



Impact of Counterparty Risk on the Reinsurance Market

Carole Bernard PhD & Mike Ludkovski PhD

To cite this article: Carole Bernard PhD & Mike Ludkovski PhD (2012) Impact of Counterparty Risk on the Reinsurance Market, North American Actuarial Journal, 16:1, 87-111, DOI: [10.1080/10920277.2012.10590634](https://doi.org/10.1080/10920277.2012.10590634)

To link to this article: <http://dx.doi.org/10.1080/10920277.2012.10590634>



Published online: 26 Nov 2012.



Submit your article to this journal [↗](#)



Article views: 109



View related articles [↗](#)



Citing articles: 5 View citing articles [↗](#)

IMPACT OF COUNTERPARTY RISK ON THE REINSURANCE MARKET

Carole Bernard* and Mike Ludkovski†

ABSTRACT

We investigate the impact of counterparty risk (from the insurer's viewpoint) on contract design in the reinsurance market. We study a multiplicative default risk model with partial recovery and where the probability of the reinsurer's default depends on the loss incurred by the insurer. The reinsurer (reinsurance seller) is assumed to be risk-neutral, while the insurer (reinsurance buyer) is risk-averse and uses either expected utility or a conditional tail expectation risk criterion. We show that generally the reinsurance buyer wishes to overinsure above a deductible level, and that many of the standard comparative statics cease to hold. We also derive the properties of stop-loss insurance in our model and consider the possibility of divergent beliefs about the default probability. Classical results are recovered when default risk is loss-independent or there is zero recovery rate. Results are illustrated with numerical examples.

1. INTRODUCTION

During the recent financial crisis, counterparty risk, credit risk, and systemic dependency between financial firms have become front-page news. A similar story played out in the reinsurance industry with reinsurers' default risk rising exactly at the time when reinsurance was most needed. Potential scenarios of large-scale natural catastrophes, nondiversifiable shifts in longevity, and systemic shocks to the reinsurance industry all contribute to this phenomenon. In this paper we investigate the impact of counterparty risk on the optimal insurance design in the reinsurance market. We propose a new one-period model to solve the optimal risk sharing when there is a probability that the reinsurer defaults on his or her contract obligations and only partly reimburses the promised insurance indemnity. Crucially, we assume that this counterparty risk is related to the losses of the reinsurance buyer. Indeed, when the reinsurance buyer has a big loss, the reinsurer is not only responsible for making a large indemnity payment, but is also likely affected by similar losses from other reinsurance buyers. Thus, large losses are commonly due to a systemic factor and cause undiversifiable stress on the reinsurer's capital.

To capture systemic effects, we assume negative stochastic dependency in the sense of Capéraà and Lefoll (1983) between the loss incurred by an insurance company X and the fraction $0 \leq \Theta \leq 1$ of indemnity actually paid out by the reinsurer (after default). We term default any event where $\Theta < 1$ and the promised indemnity is not fully paid. This means that conditional on X , Θ is a nonincreasing function. While we assume that large X makes it more likely that $\Theta < 1$, we do not introduce any direct *structural* model of such cause and effect. Thus, our model is complementary to the recent study of Biffis and Millosovich (2010), who investigate reinsurer default that is explicitly caused by the payments of the indemnity and the consequent nonperformance of the reinsurer's net assets. We believe

* Carole Bernard, PhD, is Assistant Professor in the Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, ON, N2L 3G1, Canada, c3bernar@uwaterloo.ca.

† Mike Ludkovski, PhD, is Assistant Professor in the Department of Statistics and Applied Probability University of California Santa Barbara, 5520 South Hall, CA 93106-3110, ludkovski@pstat.ucsb.edu.

that a typical reinsurance company has many policyholders, so it is unlikely that default is directly triggered by the payment of one particular indemnity.

In our model counterparty risk is interpreted as a form of “background risk” for the reinsurance buyer. Several previous papers have already studied the effects of background risk on risk-sharing agreements; see, for instance, the survey in Schlesinger (2000) and, more recently, Dana and Scarsini (2007). However, most existing literature assumes that the background risk is additive and relies on very special cases of dependence (such as Gaussian correlation or certain copula structures). In contrast, reinsurance default risk must be multiplicative and requires more general dependence structures. The early work by Doherty and Schlesinger (1990) considered a three-state model of insurance under counterparty risk with total default and was later extended to partial default by Mahul and Wright (2007). More sophisticated models of independent default risk were analyzed in Mahul and Wright (2004) and Cardenas and Mahul (2006). Finally, Cummins and Mahul (2003) considered a loss-dependent counterparty risk but with total default. We extend this strand of literature by incorporating loss-dependent probability of default, as well as partial recovery in the event of contract nonperformance. More generally, we extend Dana and Scarsini (2007) to the case when the background risk is multiplicative and not additive, and we underline that the presence of multiplicative background risk can be more complex than additive risk. This was also noted in another context by Franke et al. (2006).

As in aforementioned papers, we show that it is generally optimal for a risk-averse insurer (facing default risk of the reinsurer) to lower the reinsurance demand by increasing the optimal deductible level, while at the same time increasing the marginal insurance rate over the deductible. If there is partial recovery in case of default, there is an increase in protection of the tail risk, so that the optimal shape of the contract involves marginal overinsurance but the optimal premium is lower. However, in general, the overall shape of optimal contracts may be very complex and include decreasing indemnification, overinsurance, and counterintuitive comparative statics. Thus, most of the standard properties of optimal indemnities are rendered false. We document all these effects and illustrate them with numerical examples. We also show that partial recovery can lead to very different indemnities compared to total default, highlighting the need for proper modeling of recovery rates.

To connect our model with more realistic insurance contracts, we also investigate the properties of optimal stop-loss reinsurance in our setting. While we demonstrate that within the classical expected utility framework stop-loss indemnities are suboptimal and continue to have counterintuitive comparative statics, they can be tractable when using the increasingly popular conditional tail expectation (CTE) and Value-at-Risk (VaR) criteria. Namely, we derive explicit formulas for the impact of counterparty risk on the optimal deductible levels under the CTE/VaR risk measures. These results are then compared to the work of Cai and Tan (2007) done in the absence of counterparty risk. We find that counterparty risk again lowers the insurance demand by increasing the optimal deductible level.

In Section 2 we first set up the problem and the model for the dependency between realized losses and counterparty risk. Section 3 then investigates the effects of counterparty risk on the deductible level if a stop-loss indemnity is bought in the case of expected utility maximization and of risk minimization (where risk is measured through VaR or CTE). Through examples we show evidence that a deductible indemnity may not be optimal, and we derive general properties of the optimal risk sharing in the reinsurance market in the presence of counterparty risk in Section 4. Section 5 considers the model where the reinsurer ignores his or her own default risk, related to the setting of Cummins and Mahul (2003). All proofs are given in the Appendix.

2. FRAMEWORK

In this section we present the optimal insurance design, the assumptions of the model, and how counterparty risk is modeled.

2.1 Optimal Insurance Design

Consider an insurance company with initial endowment W . We assume a one-period setting. Let π denote the premium paid by the insurer to the reinsurance company at the beginning of the coverage

period. Let X denote the aggregate loss of the insurance company during this period and $I(X)$ the reinsurance indemnity paid by the reinsurer at the end of the coverage period. We will refer to the reinsurer as the “seller” and to the insurer as the “buyer.” The loss X is bounded almost surely by \bar{x} . We suppose that the reinsurance contract is subject to counterparty risk, that is, with a small probability the seller will not be able to pay entirely the promised benefits $I(X)$ to the insurance company.

We model this problem as follows. Let Θ represent the “recovery rate.” In case of default of the seller, it is the percentage of the indemnity that the seller can pay. The reinsurance buyer thus receives $\Theta I(X)$, where

$$\begin{cases} \Theta = 1 & \text{with prob. } p, \\ \Theta \sim \mathbb{D}(0, 1) & \text{with prob. } 1 - p. \end{cases}$$

The above equation implies that the random variable Θ has a mixed distribution, with a point mass at 1 of size p and a continuous distribution over $[0, 1)$ with density equal to $f(x) = (1 - p)g(x)$, where $g(x)$ is a density for the given recovery distribution $\mathbb{D}(0, 1)$ over $[0, 1)$. Most commonly we will assume that Θ takes on just two values, $\Theta(\omega) \in \{\theta_0, 1\}$, with $0 \leq \theta_0 \leq 1$ representing the recovery rate in the case of nonperformance. While certainly not realistic, this case is already sufficiently rich (and in particular much more complex than total default $\theta_0 = 0$) to be of theoretical interest. Moreover, given significant difficulties in empirically estimating recovery rates, it is common in financial practice to assume constant recovery rates in case of default.

