

Pre-Calculus with Python Programming

Unit 1: Function



CHAPTER 1: INTRODUCTION TO PRECALCULUS

DR. ERIC CHOU

IEEE SENIOR MEMBER

LECTURE 1

Overview



What is Pre-Calculus?

- Pre-Calculus is also called Algebra 3, or Algebra 4. It is supposed to be the last high school basic Algebra.
- Algebra: Algebra (from Arabic "al-jabr" literally meaning "reunion of broken parts") is one of the broad parts of mathematics, together with number theory, geometry and analysis. In its most general form, algebra is the study of mathematical symbols and the rules for manipulating these symbols; it is a unifying thread of almost all of mathematics.



Branches of Math

- Arithmetic
- Algebra (use of symbols to replace numbers: variables)
- Number Theory
- Probability and Statistics (use of random variables)
- Geometry (including analytical geometry)
- Calculus (use of lambda-calculus, functions)

Pre-calculus is presumably the aggregation of all the knowledge before calculus. Yet, the curriculum usually covers only Algebra 3.



Algebra 3

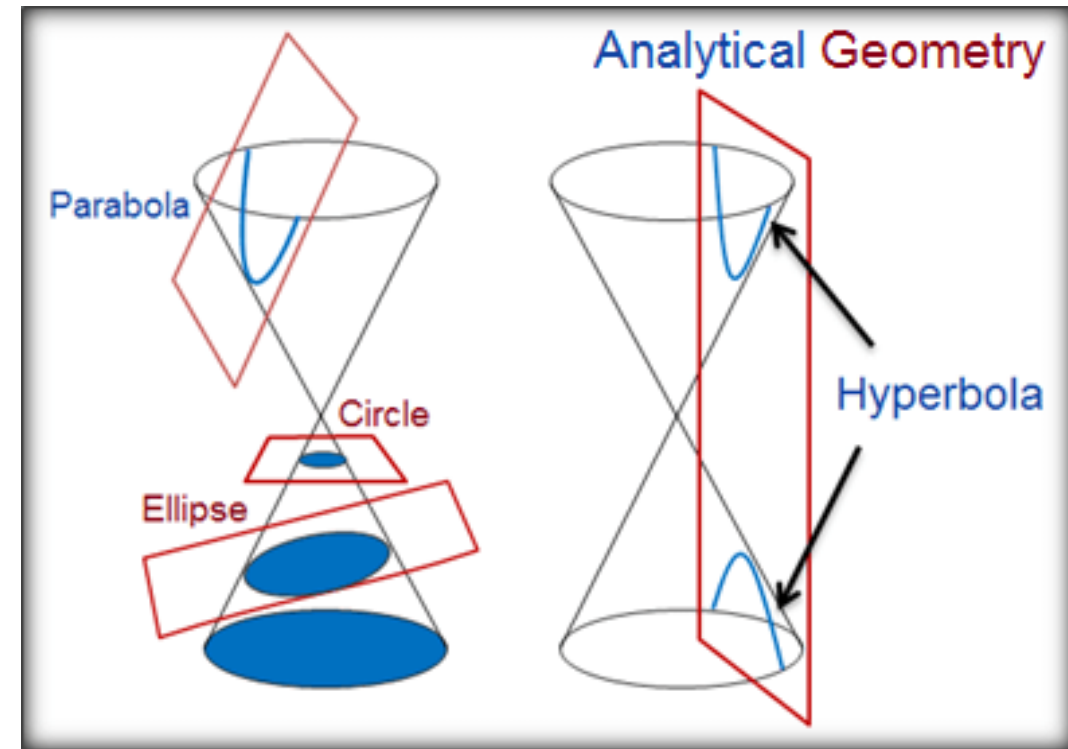
- Algebra 3 focuses on the continuation of study of Algebra and Trigonometry.
- Topics studied in this course include
 1. linear equations and inequalities,
 2. polynomials, factoring,
 3. rational expressions,
 4. trigonometric identities and
 5. functions: exponential, logarithmic, trigonometric, inverse trigonometric.

Analytical Geometry

Topic:

1. Scalar, Vector
2. Complex Number and Complex Plane
3. Linear, polynomial, exponential, logarithmic, conic, trigonometric functions and their graphing on a plane
4. Stretching/Shrinking, Translation, Rotation, Transformation
5. Matrix theory and multi-dimensional data
6. Complex plane, Cartesian plane, Polar system.
7. Trigonometry and Complex number
8. Distance and angle
9. Tangents and Normals

解析幾何





Analytical Geometry for a Line

Slope formula	$\text{slope}, m = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$ <p>Parallel lines have equal slopes. The slopes of perpendicular lines are opposite reciprocals of each other</p>
General Form	$Ax + By = C$
Slope Intercept Form	$y = mx + b$ <p>where m is the slope and b is the y-intercept</p>
Point Slope Form	$(y - y_1) = m(x - x_1)$ <p>where m is the slope</p>
Midpoint Formula	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$
Distance Formula	$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



Discrete and Continuous Aggregation

1. Sequence,
2. Approximation,
3. Asymptotic Analysis and
4. Limits Theory

Modeling
Parabolas
Systems
Inequalities
Algebra
Trigonometry
Ellipses
Conics
Polar
Circles
Equations
Hyperbolas
Parametric
Precalculus
Limits
Functions
Matrices
Graphs



frequency

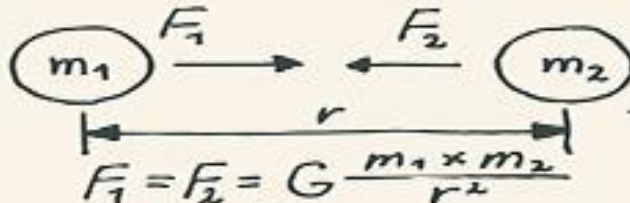
LIGHT



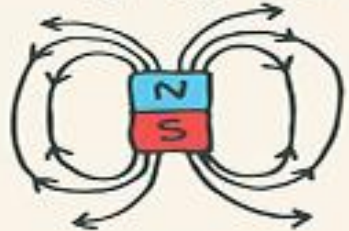
MECHANICS

$$F=ma$$

time

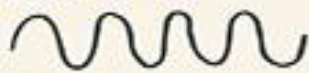


GRAVITY

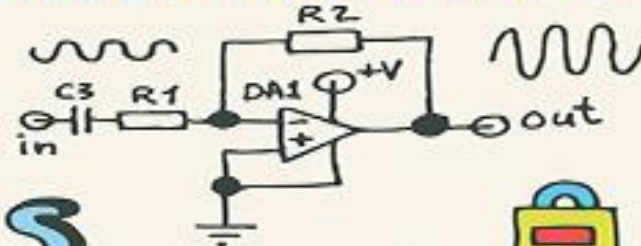


MAGNET

$$U = I \times R$$



$$V = IR$$



mass

Physics



$$I = \frac{U}{R}$$



Quantum

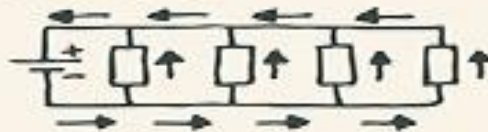
$$\exists |\epsilon$$

$$E = mc^2$$



$$R = \frac{U}{I}$$

ELECTRON





**KEEP
CALM**

AND

LEARN

PRE-CALCULUS

$$e^{ix} = \cos x + i \sin x$$

for $x = \frac{\pi}{2}$:

$$e^{i \cdot \frac{\pi}{2}} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$$

$$e^{\frac{i\pi}{2}} = i$$

$$\left(e^{\frac{i\pi}{2}}\right)^i = i^i$$

$$e^{-\frac{\pi}{2}} = i^i$$

i^i is a real number

A α	alpha	N ν	nu
B β	beta	Ξ ξ	ksi
Γ γ	gamma	O o	omicron
Δ δ	delta	Π π	pi
E ϵ	epsilon	P ρ	rho
Z ζ	zeta	Σ ς	sigma
H η	eta	T τ	tau
Θ θ	theta	Y υ	upsilon
I ι	iota	Φ ϕ	phi
K κ	kappa	X χ	chi
Λ λ	lambda	Ψ ψ	psi
M μ	mu	Ω ω	omega

LECTURE 1

Numbers, Variables and Expressions

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{A}_R \subset \mathbb{R} \subset \mathbb{C}$$

Natural Numbers

1, 2, 3, 4, ... **N**

Counting
Numbers

Whole Numbers

0, 1, 2, 3, 4, ...

Integers

..., -2, -1, 0, 1, 2, ... **Z**

Rational Numbers

$\frac{1}{2}$, $\frac{3}{5}$, $\frac{5}{6}$, $1\frac{1}{2}$, 2.25, etc. **Q**

Measuring
Numbers

Real Algebraic Numbers

$\sqrt{2}$, $-\sqrt{3}$, $1+\sqrt{3}/2$, etc. **A_R**

Estimating
Numbers

Real Numbers

e , π , e^π , $\ln 2$, $\sin(\pi/3)$, etc. **R**

Transcendental
Numbers

Irrational
Numbers

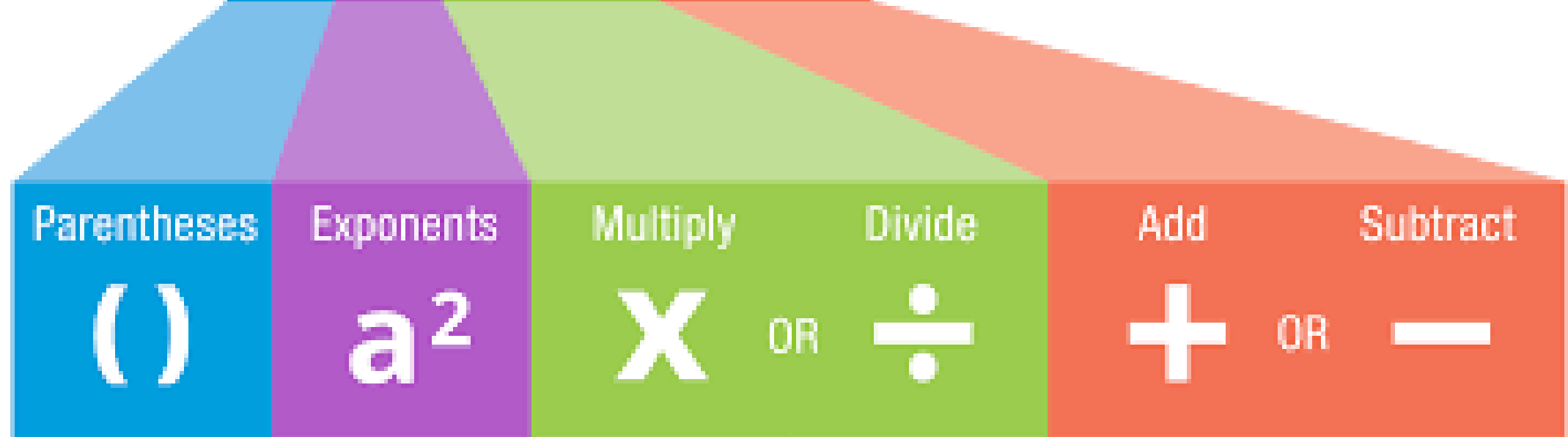
Imaginary Numbers

i , πi , $\sqrt{-2}$, etc.

Complex Numbers

$4 + 2i$, $\frac{1}{2} - \frac{1}{4}i$, $e + \pi i$, etc. **C**

P E M D A S Goodies!



LECTURE 1

Vector, Matrix, and Tensor

SCALARS AND VECTORS

A scalar quantity can be characterized by a single real number.

Examples: area, volume, time, mass, temperature.

A vector is a quantity involving both magnitude and direction and cannot be characterized completely by a single real number.

Examples: force, velocity, acceleration.

