</u> of the metric tensor, but not the full metric itself. The full metric (which is a linear combination over basis vectors) is denoted as g and the metric components as g_{ij}.

We can use the same kind of matrix *representation* for the metric components in any number of dimensions.

The first index of g_{ij} (i, in this case) stays constant along the rows of the matrix and the second index (j) stays constant along the columns:



That's how we represent the components of the metric - by collecting each component into a matrix like the one above. As for what these components actually describe, we'll get to that as soon as we discuss how the metric is related to dot products.

An important property to mention here is that the metric tensor is always **symmetric**. This means that its two indices can be freely interchanged, for example, $g_{12} = g_{21}$ - or more generally, $g_{ij} = g_{ji}$. Therefore, the metric (in 3D) actually has only 6 independent components instead of 9. A general metric could therefore be written as:

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{12} & g_{22} & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}$$

2.1. The Inverse Metric Tensor

In the previous lesson, we also saw the inverse metric tensor, g^{-1}, which takes in two covectors and produces a scalar out of them:

$$g^{-1}(\tilde{v}, \tilde{u}) = g^{ij} v_i u_j$$

The objects with upstairs indices here, g^{ij}, are the components of the inverse metric. The index placement here will become more clear when we discuss the transformation properties of tensor components. For now, it's enough to just remember that the "ordinary" metric components naturally have downstairs indices and the inverse metric components have upstairs indices.

Exactly like above, we can also represent the inverse metric components as a matrix:

j increases →

$$g^{ij} = \begin{pmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{pmatrix}$$

↓ i increases

The inverse metric is also symmetric, meaning $g^{ij} = g^{ji}$.

The inverse metric components are essentially related to the ordinary metric components by taking a *matrix inverse*. If you know linear algebra, perhaps you know how to calculate the inverse of a matrix - that will work here.

However, we won't have to do much linear algebra here, since most metrics we are dealing with, will be **diagonal metrics**. This means all the non-diagonal elements are zero:

$$g_{ij} = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix}$$

For these, the inverse metric components are simply just inverses of the diagonal elements. So, given that we have found some metric that is diagonal, the inverse metric is obtained from it simply as:

$$g^{ij} = \begin{pmatrix} g^{11} & 0 & 0 \\ 0 & g^{22} & 0 \\ 0 & 0 & g^{33} \end{pmatrix} = \begin{pmatrix} 1/g_{11} & 0 & 0 \\ 0 & 1/g_{22} & 0 \\ 0 & 0 & 1/g_{33} \end{pmatrix}$$

That's pretty much all we need at this point. Next, we'll look at what the metric actually describes and how it allows us to generalize some familiar concepts to tensor calculus - such as the dot product.

2.2. Dot Products In Terms of The Metric

Previously, we've been thinking of the metric tensor as an abstract function that acts on two vectors and produces a scalar. The way we analyzed this was to express the two vectors as linear combinations in the form $\vec{v} = v^i \vec{e}_i$ and $\vec{u} = u^j \vec{e}_j$, in which case:

$$g(\vec{v}, \vec{u}) = g\left(v^i \vec{e}_i, u^j \vec{e}_j\right) = v^i u^j g(\vec{e}_i, \vec{e}_j)$$

We then defined the quantity $g(\vec{e}_i, \vec{e}_j)$ as the components of the metric tensor, so that:

$$g(\vec{v}, \vec{u}) = g_{ij} v^i u^j$$

This is nice and all, but the real question now is "what does it actually mean for the metric tensor to act on vectors?". Sure, the metric takes vectors to scalars, but how does it actually do that in practice?

To understand this, let's think about what "producing a scalar out of two vectors" actually means - it sounds a lot like what the **dot product** does. In fact, the dot product between two arbitrary vectors written as linear combinations of their components and basis vectors can be calculated as:

$$\vec{v} \cdot \vec{u} = \left(v^i \vec{e}_i\right) \cdot \left(u^j \vec{e}_j\right) = \vec{e}_i \cdot \vec{e}_j v^i u^j$$

This is the dot product in a general coordinate system. The dot product between basis vectors here accounts for the fact that the basis vectors may not be orthogonal or of unit length in the general case (in Cartesian coordinates, we would have $\vec{e}_i \cdot \vec{e}_j = 1$ for all the basis vectors).

The interesting thing here is how similar this looks to the action of the metric on two vectors from above, $g(\vec{v}, \vec{u}) = g_{ij} v^i u^j$.

In fact - and here comes the important part - if we make the identification that $g_{ij} = \vec{e}_i \cdot \vec{e}_j$, they are *exactly* the same expression.

This is simply a definition we make - because the metric tensor acting on two vectors has all the properties of exactly what the dot product does, it's quite natural to just relate the two. This gives us a practical idea of what it actually means for the metric tensor to act on vectors - it defines and computes their dot product!

The important result of this is that the **general definition for the dot product** can be put as follows:

$$\boxed{\vec{v} \cdot \vec{u} = g(\vec{v}, \vec{u}) = g_{ij} v^i u^j}$$, where $g_{ij} = \vec{e}_i \cdot \vec{e}_j$.

This formula allow us to compute the dot product in any coordinate system if we know the vector components and the metric components (or equivalently, the basis vectors) in that coordinate system.

Example: Kinetic Energy In General Coordinate Systems

At this point, it's worth looking at an example before moving forward. For this example, we'll derive a general formula for the kinetic energy in any coordinate system - curvilinear or not!

Generally, the kinetic energy of a particle is defined in terms of the squared magnitude of its velocity vector as:

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \vec{v} \cdot \vec{v}$$

So far, this doesn't say anything about what coordinate system we might be using. For this, we need this in terms of components. We can do this using the metric and the definition of the dot product to write $\vec{v} \cdot \vec{v} = g_{ij} v^i v^j$.

With this, the kinetic energy becomes:

$$T = \frac{1}{2} m g_{ij} v^i v^j$$

This is the general formula for the kinetic energy in any coordinate system. Perhaps you haven't seen it written in this kind of general form before, so seeing some examples of using this in different coordinate systems might be useful.

The components v^i here represent the coordinate velocities, meaning simply time derivatives of the particle's coordinates:

$$v^i = \frac{dx^i}{dt} = \dot{x}^i$$

We're using the dot notation here, meaning time derivatives are represented by a dot above the quantity. This notation is quite common in physics.

First, in 2D **Cartesian coordinates**, the metric tensor components are quite simple:

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The components of the coordinate velocity in Cartesian coordinates are just:

$$v^i = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

Therefore, the kinetic energy would be - remember the Einstein sum convention here:

$$\begin{aligned} T &= \frac{1}{2} m g_{ij} v^i v^j \\ &= \frac{1}{2} m \left(g_{11} v^1 v^1 + g_{12} v^1 v^2 + g_{21} v^2 v^1 + g_{22} v^2 v^2 \right) \\ &= \frac{1}{2} m \left(1 \cdot v^1 v^1 + 0 \cdot v^1 v^2 + 0 \cdot v^2 v^1 + 1 \cdot v^2 v^2 \right) \end{aligned}$$

$$= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

Maybe this is a more familiar expression you've seen for the kinetic energy - however, it is just one example, namely the kinetic energy in Cartesian coordinates. Let's look at **polar coordinates** next. The components of the metric tensor in polar coordinates are given by (without yet justifying where these come from):

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

The coordinate velocities are just the time derivatives of the polar coordinates r and θ:

$$v^i = \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix}$$

The kinetic energy in polar coordinates would then be:

$$T = \frac{1}{2}mg_{ij}v^iv^j$$
$$= \frac{1}{2}m\left(g_{11}v^1v^1 + g_{12}v^1v^2 + g_{21}v^2v^1 + g_{22}v^2v^2\right)$$
$$= \frac{1}{2}m\left(1 \cdot v^1v^1 + 0 \cdot v^1v^2 + 0 \cdot v^2v^1 + r^2 \cdot v^2v^2\right)$$
$$= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

This is different than in the Cartesian case due to the factor r^2 here. This means that the kinetic energy in polar coordinates actually also depends on how far the particle is from the origin. The point here is that we can derive all of this by knowing the metric in the given coordinates - polar coordinates, in this case.

2.3. Interpretation of The Metric Tensor Components

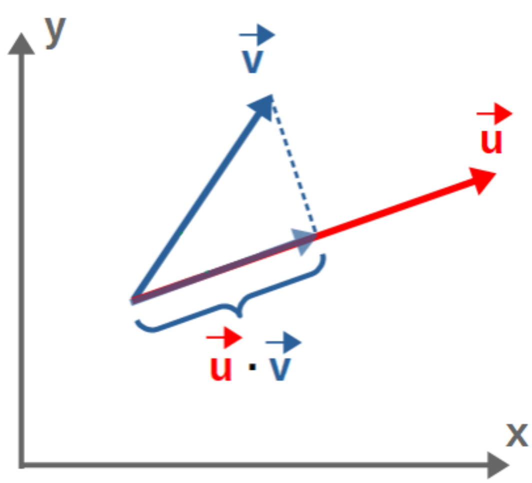
Okay, we have a general definition for the dot product in terms of the metric. However, another interesting thing about this definition we made is that it also provides a nice geometric interpretation for what the components of the metric tensor actually represent.

With what we have above, the components of the metric are given simply by:

$$g_{ij} = g(\vec{e}_i, \vec{e}_j) = \vec{e}_i \cdot \vec{e}_j$$

In other words, the various components of the metric represent **dot products between basis vectors** in a given coordinate system.

If you remember the geometric interpretation of the dot product as describing essentially the "overlap" between two vectors (how much a given vector is pointed along another vector) then interpreting the metric components is quite straigthforward.



The components of the metric tensor therefore describe the **length of each basis vector** as well as **how much the basis vectors are aligned** with one another.

More precisely, the diagonal elements of the metric represent the (squared) lengths of each basis vector:

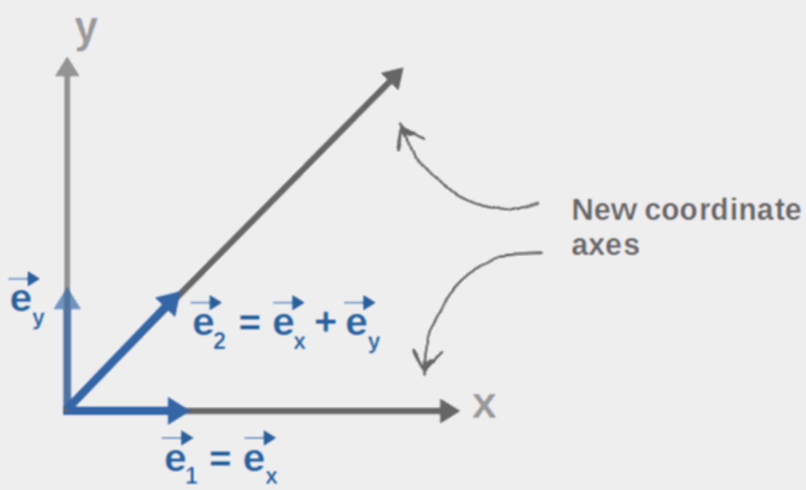
$$g_{ij} = \begin{pmatrix} g_{11} & & \\ & g_{22} & \\ & & g_{33} \end{pmatrix} = \begin{pmatrix} \vec{e}_1 \cdot \vec{e}_1 & & \\ & \vec{e}_2 \cdot \vec{e}_2 & \\ & & \vec{e}_3 \cdot \vec{e}_3 \end{pmatrix} = \begin{pmatrix} |\vec{e}_1|^2 & & \\ & |\vec{e}_2|^2 & \\ & & |\vec{e}_3|^2 \end{pmatrix}$$

The off-diagonal elements, on the other hand, represent the overlap of the basis vectors. In orthogonal coordinate systems, these are always zero, but in more general coordinate systems, they may not be:

$$g_{ij} = \begin{pmatrix} & g_{12} & g_{13} \\ g_{21} & & g_{23} \\ g_{31} & g_{32} & \end{pmatrix} = \begin{pmatrix} & \vec{e}_1 \cdot \vec{e}_2 & \vec{e}_1 \cdot \vec{e}_3 \\ \vec{e}_2 \cdot \vec{e}_1 & & \vec{e}_2 \cdot \vec{e}_3 \\ \vec{e}_3 \cdot \vec{e}_1 & \vec{e}_3 \cdot \vec{e}_2 & \end{pmatrix}$$

Let's take some arbitrary coordinate system as an example, with the basis vectors of this coordinate system defined in terms of the Cartesian basis vectors as $\vec{e}_1 = \vec{e}_x$ and $\vec{e}_2 = \vec{e}_x + \vec{e}_y$.