We assume that the buyer is a von Neumann–Morgenstern utility maximizer with utility function U . Thus, the buyer is maximizing the expected utility of his or her final wealth:

$$\mathbb{E}[U(W - \pi - X + \Theta I(X))].$$

We suppose that $U(\cdot)$ satisfies Inada’s conditions, that is, $U(0) = 0$, U is continuously differentiable and strictly increasing, U' is strictly decreasing (so that the buyer is risk averse), $\lim_{x \searrow 0} U'(x) = +\infty$, and $\lim_{x \nearrow +\infty} U'(x) = 0$. We also assume that the seller is risk neutral and that he or she simply maximizes his or her expected profit at the end of the period. In other words, the seller accepts to sell the reinsurance contract if the premium is enough to cover the expected costs. Let C represent the minimum profit (expressed as an expected value) that the seller wants to achieve by selling reinsurance to this buyer.

We assume that the seller is risk-neutral and shares the same view as the buyer about the default risk. The participation constraint can then be written as

$$C \leq \mathbb{E}[\pi - \Theta I(X)],$$

where C is a minimum expected profit. Equivalently, one has

$$\mathbb{E}[\Theta I(X)] \leq K, \tag{1}$$

where $K = \pi - C$. The expected profit of the seller is classically measured through a linear safety loading ρ as

$$K = \frac{\pi}{1 + \rho}.$$

Our problem consists of finding the optimal reinsurance arrangement $x \mapsto I(x)$ such that the expected utility of final wealth of the buyer is maximized. Additional standard constraints are that indemnities are positive and that they cannot exceed the actual loss (to avoid moral hazard), $0 \leq I(x) \leq x$. Such assumptions were adopted by most papers on optimal insurance since the pioneering works of Arrow (1963, 1971) and Raviv (1979); see also Eeckhoudt et al. (2005). To summarize, the optimal reinsurance contract solves the following optimization problem:

$$\begin{aligned} \max_{I, \pi} \{ \mathbb{E}[U(W - \pi - X + \Theta I(X))] \}: \\ \begin{cases} 0 \leq I(x) \leq x, \\ \mathbb{E}[\Theta I(X)] \leq K. \end{cases} \end{aligned} \quad (\text{A})$$

When $\Theta \equiv 1$ and there is no counterparty risk, the above problem reduces to the standard model of insurance design, whereby optimal indemnity is a stop-loss policy. Otherwise, Problem A can be seen as an extension of Cummins and Mahul (2003), where the default may happen partially instead of a total default, and as an extension of Doherty and Schlesinger (1990) and Mahul and Wright (2007), where the loss X can take more than two values.

Our objective in this paper is to solve Problem A over all possible reinsurance indemnities $I(\cdot)$ and all possible premium levels π . This optimal reinsurance problem is commonly solved via a two-step procedure. The first step is to find the optimal form of the reinsurance contract $I_\pi(\cdot)$ while keeping the reinsurance premium π fixed. The second step is to find the optimal premium level π . This is the technique adopted in many references in the literature, including Raviv (1979) and Gollier (1987, Problem 5.a), who notes that an “insurance policy is characterized by the premium π paid by the insured and by a coverage function $I(X)$ specifying the reimbursement of the insurer for each possible loss X .”

However, solving the first step of Problem A is usually challenging because it is an optimization over an infinite dimensional space. In Section 3 we therefore initially solve the second step of the problem when the indemnity shape $I(\cdot)$ is already given. Our objective is to understand the impact of counterparty risk on the reinsurance demand measured by the amount of reinsurance purchased. We investigate simple indemnities, such as the stop-loss contract $I(x) = (x - d)_+$ which depends on a single parameter, the deductible level d . Finding the optimal deductible is equivalent to finding the optimal premium because there is a bijection between π and d . In Section 4 we then solve the full Problem A by fixing π and solving for the corresponding optimal contract $I_\pi(\cdot)$. The second step of optimizing over premiums can then only be performed numerically; see, for example, Figure 2 in Section 4.4, where the expected utility is represented as a function of the premium π .

In Problem A, the insurance buyer and the insurance seller share the same beliefs about the default risk of the seller and the distribution of the recovery rate. We will discuss the presence of asymmetric information and its effects on the solution to Problem A in Section 5.

2.2 Dependency Modeling

In the optimal insurance design Problem A, there are two random variables: the aggregate loss amount X and the percentage of the indemnity paid to the buyer at the end of the period Θ . Economic considerations imply that there is dependency between these two variables. When the insurance company suffers from major losses in a given period, it is likely due to some significant natural disaster. In this case, other insurance companies also incur large losses, and the seller (who reinsures multiple buyers) is much more likely to default. Such a cascade effect would happen during a major hurricane, earthquake, or worldwide financial crisis. Therefore, Θ is more likely to be small (in particular less than 1) when X becomes very big.

DEFINITION 2.1

A random variable Θ is stochastically decreasing (respectively stochastically increasing) in another random variable X if

$$x \mapsto \mathbb{E}[f(\Theta)|X = x]$$

is nonincreasing (respectively nondecreasing) for every nondecreasing function f for which expectations exist. We denote by $\Theta \downarrow_{st} X$ (resp. $\Theta \uparrow_{st} X$) when Θ is stochastically decreasing (resp. increasing) with X .

By the above considerations, for the remainder of the paper we assume $\Theta \downarrow_{st} X$. The concept of “stochastically increasing” or “stochastically decreasing” risks could be useful in many other actuarial contexts to model dependency with a very general approach. For example, there is positive dependence among the risks of getting sick within a household (because of infectious diseases), between the risks of losing a job when two people are working in the same company or same field, between the mortality of husband and wife, or between the risk of water damage and fire for a given building. Note that this is not a symmetric relationship, and $\Theta \downarrow_{st} X$ is different from $X \downarrow_{st} \Theta$.¹

A somewhat different version of our problem is to consider $\Theta \downarrow_{st} I(X)$, which emphasizes the link between the payment of a high indemnity and the default risk of the reinsurer. The latter concept was explored in Biffis and Millosovich (2010), who studied the default risk of a reinsurer using a “structural model” in which default is predicated by a large $I(X)$. In our model we focus on the default risk due to potential systematic catastrophes that cause the reinsurer to have an exceptionally high aggregate loss. Thus, when the reinsurer goes bankrupt it is not only due to the payment of one high indemnity, but also due to systemic factors. For instance, when a natural catastrophe occurs, X is large and so are the payments of the indemnities to other buyers that are not independent anymore. Moreover, a large X might also affect other market players or the financial market, leading to a knock-on effect on reinsurer’s investments or cascading defaults of other reinsurer’s counterparties and undermining its net assets.

3. STOP-LOSS REINSURANCE WITH COUNTERPARTY RISK

It is well known that stop-loss insurance is optimal in the standard expected utility framework (Arrow 1963). However, the analysis in Section 4 will show that in general stop-loss reinsurance is no longer optimal under counterparty risk. Instead, the buyer prefers a disappearing deductible structure (where there is “marginal overinsurance” in the sense that $\partial I(x)/\partial x > 1$). Note that there is no overall “overinsurance” since the optimal indemnity is constrained to satisfy $I(x) \leq x$. For legal and moral hazard reasons, practical contracts rarely allow for marginal overinsurance. Instead, it is observed that nearly all empirical reinsurance contracts are either of the stop-loss $I(x) = (x - d)_+$ or coinsurance with deductible $I(x) = \kappa(x - d)_+$ (with or without upper-limits) types. In this section we investigate the properties of the optimal stop-loss/coinsurance levels of such contracts.

The simplest nontrivial model where the recovery rate Θ takes on two values and the loss X has a Bernoulli distribution was considered by Doherty and Schlesinger (1990) and Mahul and Wright (2007). It was shown that already in this context most of the classical comparative statics need not hold. We extend these results to more elaborate loss distributions in Section 3.1. From a different direction, Cai and Tan (2007) studied optimal stop-loss levels (without default risk) when the buyer measures losses through the Value-at-Risk (VaR) and Conditional Tail Expectation (CTE) risk criteria. Such risk measures are attractive in the reinsurance context since they focus on catastrophic losses and are insensitive to modeling assumptions about small X . In Section 3.2 we extend their methods in the presence of counterparty risk.