Variables as Arrays

In MATLAB, a **variable** is stored as an array of numbers. When appropriate, it is interpreted as a **scalar**, **vector** or **matrix**.

scalar

1×1

vector

$n \times 1$ or $1 \times n$

matrix

$n \times m$

The size of an array is specified by the number of rows and the number of columns in the array, with the number of rows indicated first.

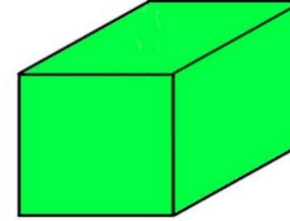
1D TENSOR /
VECTOR

5
7
4 5
1 2
- 6
3
2 2
1
6
3
- 9

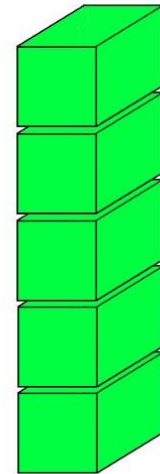
2D TENSOR /
MATRIX

- 9	4	2	5	7
3	0	1 2	8	6 1
1	2 3	- 6	4 5	2
2 2	3	- 1	7 2	6

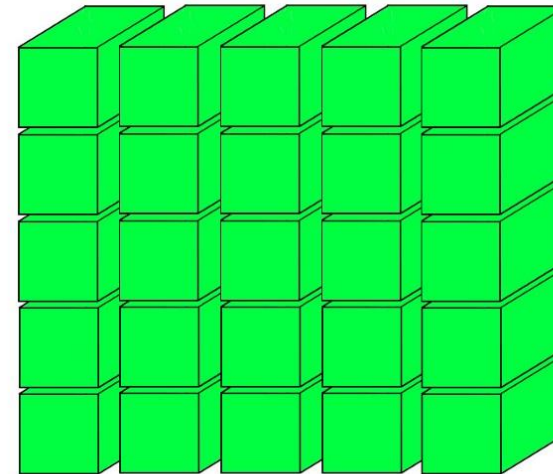
3D TENSOR /
CUBE



- 9	4	2	5	7
3	0	1 2	8	6 1
1	2 3	- 6	4 5	2
2 2	3	- 1	7 2	6



4D TENSOR
VECTOR OF CUBES



5D TENSOR
MATRIX OF CUBES

Tensors

- Python

```
# Scalar
```

```
> 3
```

```
# Matrix
```

```
> [[1, 2, 3],  
   [4, 5, 6],  
   [7, 8, 9]]
```

```
# Vector
```

```
> [1, 2, 3]
```

```
# 3-D Tensor
```

```
> [[ [0,0], [0,1], [0,2] ],  
   [ [1,0], [1,1], [1,2] ],  
   [ [2,0], [2,1], [2,2] ]]
```

- NumPy

```
# Matrix as NumPy Array
```

```
> np.array([[1, 2, 3],  
           [4, 5, 6],  
           [7, 8, 9]],  
           dtype=np.int64)
```

LECTURE 1

Variable



Independent Variable and

Java:

```
double x;
```

```
double y;
```

```
y = f(x);
```

Question: What's a variable?

Answer: A variable is an object, event, idea, feeling, time period, or any other type of category you are trying to measure. There are two types of variables-independent and dependent.

Question: What's an independent variable?

Answer: An independent variable is exactly what it sounds like. It is a variable that stands alone and isn't changed by the other variables you are trying to measure. For example, someone's age might be an independent variable. Other factors (such as what they eat, how much they go to school, how much television they watch) aren't going to change a person's age. In fact, when you are looking for some kind of relationship between variables you are trying to see if the independent variable causes some kind of change in the other variables, or dependent variables.



Independent Variable and Dependent Variable

Java:

```
double x;  
double y;  
y = f(x);
```

Question: What's a dependent variable?

Answer: Just like an independent variable, a dependent variable is exactly what it sounds like. It is something that depends on other factors. For example, a test score could be a dependent variable because it could change depending on several factors such as how much you studied, how much sleep you got the night before you took the test, or even how hungry you were when you took it. Usually when you are looking for a relationship between two things you are trying to find out what makes the dependent variable change the way it does. Many people have trouble remembering which is the independent variable and which is the dependent variable. An easy way to remember is to insert the names of the two variables you are using in this sentence in the way that makes the most sense. Then you can figure out which is the independent variable and which is the dependent variable:
(Independent variable) causes a change in (Dependent Variable) and it isn't possible that (Dependent Variable) could cause a change in (Independent Variable).

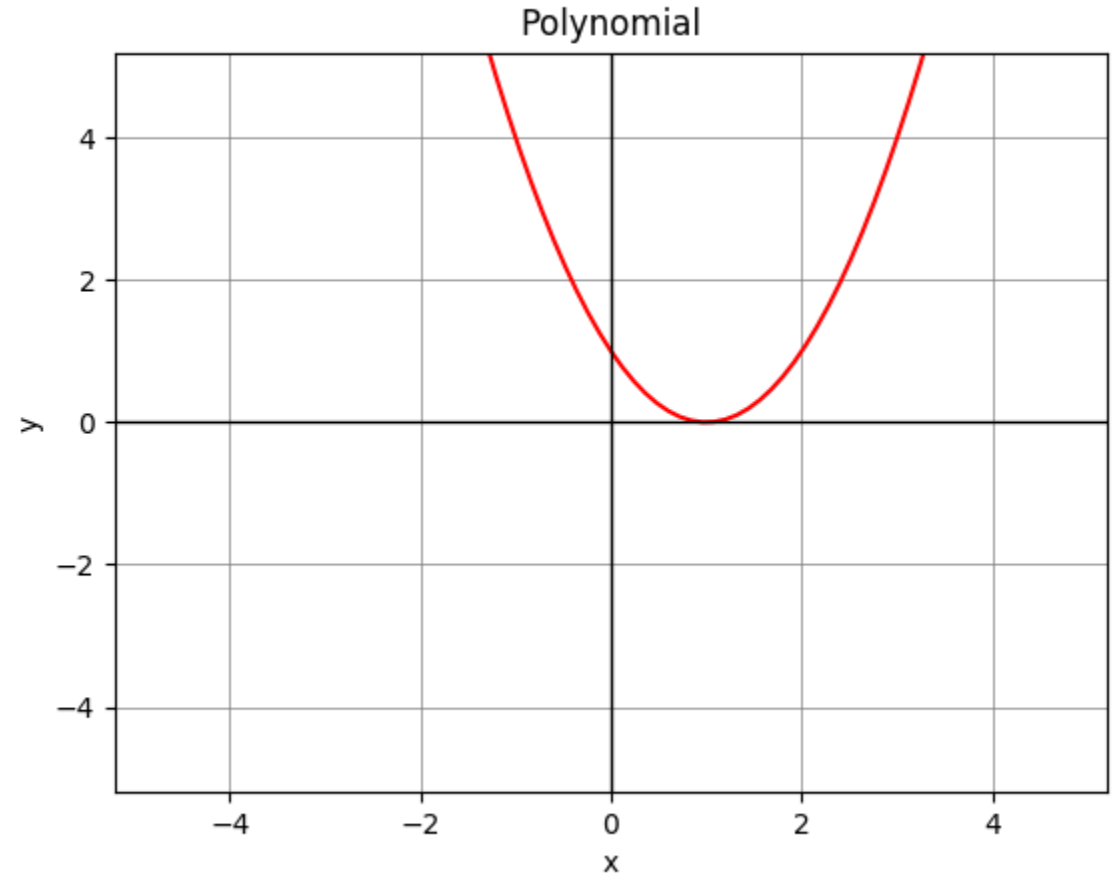
```

from pylab import *
import numpy as np

# definition of a polynomial
def polynomial(x, c):
    y = np.ndarray(len(x))
    i = 0
    for xt in x:
        yt = 0
        for z in c:
            yt = yt * xt + z
        y[i]=yt
        i+=1
    return y

```

Figure 1



prepare data

```
c = [1, -2, 1]
x = linspace(-6, 6, 101)
y = polynomial(x, c)
print(type(x), len(x))
print(type(y), len(y))
```

Independent Variable

dependent Variable

drawing graph

```
figure() # create a figure
xlim(xmin=-5.2, xmax=5.2) # set x-axis range
ylim(ymin=-5.2, ymax=5.2) # set y-axis range
plot(x, y, 'r') # create x-y plot using red dot
xlabel('x') # label x-axis
ylabel('y') # label y-axis
title('Polynomial') #
grid(linestyle='-', linewidth='0.5', color='gray')
axhline(0, color='black', lw=1.0)
axvline(0, color='black', lw=1.0)
show()
```

Variable -> Single Variable

Vector -> Array (or list)

Quantized Function ->
Array (List) of Undefined
Length in discrete domain.



Core Operation (Multiply-Shift and Add)

```
def polynomial(x, c):  
    y = np.ndarray(len(x))  
    i = 0  
    for xt in x:  
        yt = 0  
        for z in c:  
            yt = yt * xt + z  
        y[i] = yt  
        i += 1  
    return y
```

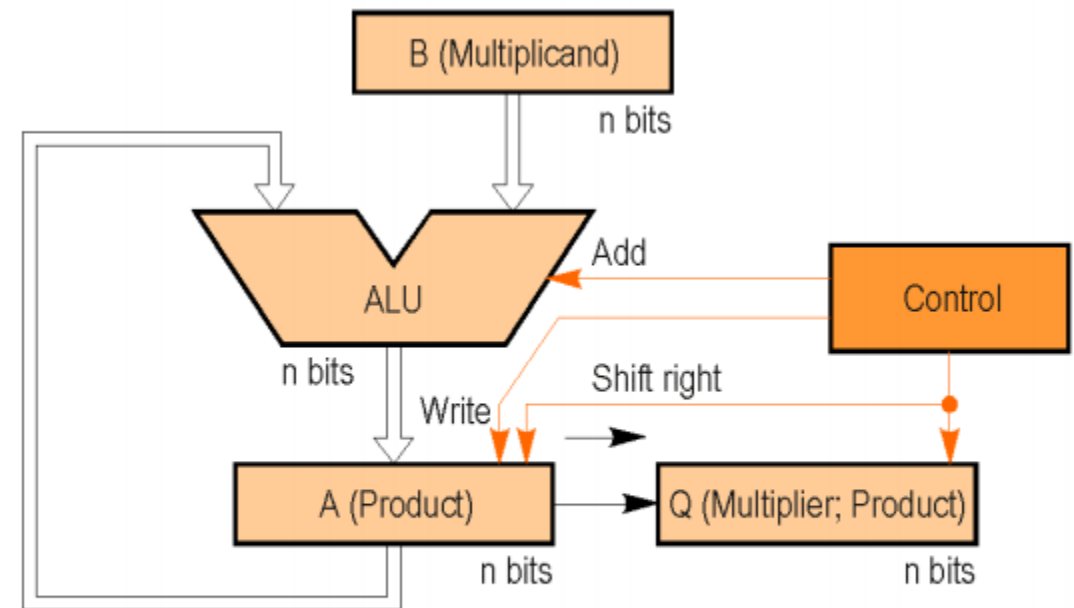


Figure 3.13. Final version of the multiplier circuit.

