Initial value problems

Consider the following situation. You're given the function $f(x) = 2x - 3$ and asked to find its derivative. This function is pretty basic, so unless you're taking calculus out of order, it shouldn't cause you too much stress to figure out that the derivative of $f(x)$ is 2.

Now consider what it would be like to work backwards from our derivative. If you're given the function $f'(x) = 2$ and asked to find its integral, it's impossible for you to get back to the original function, $f(x) = 2x - 3$. As you can see, taking the integral of the derivative we found gives us back the first term of the original function, $2x$, but somewhere along the way we lost the -3. In fact, we always lose the constant (term without a variable attached), when we take the derivative of something. Which means we're never going to get the constant back when we try to integrate our derivative. It's lost forever.

Accounting for that lost constant is why we always add C to the end of our integrals. C is called the "constant of integration" and it acts as a placeholder for our missing constant. In order to get back to our original function, and find our long-lost friend, -3 , we'll need some additional information about this problem, namely, an initial condition, which looks like this:

$$y(0) = -3$$

Problems that provide you with one or more initial conditions are called Initial Value Problems. Initial conditions take what would otherwise be an entire rainbow of possible solutions, and whittles them down to one specific solution.

Remember that the basic idea behind Initial Value Problems is that, once you differentiate a function, you lose some information about that function. More specifically, you lose the constant. By integrating $f'(x)$, you get a family of solutions that only differ by a constant.

$$\int 2\ dx = 2x - 3$$

$$\int 2\ dx = 2x + 7$$

$$\int 2\ dx = 2x - \sqrt{2}$$

Given one point on the function, (the initial condition), you can pick a specific solution out of a much broader solution set.

Example

Given $f'(x) = 2$ and $f(0) = -3$, find $f(x)$.

Integrating $f'(x)$ means we're integrating $2\ dx$, and we'll get $2x + C$, where C is the constant of integration. At this point, C is holding the place of our now familiar friend, -3, but we don't know that yet. We have to use our initial condition to find out.

To use our initial condition, $f(0) = -3$, we plug in the number inside the parentheses for x and the number on the right side of the equation for y. Therefore, in our case, we'll plug in 0 for x and -3 for y.

$$-3 = 2(0) + C$$

$$-3 = C$$

Notice that the solution would have been different had we been given a different initial condition. Now we know exactly what the full solution looks like, and exactly which one of the many possible solutions was originally differentiated. Therefore, the final answer is the function we originally differentiated:

$$f(x) = 2x - 3$$