3.1 CRRA Utility

Because in the model of Doherty and Schlesinger (1990) X is two-valued, it is not sufficiently rich to distinguish between different *functional* forms of insurance, such as co-insurance, stop-loss, etc. Accordingly, we have carried out extensive numerical experiments in the generalized model where X has either a discrete or a continuous distribution. For concreteness, we considered (1) a discrete uniform $X \sim DU(0, \bar{x})$, (2) binomial $X \sim Bin(\bar{x}, p)$, and (3) uniform $X \sim Unif(0, \bar{x})$ loss distributions with $\bar{x} = 10$. We assumed that $\Theta \in \{\theta_0, 1\}$, and the conditional default probability was taken as hyperbolic:

$$p(x) \stackrel{\Delta}{=} \mathbb{P}(\Theta = 1 | X = x) = \frac{c}{c + x}. \quad (2)$$

¹ As noted by Dana and Scarsini (2007), “stochastic increasingness” is an asymmetric dependence relationship unlike the concepts of “affiliation” or “association.”

The larger is X , the more likely the seller is to default. The utility function was the commonly used constant relative risk aversion (CRRA) power utility $U(x) = x^\gamma/\gamma$, $\gamma < 1$.

We postulated that the indemnity was of stop-loss type, $I(x) = (x - d)_+$ for some $d \geq 0$, and that the premium was computed via a linear safety loading ρ , $\pi = (1 + \rho)\mathbb{E}[\Theta I(X)]$. We then optimized over d the resulting expected utility:

$$\sup_{0 \leq d \leq \bar{x}} \int_0^{\bar{x}} [p(x)U(W - \pi - x \wedge d) + (1 - p(x))U(W - \pi - x + \theta_0(x - d)_+)]F_X(dx)$$

with

$$\pi \stackrel{\Delta}{=} \int_0^{\bar{x}} (1 + \rho)[p(x)(x - d)_+ + (1 - p(x))\theta_0(x - d)_+]F_X(dx).$$

For discrete X , the above are finite sums, while for continuous X , the chosen functional forms allow for closed-form expressions for π and $\mathbb{E}[U(W - \pi - X + \Theta(X - d)_+)]$. In either case, this is a simple one-dimensional optimization that can be straightforwardly implemented in any software package, yielding the optimal deductible level as a function of model parameters. Below we summarize the observed comparative statics.

Effect of the risk loading ρ . In the absence of default, if $\rho = 0$, then it is optimal for the buyer to buy full insurance (Mossin 1968). In the presence of default, the graph of optimal stop-loss with respect to ρ is increasing but starts at a positive value when $\rho = 0$. This confirms observations in Doherty and Schlesinger (1990) and Mahul and Wright (2007), who show that in the presence of default risk for the seller it might be optimal to buy less than full insurance even if there is no cost (pure premium). When ρ is sufficiently large, $d^* = \bar{x}$ and no insurance is purchased.

Effect of the risk aversion γ and initial wealth W . Recall that in the absence of default, the more risk-averse is the buyer, the lower the stop-loss (see, for example, Schlesinger 1981). This monotonicity is not always observed in the case when there is a possible default. In particular, similar to Doherty and Schlesinger (1990) we find that for very risk-averse buyers, $\gamma \mapsto d^*(\gamma)$ is decreasing when γ sufficiently negative. Similarly, d^* may be nonmonotone in initial wealth W . We recall that for CRRA utility as W becomes large, the effective risk aversion with respect to a fixed risk X is decreasing, so this behavior is similar to the impact of γ .

Effect of recovery rate θ_0 and probability of default $p(x)$. Larger recovery rates alleviate the counterparty risk, and therefore intuitively we expect that $d(\theta_0)$ is decreasing. However, in fact, $d(\theta_0)$ is convex and increasing for θ_0 close to 1. This occurs because the relative risk tolerance (difference between $u'(W - \pi - x + I(x))$ and $u'(W - \pi - x + \theta_0 I(x))$) is decreasing in θ_0 , and therefore for large θ_0 , the buyer is paying increasing attention to the risk loading ρ compared to the counterparty risk. However, in all our studies larger $p(x)$ (smaller probability of default) does lower optimal deductible levels. Curiously, when $p(x) = c/c + x$ and $c \mapsto 0$, the probability of default converges to 1 but the optimal stop-loss remains strictly less than \bar{x} . Therefore, buying some insurance is desirable even when probability of performance is very small. As $p(x) \mapsto 1$, $\forall x$, counterparty risk disappears, and the optimal deductible converges to that in the absence of default.

3.1.1 Other Indemnity Structures

To further understand the structure of insurance with counterparty risk, we tried other policies, namely, (a) capped stop-loss, $I(x) = (x - d)_+ \wedge m$ for deductible d and maximum payout m , and (b) co-insurance above a deductible, $I(x) = \kappa(x - d)_+$ for $\kappa \in \mathbb{R}_+$. For each case we repeated the above studies by finding optimal (d^*, m^*) and (d^*, κ^*) , respectively. Our numerical experiments showed that for CRRA utility it was never optimal to impose a cap, that is, $m^* = \bar{x}$.

Conversely, we found that proportional marginal overinsurance ($\kappa^* > 1$) is often preferred over a straight deductible, which is consistent with the general results obtained in Section 4. Optimal overinsurance was generally mild, with $\kappa^* < 1.1$, for nearly all parameter combinations we tried. Compared to having a straight stop-loss, the possibility to overinsure allows one to *increase* the deductible level, thus giving enhanced focus on the largest losses that are most important for the buyer.

The results in this section are somewhat sensitive to the distribution of X and the utility function U , though the observed lack of monotonicity in most model parameters is rather general. We conclude that little can be said about optimal stop-loss levels in such a general case. One difficulty comes from the lack of explicit expression of the optimal stop-loss level in the presence of default. Another one stems from the nontrivial interaction between counterparty risk and the expected utility framework; accordingly in the next subsection we consider coherent risk measures that allow more transparent treatment of default risk.

3.2 Conditional Tail Expectation

In this subsection we consider a continuous loss distribution F_X . The CTE (also known as the conditional or tail VaR) is a well-known risk measure that satisfies the axioms of Artzner et al. (1999) and can be viewed as generalizing the popular VaR. We recall that for a continuous random variable X ,

$$\text{CTE}_\alpha(X) \triangleq \mathbb{E}[X|X > q_{1-\alpha}(X)] = \frac{1}{\alpha} \int_{q_{1-\alpha}(X)}^{\infty} S_X(z) dz,$$

where a $(1 - \alpha)$ -quantile $q_{1-\alpha}(X)$ of the risk X is defined for our purposes as

$$q_{1-\alpha}(X) = \inf\{x : S_X(x) \leq \alpha\}.$$

Recall also the VaR $\text{VaR}_\alpha(X) \equiv q_{1-\alpha}(X)$ measure of risk and the fact that the CTE can be viewed as the average of $\text{VaR}_\beta(X)$ over $0 \leq \beta \leq \alpha$. In this subsection we assume that the buyer measures his or her risk through the VaR/CTE of the retained losses but must also contend with counterparty risk. Because VaR/CTE are cash additive, $\text{CTE}_\alpha(X + a) = \text{CTE}_\alpha(X) + a$ for any $a \in \mathbb{R}$, the initial wealth is irrelevant, and the problem reduces to

$$\inf_{I(\cdot)} \{\text{CTE}_\alpha(X - \Theta I(X)) + \pi\}. \quad (3)$$

We continue to assume linear safety loading, $\pi = (1 + \rho)\mathbb{E}[\Theta I(X)]$, constant recovery rate, $\Theta \in \{\theta_0, 1\}$ with $0 \leq \theta_0 < 1$, and stop-loss contracts, $I(X) = X - d)_+$. In this setting (3) reduces to the one-dimensional optimization:

$$\inf_{d \geq 0} \{\text{CTE}_\alpha((1 - \Theta)X + \Theta(X \wedge d)) + (1 + \rho)\mathbb{E}[\Theta(X - d)_+]\}. \quad (4)$$

Problem 4 was considered in Cai and Tan (2007) for the case without counterparty risk, $\Theta \equiv 1$. Cai and Tan showed that the optimal solution in that case is

$$\hat{d} \triangleq S_X^{-1}((1 + \rho)^{-1}), \quad (5)$$

if $(1 + \rho) < \alpha^{-1}$ and $\hat{d} = +\infty$ otherwise.

For a fixed sensitivity level α , define the two levels $\underline{a} < \bar{a}$ via

$$\int_{\underline{a}}^{\infty} (1 - p(x))F_X(dx) := \alpha; \quad \int_{\bar{a}}^{\infty} F_X(dx) := \alpha.$$

Note that \bar{a} is simply a $(1 - \alpha)$ -quantile of X . Depending on the deductible level d , three cases are possible regarding the retained loss $Z \triangleq X - \Theta(X - d)_+$:

1. $d < \underline{a}$. The right tail of Z is composed entirely from default losses above the deductible.
2. $\underline{a} < d < \bar{a}$. The right tail of Z is composed from the deductible and default losses above.

3. $d > \bar{a}$. The right tail of Z is composed from uncovered losses above \bar{a} , covered losses at the deductible level, and default losses.