LECTURE 1

Algebraic Expression

Properties of Real Numbers

Commutative Properties	<p>Changing the order of the terms in addition or multiplication does not affect the sum or product.</p> <p>The Commutative Property of Addition states: $a + b = b + a$.</p> <p>The Commutative Property of Multiplication states: $ab = ba$.</p>
Associative Properties	<p>Changing the grouping of numbers in addition or multiplication does not affect the sum or product.</p> <p>The Associative Property of Addition states: $(a + b) + c = a + (b + c)$.</p> <p>The Associative Property of Multiplication states: $(ab)c = a(bc)$.</p>
Distributive Property	<p>Multiply the factor outside a set of parentheses by every term inside the parentheses when simplifying or evaluating.</p> <p>The Distributive Property states: $a(b + c) = ab + ac$.</p>

Identity Properties	<p>The identity for an operation is the value that does not change the value of an expression for that operation. Zero is the additive identity; adding zero to a number does not change its value. One is the multiplicative identity; multiplying a number by one does not change its value.</p> <p>The Identity Property of Addition states: $n + 0 = n$.</p> <p>The Identity Property of Multiplication states: $n \cdot 1 = n$.</p>
Inverse Properties	<p>The inverse of a number for an operation is the value that combines with the number to give the identity for the operation as a result.</p> <p>The Inverse Property of Addition states: $n + (-n) = 0$, with $-n$ being the additive inverse and zero being the additive identity.</p> <p>The Inverse Property of Multiplication states: $n \cdot \frac{1}{n} = 1, n \neq 0$, where $\frac{1}{n}$ is the multiplicative inverse and one is the multiplicative identity.</p>



Simplification of Algebraic Expression

Example

Simplify the following expression: $4(x^2 + 3x) + 2(3x^2 + 5x)$.

$$4(x^2 + 3x) + 2(3x^2 + 5x)$$

$4 \cdot x^2 + 4 \cdot 3x + 2 \cdot 3x^2 + 2 \cdot 5x$ Start by using the Distributive Property twice to remove the parentheses.

$$4x^2 + 12x + 6x^2 + 10x$$
 Multiply.

$$4x^2 + 6x^2 + 12x + 10x$$
 Use the Commutative Property to put like terms together.

$$10x^2 + 22x$$
 Add to combine like terms.

Note: Remember that if there is a negative sign outside a set of parentheses, you need to distribute the negative throughout the terms inside the parentheses. For example, $-(2x^2 + 3) = -2x^2 - 3$.

Properties of Exponents and Square Roots

The following are the properties of exponents and square roots where a and b are real numbers or algebraic expressions and m and n are real numbers.

Zero Exponent Property	$a^0 = 1$, where $a \neq 0$
Negative Exponent Property	$a^{-n} = \frac{1}{a^n}$, where $a \neq 0$
Power Rule for Products	$(ab)^n = a^n b^n$
Power Rule for Quotients	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, where $b \neq 0$
Power Rule for Powers	$(a^n)^m = a^{nm}$
Product Rule for Powers	$a^n \cdot a^m = a^{n+m}$
Quotient Rule for Powers	$\frac{a^n}{a^m} = a^{n-m}$, where $a \neq 0$
Square Root Rule for Products	$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, where $a \geq 0$ and $b \geq 0$
Square Root Rule for Quotients	$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$, where $a \geq 0$ and $b > 0$

Examples

Simplify the following expressions.

1. $(-4x^3)^2$

Answer:

$$(-4x^3)^2$$

$$(-4)^2 (x^3)^2 \quad \text{Use the Power Rule for Products.}$$

$$16x^{3 \cdot 2} \quad \text{Use the Power Rule for Powers.}$$

$$16x^6 \quad \text{Simplify.}$$

2. $\frac{15x^6y^3}{5x^3y}$

Answer:

$$\frac{15x^6y^3}{5x^3y}$$

$$3x^{6-3}y^{3-1} \quad \text{Use the Quotient Rule for Powers.}$$

$$3x^3y^2 \quad \text{Simplify.}$$

$$3. (3x^4y^5)(-7x^2y^2)$$

Answer:

$$(3x^4y^5)(-7x^2y^2)$$

$$3(-7)(x^4x^2)(y^5y^2) \quad \text{Group factors with the same base.}$$

$$-21x^{4+2}y^{5+2} \quad \text{Use the Product Rule for Powers.}$$

$$-21x^6y^7 \quad \text{Simplify.}$$

$$4. \left(\frac{6x^8}{y}\right)^{-2}$$

Answer:

$$\left(\frac{6x^8}{y}\right)^{-2}$$

$$\frac{(6x^8)^{-2}}{y^{-2}} \quad \text{Use the Power Rule for Quotients.}$$

$$\frac{6^{-2}(x^8)^{-2}}{y^{-2}} \quad \text{Use the Power Rule for Products.}$$

$$\frac{6^{-2}x^{-16}}{y^{-2}} \quad \text{Use the Product Rule for Powers.}$$

$$\frac{y^2}{6^2x^{16}} \quad \text{Use the Negative Exponent Property.}$$

$$\frac{y^2}{36x^{16}} \quad \text{Simplify.}$$

$$5. \sqrt{\frac{54y^4}{9y^2}}$$

Answer:

$$\sqrt{\frac{54y^4}{9y^2}}$$

$$\sqrt{\frac{54y^4}{9y^2}} \quad \text{Use the Square Root Rule for Quotients.}$$

$$\sqrt{6y^2} \quad \text{Use the Quotient Rule for Powers.}$$

$$\sqrt{6}\sqrt{y^2} \quad \text{Use the Square Root Rule for Products.}$$

$$y\sqrt{6} \quad \text{Simplify.}$$

Now you try. Simplify the following expressions:

$$1. (-4x^3y^2z^5)(-8x^4yz^3)$$

$$2. \left(\frac{24x^2y}{12xy^2}\right)^{-3}$$

$$3. \sqrt{10y} \cdot \sqrt{5y}$$



Algebraic Equations

An **algebraic equation** is a statement that two algebraic expressions are equal, or that an algebraic expression is equal to a constant.

The **solution(s)** of an equation is the value or values that make(s) the equation true. The goal when solving an algebraic equation is to simplify and isolate the variable on one side of the equation using properties of equality and inverse operations.



Linear Equations

1. To solve the equation $5(3x - 2) + 6 = 2 - (3x + 4)$, start by simplifying the expressions on both sides of the equal sign. Then, use inverse operations and properties of equality to get the variable on one side and the constants on the other.

$$5(3x - 2) + 6 = 2 - (3x + 4)$$

$$15x - 4 = -2 - 3x$$

Simplify both sides.

$$18x = 2$$

Use properties of equality to move variables on one side and constants on the other.

$$x = \frac{1}{9}$$

Use inverse operations to isolate the variable.



Linear Equations

2. To solve an equation that involves fractions, such as $\frac{x+6}{4} + \frac{14-x}{3} = 6$, find the least common denominator of the fractions and multiply both sides of the equation by that number. In this situation, the least common denominator is 12, so multiply both sides of the equation by 12. This will result in an equivalent equation that is a multiple of the original equation with no fractions. Once the equation is rewritten without fractions, use inverse operations and properties of equality to get the variable on one side and the constants on the other.

$$12\left(\frac{x+6}{4} + \frac{14-x}{3}\right) = (12)6 \quad \text{Multiply both sides by the least common denominator.}$$

$$3(x+6) + 4(14-x) = 72 \quad \text{Distribute to get rid of the fractions.}$$

$$3x + 18 + 56 - 4x = 72 \quad \text{Distribute.}$$

$$-x + 74 = 72 \quad \text{Combine like terms.}$$

$$-x = -2 \quad \text{Use inverse operations.}$$

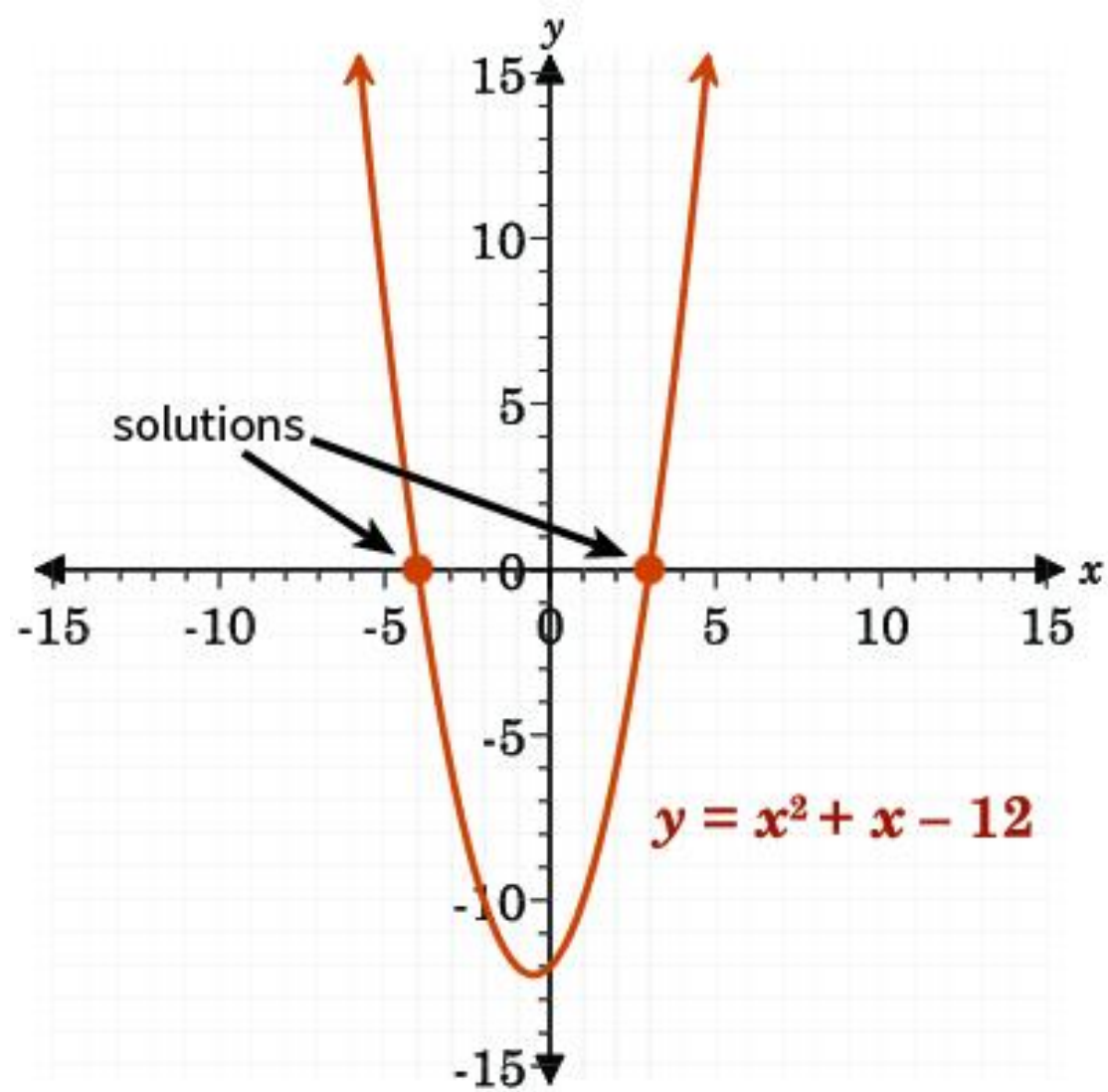
$$x = 2 \quad \text{Multiply both sides by } -1.$$



Quadratic Equations

Many real-world situations are modeled using second-degree equations, or quadratic equations, which you have studied in previous courses. For example, the equation used to represent projectile motion is a second-degree equation. The solutions to a quadratic equation can be found algebraically or graphically.

Graphically, the solutions to a quadratic equation will be the points where the graph crosses the x -axis. The following graph is of the function $y = x^2 + x - 12$. Since the graph crosses the x -axis at -4 and 3 , the solutions to the equation $x^2 + x - 12 = 0$ are $x = -4$ or $x = 3$.