This is how our coordinate system would look like:



The components of the metric in this new coordinate system could then be calculated as:

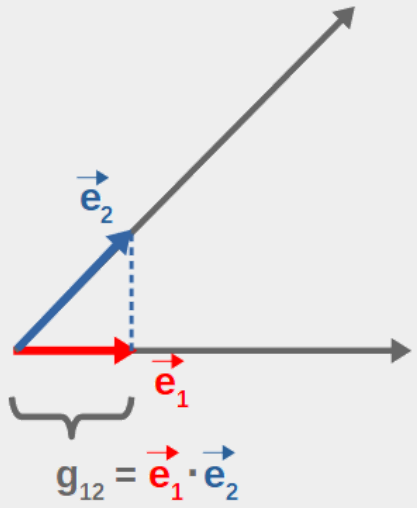
$$\begin{cases} g_{11} = \vec{e}_1 \cdot \vec{e}_1 = \vec{e}_x \cdot \vec{e}_x = 1 \\ g_{12} = \vec{e}_1 \cdot \vec{e}_2 = \vec{e}_x \cdot (\vec{e}_x + \vec{e}_y) = 1 + 0 = 1 \\ g_{21} = \vec{e}_2 \cdot \vec{e}_1 = \vec{e}_1 \cdot \vec{e}_2 = 1 \\ g_{22} = \vec{e}_2 \cdot \vec{e}_2 = (\vec{e}_x + \vec{e}_y) \cdot (\vec{e}_x + \vec{e}_y) = 1 + 0 + 0 + 1 = 2 \end{cases}$$

We can collect these components into a matrix:

$$g_{ij} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

What do these component values mean? Well, the components g_{11} and g_{22} describe the lengths of our basis vectors in the new coordinate system, so $g_{11} = |\vec{e}_1|^2 = 1$ and $g_{22} = |\vec{e}_2|^2 = 2$. We can see here that the basis vector $\vec{e}_2$ is not of unit length anymore, which is expected from how we defined it.

The off-diagonal components g_{12} and g_{21} (which are generally always equal due to the symmetry of the metric) represent the alignment of the basis vectors, since they are not orthogonal anymore in this new coordinate system. More precisely, they describe the projections of the basis vectors onto each other - which is what the dot product does in general:



Another interesting example would be polar coordinates. We already know that the polar basis vectors are orthogonal, so the off-diagonal metric components are simply $g_{12} = g_{21} = \vec{e}_r \cdot \vec{e}_\theta = 0$. We also know that the r-basis vector has unit length, but the θ-basis vector has a coordinate-dependent length of r. The diagonal metric components are therefore:

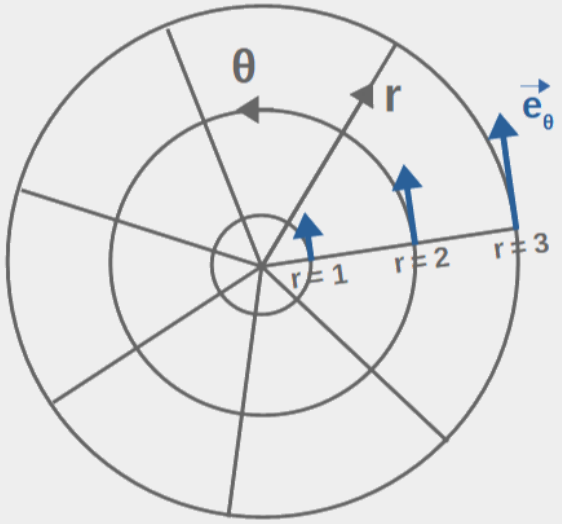
$$g_{11} = \vec{e}_r \cdot \vec{e}_r = 1$$
$$g_{22} = \vec{e}_\theta \cdot \vec{e}_\theta = r^2$$

The metric in polar coordinates can then be written as the following matrix:

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

How do we interpret this? Well, the off-diagonal elements being zero corresponds to the fact that there is no "overlap" between the polar basis vectors - they are orthogonal everywhere.

The g_{11}-component tells us that the radial basis vector $\vec{e}_1 = \vec{e}_r$ naturally has unit length. On the other hand, the g_{22}-component tell us that the basis vector $\vec{e}_2 = \vec{e}_\theta$ has a length of r, meaning its length scales with the radial distance to the origin:



This is one of the interesting things about the polar coordinate system - the basis vectors are not constant. The metric tensor encodes all of this into its various components.

The way in which the metric describes the lengths and alignment of basis vectors is, in fact, enough to define pretty much the entire geometry of a given coordinate system. It allows us to calculate distances, angles or any other quantities we wish.

This also carries out to curved spaces and geometries - so far, we've only been looking at curved coordinates, but in a flat space - which is why the metric is such an important object in differential geometry. In general relativity, the metric is used to similarly define the geometry of *spacetime* (more on this at the end of this lesson).

3. How The Metric Tensor Defines Lengths & Angles

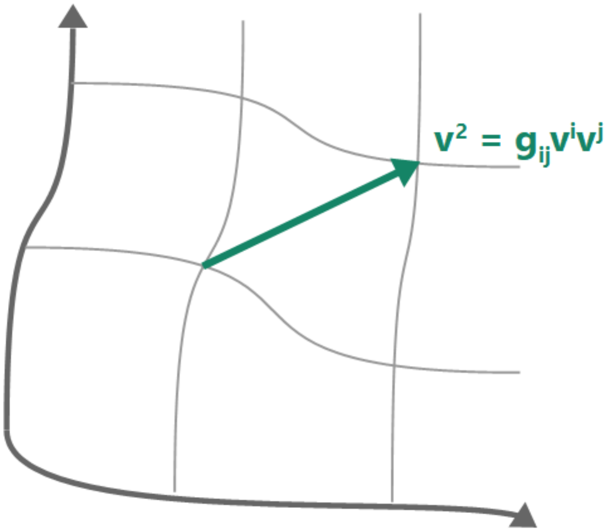
When you begin learning about the metric tensor, one of the first things you here is how the metric defines **lengths** and **angles** in a geometry or coordinate system. The underlying reason for this comes simply from the fact that the metric defines the dot product.

Then, if we have a dot product, we can calculate the length of any vector and also the angle between any two vectors.

We already saw earlier how the metric defines the length of a vector (or its square), $v^2 = \vec{v} \cdot \vec{v}$:

$$v^2 = \vec{v} \cdot \vec{v} = g(\vec{v}, \vec{v}) = g_{ij} v^i v^j$$

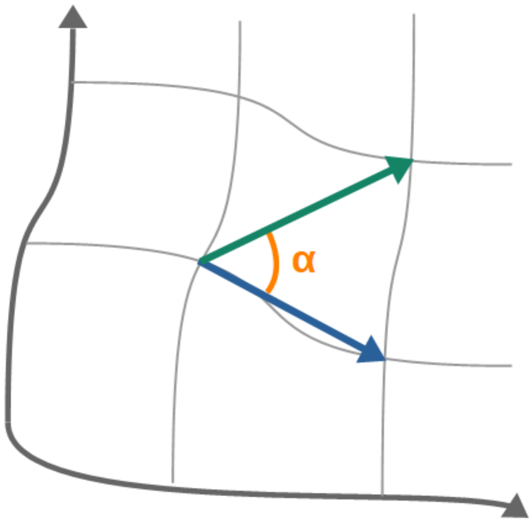
This formula works in any coordinate system or curved space, given that we know the metric components in that coordinate system.



The metric also defines the angle α between two vectors, which again comes from the dot product. More precisely, the dot product can be calculated as $\vec{v} \cdot \vec{u} = vu \cos \alpha$, so:

$$\begin{aligned}
&\vec{v} \cdot \vec{u} = vu \cos \alpha \\
&\Rightarrow \quad g_{ij} v^i u^j = \sqrt{g_{ij} v^i v^j} \sqrt{g_{ij} u^i u^j} \cos \alpha \\
&\Rightarrow \quad \alpha = \arccos \left(\frac{g_{ij} v^i u^j}{\sqrt{g_{ij} v^i v^j} \sqrt{g_{ij} u^i u^j}} \right)
\end{aligned}$$

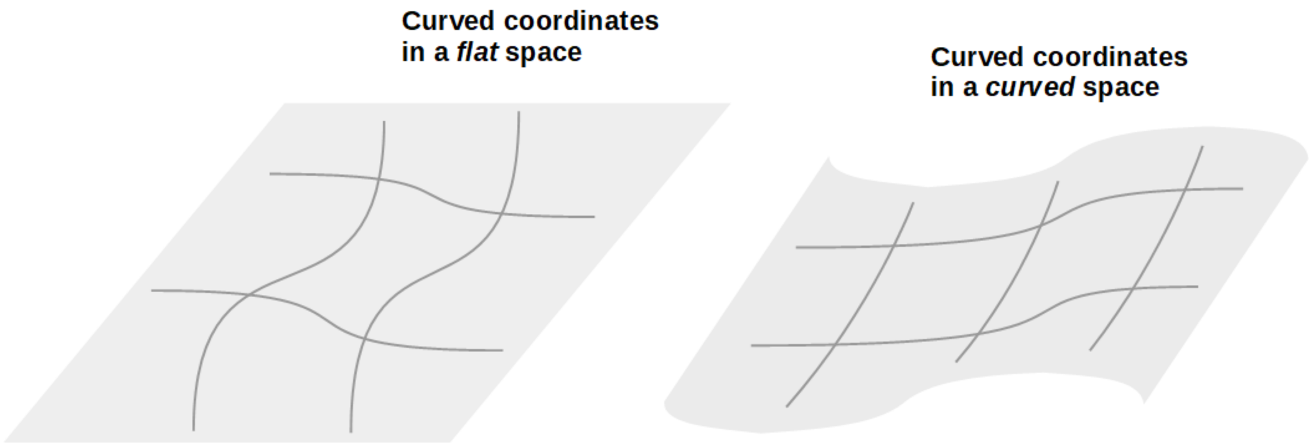
This is the formula for the angle between two vectors that works in any coordinate system or curved geometry.



Now, I've been mentioning the term 'curved space' a few times here. We haven't exactly defined what a curved space is yet. This goes into the realm of **differential geometry**, which we will be discussing in a lot more detail later.

So far, we've only been discussing **curvilinear coordinates**, but in an underlying *flat* space. This means that while our coordinate axes may be curved, the space in which the coordinate system is located in is flat.

However, we could also have the underlying space itself be *curved*, which would also result in the coordinate lines getting curved.