Generally, we expect that $p(x)$ is close enough to 1 such that the unconditional probability of default $\int_0^\infty (1 - p(x))F_X(dx)$ is less than α ; in that case $\underline{a} = 0$ and the first case is ruled out.

For any deductible level d , define $V(d) = \text{VaR}_\alpha(X - \Theta(X - d)_+)$ and $C(d) = \text{CTE}_\alpha(X - \Theta(X - d)_+)$. Then we find

$$V(d) = \max(\underline{a}, \min(d, \bar{a})), \quad (6)$$

and using $R(x) \triangleq \{(1 - \theta_0)x + \theta_0 d\}(1 - p(x))$,

$$C(d) = \begin{cases} \frac{1}{\alpha} \int_{\underline{a}}^\infty R(x)F_X(dx), & d \leq \underline{a}; \\ \frac{1}{\alpha} \left\{ \int_d^\infty R(x)F_X(dx) + d \left[\alpha - \int_d^\infty (1 - p(x))F_X(dx) \right] \right\}, & \underline{a} < d < \bar{a}; \\ \frac{1}{\alpha} \left\{ \int_{\bar{a}}^d xF_X(dx) + \int_d^\infty \{dp(x) + R(x)\}F_X(dx) \right\}, & d \geq \bar{a}. \end{cases} \quad (7)$$

Moreover, the corresponding premium is given by

$$\pi(d) = (1 + \rho) \int_d^\infty \{(x - d)p(x) + \theta_0(x - d)(1 - p(x))\}F_X(dx). \quad (8)$$

The optimization problem is now to find d_* that minimizes $I_C(d) \triangleq C(d) + \pi(d)$ and $I_V(d) \triangleq V(d) + \pi(d)$, respectively. We observe that $\pi(\cdot)$ is a decreasing convex function of d , while $V(\cdot)$ is an increasing piecewise linear function. Moreover, $C(\cdot)$ is increasing linear on $[0, \underline{a}]$, increasing convex on $[\underline{a}, \bar{a}]$, and increasing concave on $[\bar{a}, \infty)$. While $C(\cdot)$ is defined piecewise, it can be checked that it is in fact continuously differentiable on \mathbb{R}_+ .

The next proposition gives the solution for the CTE risk criterion.

Proposition 3.1

If $(1 + \rho) < \alpha^{-1}$, then $C(d) + \pi(d)$ has a unique global minimum at $\underline{a} < d_*^C < \bar{a}$ given implicitly by

$$1 = \int_{d_*^C}^\infty \left\{ \left[\frac{1 - \theta_0}{\alpha} + \theta_0(1 + \rho) \right] (1 - p(x)) + (1 + \rho)p(x) \right\} F_X(dx). \quad (9)$$

Otherwise, no insurance is purchased and $d_*^C = +\infty$.

For the VaR criterion, $V(d)$ is not differentiable at \underline{a} or \bar{a} , leading to an additional corner case. We again verify that when $(1 + \rho) > \alpha^{-1}$, $V(d) + \pi(d)$ is decreasing throughout, and hence $d_*^V = \infty$. Otherwise, we have the following proposition.

Proposition 3.2

Suppose $(1 + \rho) < \alpha^{-1}$. Then

1. If $\rho < \alpha(1 - \theta_0)/1 - \alpha(1 - \theta_0)$, then $d_*^V = \underline{a}$.
2. Else, either $d_*^V = +\infty$ or $\underline{a} < d_*^V < \bar{a}$, and d_*^V solves

$$1 = (1 + \rho) \int_{d_*^V}^\infty \{p(x) + \theta_0(1 - p(x))\}F_X(dx). \quad (10)$$

As immediate corollaries from (9) and (10) we have the following comparative statistics.

- If the risk loading ρ increases, then d_* increases; as in the classical case, large risk loading increases the deductible. Furthermore, when $\rho = 0$, $d_*^C > 0$ is strictly positive and full insurance is no longer optimal.
- If the recovery rate θ_0 increases, or if the compliance probability $p(x)$ increases uniformly on any $[x_0, \infty)$, then d_*^C decreases while d_*^V increases. The latter phenomenon contradicts anecdotal evidence that higher counterparty risk discourages reinsurance buyers. This counterintuitive effect of recovery rate θ_0 and default probability $p(x)$ on d_*^V may suggest that VaR is *not* a good risk measure to use when analyzing defaultable insurance demand.
- If the sensitivity level α decreases, then d_*^C unambiguously increases, so smaller α makes the buyer more sensitive to default risk and reduces demand for reinsurance. However, if $\rho > \alpha(1 - \theta_0)/(1 - \alpha(1 - \theta_0))$, then d_*^V is given in (10) and is not impacted by changing the VaR level α (else, $d_*^V = \underline{a}$ increases).

The next proposition, proven in Appendix A.3, compares the optimal deductible levels.

Proposition 3.3

Suppose that d_*^C and d_*^V are finite. Then we have $d_*^C \geq \max(d_*^V, \hat{d})$.

Thus, compared to no-default risk or when using the coarser VaR criterion, a CTE-minimizing buyer will purchase less stop-loss insurance. Note that it is possible that $d_*^C < \infty$ while $d_*^V = +\infty$. To see more transparently the relationship between d_*^C , d_*^V , and \hat{d} , consider the case where $p(x) = p_0$ is a constant, independent of x . Direct computation shows that in such a setting, optimal deductibles are given by the quantiles

$$d_*^C = S_X^{-1} \left(\frac{1}{1 + \rho + (1 - p_0)(1 - \theta_0)(\alpha^{-1} - (1 + \rho))} \right),$$

$$d_*^V = S_X^{-1} \left(\frac{1}{(1 + \rho)(1 - (1 - p_0)(1 - \theta_0))} \right),$$

so that the second term in the denominators is precisely the difference with $\hat{d} = S_X^{-1}(1/(1 + \rho))$. We also note that in this case $d_*^V < \hat{d} < d_*^C$.

3.2.1 Illustration

To numerically illustrate the above results, we consider $X \sim \text{Exp}(m)$ and $p(x) = \mathbb{P}(\Theta = 1 | X = x) = p_0 + (1 - p_0)e^{-\mu x}$. In that case the integrals in (9) and (10) can be evaluated explicitly, and we find

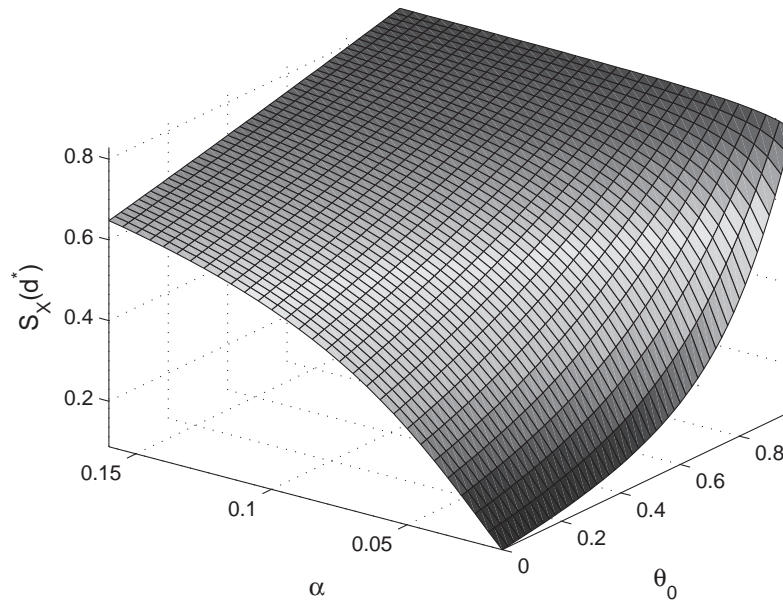
$$1 = (1 + \rho)(\theta_0(1 - p_0) + p_0)e^{-md^V} + (1 + \rho)(1 - \theta_0)(1 - p_0) \frac{m}{m + \mu} e^{-(m+\mu)d^V}, \quad (11)$$

$$1 = \left\{ \left(\theta_0(1 + \rho) + \frac{1 - \theta_0}{\alpha} \right) (1 - p_0) + (1 + \rho)p_0 \right\} e^{-md^C} + \left(1 + \rho - \frac{1}{\alpha} \right) (1 - \theta_0)(1 - p_0) \frac{m}{m + \mu} e^{-(m+\mu)d^C}. \quad (12)$$

For comparison we recall from Cai and Tan (2007) that $\hat{d} = S_X^{-1}((1 + \rho)^{-1}) = 1/m \log((1 + \rho))$ or $1 = (1 + \rho)e^{-m\hat{d}}$ is optimal without default risk. This is exactly a special case of (12) when $p_0 = 1$.