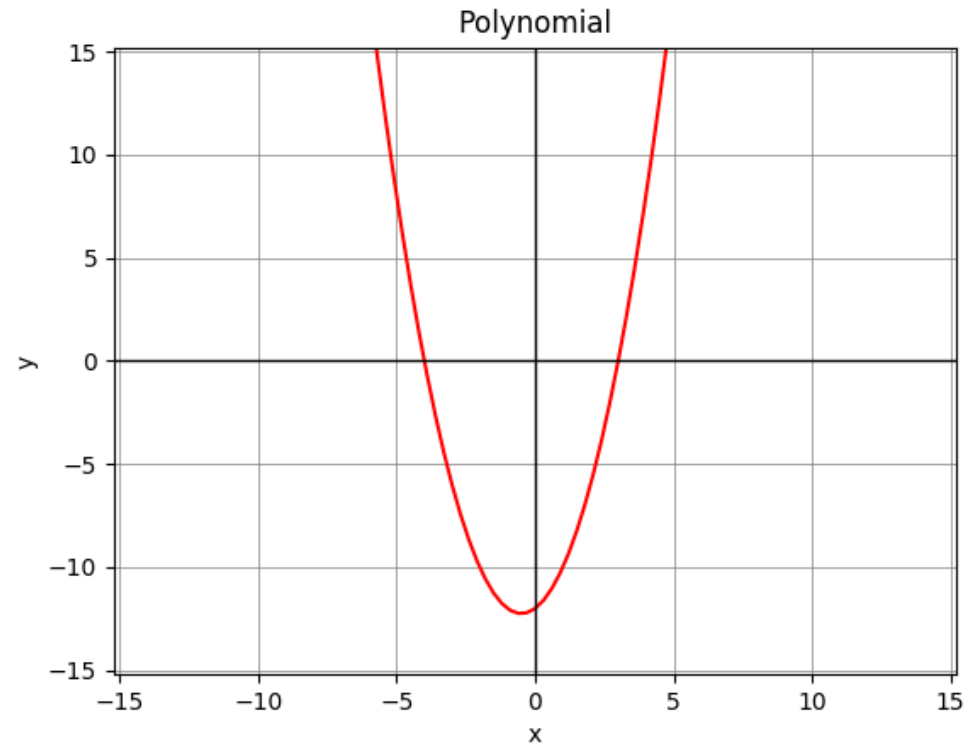


Demo Program: Quadratic Function

quadratic.py

Go PyCharm!!!

Figure 1



The Quadratic Formula

The solutions to any quadratic equation in the form $ax^2 + bx + c = 0$ can be found using the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example

Solve the quadratic equation $x^2 + 6x + 4 = 0$.

This quadratic equation cannot be factored. Either technique, completing the square or the quadratic formula, can be used to solve this equation. Both techniques will be demonstrated here.

Complete the following activities. Be sure to show all work.

1. Simplify the following expressions:

a. $2(4x - 3) - (5x + 7)$

b. $(3x^2y^3)(-4x^{-3}y^3)$

c. $\frac{\sqrt{32y^3}}{\sqrt{4y}}$

2. Solve the following equation: $\frac{x+7}{4} + \frac{13-x}{2} = 7.$

3. Solve the following equation: $x^2 + x - 9 = -x + 6.$

4. Solve the following equation: $x^2 - 6x + 3 = 0.$

LECTURE 1

Relation and Function



Representing Relations and Functions

- A **relation** is a relationship between the values in two sets of data—one set provides the input values for the relation, and the other set is the output values. A **function** is a specific type of relation for which each value in the set of inputs is paired with exactly one value in the set of outputs.
- The **domain** of a relation or function is the term that refers to the set of input values. The **range** is the term that refers to the set of output values. The domain and range of a relation or function can be shown in sets, ordered pairs, tables, or graphs.

Examples

Identify the domain and range of each relation.

1.

x	1	2	3	4
y	2	4	6	8

The x -values are the inputs. The y -values are the outputs.

domain: $\{1, 2, 3, 4\}$

range: $\{2, 4, 6, 8\}$

2. $\{(-1, -2.5), (-1, 1), (2, -2.5), (3, 5)\}$

Each ordered pair is in the format (x, y) .

The x -values are $-1, -1, 2, 3$.

The y -values are $-2.5, 1, -2.5$, and 5 .

domain: $\{-1, 2, 3\}$

range: $\{-2.5, 1, 5\}$

The domain and range are generally written in ascending order.

Note that if an x -value or a y -value repeats, it is still only included once in the domain or range.



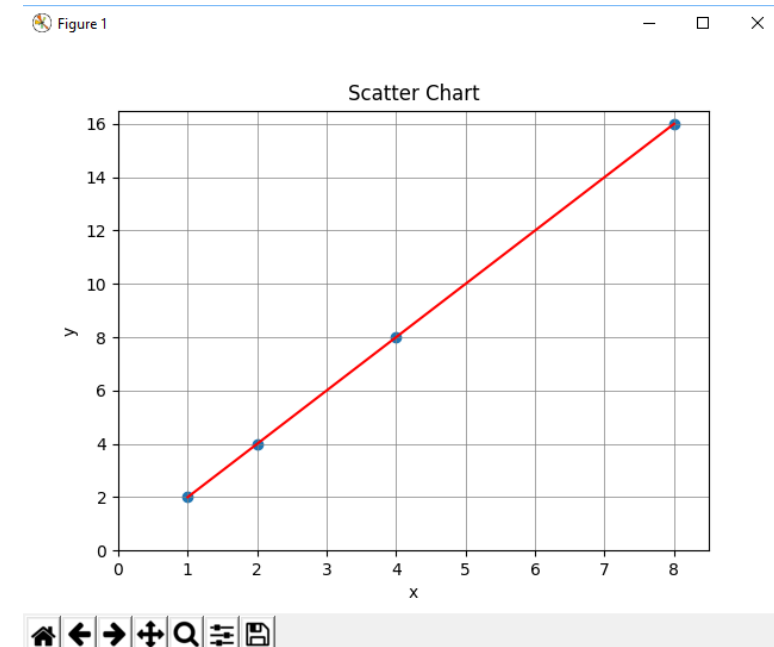
Scatter + Line Plot (Example 1)

scatter.py

```
from pylab import *
import numpy as np

x = [1, 2, 4, 8]
y = [2, 4, 8, 16]

figure()
scatter(x, y)
plot(x, y, color='r', linestyle='-')
xlim(xmin=0, xmax=8.5)          # set x-axis range
ylim(ymin=0, ymax=16.5)        # set y-axis range
xlabel('x')                     # label x-axis
ylabel('y')                     # label y-axis
title('Scatter Chart')          #
grid(linestyle='-', linewidth='0.5', color='gray')
show()
```

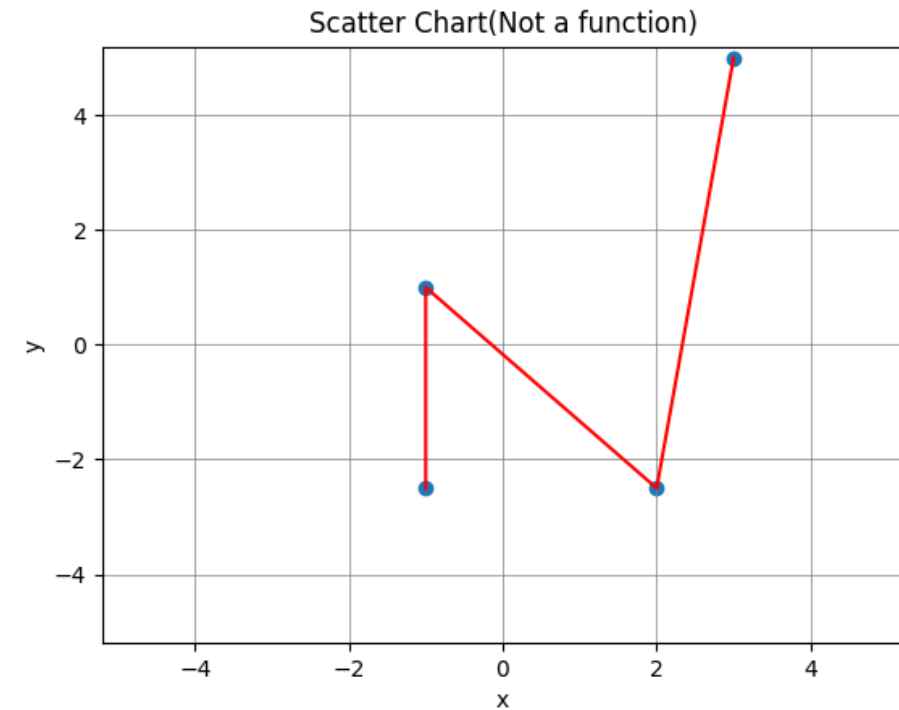


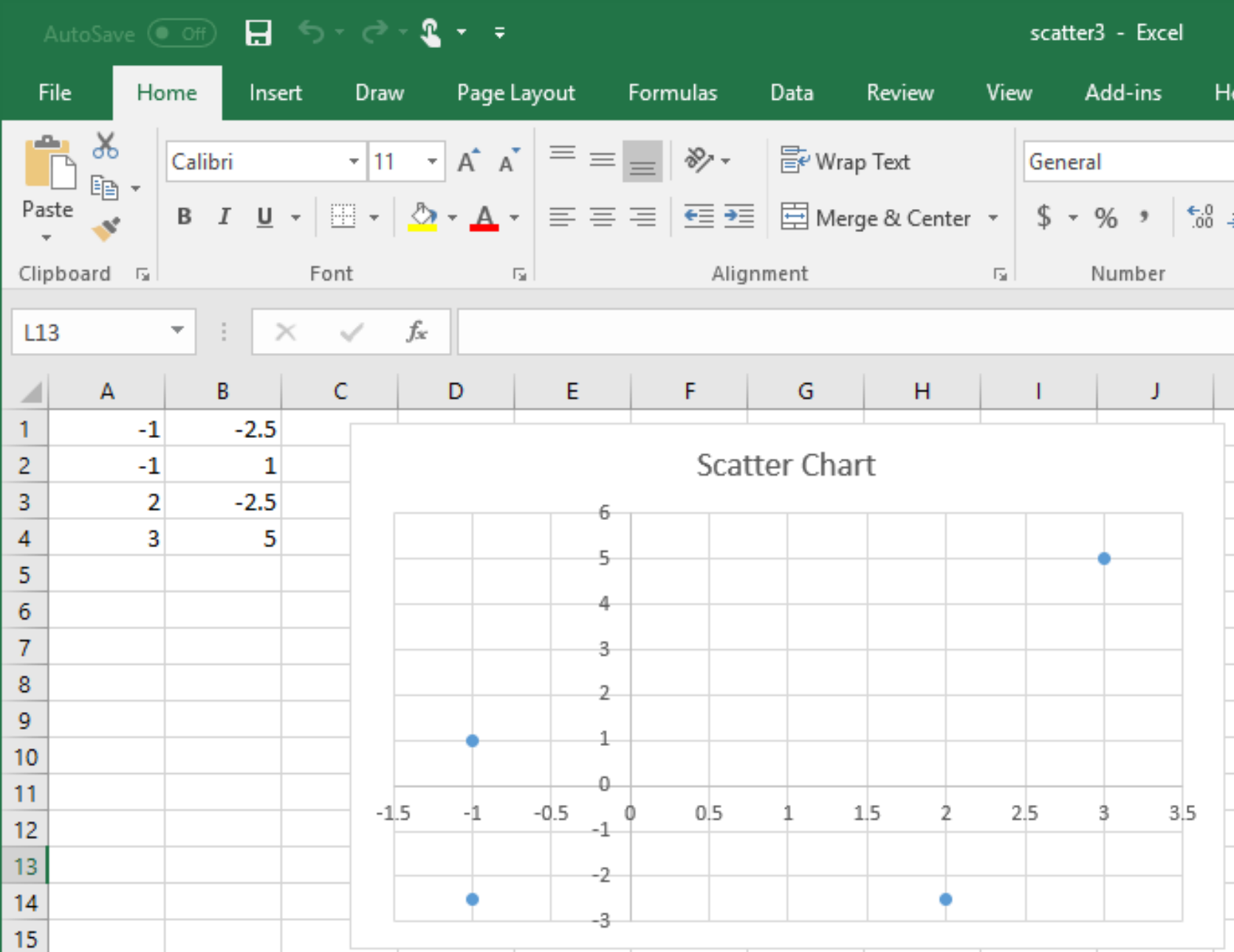


Try Example 2?