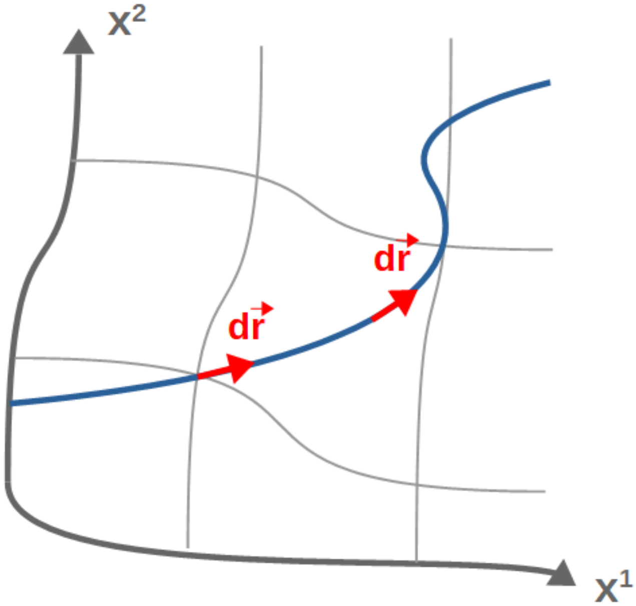
The nice thing is that the formulas we have above apply in either case - so, the metric can be used to define the geometry of a curved space itself as well. We can then compute distances and angles in pretty much the same way as we would in a flat space.

3.1. Line Elements

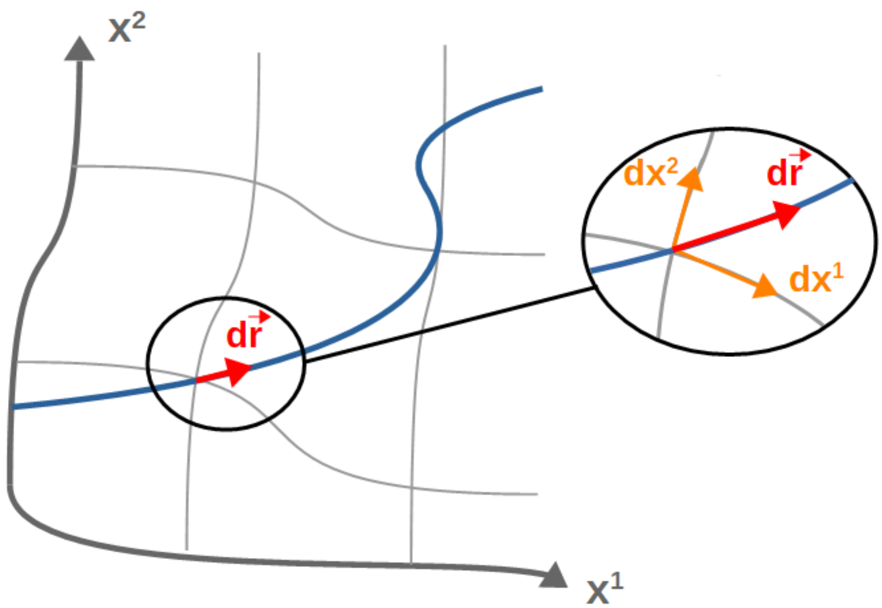
So far, we've looked at how the metric allows us to calculate the length of a vector in any coordinate system. However, it also allows us to calculate the **length along any curve** in any coordinate system or on any surface (given that we can find a metric for the surface, of course.)

The general way to do this would be to break down the given curve into small, infinitesimal displacement pieces.

We can describe each of these displacement pieces by a displacement vector $d\vec{r}$, which points along the curve at each point:



Since these displacement vectors are, of course, vectors, we can break them down into their components with respect to each of the coordinate and basis vectors. The components of these will be the displacements dx^i in each coordinate direction:



Therefore, we can express the displacement vector at each point as $d\vec{r} = dx^i \vec{e}_i$. Then, the little "piece of length" along the curve at each point, call it ds, will be the magnitude of this displacement vector. We can calculate this (or the square of the magnitude) using the dot product as:

$$ds^2 = d\vec{r} \cdot d\vec{r} = \left(\vec{e}_i dx^i\right) \cdot \left(\vec{e}_j dx^j\right) = \vec{e}_i \cdot \vec{e}_j dx^i dx^j$$

Here, we have the components of the metric tensor, $g_{ij} = \vec{e}_i \cdot \vec{e}_j$, so we get:

$$\boxed{ds^2 = g_{ij} dx^i dx^j}$$

This particular quantity ds^2 is called a **line element**. It describes an *infinitesimal length element* in any coordinate system (and in any curved space as well, actually).

The crucial part here are the metric components g_{ij}, which allow us to calculate the distance element from the coordinate displacements dx^i.

The line element also provides another way to think about the metric tensor. Essentially, the metric "converts" coordinate displacements to actual lengths - it takes us from the coordinate displacements dx^i to an actual length ds.

It does this by including the relevant "conversion factors", which are the metric components g_{ij} here. These metric components ensure that the units of the line element are actually those of distance, since displacements in coordinate themselves won't always be directly related to distances.

The metric then tags on these factors g_{ij}, which results in something that actually does describe distance - this is also (probably) where the name *metric* tensor comes from.

In the example below, you'll see a practical example of how all of this works.

Example: Line Element In Polar Coordinates

Let's look at the line element in polar coordinates as an example. We already have the metric components from earlier as:

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Therefore, the line element is simply:

$$\begin{aligned} ds^2 &= g_{ij} dx^i dx^j \\ &= g_{11} dx^1 dx^1 + g_{12} dx^1 dx^2 + g_{21} dx^2 dx^1 + g_{22} dx^2 dx^2 \\ &= 1 \cdot dr dr + 0 \cdot dr d\theta + 0 \cdot d\theta dr + r^2 \cdot d\theta d\theta \\ &= dr^2 + r^2 d\theta^2 \end{aligned}$$

Notice how straightforward calculating the line element is if we know the metric components already.

Another way would be to actually calculate the line element directly from the coordinate relations $x = r\cos\theta$ and $y = r\sin\theta$. Then, we can use the multivariable chain rule to get the differentials dx and dy in terms of dr and $d\theta$ as:

$$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta = \frac{\partial}{\partial r}(r\cos\theta) dr + \frac{\partial}{\partial \theta}(r\cos\theta) d\theta = \cos\theta dr - r\sin\theta d\theta$$

$$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta = \frac{\partial}{\partial r}(r\sin\theta) dr + \frac{\partial}{\partial \theta}(r\sin\theta) d\theta = \sin\theta dr + r\cos\theta d\theta$$

We then calculate the line element from the Cartesian line element $ds^2 = dx^2 + dy^2$ using these. With a bunch of the terms cancelling, we get:

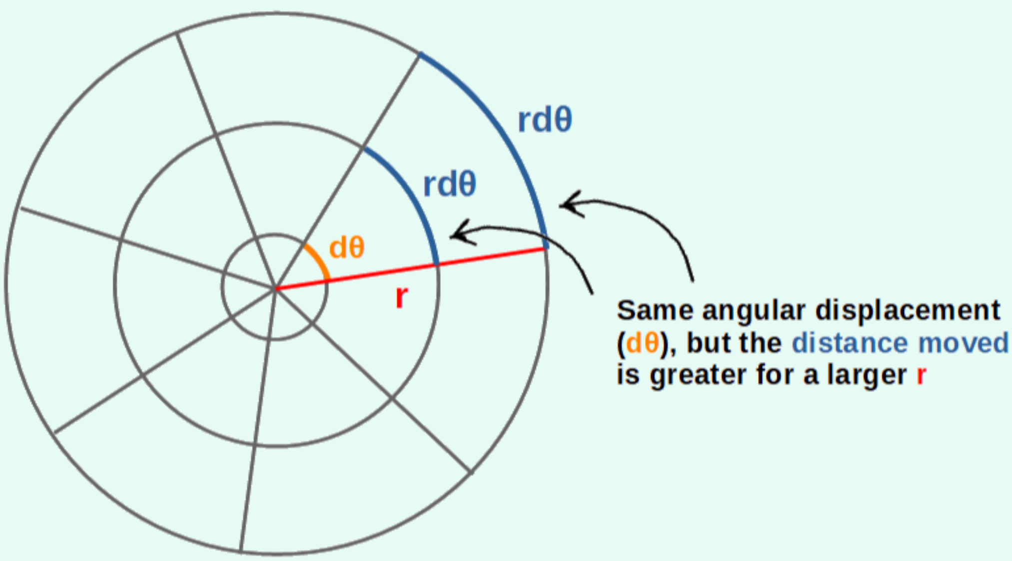
$$ds^2 = dx^2 + dy^2 = (\cos\theta dr - r\sin\theta d\theta)^2 + (\sin\theta dr + r\cos\theta d\theta)^2 = dr^2 + r^2 d\theta^2$$

Here, we essentially went in the opposite direction - we first calculate the line element, from which we can then identify the metric components as the prefactors in front of the coordinate displacements. The method you *should* use will, of course, depend on the specific situation.

Now, for some interpretation. First of all, because the off-diagonal elements of the metric are zero, this means that the coordinate displacements dr and $d\theta$ are completely **independent** - a displacement in the angular direction doesn't affect the radial direction and vice versa.

Also, we see that the line element contains a simple dr^2-term with no additional prefactors. This means that displacements in the radial direction are always of the "same size" in distance, no matter where we are in the coordinate system.

On the other hand, the angular displacement term with $d\theta^2$ contains an additional factor of r^2. This means that a given displacement in the angular direction actually corresponds to a larger (physical) distance moved the further we are in the radial direction. This is illustrated in the picture below:



In this example, we also see how the metric components act as the "conversion factors" from coordinate displacements to physical distances. The radial displacement dr is already directly a length (it also has the units of length), so the conversion factor is just 1.

On the other hand, the angular displacement $d\theta$ does not have anything to do with distance a priori, but the metric converts it to one by tagging on a factor of r. The combination $rd\theta$ is then indeed an actual distance (and has units of length).

The line element itself is absolutely crucial in differential geometry and general relativity, as it completely defines the notion of distance in a given coordinate system or geometric space. Because of this, especially in general relativity, the line element is often taken as the **fundamental object of study** instead of the metric components directly.

In fact, the line element is so fundamental that it itself is often referred to as the **metric**. This terminology is based on the fact that the coordinate displacements dx^i uniquely define each direction in a given coordinate system - exactly like basis vectors - so it's possible to alternatively represent the basis itself using these.

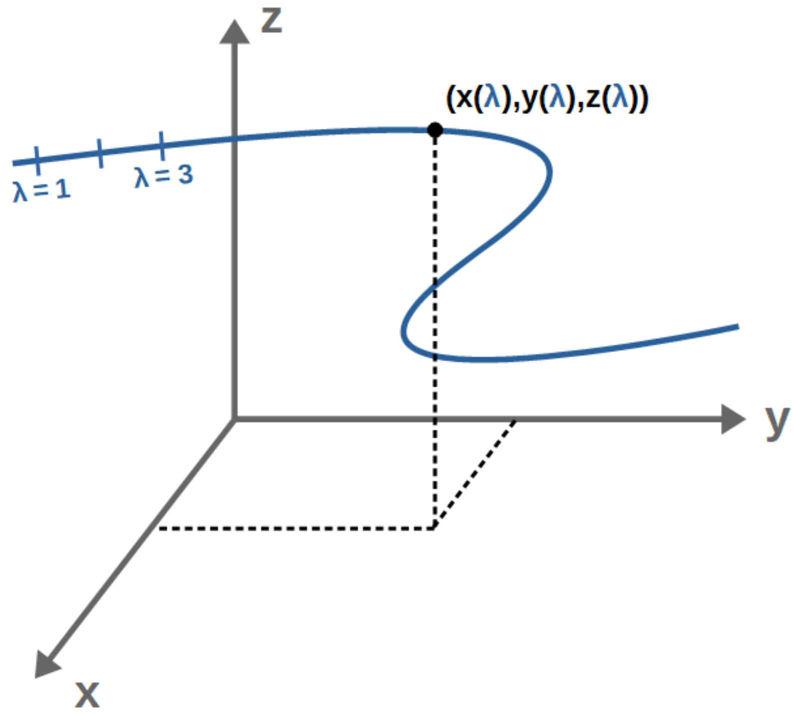
The line element $ds^2 = g_{ij}dx^i dx^j$ would then be a perfectly equivalent representation of the metric tensor as the linear combination $g = g_{ij}\tilde{e}^i \otimes \tilde{e}^j$, so it really does deserve to be called the 'metric'. Anyway, this is just a piece of terminology to be aware of - the terms 'metric' and 'line element' are often used interchangeably.