Figure 1 shows the fraction (quantile) of all losses covered by the reinsurance as a function of the recovery rate θ_0 and the loss sensitivity α of the reinsurance buyer. In the absence of default, $S_X(\hat{d}) = (1 + \rho)^{-1}$ of losses are above the deductible and hence subject to reinsurance. Lower recovery rates or increased tail sensitivity reduce reinsurance demand, that is, lower $S_X(d_*^C)$. We observe in Figure 1 that the impact is most dramatic when $\alpha < 0.05$.

Figure 1
Comparative Statics for Optimal Deductible Level for CTE Risk Metric



Note: In this example $X \sim \text{Exp}(m)$, $p_0 = 0$, $p(x) = e^{-\mu x}$, $\rho = 0.2$ with $m = 0.01$ and $\mu = 0.0005$. We show the percent of losses that are above the deductible, $S_X(d)$, as a function of the recovery rate $0 \leq \theta_0 < 1$ and CTE level $0.01 \leq \alpha \leq 0.15$. In the absence of default, $S_X(d) = 1/(1 + \rho)$, which is the value in the upper-right corner of the figure.

3.3 Other Insurance Structures

We may also consider the case where there is co-insurance above the deductible, $I(x) = \kappa(x - d)_+$, with $0 < \kappa \leq 1$. For brevity, we describe only the resulting changes in the main case (2) of the CTE criterion. For any $b > d$, the total probability that losses exceed b is

$$\ell(b) := \int_{b-\kappa\theta_0 d/1-\kappa\theta_0}^{\infty} (1 - p(x))F_X(dx) + \int_{b-\kappa d/1-\kappa}^{\infty} p(x)F_X(dx),$$

which correspond to default on the event $\{(1 - \kappa\theta_0)X + \kappa\theta_0 d > b\}$ or seller compliance on the event $\{(1 - \kappa)X + \kappa d > b\}$. Therefore, let $b(d; \kappa)$ be such that $\ell(b(d; \kappa)) = \alpha$. Then the second line of (7) becomes

$$C(d; \kappa) = \frac{1}{\alpha} \left\{ \int_{b(d; \kappa) - \kappa\theta_0 d/1 - \kappa\theta_0}^{\infty} \{(1 - \theta_0\kappa)x + \theta_0\kappa d\}(1 - p(x))F_X(dx) + \int_{b(d; \kappa) - \kappa d/1 - \kappa}^{\infty} (\kappa d + (1 - \kappa)x)p(x)F_X(dx) \right\},$$

with

$$\pi(d; \kappa) = (1 + \rho) \int_d^{\infty} \{\kappa(x - d)p(x) + \theta_0\kappa(x - d)(1 - p(x))\}F_X(dx).$$

The total quantity to be minimized is now $I_c(d; \kappa) = C(d; \kappa) + \pi(d; \kappa)$. Numerical experiments suggest that $I_c(d; \kappa)$ is decreasing in κ for $\kappa \in (0, 1)$ and the buyer would like to *overinsure*. If $\kappa > 1$, then retained losses are not monotone in original losses, and finding the quantiles of $X - \Theta I(X)$ becomes even more complicated. These observations highlight the fact that the straight stop-loss indemnity is

not optimal in the presence of counterparty risk. In the next section we derive general properties of the shape of the optimal reinsurance contract.

4. THEORY

We first give some general properties of the optimal indemnity solving Problem A and discuss some special cases for a better understanding of the impact of counterparty risk on the optimal risk sharing in the reinsurance market.

4.1 Characterization of the Optimal Reinsurance Policy

To solve Problem A we adopt the variational formulation and apply conditioning. Given a premium π and $\lambda > 0$, we first solve the following auxiliary problem for the optimal indemnity using a state-by-state optimization:

$$\max_{0 \leq y \leq x} \{ \mathbb{E}[U(W - \pi - x + \Theta y) | X = x] - \lambda y \mathbb{E}[\Theta | X = x] \}. \quad (13)$$

Thanks to the concavity of U , the objective function to be maximized is a concave function of y . The optimum over the real line is characterized by the zero of the first derivative with respect to y . Denote by $I_\lambda^*(x)$ the optimal solution, then $I_\lambda^*(x)$ is equal to

$$I_\lambda^*(x) = \max(\min(y_{\lambda,x}, x), 0), \quad (14)$$

where $y_{\lambda,x}$ is the unique solution to

$$\mathbb{E}[\Theta U'(W - \pi - x + \Theta y_{\lambda,x}) | X = x] - \lambda \mathbb{E}[\Theta | X = x] = 0. \quad (15)$$

Note that (15) has a unique solution because of the Inada conditions given in Section 2.1 and because U is strictly concave, and therefore the first derivative is strictly decreasing and continuous with respect to y .

Proposition 4.1

If there exists $\lambda > 0$ such that $I_\lambda^*(x)$ given by (14) satisfies the constraint $\mathbb{E}[\Theta I_\lambda^*(X)] = K$, then $I_\lambda^*(x)$ is an optimal solution to Problem A.

The dependence structure between X and Θ has an important impact on the optimal risk sharing given by (14), allowing for complex shapes of $I_\lambda^*(x)$ and possibly decreasing indemnities. This is very different from existing literature on optimal insurance design. This particularity stems from the fact that when $\Theta \downarrow_{st} X$, the function

$$(x, y) \mapsto \mathbb{E}[U(W - \pi - x + \Theta y) | X = x]$$

is generally *not* supermodular,² and therefore we cannot guarantee that there exists a nondecreasing optimal solution of Problem A using supermodularity (as was done, for example, by Bernard and Tian [2009] in other contexts). This is a major difference between the presence of an additive and a multiplicative background risk. In the presence of an additive background risk Y , Dana and Scarsini (2007) proved that

$$(x, y) \mapsto \mathbb{E}[U(W - \pi - x - Y + y) | X = x]$$

is supermodular when $Y \downarrow_{st} X$ and used it to prove that the optimal indemnity is nondecreasing.

Such a supermodularity property does not hold with a multiplicative background risk. Moreover, since $y_{\lambda,x}$ is defined implicitly, its shape is not straightforward to understand. Therefore, we now review

² That is, the cross-derivative with respect to x and y may be negative. See definition 3.1 of a supermodular function (when it is C^2) by Dana and Scarsini (2007, p.157).

special cases. First, we study the case when the default risk of the seller is independent of the loss incurred by the buyer.

4.2 Independence of X and Θ

Proposition 4.2

Under the assumption of independence between Θ and X and when $K \in (0, \mathbb{E}[X])$, the optimal solution to Problem A exists, is unique, and is nondecreasing with respect to X . Moreover, $\partial y_{\lambda,x}/\partial x \geq 1$, where $y_{\lambda,x}$ is implicitly defined by (15).

In the absence of default risk, $\Theta \equiv 1$ and the optimal indemnity is a stop-loss. In this case, when the indemnity is nonzero, its slope is equal to one. In the presence of an *independent* default risk, the marginal insurance rate $\partial I/\partial x$ is bigger than one, and there is therefore a higher demand for insurance above the deductible. This is consistent with Cardenas and Mahul (2006, Corollary 1). We will see in the following that a dependence structure between the loss of the buyer and the default of the seller fundamentally changes the optimal risk sharing, and a nondecreasing indemnity is no longer necessarily optimal.

4.3 Bernoulli Recovery Rate

We now assume that Θ and X are dependent and that the recovery rate follows a Bernoulli distribution. Therefore we need only to study for $x \geq 0$,

$$p(x) \triangleq \mathbb{E}[\Theta = 1 | X = x].$$

We assume that p is not constant, which ensures that Θ and X are not independent. Due to the assumption on the dependence structure (see Section 2.2), p is nonincreasing with respect to x . In other words, it is more likely to observe a default of the seller when there is a big loss x .

Assume first that p is nonincreasing and differentiable.

Proposition 4.3

Assume that Θ can take only two possible values 1 and $\theta_0 \in [0, 1)$ and that $p(x)$ takes values in $(0, 1)$.