Go Ahead!

Figure 1





Excel Scatter Chart

Ch1_Introduction - [C:\Eric_Chou\Python Course\PreCalculus with Python Programming\PyDev\U1_Functions\Ch1_Introduction] - ...Quadratic\scatter3x.py - PyCharm Community Edition 2017.2.1

File Edit View Navigate Code Refactor Run Tools VCS Window Help

Ch1_Introduction > Quadratic > scatter3x.py

Project: C:\Eric_Chou\Python Course\PreCalculus with Python Programming

- Ch1_Introduction
 - Coordinates
 - CartesianPlane.py
 - ComplexNumber.py
 - Polar.py
 - Polynomial.py
 - Quadratic
 - quadratic.py
 - Scatter.py
 - scatter2.py
 - scatter3.csv
 - scatter3.py
 - scatter3.xlsx
 - scatter3x.py
 - External Libraries

```
1 from pylab import *
2 import numpy as np
3
4 x=[]
5 y=[]
6 for line in open("scatter3.csv", "r"):
7     t = line.rstrip().split(',')
8     print(t)
9     x.append(float(t[0]))
10    y.append(float(t[1]))
11
12 figure()
13 scatter(x, y)
14 plot(x, y, color='r', linestyle='--')
15 xlim(xmin=-5.2, xmax=5.2) # set x-axis range
16 ylim(ymin=-5.2, ymax=5.2) # set y-axis range
17 xlabel('x') # label x-axis
18 ylabel('y') # label y-axis
19 title('Scatter Chart(Not a function)') #
20 grid(linestyle='--', linewidth='0.5', color='gray')
21 show()
```

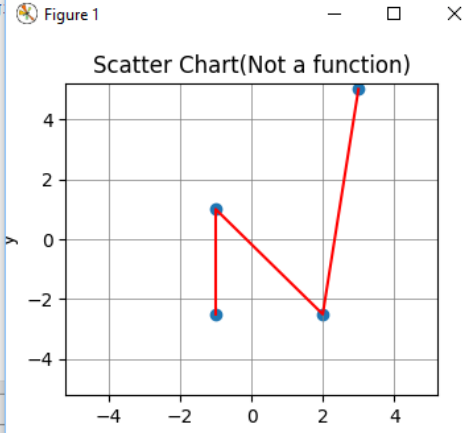
Run: scatter3 scatter3x scatter2 Scatter scatter2 scatter2 scatter2 scatter2 scatter3

C:\Python\Python36\python.exe "C:\Eric_Chou\Python Course\PreCalculus with Python Programming\PyDev\U1_Functions\Ch1_Introduction\Quadratic\scatter3x.py"

```
['-1', '-2.5']
['-1', '1']
['2', '-2.5']
['3', '5']
```

Figure 1

Scatter Chart(Not a function)



x	y
-1	-2.5
-1	1
2	-2.5
3	5

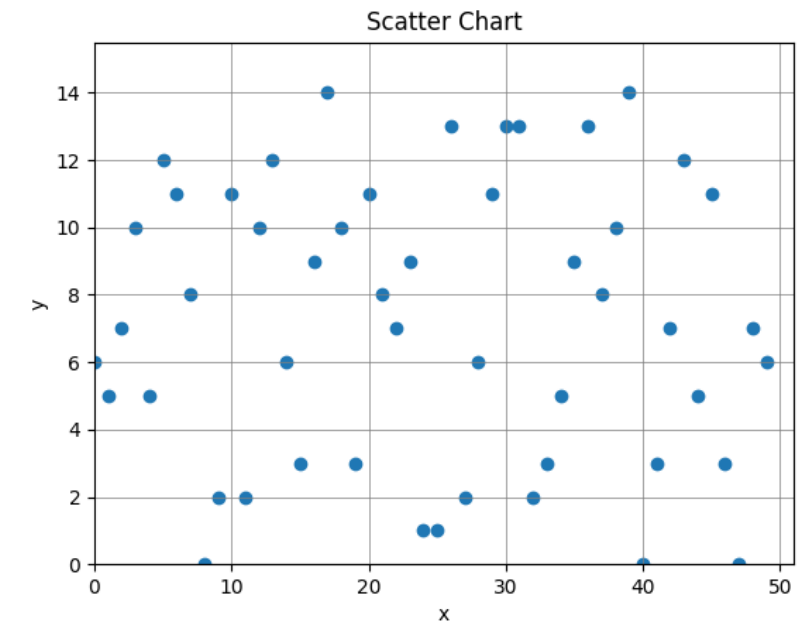


PyLab Scatter Chart

scatter2.py

```
from pylab import *
import numpy as np
n = 50
x = np.linspace(0, n-1, n)
y = np.ndarray(n)
for i in range(n):
    y[i] = int(random()*15)
figure()
scatter(x, y)
#plot(x, y, color='r', linestyle='-')
xlim(xmin=0, xmax=n+1)          # set x-axis range
ylim(ymin=0, ymax=15.5)         # set y-axis range
xlabel('x')                      # label x-axis
ylabel('y')                      # label y-axis
title('Scatter Chart')          #
grid(linestyle='-', linewidth='0.5', color='gray')
show()
```

Figure 1



LECTURE 1

Complex Number

Complex Number as Roots

QUADRATIC FUNCTIONS

● Quadratic Equations and Roots Containing "i":

In relation to quadratic equations, imaginary numbers (and complex numbers) occur when the value under the radical portion of the quadratic formula is negative. When this occurs, the equation has no roots (zeros) in the set of real numbers. The roots belong to the set of complex numbers, and will be called "**complex roots**" (or "**imaginary roots**"). These complex roots will be expressed in the form $a + bi$.

Consider this example:

Find the roots: $x^2 + 4x + 5 = 0$

This quadratic equation is not factorable, so we apply the quadratic formula. Notice that after combining the values, we are left with a negative value under the square root radical. This negative square root creates an **imaginary number**.

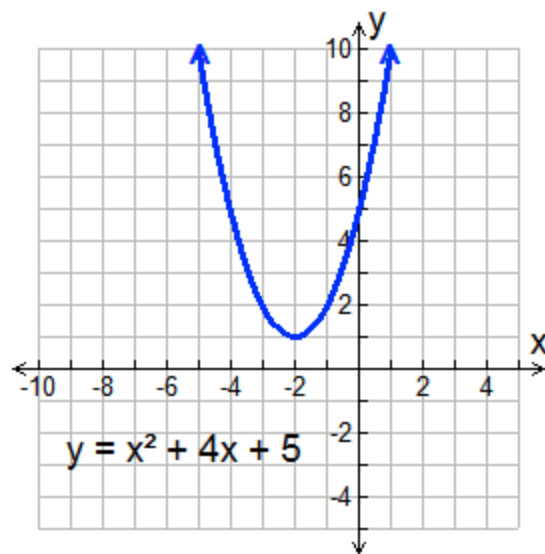
$$x^2 + 4x + 5 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm 2i}{2} = -2 \pm i$$



The graph of this quadratic function shows that there are **no real roots (zeros)** because the graph does not cross the x -axis. Such a graph tells us that the roots of the equation are complex numbers, and will appear in the form $a + bi$.

The Need for Imaginary Numbers

You have learned that the product of two numbers with the same sign is always positive. You have also learned that the square root of a number is the quantity whose square is equal to that number. Within the constraints of these two principles, how can a negative number have a square root?

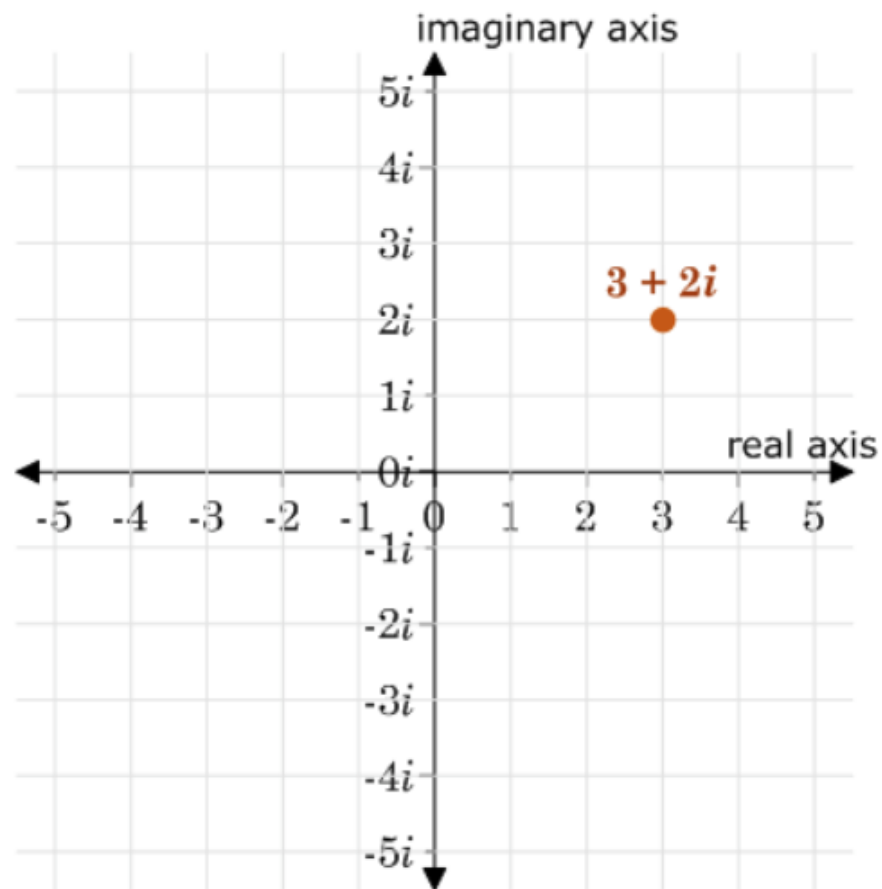
If you were to solve the equation $x^2 + 2x + 5 = 0$ with the quadratic formula, the solution would include the square root of a negative number. The need for imaginary numbers arose from situations like this.

The **imaginary unit**, i , is the number whose square equals -1 . It was defined in order to have a standard way to write imaginary numbers. For any real positive number n , $\sqrt{-n} = i\sqrt{n}$.

Imaginary numbers can be simplified using the definition of i and the properties of operations.