3.2. The Generalized Arc Length Formula

Let's look at one of the applications of the metric tensor more deeply - calculating distances along curves in any coordinate system or geometry. More precisely, we'll see how the metric tensor defines the notion of arc length in a very general way.

If you recall from *Part 2: Vector Calculus*, we looked at calculating distances along parametric curves. The idea here was to express the coordinates of a given curve as functions of some curve parameter, which we will call λ here (in Part 2, this was labeled as t). Namely, we always dealt with the **Cartesian coordinates** of the curve in the form:

$$\begin{cases} x = x(\lambda) \\ y = y(\lambda) \\ z = z(\lambda) \end{cases}$$



One way to visualize a parameterized curve is to think of λ as a "tickmark" that runs along the curve and labels all points on the curve. Of course, we can have all kinds of other parameterizations that cannot necessarily be visualized as easily.

What the metric tensor allows us to easily do, is to parameterize the curve in **any coordinate system** - so, in other coordinates than Cartesian as well.

This is hugely powerful, because in many cases, a given curve will be much simpler to express parametrically in coordinates other than Cartesian.

What makes this even more powerful is that the ideas we discuss here for calculating distance along curves also generalizes quite straightforwardly to curved spaces. Thus, the tools we look at will allow us to calculate distances in, for example, any curved spacetime in general relativity.

Okay, the basic idea here is quite simple - instead of expressing the Cartesian coordinates of the curve in terms of the parameterization, we express some other, arbitrary set of coordinates in terms of this parameterization.

So, we have a set of coordinates we choose to describe the curve with, x^i, and we write them in terms of a suitable curve parameter λ:

$$x^i = x^i(\lambda) \quad \Rightarrow \quad \begin{cases} x^1 = x^1(\lambda) \\ x^2 = x^2(\lambda) \\ x^3 = x^3(\lambda) \\ \ldots \end{cases}$$

The nice thing now is that we already have a formula used to calculate distance in arbitrary coordinate systems - the line element, $ds^2 = g_{ij}dx^i dx^j$. If the coordinates x^i are now functions of λ, we can calculate coordinate displacements as:

$$dx^i = \frac{dx^i}{d\lambda}d\lambda$$

The line element is therefore:

$$ds^2 = g_{ij}dx^i dx^j = g_{ij}\frac{dx^i}{d\lambda}d\lambda\frac{dx^j}{d\lambda}d\lambda = g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}d\lambda^2$$

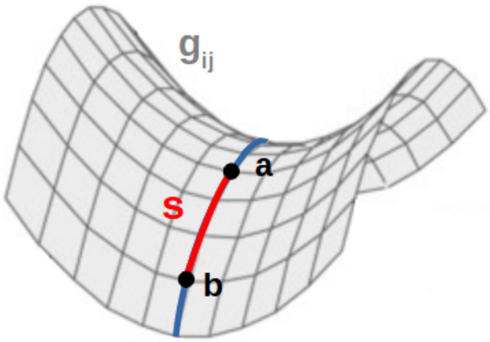
Now, here comes the important thing - this ds describes a small piece of length along the curve we are looking at. The actual **arc length**, which we'll call s, along the curve between any two points a and b, is then obtained by integrating ds:

$$s = \int_a^b ds \quad \Rightarrow \quad \boxed{s = \int_a^b \sqrt{g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}}d\lambda}$$

Notice that this also has the form of a functional, which we discussed in Part 3: Calculus of Variations. Indeed, if we were to apply the Euler-Lagrange equation to this functional, we would obtain the so-called geodesic equation - but more on that in a later lesson!

This is a fully general formula for calculating the arc length along any curve in any coordinate system - provided, of course, that we are able to parameterize the curve properly.

The nice thing about this formula is that it also applies in curved spaces for curves along curved geometries. We could, for example, use this to calculate the distance an object moves in any curved spacetime in general relativity.



Below, you'll find a simple example of actually using the above arc length formula.

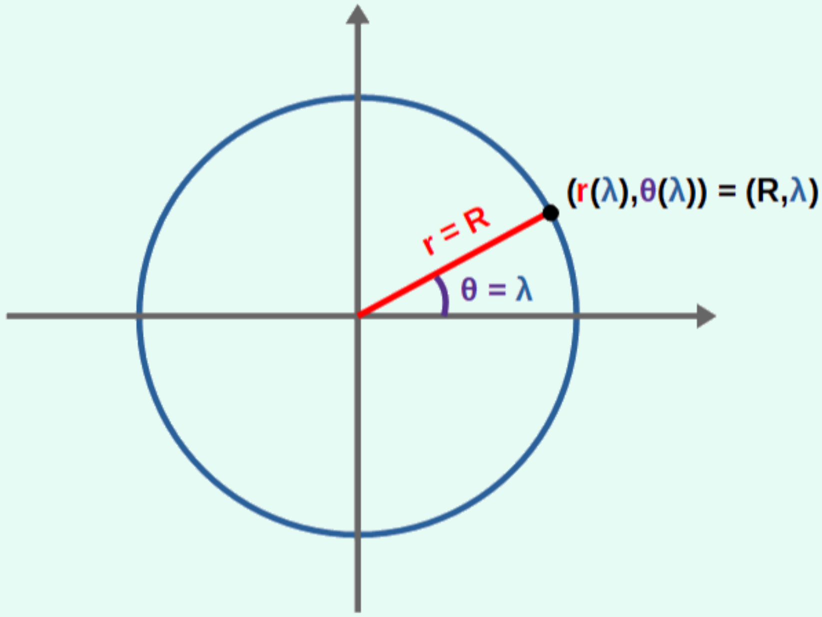
Example: Calculating Arc Length In Polar Coordinates

In this example, we will calculate the arc length along a circle or radius R in polar coordinates. In the vector calculus lessons, we did this by parameterizing the circle in terms of its Cartesian coordinates as:

$$\begin{cases} x(\lambda) = R\cos\lambda \\ y(\lambda) = R\sin\lambda \end{cases}$$

With the power of the metric and line elements at our disposal, we can now do this using a much simpler parameterization. In polar coordinates, the radius of the circle is a constant, so $r = R$. The angular coordinate θ, on the other hand, just increases linearly around the circle, so we could write $\theta = \lambda$, with $0 \leq \lambda \leq 2\pi$. Therefore, we can parameterize the circle in polar coordinates simply as:

$$\begin{cases} r(\lambda) = R \\ \theta(\lambda) = \lambda \end{cases}$$



Clearly this is much simpler than the Cartesian parameterization.

Now, although in this case, calculating the arc length around a circle would not be too difficult using Cartesian coordinates either, the advantage of using different coordinate systems becomes even more evident for more complicated curves.

Anyway, we now have our coordinates in terms of the curve parameter λ as $x^i = (r, \theta) = (R, \lambda)$. The next step would be to compute the derivatives of these:

$$\frac{dx^1}{d\lambda} = \frac{dr}{d\lambda} = \frac{dR}{d\lambda} = 0$$
$$\frac{dx^2}{d\lambda} = \frac{d\theta}{d\lambda} = \frac{d\lambda}{d\lambda} = 1$$

These are clearly much simpler than the derivatives of trigonometric functions we would've had to dealt with in Cartesian coordinates. Using these, we can then calculate the arc length along the circle (so from $\lambda = 0$ to $\lambda = 2\pi$):

$$s = \int_a^b \sqrt{g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}}d\lambda$$

Writing out the summations over both i and j here, we get using the metric components in polar coordinates ($g_{11} = 1$, $g_{12} = g_{21} = 0$ and $g_{22} = r^2$):

$$
\begin{aligned}
s &= \int_0^{2\pi} \sqrt{g_{11}\frac{dx^1}{d\lambda}\frac{dx^1}{d\lambda} + g_{12}\frac{dx^1}{d\lambda}\frac{dx^2}{d\lambda} + g_{21}\frac{dx^2}{d\lambda}\frac{dx^1}{d\lambda} + g_{22}\frac{dx^2}{d\lambda}\frac{dx^2}{d\lambda}}d\lambda \\
&= \int_0^{2\pi} \sqrt{1 \cdot 0^2 + R^2 \cdot 1^2}d\lambda \\
&= \int_0^{2\pi} Rd\lambda \\
&= 2\pi R
\end{aligned}
$$

Quite a simple calculation, right? What specifically made this calculation simple was choosing the right coordinates from the get-go.

The "right" coordinates, of course, depend on the problem and they are generally the ones that make the parameterization of the given curve as simple as possible. In this case, those were polar coordinates.

Using this formalism, we can also derive general formulas for the arc length of any curve in any coordinate system explicitly.

To see what I mean by this, let's look at polar coordinates again.

Usually, we express all curves in polar coordinates by specifying the coordinate r as function of θ, so a general curve can be described by some function $r(\theta)$ - similarly like we do when expressing curves in Cartesian coordinates as $y(x)$.

Then, parameterizing such a curve is simple - we could set $\theta = \lambda$ and $r = r(\lambda)$. This is generally suitable for describing any curve in polar coordinates, because we can always choose the function $r(\lambda)$ accordingly.

Using this kind of general parameterization, our coordinates are $x^i = (r, \theta) = (r(\lambda), \lambda)$.

The arc length of any curve using the metric in polar coordinates is then:

$$
\begin{aligned}
s &= \int_a^b \sqrt{g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}}d\lambda \\
&= \int_a^b \sqrt{g_{11}\frac{dx^1}{d\lambda}\frac{dx^1}{d\lambda} + g_{12}\frac{dx^1}{d\lambda}\frac{dx^2}{d\lambda} + g_{21}\frac{dx^2}{d\lambda}\frac{dx^1}{d\lambda} + g_{22}\frac{dx^2}{d\lambda}\frac{dx^2}{d\lambda}}d\lambda \\
&= \int_a^b \sqrt{1\cdot\frac{dr}{d\lambda}\frac{dr}{d\lambda} + r^2\cdot\frac{d\theta}{d\lambda}\frac{d\theta}{d\lambda}}d\lambda \\
&= \int_a^b \sqrt{\left(\frac{dr(\lambda)}{d\lambda}\right)^2 + r^2\left(\frac{d\lambda}{d\lambda}\right)^2}\, d\lambda \\
&= \int_a^b \sqrt{r^2 + \left(\frac{dr(\lambda)}{d\lambda}\right)^2}\, d\lambda
\end{aligned}
$$

If you want, you can express this again in terms of $\theta = \lambda$, so:

$$
s = \int_a^b \sqrt{r^2 + \left(\frac{dr(\theta)}{d\theta}\right)^2}\, d\theta
$$

This is the general formula for the arc length of any curve $r(\theta)$ in polar coordinates. For example, you could try plugging in $r(\theta) = R$ like we had for the circle from earlier, or something like $r(\theta) = a\theta$, which describes a curve known as the Archimedean spiral.

The reason we discussed the generalized arg length formula here is that, firstly, it shows a pretty useful example application of what the metric tensor can be used for.

Secondly, the above arc length formula is actually going to be really important when we get to discussing geodesics - the paths of shortest distance, which are used all the time in general relativity.

4. How Do We Actually Find Metric Tensors?

So far, we've looked at how to interpret the components of a metric and what we can do with them, assuming someone has given us a metric in the first place. The question now is - how do we actually calculate or find them?

Our discussion here will be mostly about finding metric tensors in **flat space**. It turns out that most of what we discuss is also applicable to *curved spaces*, but with some additional caveats.