- When $\theta_0 = 0$ (no recovery), the optimum is a stop-loss contract $I_\lambda^*(x) = (x - W + \pi + [U']^{-1}(\lambda))^+$ for $\lambda > 0$.
- When $0 < \theta_0 < 1$,

$$I_\lambda^*(x) = \begin{cases} 0 & \text{if } x \leq W - \pi - [U']^{-1}(\lambda) \\ \min(y_{\lambda,x}, x) & \text{if } x > W - \pi - [U']^{-1}(\lambda) \end{cases} \quad (16)$$

and

$$A(x) \frac{\partial y_{\lambda,x}}{\partial x} = 1 + \frac{p'(x)}{p^2(x)} \theta_0 \frac{U'(W - \pi - x + \theta_0 y_{\lambda,x}) - \lambda}{U''(W - \pi - x + y_{\lambda,x})} + \frac{1 - p(x)}{p(x)} \theta_0 \frac{U''(W - \pi - x + \theta_0 y_{\lambda,x})}{U''(W - \pi - x + y_{\lambda,x})}, \quad (17)$$

with

$$A(x) = 1 + \frac{\theta_0^2 U''(W - \pi - x + \theta_0 y_{\lambda,x})(1 - p(x))}{p(x) U''(W - \pi - x + y_{\lambda,x})} \geq 1 \quad (18)$$

and where $\partial y_{\lambda,x}/\partial x$ at the deductible $x = W - \pi - [U']^{-1}(\lambda)$ is strictly greater than 1. Locally the optimal indemnity exceeds the stop-loss contract at the deductible level.

Note that the case $\theta_0 = 1$ corresponds to the no-default case studied by Arrow (1963) for which the stop-loss contract is optimal.

In general, it is not clear from Proposition 4.3 whether I_λ^* is nondecreasing. However, there are several sufficient conditions to ensure that the optimal solution is such that I_λ^* is locally nondecreasing, as given in the following corollary.

Corollary 4.1

The optimal indemnity is locally nondecreasing in the neighborhood of x_0 , that is, $\partial y_{\lambda,x}/\partial x|_{x=x_0} \geq 0$, if one of the following conditions is true:

- The probability of default is insensitive to local changes in loss levels, $p'(x_0) = 0$.
- $p(x_0) \neq 0$ and the following inequality holds:

$$-\frac{U''(W - \pi - x_0 + y_{\lambda,x_0})}{U'(W - \pi - x_0 + y_{\lambda,x_0})} > \frac{p'(x_0)}{p(x_0)}. \quad (19)$$

- The map $t \mapsto \varpi(t) \triangleq tU'(W - \pi - x_0 + ty_{\lambda,x_0})$ is such that

$$\varpi'(t) \leq 0 \text{ on } [\theta_0, 1]. \quad (20)$$

Let us give an example. We assume an exponential CARA utility $U(x) = -e^{-\gamma x}$ and hyperbolic no-default probability $p(x) = c/(c + x)$. Condition (19) reduces to $\gamma > (c + x)^{-1}$ and is therefore satisfied for large losses $x > \gamma^{-1} - c$. So for large x , the optimal contract is *increasing*. Rearranging the expression for the derivative of $\varpi(t)$, we find that Condition (20) is equivalent to

$$\varpi'(t) < 0 \Leftrightarrow -\frac{U''(W - \pi - x + ty_{\lambda,x})}{U'(W - \pi - x + ty_{\lambda,x})} > \frac{1}{\theta_0 y_{\lambda,x}}, \quad (21)$$

which again reduces to a statement about the absolute risk aversion of U . For CARA risk preferences, (21) becomes $\gamma > (\theta_0 y_{\lambda,x})^{-1}$.

A complementary result is given in the following proposition, which relies on the supermodularity property.

Proposition 4.4

Assume that Θ follows a Bernoulli distribution. The optimal solution to Problem A is nondecreasing if for all $y \in (0, \bar{x})$

$$x \mapsto \theta_0 U'(W - \pi - x + \theta_0 y) + p(x)(U'(W - \pi - x + y) - \theta_0 U'(W - \pi - x + \theta_0 y))$$

is nondecreasing.

The condition in Proposition 4.4 does not directly involve the implicit $y_{\lambda,x}$ and can be easily verified (numerically or analytically). For instance, for CARA risk preferences $U(x) = -e^{-\gamma x}$ and exponential $p(x) = e^{-\mu x}$ it reduces to $\gamma > \mu(1 - \theta_0)$. This is a mild assumption about the risk aversion of the buyer, since typically we would have $\gamma \gg \mu$.

Assume now that p can take only two values, 1 and $p_0 < 1$. Since p is nonincreasing, there exists x_0 , such that $p(x) = 1$ for all $x < x_0$, and $p(x) = p_0 \forall x \geq x_0$.

Proposition 4.5

Suppose $p(x)$ takes only two values, then

- When $\theta_0 = 0$ or $\theta_0 = 1$, the optimal indemnity is a stop-loss;
- When $0 < \theta_0 < 1$, the optimal indemnity is nondecreasing and $\partial I^*/\partial x \geq 1$.

Thus, with partial recovery, optimal indemnity requires marginal overinsurance, that is, increase in coverage that is strictly higher than the increase in loss level.

REMARK 4.1

The case $\theta_0 = 0$ is the full-default model considered by Cummins and Mahul (2003). It is consistent with their results, namely, that stop-loss insurance is optimal in the presence of a total default risk when the insurance buyer and the insurance seller have the same beliefs about the default probability.

REMARK 4.2

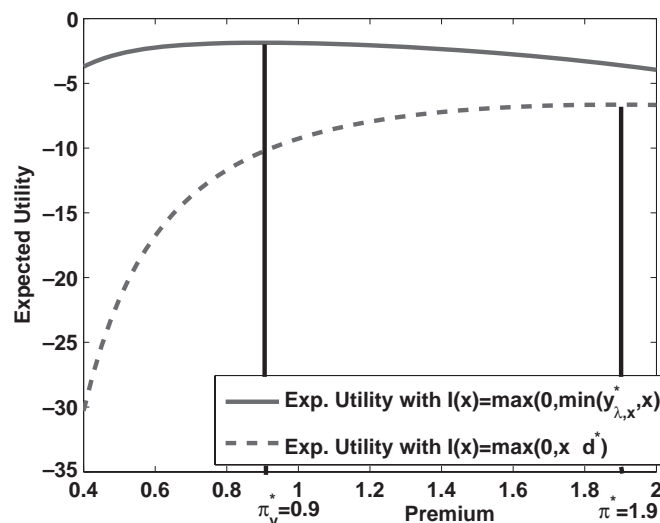
Under the assumptions of Proposition 4.5 there is one jump in $p(x)$ at x_0 so that Proposition 4.3 does not apply directly. As shown in the proof of Proposition 4.5 the effect of this jump is a strict increase in the slope $\partial y_{\lambda,x}/\partial x$ at the level x_0 . If it is optimal to buy insurance around the value x_0 where a jump is observed, a strict increase in the marginal rate of insurance will be observed. Precisely, if x_0 is larger than the stop-loss level of the optimal indemnity, then the optimal contract is first a linear stop-loss with slope 1, and when x_0 is reached, the marginal demand for insurance becomes strictly bigger.

4.4 Illustration

Given π , it is possible to solve the implicit equation (15) to derive $y_{\lambda,x}$ with λ , so that the premium constraint is met. Similar to Figure 1, let us consider an exponential loss $X \sim \text{Exp}(m)$, with probability of recovery $p(x) = c/c + x$. Figure 2 compares the expected utility as a function of the premium π spent to buy the reinsurance contract for the optimal indemnity given in (14) and a stop-loss indemnity investigated in Section 3. We observe that the resulting reinsurance premia are very different and the associated loss in expected utility is significant. If the buyer can choose the shape of the reinsurance indemnity, then he or she needs to spend much less (and can gain much more utility) than if he or she has to insure through a deductible indemnity.