The Complex Number Plane





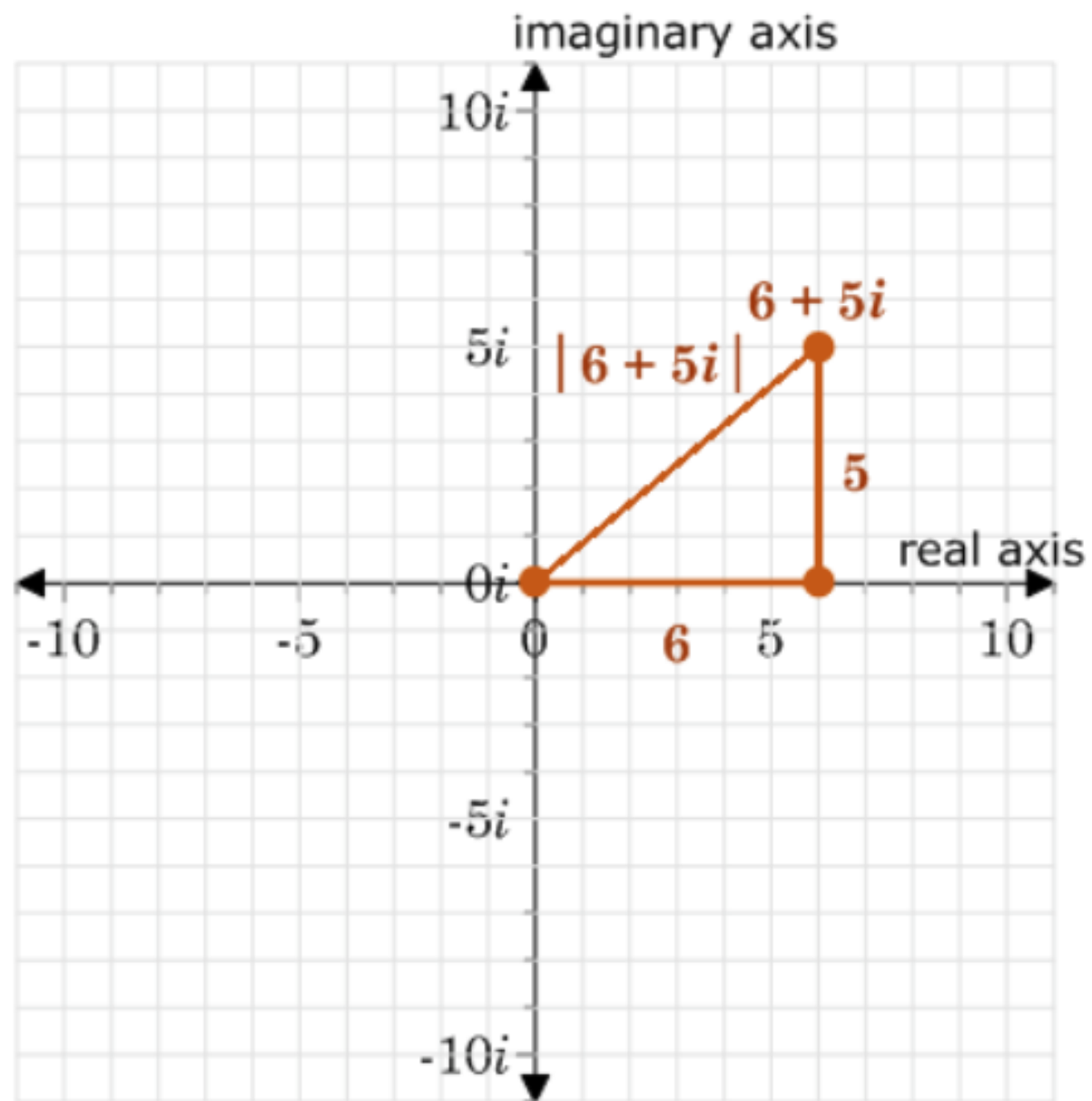
Absolute Value of Complex Numbers

The **absolute value of a complex number** is its distance from the origin on the complex plane. The absolute value of $a + bi$ is written as $|a + bi|$.

You can find a distance on the complex plane the same way you would find a distance on the Cartesian plane: using the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Since the coordinates of the origin are $(0, 0)$, the distance from the origin to the point (x, y) is $d = \sqrt{x^2 + y^2}$.

On the complex plane, the complex number $a + bi$ is represented by the point (a, b) . Since the absolute value of a complex number is its distance from the origin, $|a + bi| = d = \sqrt{x^2 + y^2} = \sqrt{a^2 + b^2}$.

You can also think of the distance from any point on the complex plane to the origin as the length of the hypotenuse of a right triangle whose legs are the lengths of the real part and the coefficient of the imaginary part. In other words, for the complex number $a + bi$, the distance from $(0, 0)$ to the point is the length of the hypotenuse of a right triangle with legs of length a units and b units.

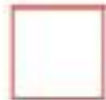


Complex Number as a Vector

2D POINTS



Complex Addition and Subtraction



COMPLEX NUMBERS

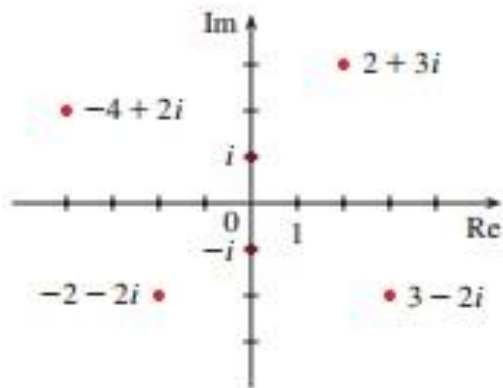


FIGURE 1

Complex numbers as points in the Argand plane

A **complex number** can be represented by an expression of the form $a + bi$, where a and b are real numbers and i is a symbol with the property that $i^2 = -1$. The complex number $a + bi$ can also be represented by the ordered pair (a, b) and plotted as a point in a plane (called the Argand plane) as in Figure 1. Thus, the complex number $i = 0 + 1 \cdot i$ is identified with the point $(0, 1)$.

The **real part** of the complex number $a + bi$ is the real number a and the **imaginary part** is the real number b . Thus, the real part of $4 - 3i$ is 4 and the imaginary part is -3 . Two complex numbers $a + bi$ and $c + di$ are **equal** if $a = c$ and $b = d$, that is, their real parts are equal and their imaginary parts are equal. In the Argand plane the horizontal axis is called the real axis and the vertical axis is called the imaginary axis.

The sum and difference of two complex numbers are defined by adding or subtracting their real parts and their imaginary parts:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$



Conjugate

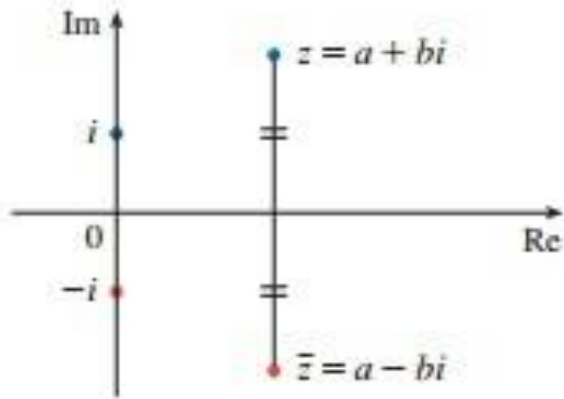


FIGURE 2

The geometric interpretation of the complex conjugate is shown in Figure 2: \bar{z} is the reflection of z in the real axis. We list some of the properties of the complex conjugate in the following box. The proofs follow from the definition and are requested in Exercise 18.

Properties of Conjugates

$$\overline{z + w} = \bar{z} + \bar{w}$$

$$\overline{zw} = \bar{z} \bar{w}$$

$$\overline{z^n} = \bar{z}^n$$



Absolute Value of Complex Numbers

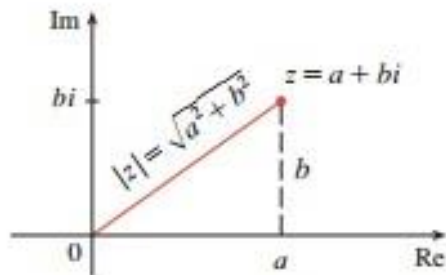


FIGURE 3

The **modulus**, or **absolute value**, $|z|$ of a complex number $z = a + bi$ is its distance from the origin. From Figure 3 we see that if $z = a + bi$, then

$$|z| = \sqrt{a^2 + b^2}$$

Notice that

$$z\bar{z} = (a + bi)(a - bi) = a^2 + abi - abi - b^2i^2 = a^2 + b^2$$

and so

$$z\bar{z} = |z|^2$$

This explains why the division procedure in Example 2 works in general:

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2}$$

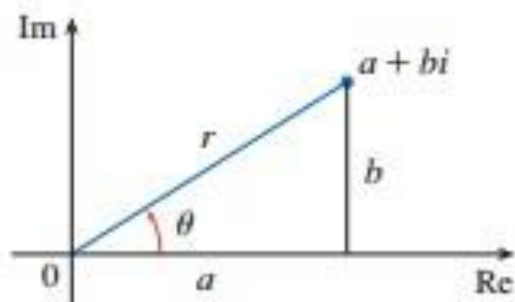
Since $i^2 = -1$, we can think of i as a square root of -1 . But notice that we also have $(-i)^2 = i^2 = -1$ and so $-i$ is also a square root of -1 . We say that i is the **principal square root** of -1 and write $\sqrt{-1} = i$. In general, if c is any positive number, we write

$$\sqrt{-c} = \sqrt{c} i$$

Complex Number as a Phasor

POLAR FORM

<https://www.electronics-tutorials.ws/accircuits/phasors.html>



POLAR FORM

We know that any complex number $z = a + bi$ can be considered as a point (a, b) and that any such point can be represented by polar coordinates (r, θ) with $r \geq 0$. In fact,

$$a = r \cos \theta \quad b = r \sin \theta$$

as in Figure 4. Therefore, we have

Thus, we can write any complex number z in the form

$$z = r(\cos \theta + i \sin \theta)$$

where $r = |z| = \sqrt{a^2 + b^2}$ and $\tan \theta = \frac{b}{a}$

The angle θ is called the **argument** of z and we write $\theta = \arg(z)$. Note that $\arg(z)$ is not unique; any two arguments of z differ by an integer multiple of 2π .



Complex Multiplication

The polar form of complex numbers gives insight into multiplication and division. Let

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \quad z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

be two complex numbers written in polar form. Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \end{aligned}$$

Therefore, using the addition formulas for cosine and sine, we have

1

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

This formula says that *to multiply two complex numbers we multiply the moduli and add the arguments.* (See Figure 6.)

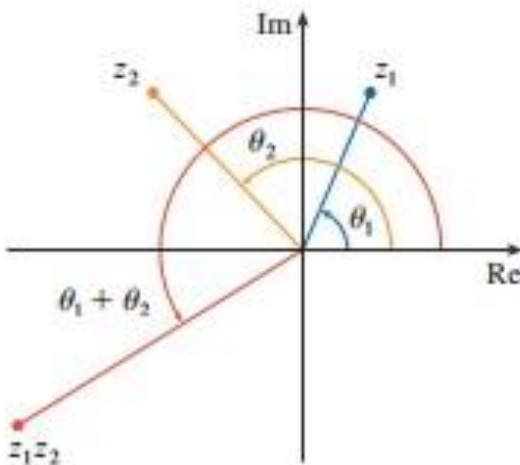


FIGURE 6



Complex Division

A similar argument using the subtraction formulas for sine and cosine shows that *to divide two complex numbers we divide the moduli and subtract the arguments.*

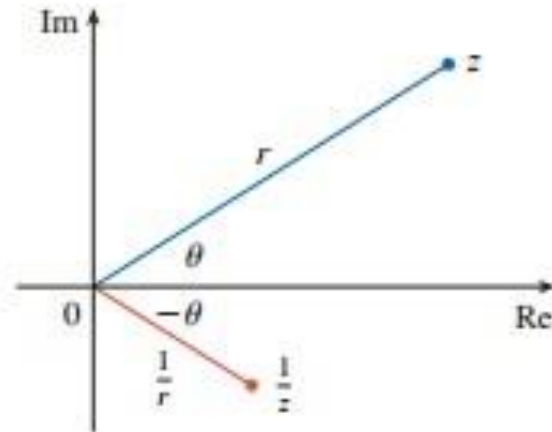


FIGURE 7

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)] \quad z_2 \neq 0$$

In particular, taking $z_1 = 1$ and $z_2 = z$, (and therefore $\theta_1 = 0$ and $\theta_2 = \theta$), we have the following, which is illustrated in Figure 7.

$$\text{If } z = r(\cos \theta + i \sin \theta), \text{ then } \frac{1}{z} = \frac{1}{r} (\cos \theta - i \sin \theta).$$

COMPLEX EXPONENTIALS

We also need to give a meaning to the expression e^z when $z = x + iy$ is a complex number. The theory of infinite series as developed in Chapter 8 can be extended to the case where the terms are complex numbers. Using the Taylor series for e^x (8.7.12) as our guide, we define

$$\boxed{4} \quad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

and it turns out that this complex exponential function has the same properties as the real exponential function. In particular, it is true that

$$\boxed{5} \quad e^{z_1+z_2} = e^{z_1}e^{z_2}$$

If we put $z = iy$, where y is a real number, in Equation 4, and use the facts that

$$i^2 = -1, \quad i^3 = i^2i = -i, \quad i^4 = 1, \quad i^5 = i, \quad \dots$$

we get

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \cdots \\ &= 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} + \cdots \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right) \\ &= \cos y + i \sin y \end{aligned}$$