The nice thing about flat spaces is that we can always use Cartesian coordinates as a "reference", which allows for some pretty quick ways of calculating the metric components in any other coordinates. In curved spaces, however, using Cartesian coordinates is not possible, so we need some different methods.

Here, we'll be discussing three different ways of calculating the metric components in any coordinate system:

1. The straightforward way directly from the **basis vectors** as $g_{ij} = \vec{e}_i \cdot \vec{e}_j$.
2. From the **Jacobian matrix** for the transformation *from Cartesian* to our coordinate system of interest.
3. From the **line element** using a set of coordinate transformation equations.

The underlying theme with all of these methods is that we need information about the coordinate system we want to find the metric for. Either, we need to know about the relation between the basis vectors, or we need a set of coordinate transformation equations.

For the first method, we've pretty much covered it already, so we don't need to go into an extensive discussion on it.

The way this method works is extremely simple - if we know the basis vectors explicitly in a given coordinate system, we can compute the components of the metric in that coordinate system by taking all combinatations of dot products between them:

$$g_{ij} = \vec{e}_i \cdot \vec{e}_j$$

Often, these basis vectors will be expressed in terms of the Cartesian unit basis vectors because the dot products between these are so simple. For example, in polar coordinates, we have:

$$\begin{cases} \vec{e}_r = \cos\theta \vec{e}_x + \sin\theta \vec{e}_y \\ \vec{e}_\theta = -r\sin\theta \vec{e}_x + r\cos\theta \vec{e}_y \end{cases}$$

We could then take the dot products between these like we already did previously. Now, we don't always *have to* express these basis vectors in terms of the Cartesian basis.

This is just the simplest way in many cases, but all we really need are some expressions that allow us to calculate the dot products between all the basis vectors.

Now, for complicated coordinate systems, finding the basis vectors first in order to get the metric can be quite cumbersome.

We would need to, for example, calculate the Jacobian matrix and then transform the basis vectors using the covariant transformation rule - this can be a lot of work. Even in the (still fairly simple) spherical coordinate system, this gets a little tedious.

Luckily, there are more efficient methods that don't require us to calculate so many things first.

4.1. Calculating Metric Components From The Jacobian Matrix

One way to find metric tensor components is actually by using the Jacobian matrix directly. This is quite nice, because we've already looked at how to find Jacobian matrices for various coordinate transformations, so that turns out handy here as well.

One thing to note before we discuss anything else is that the Jacobian matrix is always defined for a **coordinate transformation** - it is defined between two coordinates systems, not just for an individual coordinate system.

However, the components of a metric are always defined in a **single coordinate system** - in another coordinate system, these components would be different. So, metric components are associated with a coordinate system, while Jacobian matrices are associated with a coordinate transformation.

However, there is a way in which we can get the metric components from the Jacobian matrix and that is by assuming the Jacobian is always describing a transformation from Cartesian to whatever coordinate system we want. Doing this, it turns out that we are then able to get the metric components directly in our coordinate system of interest.

Without futher ado, here is the formula:

$$g_{ij} = \sum_k \Lambda^k_{\,i} \Lambda^k_{\,j}$$

The sum over k here goes over all the "unbarred" coordinates. We need the summation sign explicitly, as the Einstein summation convention would not apply here.

The above formula is handy if we have the Jacobian matrix already calculated. However, if we don't, there are usually faster ways to find the metric (which we will discuss soon).

But first, where does this formula come from? Well, one way to understand it is from the transformation rules of tensor components. However, since we have not talked about those yet, we can look at it from another perspective.

The way to do this is to consider the transformation of the Cartesian basis vectors. So, we begin from Cartesian coordinates and transform the basis vectors to whatever our new coordinate system is using the definition:

$$\vec{e}_i = \Lambda^k_i \hat{x}_k$$

Here, $\hat{x}_k$ denote the Cartesian basis vectors specifically, so $\hat{x}_1 = \hat{x}$, $\hat{x}_2 = \hat{y}$ and $\hat{x}_3 = \hat{z}$. This is just to make it explicit that these are the Cartesian basis vectors.

The $\vec{e}_i$ here are the basis vectors in the new coordinate system we are interested in. Therefore, we could calculate the metric in this new coordinate system as:

$$g_{ij} = \vec{e}_i \cdot \vec{e}_j = \Lambda^k_i \hat{x}_k \cdot \Lambda^m_j \hat{x}_m = \Lambda^k_i \Lambda^m_j \hat{x}_k \cdot \hat{x}_m$$

We need a **different summation index** here as the sums over k and m should be independent of one another

Since the Cartesian basis vectors are always bot orthogonal and have unit length, the dot product $\hat{x}_k \cdot \hat{x}_m$ can be expressed as:

$$\hat{x}_k \cdot \hat{x}_m = \begin{cases} 0 & , k \neq m \\ 1 & , k = m \end{cases}$$

Therefore, what we can do here is to just replace, for example, the sum over the index m with just the sum over k. This is because the sum only contains non-zero components if $k = m$. A valid way to write the above expression would then be:

$$g_{ij} = \sum_k \Lambda^k_i \Lambda^k_j$$

That's essentially where this formula comes from. The only assumption here is that the Jacobian matrix Λ^k_i has to be for the coordinate transformation from Cartesian to whatever coordinates we are interested in - so, the "old coordinates" in the Jacobian here need to be the Cartesian coordinates.

As a quick example of using this formula, let's calculate the metric in polar coordinates - once again. We already have the Jacobian for the Cartesian-to-polar transformation from Lesson 2 as:

$$\Lambda^j_i = \begin{pmatrix} \Lambda^1_1 & \Lambda^1_2 \\ \Lambda^2_1 & \Lambda^2_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

The metric components are then:

$$g_{11} = \sum_k \Lambda^k_1 \Lambda^k_1 = \underbrace{\Lambda^1_1}_{\cos\theta} \Lambda^1_1 + \underbrace{\Lambda^2_1}_{\sin\theta} \Lambda^2_1 = \cos^2\theta + \sin^2\theta = 1$$

$$g_{12} = g_{21} = \sum_k \Lambda^k_1 \Lambda^k_2 = \underbrace{\Lambda^1_2}_{-r\sin\theta} \underbrace{\Lambda^1_1}_{\cos\theta} + \underbrace{\Lambda^2_1}_{\sin\theta} \underbrace{\Lambda^2_2}_{r\cos\theta} = 0$$

$$g_{22} = \sum_k \Lambda^k_2 \Lambda^k_2 = \underbrace{\Lambda^1_2}_{-r\sin\theta} \Lambda^1_2 + \underbrace{\Lambda^2_2}_{r\cos\theta} \Lambda^2_2 = r^2\sin^2\theta + r^2\cos^2\theta = r^2$$

These are the usual metric components in polar coordinates we've already seen before. The point of this calculation was to illustrate how we can get the metric components from the Jacobian matrix directly.

Using matrix notation, we could do the above calculation really fast, for example, by using a calculator that automatically does matrix multiplication (I've done the calculation here using Wolfram Alpha).

The way this works is that we multiply the transpose of the Jacobian matric with the Jacobian matrix itself using matrix multiplication. In our case, the metric components would be obtained in matrix notation as:

$$g = \Lambda^T \Lambda = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}^T \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

4.2. Calculating Metric Components From The Line Element

It is also possible to compute the metric for a given coordinate system directly from the coordinate transformation equations. The easiest way is to have the coordinate transformation be between Cartesian coordinates and the coordinate system of interest, although this method will also work between any other two coordinate systems.

The way this works is that we essentially "transform" the line element with the coordinate relations and read off the components of the metric from the new line element. This is done by calculating expressions for the coordinate displacements in terms of the new coordinates using the multivariable chain rule. As a step-by-step framework, we could put this as follows:

1. **Define a set of coordinate relations** between the Cartesian coordinates x, y and z and your new coordinates of interest.
2. **Calculate the Cartesian coordinate displacements** dx, dy and dx in terms of the new coordinates. This can be done by using the multivariable chain rule.
3. **Plug these into the line element** $ds^2 = dx^2 + dy^2 + dz^2$ and **read off the metric components** from the resulting expression.

It might not be very clear from this yet how to actually do the calculation in practice. Therefore, let's look at an example - this will be for the classic coordinate system, which we've already found the metric for a million times in this lesson. However, it's good to see the same result obtained by several different methods, so we have a way to compare them.

We'll begin from the coordinate relations $x = r\cos\theta$, $y = r\sin\theta$. We can get the coordinate displacements dx and dy from these by thinking of x and y here as functions of r and θ - which they are. So:

$$x = x(r, \theta) = r\cos\theta$$
$$y = y(r, \theta) = r\sin\theta$$

We can then apply the multivariable chain rule or the formula for the total differential (see the lesson on partial derivatives in Part 2 if you need a refresher in this).

These give us:

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta$$
$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta$$

Plugging the above coordinate relations now into these, we get:

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta = \cos\theta dr - r\sin\theta d\theta$$
$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta = \sin\theta dr + r\cos\theta d\theta$$

Then, we know that the line element in 2D Cartesian coordinates is $ds^2 = dx^2 + dy^2$. To get this in terms of polar coordinates, we simply plug the above expressions for dx and dy into this:

$$ds^2 = dx^2 + dy^2$$
$$= (\cos\theta dr - r\sin\theta d\theta)^2 + (\sin\theta dr + r\cos\theta d\theta)^2$$
$$= \cos^2\theta dr^2 + r^2\sin^2\theta d\theta^2 - 2r\sin\theta\cos\theta dr d\theta + \sin^2\theta dr^2 + r^2\cos^2\theta d\theta^2 + 2r\sin\theta\cos\theta dr d\theta$$
$$= \left(\cos^2\theta + \sin^2\theta\right)dr^2 + r^2\left(\cos^2\theta + \sin^2\theta\right)d\theta^2$$
$$= dr^2 + r^2 d\theta^2$$

Here, the cross-terms cancelled each other and we've used the fact that $\cos^2\theta + \sin^2\theta = 1$.

Now, what do we do with this result? Well, first of all - this is the line element in polar coordinates! If this is specifically what we wanted to calculate, then we're pretty much done.

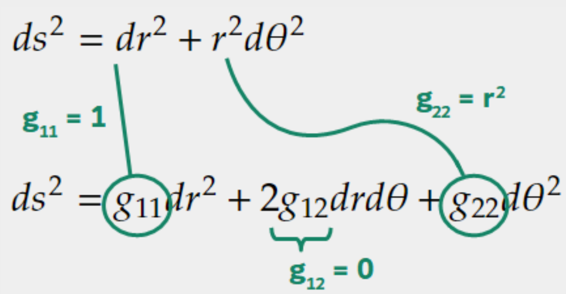
To read off the metric components from this, let's think about how the line element is generally obtained from the metric - using the general formula $ds^2 = g_{ij}dx^i dx^j$.

In polar coordinates, this would result in:

$$
\begin{aligned}
ds^2 &= g_{ij}dx^i dx^j \\
&= g_{11}dx^1dx^1 + 2g_{12}dx^1dx^2 + g_{22}dx^2dx^2 \\
&= g_{11}dr^2 + 2g_{12}drd\theta + g_{22}d\theta^2
\end{aligned}
$$

We've used here the fact that the metric is symmetric, which means that $g_{21} = g_{12}$ *and we can write the cross terms as* $g_{12}dx^1dx^2 + g_{21}dx^2dx^1 = 2g_{12}dx^1dx^2$. *This applies generally to all metrics.*

We can now compare this with the expression from above and identify all the metric components as follows:



So, we can identify the usual $g_{11} = 1$, $g_{12} = g_{21} = 0$ and $g_{22} = r^2$ - quite simple, isn't it?

In most cases, I personally prefer calculating the metric this way compared to calculating it from the Jacobian matrix.