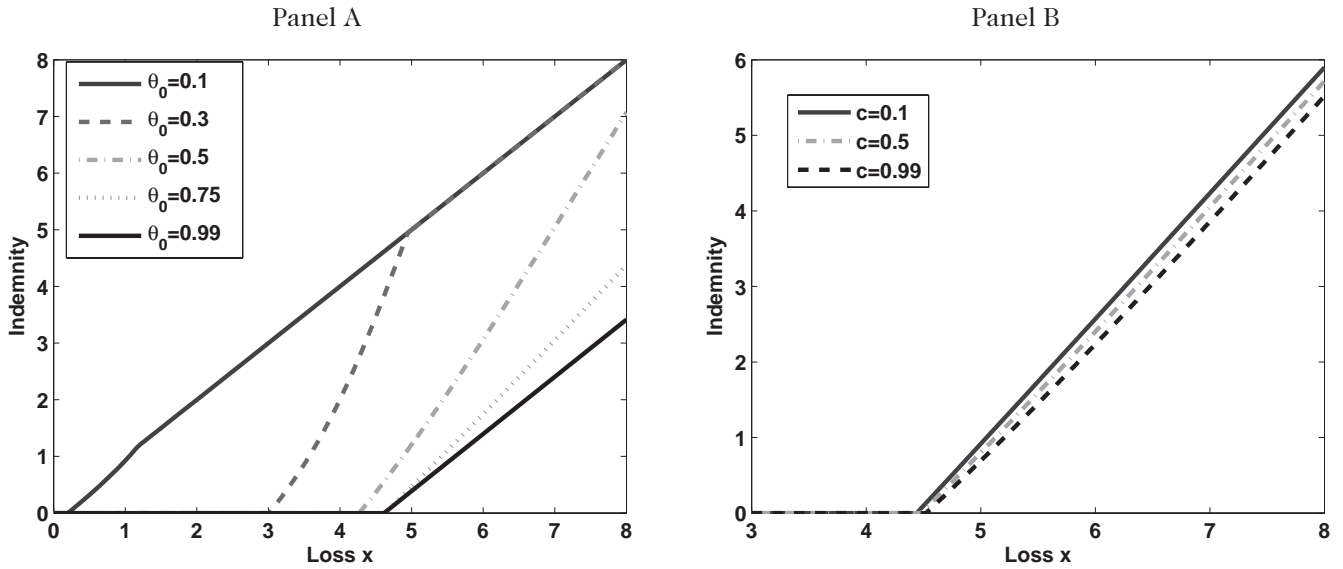
In Figure 3 we further assume that $\pi = 1$ and study the effect of an increase in default risk on the optimal shape of the contract. More precisely, we investigate the effect of varying the recovery rate θ_0 and the probability-of-default parameter c . Recalling the definition of $p(x)$ in (2), higher c increases the probability of seller performance. From Figure 3, we find that as $\theta_0 \uparrow 1$, the marginal insurance rate goes to 1, and the optimal indemnity is a stop-loss contract as in the standard framework. At the same time, lower probability of default leads to a lower marginal insurance rate above the deductible.

Figure 2
Expected Utility as a Function of the Premium π



Note: The solid line corresponds to the expected utility when the indemnity has the optimal shape given by (14). The dotted line corresponds to the expected utility when the indemnity is a stop loss and the deductible level is such that the premium constraint is met. The parameters are $m = 0.3$, $c = 0.9$, $\rho = 0.2$, $W = 5$ with recovery rate $\theta_0 = 0.6$, with CARA utility $U(x) = -e^{-\gamma x}$ with $\gamma = 2.75$.

Figure 3
Effect of Counterparty Risk on the Shape of the Optimal Indemnity



Note: Panel A displays the effect of the recovery rate θ_0 on the shape of the optimal indemnity. Panel B displays the effect of the parameter c of the conditional probability of the seller to be solvent. Parameters are the same as in Figure 2 with $\pi = 1$.

5. PRESENCE OF ASYMMETRIC INFORMATION

Problem A studied so far assumes that the buyer and the seller have the same beliefs about the default risk of the seller. However, it is commonly accepted that the reinsurance seller is more optimistic than the insurance buyer about his or her own default risk. Indeed, buyers often believe that sellers underestimate their likelihood of nonperformance. In the extreme case where the seller ignores the default risk to calculate his or her expected cost, the participation constraint becomes

$$C \leq \mathbb{E}[\pi - I(X)].$$

Equivalently, one has

$$\mathbb{E}[I(X)] \leq K, \quad (22)$$

where $K = \pi - C$ instead of (1). Similarly as before, we will use a linear loading factor ρ , such that $K = \pi/(1 + \rho)$. In this case the optimal reinsurance contract solves the following optimization problem:

$$\max_{I, \pi} \{ \mathbb{E}[U(W - \pi - X + \Theta I(X))] \} \quad (\text{AS})$$

subject to

$$\begin{cases} 0 \leq I(x) \leq x, \\ \mathbb{E}[I(X)] \leq \frac{\pi}{1 + \rho}. \end{cases}$$

REMARK 5.1

In special cases, Problem AS is identical to Problem A studied so far. For example, when $\Theta \equiv 1$ (no counterparty risk), or when Θ and X are independent (because then $\mathbb{E}[\Theta I(X)] = \mathbb{E}[\Theta] \mathbb{E}[I(X)]$, and the constraint (22) can therefore be rewritten in a similar way to (1) but with a different constant K).

In the presence of a dependency structure between the loss X incurred by the buyer and the default risk of the seller (measured by the recovery rate Θ), the solutions to Problems A and AS can be very

different. Even so, in the literature on optimal insurance design under default risk, the optimization problem is usually set as in (AS); see, for example, Doherty and Schlesinger (1990) or Cummins and Mahul (2003).

5.1 Solving Problem AS

In the general case, the optimal solution is defined implicitly by

$$\mathbb{E}[\Theta U'(W - \pi - x + \Theta z_{\lambda, x}) | X = x] - \lambda = 0, \quad (23)$$

where the Lagrange multiplier λ is such that the participation constraint of the seller is satisfied. We now consider some further special cases in parallel to Section 4.3.

Proposition 5.1

Assume that $p(x)$ is nonincreasing, differentiable, and takes value in $(0, 1)$. Assume also that the recovery rate Θ is Bernoulli with $\theta_0 \in [0, 1)$.

When $\theta_0 = 0$, the optimal solution to Problem AS is not a stop-loss contract anymore, and it is given by

$$\max \left(0, x - W + \pi + [U']^{-1} \left(\frac{\lambda}{p(x)} \right) \right).$$

When $\theta_0 \in (0, 1)$, the optimal solution to Problem AS is given by

$$x \mapsto I_{\lambda}^*(x) = \max(\min(z_{\lambda, x}, x), 0), \quad (24)$$

where

$$\frac{\partial z_{\lambda, x}}{\partial x} A(x) = 1 - \frac{p'(x)}{p^2(x)} \frac{\lambda - \theta_0 U'(W - \pi - x + \theta_0 z_{\lambda, x})}{U''(W - \pi - x + z_{\lambda, x})} + \frac{1 - p(x)}{p(x)} \theta_0 \frac{U''(W - \pi - x + \theta_0 z_{\lambda, x})}{U''(W - \pi - x + z_{\lambda, x})}, \quad (25)$$

with $A(x)$ given by (18).

Note that when $p(x)$ can take only two values, 1 and $p_0 < 1$, the results are similar to the ones obtained in Proposition 4.5.

REMARK 5.2

We can also redo the problem of stop-loss reinsurance with the CTE risk criterion from Section 3.2. If the resulting optimal stop-loss level \tilde{d} is finite, then it is implicitly given through

$$1 =: \int_{\tilde{d}}^{\infty} \left\{ (1 + \rho) + \frac{1}{\alpha} (1 - \theta_0)(1 - p(x)) \right\} F_X(dx).$$

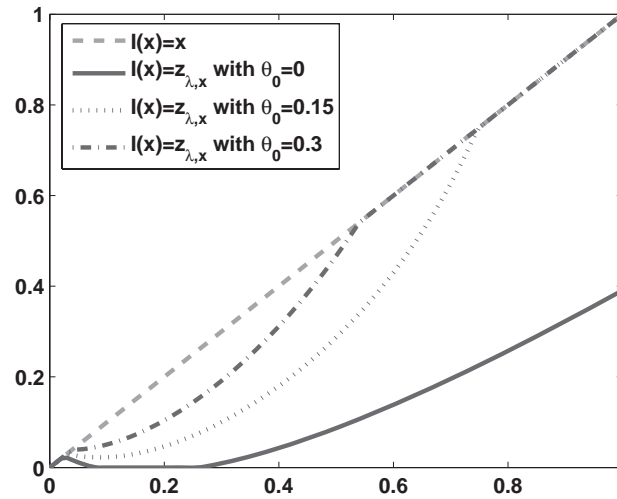
Comparing with (9), it is easy to check that $\tilde{d} > d^*$ so that demand for insurance is intuitively lower if counterparty risk is not fairly priced. Moreover, if $\theta_0 \alpha^{-1} < (1 + \rho)$, then depending on other model parameters it is possible that $\tilde{d} = +\infty$, and no insurance is purchased at all.

5.2 Comparison between Problems A and AS

Recall that the only difference between Problems A and AS is that the insurance buyer takes into account his or her own default risk in its participation constraint. We now make a few remarks about the difference of the respective optimal risk-sharing arrangements.