Euler's Formula

Here we have used the Taylor series for $\cos y$ and $\sin y$ (Equations 8.7.17 and 8.7.16). The result is a famous formula called **Euler's formula**:

6

$$e^{iy} = \cos y + i \sin y$$

Combining Euler's formula with Equation 5, we get

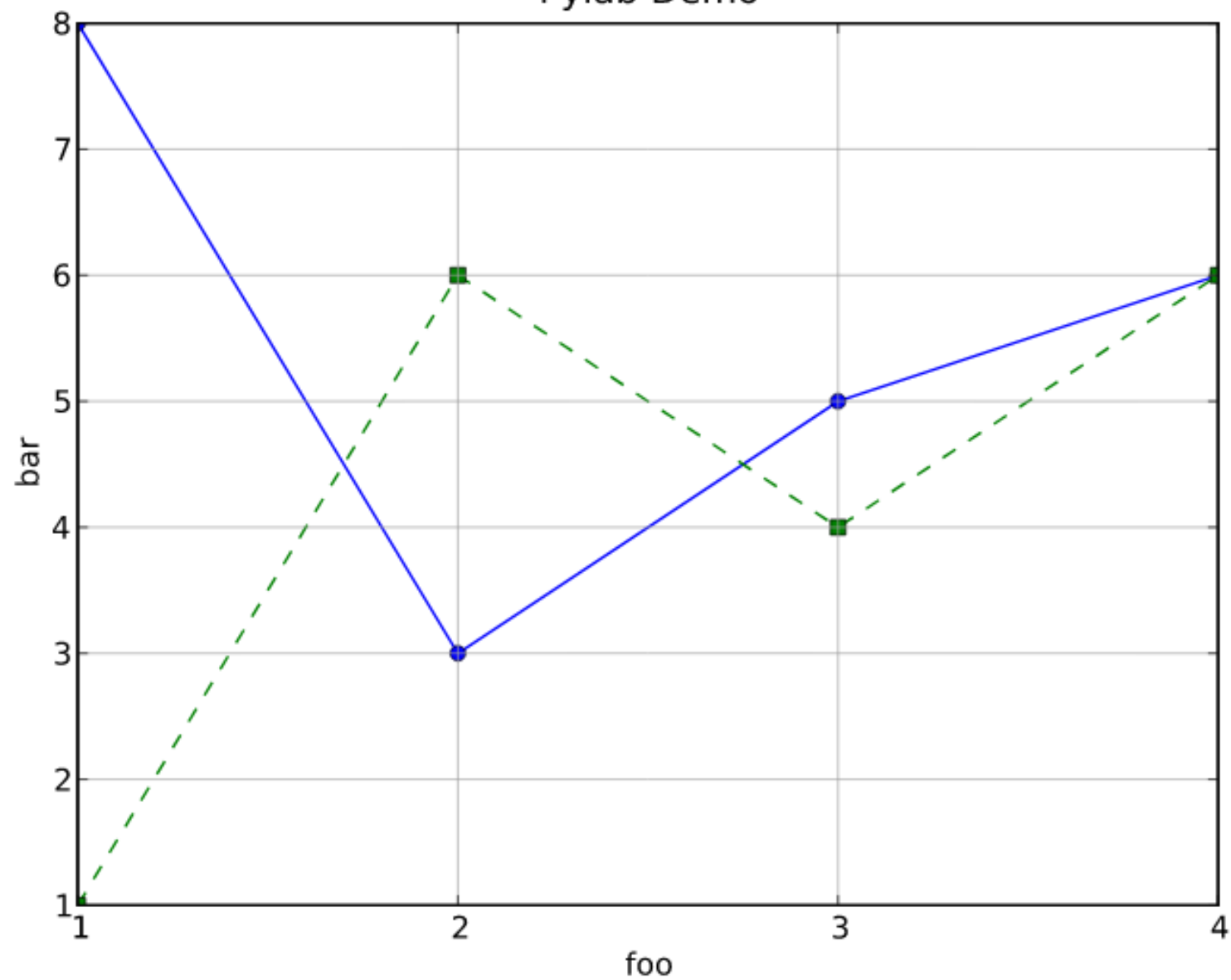
7

$$e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)$$

LECTURE 1

PyLab (Matlab-Like Python)

Pylab Demo

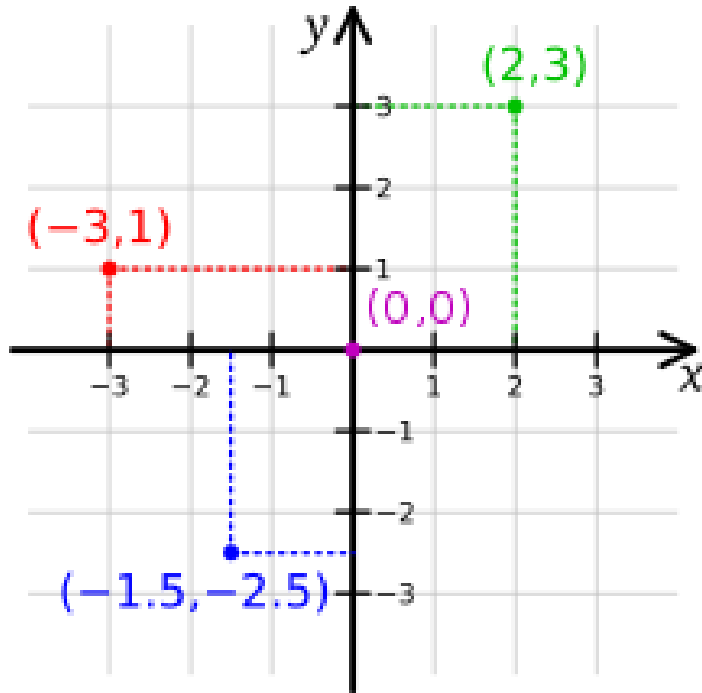


LECTURE 1

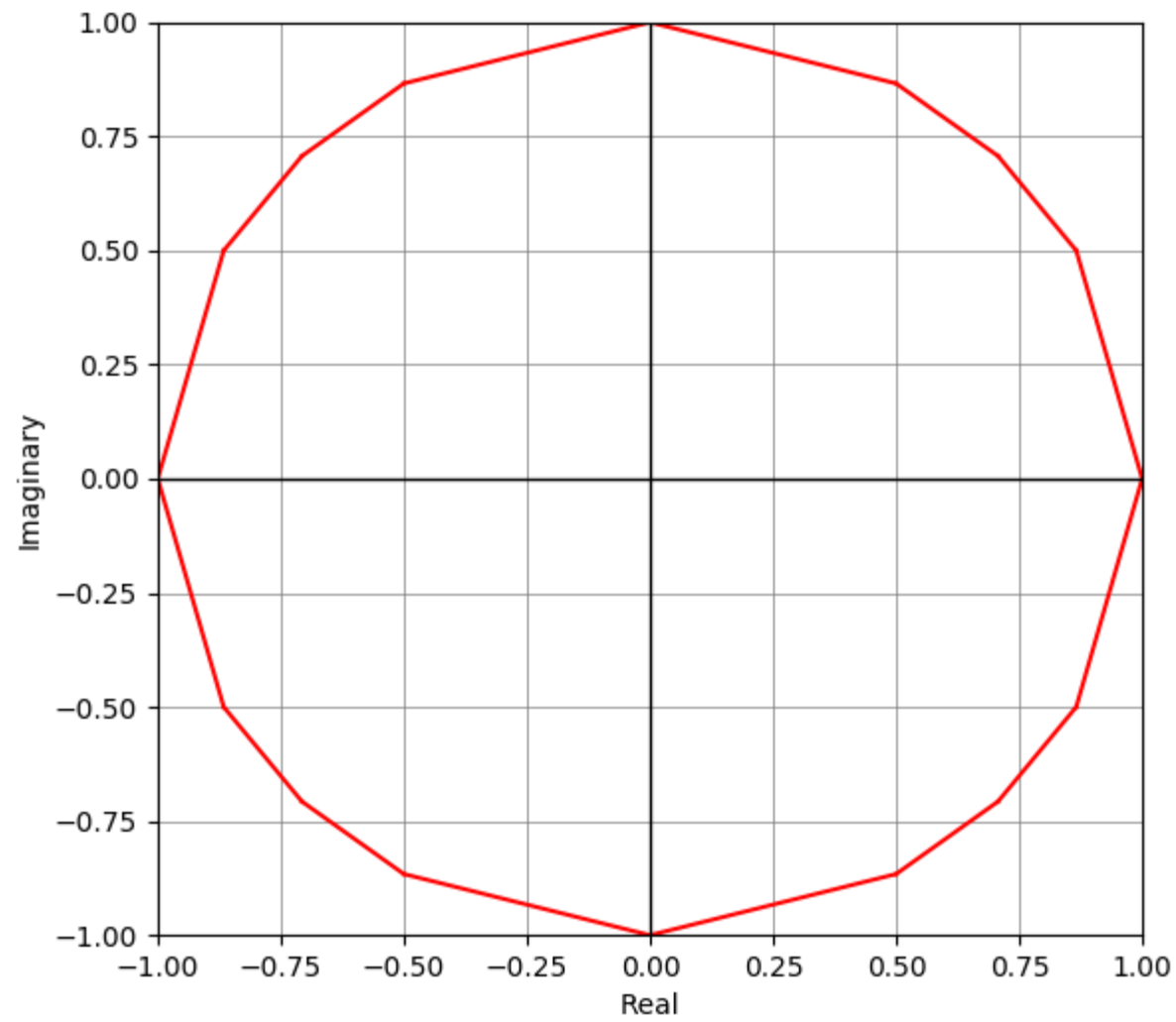
Cartesian Plane



Cartesian Plane



- A **Cartesian coordinate system** is a coordinate system that specifies each point uniquely in a plane by a pair of numerical coordinates, which are the signed distances to the point from two fixed perpendicular directed lines, measured in the same unit of length.
- Each reference line is called a *coordinate axis* or just *axis* (plural *axes*) of the system, and the point where they meet is its origin, at ordered pair $(0, 0)$. The coordinates can also be defined as the positions of the perpendicular projections of the point onto the two axes, expressed as signed distances from the origin.



```

from pylab import *
import numpy as np

def cartesian(a): # scatter chart
    figure()
    size=len(a)
    for x in range(len(a)):
        plot([a[x][0], a[(x+1)%size][0]], [a[x][1], a[(x+1)%size][1]], 'r-', label='python')
    limit=np.max(np.ceil(np.absolute(a))) # set limits for axis
    xlim((-limit, limit))
    ylim((-limit, limit))
    ylabel('Imaginary')
    xlabel('Real')
    grid(linestyle='-', linewidth='0.5', color='gray')
    axhline(0, color='black', lw=1.0)
    axvline(0, color='black', lw=1.0)
    show()
    #close()

a = [(1, 0), (sqrt(3)/2, 1/2), (sqrt(2)/2, sqrt(2)/2), (1/2, sqrt(3)/2), (0, 1),
      (-1/2, sqrt(3)/2), (-sqrt(2)/2, sqrt(2)/2), (-sqrt(3)/2, 1/2), (-1, 0),
      (-sqrt(3)/2, -1/2), (-sqrt(2)/2, -sqrt(2)/2), (-1/2, -sqrt(3)/2), (0, -1),
      (1/2, -sqrt(3)/2), (sqrt(2)/2, -sqrt(2)/2), (sqrt(3)/2, -1/2), (1, 0)
      ]
cartesian(a)

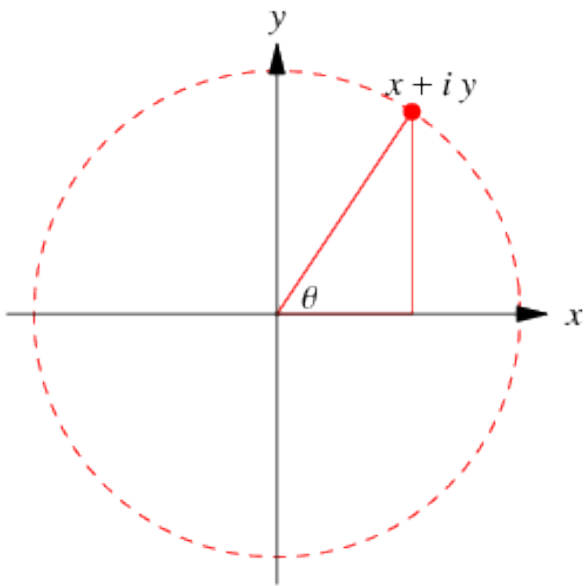
```