This is because both methods require knowing the **coordinate relations** anyway (in order to get the Jacobian in the first place) and if we have those, this method doesn't require remembering the formula for the Jacobian matrix and where all the indices go - it's just simpler.

An added benefit of this method is that we get the line element directly and sometimes, this is actually what we care about instead of the individual metric components.

4.3. Metric Tensors For Curved Spaces

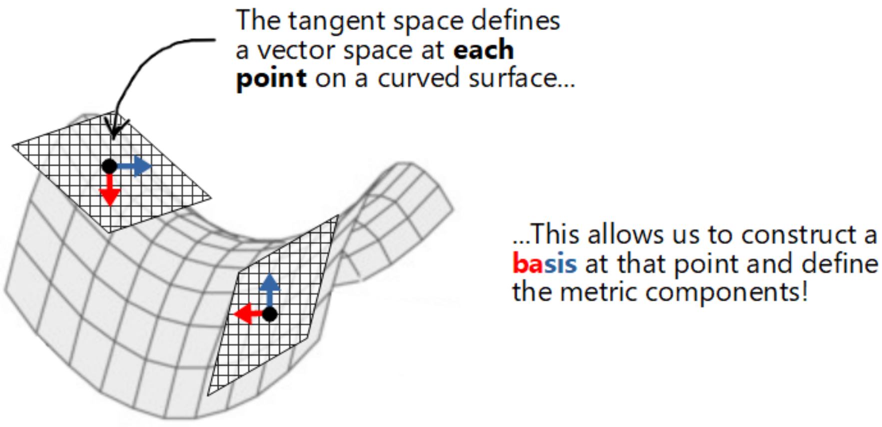
So far, the calculations we've done have been for flat spaces, but using curved coordinates like the polar coordinate system. A key element behind this was that we've been able to transform our coordinates from the **Cartesian coordinate system** to whatever our end coordinates of interest are.

While we haven't really talked about what curved spaces even are, one of the issues with them is that we cannot define Cartesian coordinates in a **curved geometry**. This is simply because Cartesian coordinates are rectangular, but a curved geometry will always result in some curvature of the coordinate lines - and those would not be Cartesian anymore.

The main issue for us then becomes - how do we define coordinates for a curved surface in such a way that we can compute the metric components for them?

The first thing to understand in relation to this is the idea of **tangent spaces**. For curved surfaces, we define everything - vectors, basis vectors, tensors and so on - in the tangent space for the surface.

The idea here is that we cannot define vectors directly on a curved surface - intuitively, how would you draw a straight arrow in a curved space? However, what we can do is attach a tangent space on the surface at each point and define things like vectors to live in this tangent space:



For curved geometries, the tangent space will always be different at each point. However, constructing such a tangent space at each point allows us to define and construct all the important objects, such as vectors and tensors, on the surface in a meaningful way.

In the context of this lesson, the tangent space - or more specifically, basis vectors defined in the tangent space - allow us to define the **metric components in the tangent space** just like we did earlier.

The only difference here compared to flat spaces is that the tangent space will be different at each point, so the metric tensor components will also vary from point to point (we would talk about a *metric tensor field*).

However, this is no problem at all because the metric components are already not constant in curvilinear coordinate systems.

Okay, but the question now is "how do we then find these metric components for the tangent space of a given surface?". Well, we need a coordinate system for that in order to define the components.

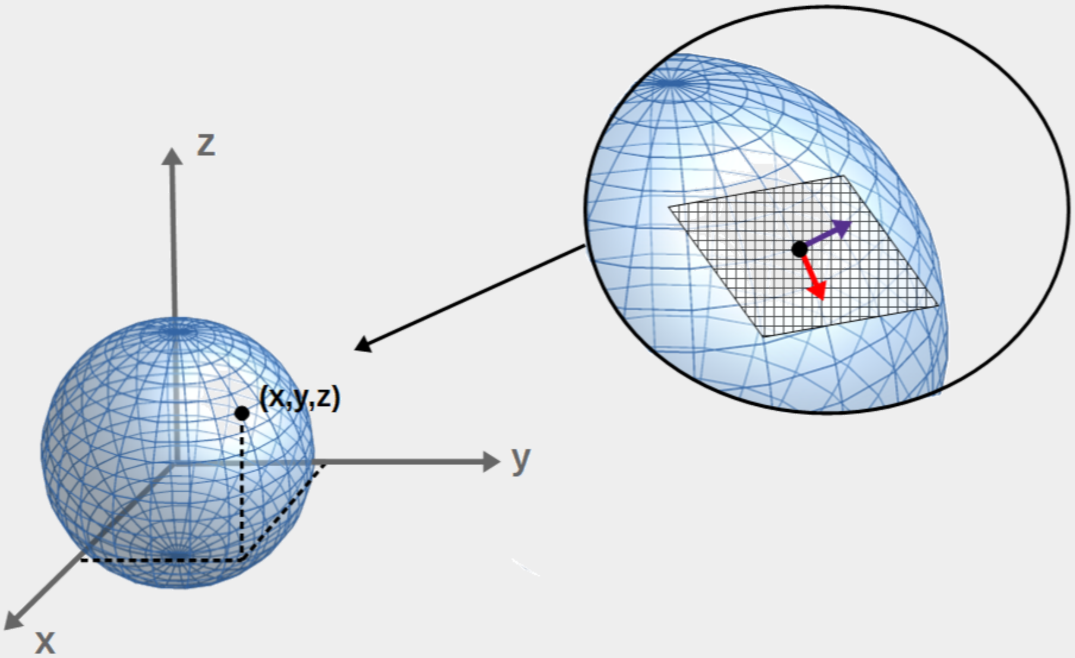
One way to define coordinates for the surface and actually get explicit expressions for them that we can calculate the metric components from, would be to essentially **embed the curved surface in an external Euclidean space**.

This allows us to attach coordinates on the surface, but express them in terms of the external space using, for example, Cartesian coordinates. This way, we can calculate the metric exactly the same way as we did earlier in this lesson.

Let's see how this actually works through an example. For this example, we'll take the surface of a sphere as our curved geometry.

Since this is a curved space, there is no way to attach Cartesian coordinates on the sphere's surface.

However, what we can do is embed the sphere into a 3D Euclidean space and define coordinates for the sphere in terms of the "outer" space, say using Cartesian coordinates:



Expressing the coordinates of the sphere in terms of the "external" flat space allows us to easily find the ***basis vectors*** *in the tangent space in terms of the* ***Cartesian coordinates*** *of the "external" space.*

How do we then define coordinates for the surface of the sphere that specifically only "pick out" points that actually lie on the sphere? Well, one way would be by using spherical coordinates (r, θ and φ) and setting the radial coordinate r to be constant.

That way, we are only describing points that are located at a constant distance from the origin - which describes a sphere!

We will choose $r = 1$ for simplicity here. The Cartesian coordinates of the sphere can then be expressed in terms of the two spherical angles, θ and φ, as:

$$x = \sin\theta\cos\varphi$$
$$y = \sin\theta\sin\varphi$$
$$z = \cos\theta$$

Note; we are using a different convention here than the one presented in Part 1 of the course - with θ and φ swapped. This convention leads to a more usual form for the metric, which is why we are using it here.

These coordinate relations describe all pointson the surface of the sphere in terms of the coordinates of the "external space". They also allow us to describe the basis vectors for the tangent space of the surface, which then define the metric components.

With the above relations, we are now able to calculate the metric components in the usual way - in other words, using tools we developed in flat space, but for curved space like the surface of a sphere. Let's calculate the coordinate differentials now like we did earlier:

$$dx = \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \varphi} d\varphi = \cos\theta \cos\varphi d\theta - \sin\theta \sin\varphi d\varphi$$
$$dy = \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \varphi} d\varphi = \cos\theta \sin\varphi d\theta + \sin\theta \cos\varphi d\varphi$$
$$dz = \frac{\partial z}{\partial \theta} d\theta + \frac{\partial z}{\partial \varphi} d\varphi = -\sin\theta d\theta$$

The line element in the "external space" in Cartesian coordinates is $ds^2 = dx^2 + dy^2 + dz^2$ as usual. If we now plug the above displacements into this, we will get the line element on the surface of the sphere:

$$ds^2 = dx^2 + dy^2 + dz^2$$
$$= (\cos\theta \cos\varphi d\theta - \sin\theta \sin\varphi d\varphi)^2 + (\cos\theta \sin\varphi d\theta + \sin\theta \cos\varphi d\varphi)^2 + (-\sin\theta d\theta)^2$$

The cross-terms resulting from writing out these squares will cancel out here. We can also use the identity $\cos^2\varphi + \sin^2\varphi = 1$ (and the same for θ as well) to get:

$$ds^2 = \cos^2\theta \cos^2\varphi d\theta^2 + \sin^2\theta \sin^2\varphi d\varphi^2 - 2\sin\theta \sin\varphi \cos\theta \cos\varphi d\theta d\varphi$$
$$+\cos^2\theta \sin^2\varphi d\theta^2 + \sin^2\theta \cos^2\varphi d\varphi^2 + 2\sin\theta \sin\varphi \cos\theta \cos\varphi d\theta d\varphi + \sin^2\theta d\theta^2$$
$$= \left(\cos^2\varphi + \sin^2\varphi\right)\cos^2\theta d\theta^2 + \sin^2\theta d\theta^2 + \left(\sin^2\varphi + \cos^2\varphi\right)\sin^2\theta d\varphi^2$$
$$= \left(\cos^2\theta + \sin^2\theta\right) d\theta^2 + \sin^2\theta d\varphi^2$$
$$= d\theta^2 + \sin^2\theta d\varphi^2$$

From these, we can read off the metric components and put them into a matrix as follows:

$$ds^2 = \underbrace{1}_{g_{11}} d\theta^2 + \underbrace{\sin^2\theta}_{g_{22}} d\varphi^2 \quad \Rightarrow \quad g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

This is the **metric for the unit sphere**. The interesting thing about this calculation is that we were able to do it using pretty much exactly the same tools we used in flat space, even though our goal was to describe a curved surface all along.

The key point is that this works if we are able to embed the surface into a higher-dimensional sphere and define coordinates for the surface in terms of this higher-dimensional space.

When we do this, we begin by coordinates in a flat space as usual - the "external" space. Then, we apply a constraint to these coordinates and we end up with *one less* coordinate, but which now describe a curved surface embedded into the "external" space. We can then calculate the metric in the usual way and eliminate any reference to the "external" space afterwards!

In this particular example, we began from Cartesian coordinates and applied the constraint $r = 1$. This ends up describing points that lie on a curved surface - even though our original coordinates were only valid for a flat geometry. This had the effect of reducing the needed coordinates to only two (θ and φ) from the original three coordinates (x, y and z).

We'll talk about curved spaces much much more in later lessons, so if you everything was not exactly clear yet, don't worry.

The main point was to highlight a general patter we'll see a lot throughout this course - we first develop a concept in *flat space* (for curvilinear coordinates) and then generalize that to curved spaces later on, typically with the addition of a few complications.

One of the things you might see from the example above is that in order to compute the metric for a curved space, we need to know something about **how coordinates are defined on the surface**. In the example above, we had to know that all points on the unit sphere have $r = 1$ and can otherwise be described using the two spherical angles, θ and φ.

Another common thing you'll see in differential geometry is that we simply just pick a metric out of thin air or assume one is given to us, and then work with it.

Basically, instead of beginning with some surface and finding the metric for it, we begin by coming up with a metric and then ask "what kind of surface or space does this metric describe?".

However, in **general relativity**, the metric for any spacetime is always calculated in the same way and that's from a specific equation - the **Einstein field equations**.

The Einstein field equations provide a way to calculate the metric for any spacetime, given that we know the energy and matter distribution in the spacetime.

So, we always start from the Einstein field equations by specifying an energy-momentum tensor and then solve the resulting equations, which gives us the components of the metric. This metric then tells us everything we need to know about the given spacetime.