LECTURE 1

Complex Plane (Argand Diagram)

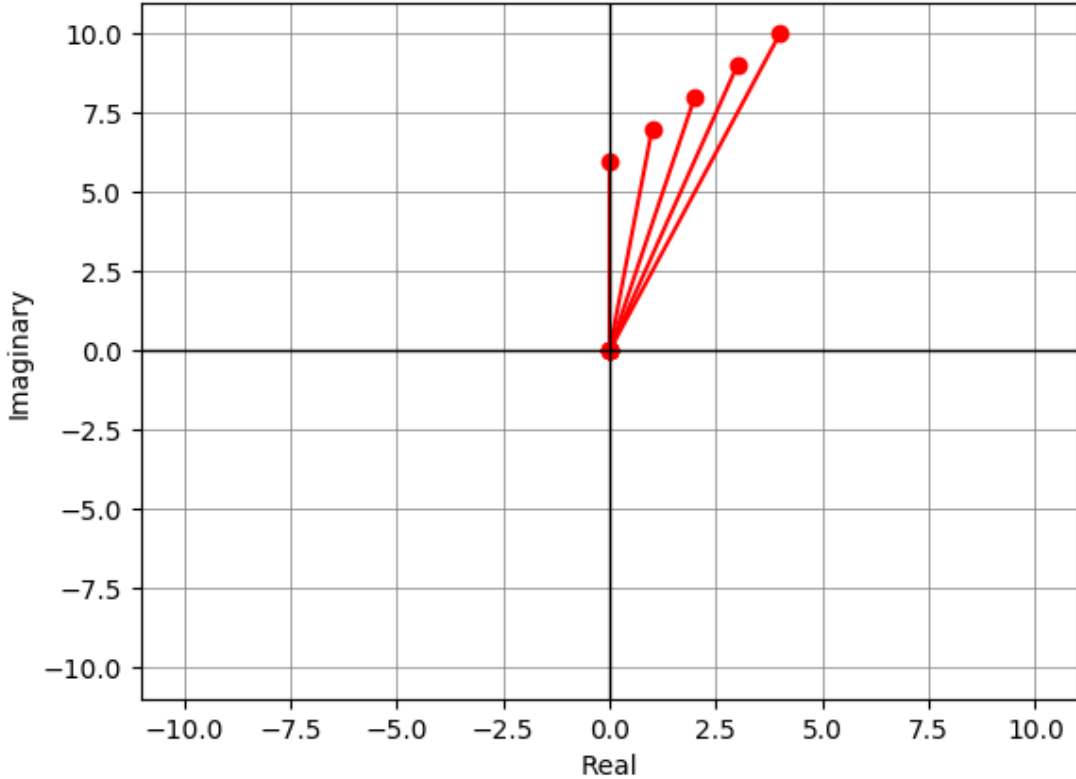


Argand Diagram (Complex Plane)



- An Argand diagram is a plot of complex numbers as points
 $z = x + i y$
- in the complex plane using the x-axis as the real axis and y-axis as the imaginary axis. In the plot above, the dashed circle represents the complex modulus $|z|$ of z and the angle θ represents its complex argument.
- While Argand (1806) is generally credited with the discovery, the Argand diagram (also known as the Argand plane) was actually described by C. Wessel prior to Argand. Historically, the geometric representation of a complex number as a point in the plane was important because it made the whole idea of a complex number more acceptable.

Figure 1



ComplexNumber.py:

```
[0.+6.j, 1.+7.j, 2.+8.j, 3.+9.j, 4.+10.j]
```

```
from pylab import *
import numpy as np
def argand(a): # scatter chart
    figure()
    for x in range(len(a)):
        plot([0, a[x].real], [0, a[x].imag], 'ro-', label='python')
    limit=np.max(np.ceil(np.absolute(a))) # set limits for axis
    xlim((-limit, limit))
    ylim((-limit, limit))
    ylabel('Imaginary')
    xlabel('Real')
    grid(linestyle='-', linewidth='0.5', color='gray')
    axhline(0, color='black', lw=1.0)
    axvline(0, color='black', lw=1.0)
    show()
```

```
a = [0.+6.j, 1.+7.j, 2.+8.j, 3.+9.j, 4.+10.j]
argand(a)
```

LECTURE 1

Polar Plane



Polar System

- In mathematics, the polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a reference point and an angle from a reference direction.
- The reference point (analogous to the origin of a Cartesian coordinate system) is called the pole, and the ray from the pole in the reference direction is the polar axis. The distance from the pole is called the radial coordinate or radius, and the angle is called the **angular coordinate, polar angle, or azimuth**.



Polar System

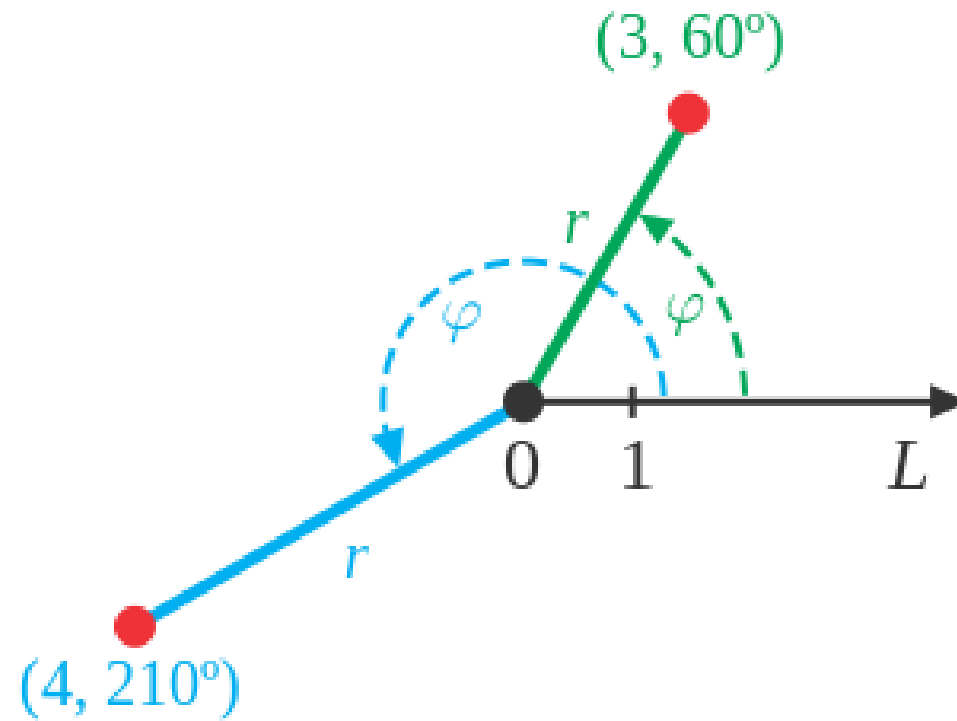
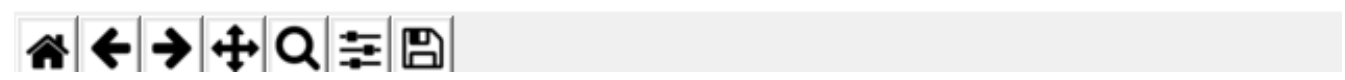
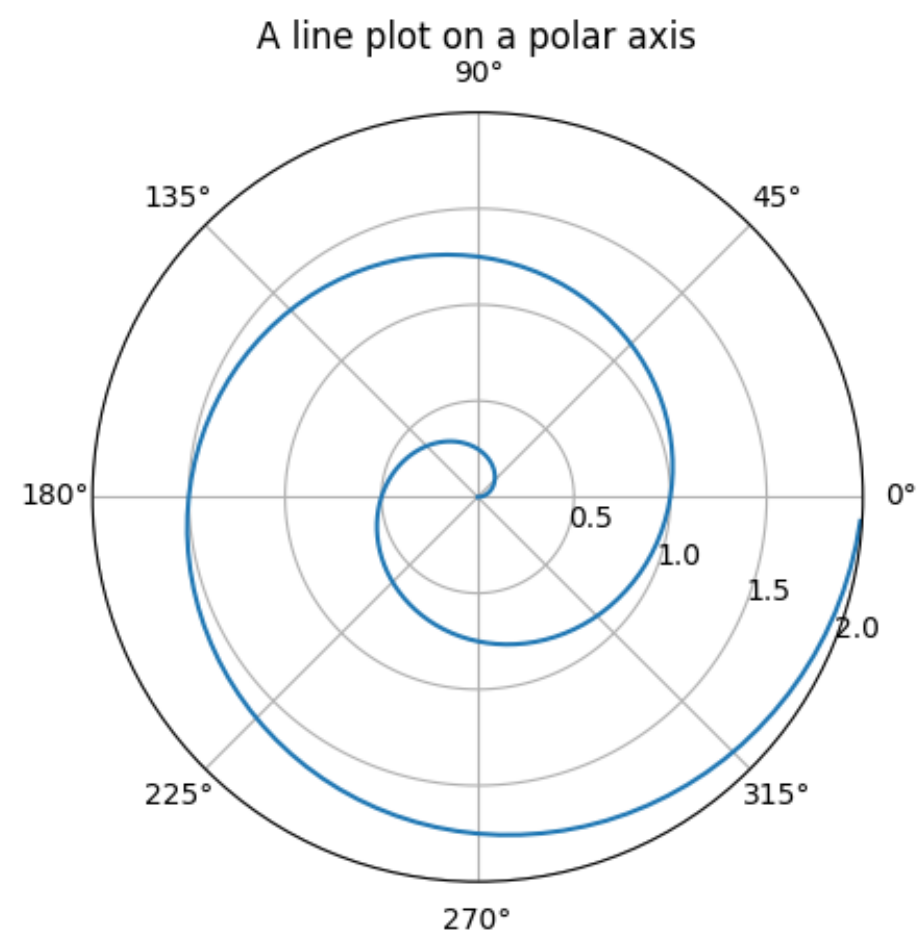


Figure 1



Polar.py

```
import numpy as np
from pylab import *

t = np.arange(0, 2, 0.01)
theta = 2 * np.pi * t      # np.pi is the pi value

ax = subplot(111, projection='polar')
ax.plot(theta, t)
ax.set_rmax(2)
ax.set_rticks([0.5, 1, 1.5, 2]) # less radial ticks
ax.set_rlabel_position(-22.5) # get radial labels away from
plotted line
ax.grid(True)

ax.set_title("A line plot on a polar axis", va='bottom')
show()
```

LECTURE 1

Summary of Chapter 1



Summary

- Algebra study the use of symbols in mathematics.
- It includes the study of variable (independent/dependent), their domain field (including number set).
- The functions for the construction of dependent variables. The oddity, evenness, symmetry of functions. Algebra also study the translation of functions.
- Functions to be studied in algebra 3: polynomials, rational function, exponential/logarithmic, and trigonometric.
- Analytical geometry is used to visualize the functional data. It is also used to study the pole/zero for a function. Pole and Limits are used to study the asymptotic behavior of functions. Zeros are used to study the cross-over points for a function on an axis.