Therefore, in general relativity, we won't have to "hand-wave" our way to a metric from thin air.

It also would not make sense to "embed" a spacetime into a higher-dimensional spacetime, as the spacetime we live in is four-dimensional by nature. So, take the above example with a grain of salt!

In general relativity, we instead always begin by specifying the energy-momentum tensor for our spacetime and the Einstein field equations then tell us what the metric should be - without needing to reference any higher dimensions.

This is the idea behind the second part of the hugely famous phrase "*spacetime tells matter how to move, matter tells spacetime how to curve*". The first part will become clear when we talk about *geodesics*.

5. Application: Metric Tensors & Line Elements In General Relativity

In this last section, we will discuss how the metric tensor is used specifically in general relativity. Now, the metric appears in nearly any equation or application of general relativity in some form, so here we'll just look at the very basics - why the metric is important for general relativity in the first place and what it describes.

5.1. The Metric Tensor of Spacetime

Back in Lesson 1, we discussed the idea of spacetime and that everything in relativity is described in spacetime.

Fundamentally, spacetime can be thought of as a **surface** (a more precise word would be 'manifold') just like a flat plane of the surface of a sphere like we discussed earlier. The difference is that the spacetime manifold does not only describe space, but also *time*.

Now, just like we do with any surface in order to describe it, we assign **coordinates to spacetime**. These include *three spatial coordinates* and *one time coordinate*.

For example, if we were to use Cartesian coordinates for our spatial coordinate axes, then we could label all the coordinates in our spacetime as:

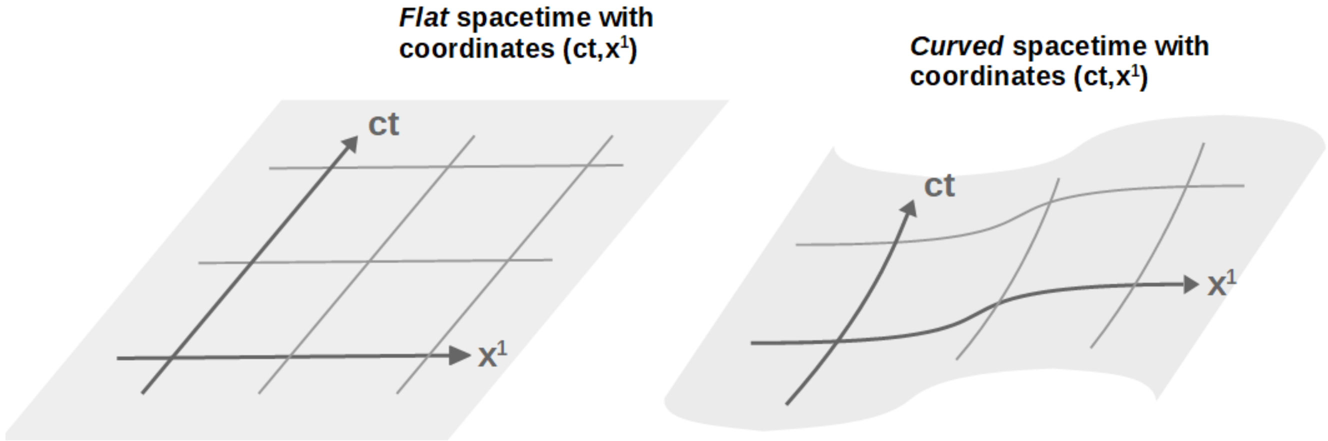
$$x^{\mu} = \left(x^0, x^1, x^2, x^3\right) = (ct, x, y, z)$$

On the other hand, we could also use something like spherical coordinates, in which case our spacetime coordinates would be:

$$x^{\mu} = \left(x^0, x^1, x^2, x^3\right) = (ct, r, \theta, \varphi)$$

The manifold of spacetime can be flat like ordinary Euclidean space, or curved like the surface of a sphere. Flat spacetimes are described by special relativity and are much simpler, while curved spacetimes belong to the realm of general relativity.

Now, it's important to distinguish between **coordinates** and **spacetime** - spacetime is the "physical thing", while coordinates are just a tool *we* assign to a given spacetime in order to describe it mathematically.



Here, we have an illustration of flat and curved spacetimes. In general, a spacetime should really be four-dimensional but drawing one, well, would be pretty difficult. Note also that the coordinates used to describe a given spacetime are not unique by any means - we can have infinitely many different coordinate systems that describe a given spacetime perfectly well.

Now, how do we describe the **geometry of a given spacetime**? Well, perhaps the answer is clear after this lesson - using the **metric tensor**! Once we have a spacetime described by a set of suitable coordinates and a metric for the spacetime expressed in these coordinates, we can calculate pretty much everything we would want in general relativity.

Because in relativity, we generally use four coordinates to describe spacetime, the metric will actually have $4 \times 4 = 16$ components. Due to its symmetry, however, the number is reduced to 10 independent components. We often represent these as a 4x4 matrix as follows:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}$$

Remember, spacetime quantities are represented by Greek indices, so here, both μ and ν can take on values between 0 to 3.

An important rule regarding spacetime metrics is that the time and space components along the diagonal need to have opposite signs - in other words, one needs to have a minus sign, while the other has a plus sign. For example, if we have g_{00} as negative, then the components g_{11}, g_{22} and g_{33} need to be positive.

It doesn't really matter whether you pick the time or space components as negative and the others as positive, as long as they have opposite signs. This "choice" of signature is called the **sign convention** and there are two possible sign conventions:

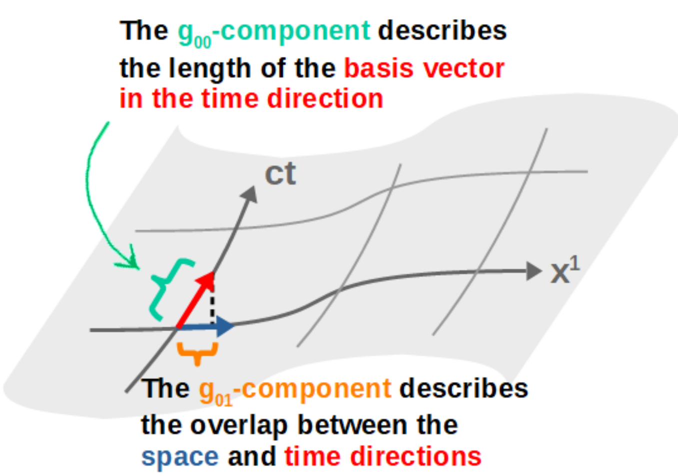
- The $(-,+,+,+)$ convention (or "mostly pluses") - with this convention, you would have g_{00} with a minus sign and g_{11}, g_{22}, g_{33} with plus signs. **In this course, we will be using this convention**.

- The $(+,-,-,.)$ convention (or "mostly minuses") - with this convention, you would have g_{00} with a plus sign and g_{11}, g_{22}, g_{33} with minus signs instead.

Okay, but what does the spacetime metric actually describe? Well, it's not so different than the metric for any other geometric space or coordinate system - it describes the **lengths and angles between the basis vectors**, which specifies the structure of the coordinate system we are using.

The main difference is that we also have time components in the metric now as well. These are the components with 0's in their indices. The g_{00}-component would describe the length of a basis vector pointing in the "time direction" and the components g_{01}, g_{02} and others would describe the "overlap" of the time direction with the space directions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}$$



A question you might still have is - why do we care about curved spacetimes in the first place? Well, the reason is that curved spacetimes are used to describe the effects of gravity in general relativity - maybe you've heard the phrase "gravity is not a force in general relativity" before.

Indeed, gravity itself is a manifestation of the curvature of spacetime (this will become more clear when we eventually get to discuss curvature), caused by matter and energy. The metric tensor is then essentially what describes the gravitational field in general relativity - actually, the gravitational potential.

A large part of general relativity is based on understanding differential geometry and curved spaces, which is why we are developing the tools to describe them in this course.

5.2. Where Do Spacetime Metrics Come From?

Next, where do metric tensors for different spacetimes come from? How do we calculate them in the context of general relativity? The answer is actually quite simple - we solve the **Einstein field equations**.

One of the main problems in differential geometry is finding the metric for a given geometric space. However, in general relativity, we have the Einstein field equations, which always give us metric tensor components as their solutions.

The Einstein field equations look as follows:

$$\boxed{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}}$$

You can immediately see the metric tensor $g_{\mu\nu}$ appearing on the left-hand side here. The two other objects on the left, $R_{\mu\nu}$ and R, are called the **Ricci tensor** and **Ricci scalar** and they are actually built out of the metric and its first and second partial derivatives. We'll talk about these objects much more in the lessons on curvature tensors.

On the right, we have a certain proportionality constant κ and the **energy-momentum tensor** $T_{\mu\nu}$. The energy-momentum tensor completely describes the matter and energy content in a given spacetime. For example, by specifying a certain energy-momentum tensor, we can describe a stationary massive black hole or even a rotating one.

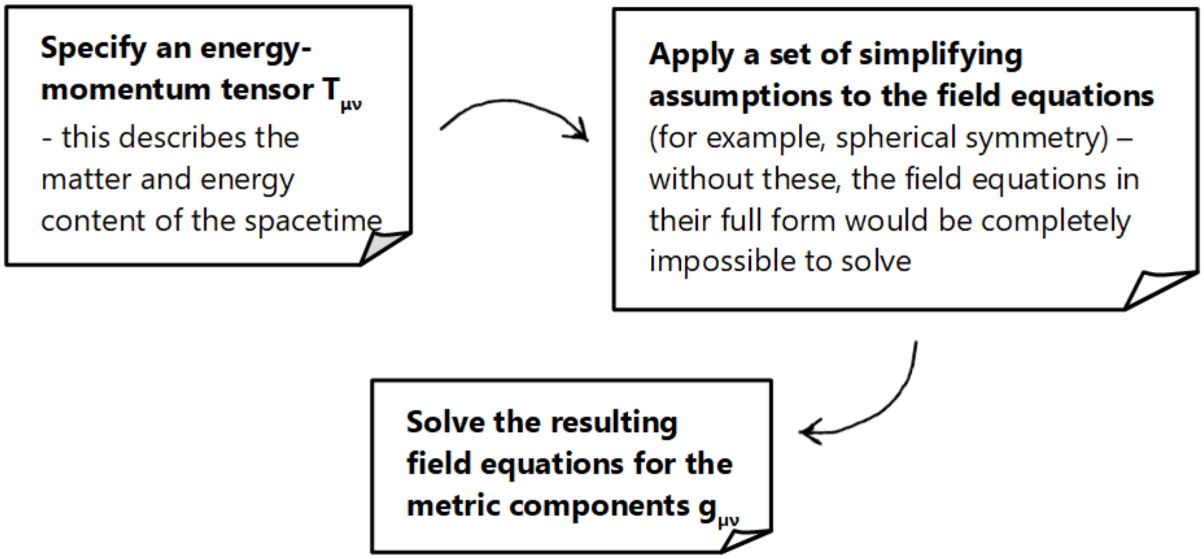
Essentially, the right-hand side here describe the "matter" part of a given spacetime and the left-hand side describes the "curvature" or "geometry" part of the spacetime, since it contains the metric. Indeed, this is exactly where the phrase "*spacetime tells matter how to move, matter tells spacetime how to curve*" is based on.

More importantly, however, the Einstein field equations are differential equations for the metric components of a given spacetime. This is because the objects on the left-hand side are all built out of the metric and its partial derivatives. The full expressions are quite complicated, but the field equations are basically of the form:

$$\partial^2 g_{\mu\nu} + \partial g_{\mu\nu} + g_{\mu\nu} \sim \kappa T_{\mu\nu}$$

These are **partial differential equations for the metric components** - for each index combination $\mu\nu$, we have one partial differential equation! Solving all of these equations would then yield all the metric components and therefore, also a full description of the given spacetime.

The procedure for solving the field equations is always more or less as follows:



The point here is that in general relativity, we have a straightforward way of finding the metric tensor components for any spacetime - provided that we are able to solve the Einstein field equations, of course.

Next, we'll dive into a couple different examples of metric tensors you are likely to encounter in both special and general relativity!

5.3. Example: The Minkowski Metric of Special Relativity

The first example we'll look at belongs to the realm of **special relativity**. This is called the **Minkowski metric** and it is actually the simplest possible metric we can have in relativity:

- The Minkowski metric describes a **flat spacetime**. Thus, it is only valid in special relativity.
- Specifically, the Minkowski metric is a **solution to the Einstein field equations** with $T_{\mu\nu} = 0$ and the spacetime being flat.
- We can have many different flat spacetime metrics, but the Minkowski metric is the **simplest one** - it describes the basic geometric structure of special relativity.

In fact, the Minkowski metric is the closest relativistic thing we have to the metrics we've looked at earlier in this lesson. The only difference is that we include an additional **time dimension**, however, the spatial dimensions remain the exact same as in the flat geometries we looked at earlier.

The spatial dimensions can then be described using whatever coordinate we wish, for example by Cartesian or polar coordinates. In Cartesian coordinates, the Minkowski metric looks as follows:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It's common to represent the Minkowski metric by the symbol η - typically, g is reserved for curved spacetime metrics in general relativity. However, this is just notation and nothing more.

We could also represent the Minkowski metric components in some different coordinate system, for example, in spherical coordinates:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

This describes the exact same flat spacetime as the one above, just in different coordinates. Which one you choose will depend on the purpose of specifying the metric - for example, if you're trying to describe rotational motion in special relativity, representing the Minkowski metric in spherical coordinates instead of Cartesian is (probably) a good idea.

Notice that the spatial components - all other components apart from the first row and column - are the same as those we already saw earlier. The interesting feature here are the time components, namely $\eta_{00} = -1$. Why is it negative?

Well, it comes back to what I told you earlier about different **sign conventions** for the metric components. Here, you can see that we are using the $(-,+,+,+)$ convention.

The reason the space and time components have different signs is, well, just because space and time behave quite differently and the spacetime they form is **not a Euclidean geometry**. What you'll find is that minus sign here is consistent with the form of the Lorentz transformations we looked at in Lesson 2.

For last, we could calculate the **line elements** using the Minkowski metric above. In Cartesian coordinates, we get:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

Remember, the time coordinate also has a factor of c by definition, $x^0 = ct$, meaning also that $dx^0 = cdt$. The sums over μ and ν here go from 0 to 3.

The same line element in spherical coordinates would instead be:

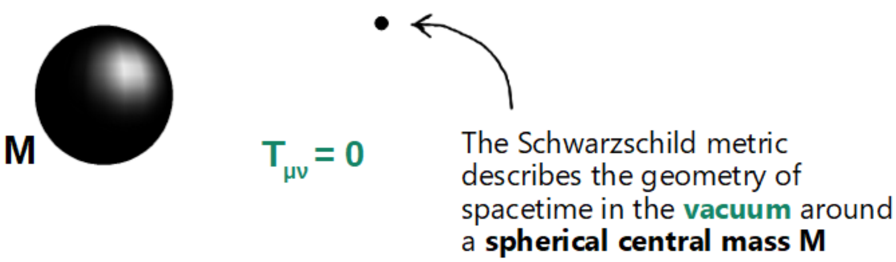
$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

5.4. Example: The Schwarzschild Metric

So far, we've looked at some examples of the metric tensor in the flat spacetime of special relativity. Now it's time to look at one of the most common metrics in general relativity - in other words, in *curved spacetime*.

The metric we'll look at is called the **Schwarzschild metric**. It describes the spacetime around a stationary, spherically symmetric central mass (with mass M). The most common application of this metric is for describing non-rotating **black holes**.

Specifically, the Schwarzschild metric is a vacuum solution to the Einstein field equations. This means a solution to the equations with $T_{\mu\nu} = 0$. Of course, the central mass does "contain matter", however, the Schwarzschild solution specifically describes the **spacetime outside the central mass** - which is just empty spacetime.



Even though there is no matter outside the central mass, the Schwarzschild spacetime around the central mass is still curved. Without further ado, here is the full Schwarzschild metric tensor - again, it can be derived by solving the vacuum Einstein field equations:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2GM}{c^2 r}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2GM}{c^2 r}\right)^{-1} & & 0 \\ 0 & 0 & r^2 & \\ 0 & 0 & & r^2 \sin^2\theta \end{pmatrix}$$

Here, M is the mass of the central object (black hole, for example), G is the gravitational constant and c is the speed of light.

This metric is represented using the so-called **Schwarzschild coordinates**. These are close to ordinary spherical coordinates, but not quite the same. The Schwarzschild coordinates consist of:

- **A time coordinate t** (or ct)- this is defined as the time measured by a far away observer that is not affected by gravity.
- **A "radial" coordinate r** - this is similar to the radial coordinate in the spherical coordinate system, but due to our spacetime being curved, it doesn't actually describe a "true" physical radial distance.
- **Two angular coordinates, θ and φ** - these are exactly the same as the two angles in the spherical coordinate system.

To better understand the geometry of the Schwarzschild spacetime and its metric, let's write out the line element in Schwarzschild coordinates using the above metric components. Here's the result:

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \frac{1}{1 - \frac{2GM}{c^2 r}}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

First of all, the angular terms in this line element look exactly the same as in the flat spacetime metric in spherical coordinates.

Therefore, the angular coordinates have exactly the same geometric interpretation as the two spherical angular coordinates like stated above.

The interesting parts are the dt- and dr-terms, which look very different to flat spacetime due to our spacetime here being curved.

As said above, the time coordinate t describes the **time measured by a far away observer** - thus, dt describes a small time interval in the case of no gravity (spacetime curvature) being present.

The *true amount* of time passed in the actual vicinity of the black hole (or whatever central mass), however, is NOT dt but instead the term appearing in the line element. Remember, the line element is what describes actual physical distance in spacetime.

Therefore, actual time intervals in the Schwarzschild spacetime are scaled by the metric component $g_{00} = -\left(1 - 2GM/c^2 r\right)$ like we see in the line element term $g_{00}c^2dt^2$. But what does this mean?

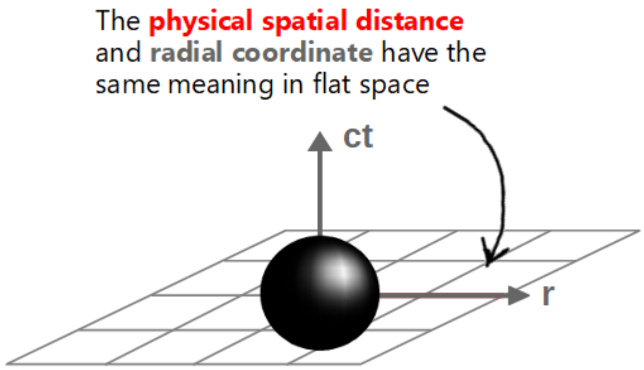
Well, we can see that for smaller r (in other words, when we're closer to the black hole), the metric component g_{00} is smaller. This means that any given coordinate displacement dt (time interval measured by a far away observer) actually corresponds to a *smaller* physical distance in spacetime.

In other words, the closer one gets to the black hole, the *smaller* each time interval gets when compared to a distant observer that doesn't experience any gravity.

So, time measured by an observer closer to the black hole would appear to *slow down* - this phenomena is called **gravitational time dilation**.

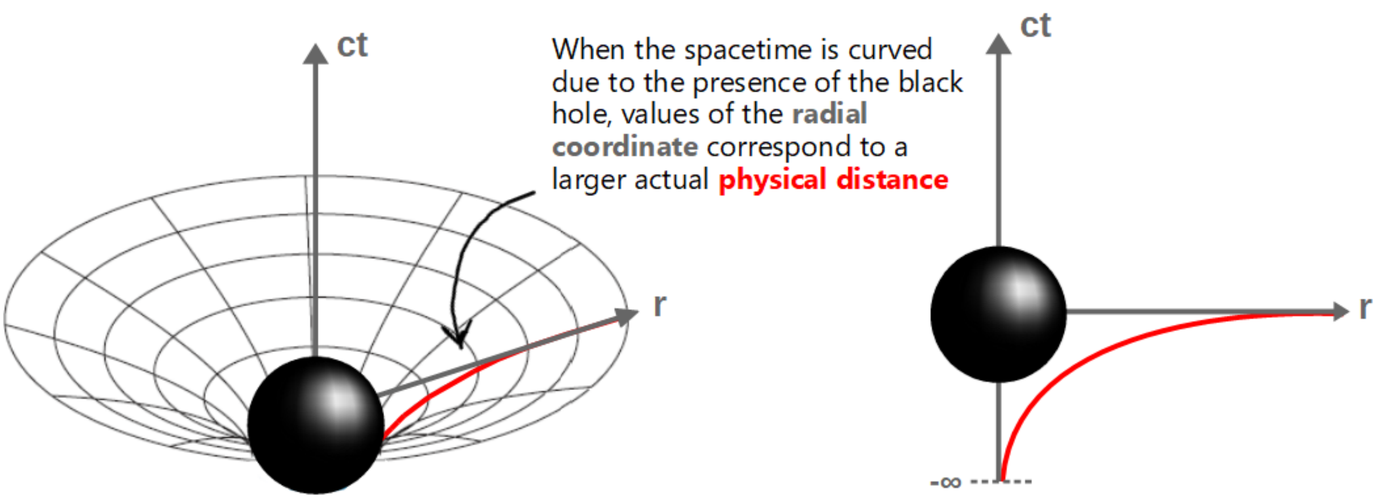
Now, I also mentioned that the coordinate r is not exactly the true physical radial distance to the central black hole. To understand why, we can first look at the situation in flat spacetime (we'll leave the angular coordinates out right now).

In flat spacetime, the **radial coordinate** is indeed the same as the physical distance in space to the black hole:



However, the Schwarzschild spacetime is NOT flat but curved. Due to this curvature, the r-coordinate is not actually the same as the true physical distance to the center *in spacetime*.

A crude way to visualize this is that curvature caused by the black hole essentially "bends" the spacetime in a way that results in actual distances being longer than the coordinate distance:



Word of caution; please take this visualization with a HUGE grain of salt. First of all, it only depicts the radial and time coordinates, but not the entire curved spacetime manifold (which would be four-dimensional and much more complicated). Secondly, there should also be curvature in the time direction that's not illustrated here. The purpose of this visualization is to simply give you an elementary idea of why the r-coordinate does not describe a true physical radius.

We can see where this intepretation comes from by looking at the metric component g_{11}. A true physical radial displacement in the Schwarzschild spacetime is not dr, but the following term in the line element (remember, the line element is what describes actual distance in a given spacetime):

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \underbrace{\frac{1}{1 - \frac{2GM}{c^2 r}}dr^2}_{\text{True physical distance}} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

From this, we see that at smaller r (closer to the black hole), a given radial coordinate displacement dr corresponds to a *larger* physical distance in spacetime since the term $g_{11} = (1 - r_s/r)^{-1}$ is larger.

Therefore, closer to the black hole, a given radial displacement dr corresponds to a larger actual distance in space and we cannot naively interpret r as a "distance to the origin" like we would in ordinary spherical coordinates.

We've now looked at a couple examples of different metric tensors in both special and general relativity. Hopefully, this has helped you get an understanding of what the metric and its components actually represent.

To sum it up, here is essentially what you should take away from this lesson, summarized into one sentence:

The metric tensor describes how coordinates and coordinate displacements are converted into actual lengths and physical distances in a given spacetime.