First, recall that if $\theta_0 = 0$ (case of a total default), then the optimal solution to Problem A is a stop-loss contract. However, the optimal indemnity for Problem AS is given as

Figure 4
Optimal Indemnities with Asymmetric Default Risk

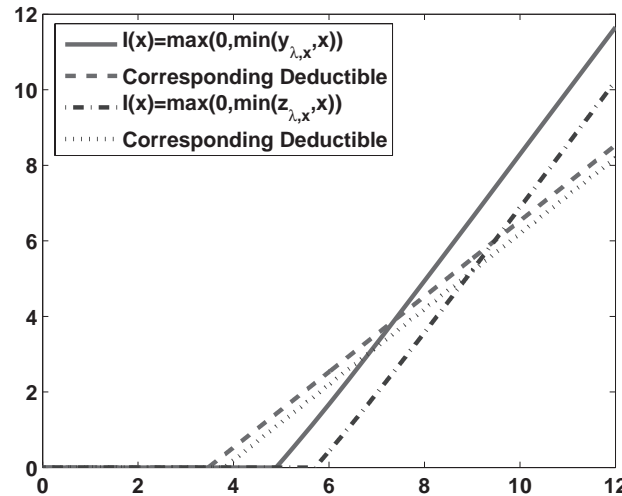


Note: The dashed line corresponds to $x \mapsto x$; the other lines correspond to $x \mapsto \max(\min(x, z_{\lambda,x}), 0)$ with different recovery rates θ_0 .

$$I_{\lambda}^*(x) = \max \left(0, \min \left(x - W + \pi + [U']^{-1} \left(\frac{\lambda}{p(x)} \right), x \right) \right),$$

and this function of x may take very complex shapes. As an illustrative example, suppose that $W = 3$, $\pi = 1$, $\lambda = 0.01$, $c = 0.2$, and $p(x) = c/(c + x)$ with CARA utility $U(x) = -e^{-\gamma x}$ with $\gamma = 2.75$. Note that the optimal indemnity shape is independent of the distribution of X . For total default $\theta_0 = 0$, Figure 4 shows that the optimal indemnity is then first equal to full insurance, then is decreasing and

Figure 5
Numerical Comparison of Problems A and AS



Note: The solid line corresponds to the optimal indemnity $\max(0, \min(y_{\lambda,x}, x))$ with λ such that the premium level is optimal equal to 0.9. The dotted line is then the corresponding deductible that can be bought with premium 0.9. In dotted-hatched it is the optimal indemnity $\max(0, \min(z_{\lambda,x}, x))$ with λ such that the premium level is optimal equal to 1.275. The hatched line is the corresponding deductible when the seller ignores its default risk to calculate the premium. Parameters are the same as in Figure 2.

equal to $\mathfrak{s}_{\lambda, x}$, then is equal to 0, and finally is equal to $\mathfrak{s}_{\lambda, x}$. For $\theta_0 > 0$ other indemnity shapes may be obtained; see Figure 4. Even though the above parameters have been chosen specifically for this case, this example shows how complex and unrealistic the optimal insurance contract can be.

In general, like in Section 4.4, we may solve the implicit equation (23) to derive $\mathfrak{s}_{\lambda, x}$ and then find λ so that the premium constraint is met. Figure 5 compares the optimal indemnities of Problems A and AS for $\theta_0 > 0$. When the premium is not fairly priced (that is, the seller ignores his or her own default risk and therefore overestimates the premium), the optimal premium level increases. For the parameters of Figure 2, it is equal to 1.275 instead of the optimal premium level of 0.9 found in Section 4.4 when the premium incorporates the seller's default risk. As shown in Figure 5, more coverage is purchased when reinsurance is fairly priced. Note that both optimal indemnities (y^* and \mathfrak{s}^*) involve marginal overinsurance above the deductible level.

6. CONCLUSION

Given the increased visibility of default risk in insurance, qualitative investigation of counterparty default on optimal reinsurance is a highly topical subject for the wider actuarial community. Our results extend the standard theory of insurance design to take into account systemic credit risk by introducing a multiplicative loss-dependent background risk. Our theoretical analysis highlights the complexity of optimal risk sharing in practice where one indeed observes both counterparty risk and asymmetric information.

Reinsurance becomes unreliable in the presence of counterparty risk. We show two important effects: namely, that it increases the reinsurance demand in the tail (the optimal shape involves marginal overinsurance above a stop-loss level), but at the same time decreases the optimal premium level (that is, the amount spent by the buyer in the reinsurance market). Overall, we believe our analysis demonstrates that the classical expected utility framework is intractable and therefore inadequate in studying risk sharing with counterparty risk as currently needed by practitioners. It remains an open question which alternative frameworks are better suited for this task. While we have shown that risk measures such as CTE can be more tractable, their cash additivity might be practically undesirable, completely removing buyer capital reserves from the model.

The model in this paper has been static, that is, one-period. Given that most reinsurance contracts are signed for several years, it is of interest to further analyze dynamic multiperiod models. For instance, at the height of the credit freeze in 2008, there was discussion of reinsurance contracts with the premium tied to the credit rating of the seller. With multiple periods it would also be possible to set up vested reinsurance accounts, so that (a portion of) the premia paid so far is guaranteed to be available once a claim is filed.

APPENDIX

PROOFS

A.1 PROOF OF PROPOSITION 3.1

PROOF

Direct computation shows that $C'(\bar{a}) > \pi'(\bar{a})$ if and only if $(1 + \rho) < \alpha^{-1}$. In the former case, it also follows that $C''(x) < \pi''(x)$ for $x > \bar{a}$, so that $C(d) + \pi(d)$ is increasing and concave on (\bar{a}, ∞) . Moreover, $C'(\underline{a}) < \pi'(\underline{a})$, so that $C(d) + \pi(d)$ is decreasing on $(0, \underline{a})$. It immediately follows that the global minimum of I_C is on $[\underline{a}, \bar{a}]$, and some algebra leads to the first-order condition (9).

In the second case, it follows that I_C is decreasing on $(0, +\infty)$, and its global minimum is at $+\infty$. Note that when $1 + \rho = \alpha^{-1}$, then I_C is constant on $(\bar{a}, +\infty)$ and so does not have a unique minimum. In that case we may again take by convention $d_C^* = +\infty$. \square

A.2 PROOF OF PROPOSITION 3.2

PROOF

A simple check shows that $V(d) + \pi(d)$ is always decreasing on $[0, \underline{a}]$. If ρ is small enough, namely, as in case (1), then we find that $V'(\underline{a}+) + \pi'(\underline{a}) > 0$, so that I_V is increasing on $[\underline{a}, \bar{a})$, and hence \underline{a} is the candidate for global minimum. In either case, I_V is convex on $[\underline{a}, \bar{a}]$ and $V'(\bar{a}-) + \pi'(\bar{a}-) > 0$, so there is a unique local minimum in (\underline{a}, \bar{a}) that must satisfy the first-order condition of (10). This local minimum may or may not be the global minimum. Indeed, I_V is decreasing on (\bar{a}, ∞) , so that $+\infty$ is another candidate. The answer depends on comparing

$$I_V(d^*) = d^* + \pi(d^*) \text{ vs. } I_V(+\infty) = \bar{a},$$

which in turn depends on ρ . We thus observe a phase transition as ρ increases: for small ρ , the local minimum on (\underline{a}, \bar{a}) is the global minimum and $d^* < \bar{a} < \infty$; for ρ close to $\alpha^{-1} - 1$ the global minimum is at $+\infty$, with an abrupt transition at some critical $\tilde{\rho}$ (given only implicitly). \square

A.3 PROOF OF PROPOSITION 3.3

PROOF

If $d_V^* = +\infty$, there is nothing to prove. Otherwise, we note that on $[\underline{a}, \bar{a}]$,

$$C'(d) = 1 - \frac{1 - \theta_0}{\alpha} \int_d^\infty (1 - p(x)) F_X(dx) < 1 = V'(d), \quad (\text{A.1})$$

which implies that the global minimum of $C(d) + \pi(d)$ is to the right of the global minimum of $V(d) + \pi(d)$. Since d_V^* is decreasing in θ_0 and $\theta_0 = 1$ corresponds to no default, it follows that $d_V^* > d$. \square

A.4 PROOF OF PROPOSITION 4.1

PROOF

For any indemnity $I(X)$ satisfying $\mathbb{E}[\Theta I(X)] \leq K$ and $0 \leq I(X) \leq X$, we have by optimality of $I_\lambda^*(X)$ that for all $x \in (0, \bar{x})$

$$\begin{aligned} \mathbb{E}[U(W - \pi - x + \Theta I(x)) | X = x] - \lambda I(x) \mathbb{E}[\Theta | X = x] \\ \leq \mathbb{E}[U(W - \pi - x + \Theta I_\lambda^*(x)) | X = x] - \lambda I_\lambda^*(x) \mathbb{E}[\Theta | X = x]. \end{aligned} \quad (\text{A.2})$$

Integrating over x and using the fact that $\mathbb{E}[\Theta I_\lambda^*(X)] = K$ and $\mathbb{E}[\Theta I(X)] \leq K$, then

$$\mathbb{E}[U(W - \pi - X + \Theta I(X))] \leq \mathbb{E}[U(W - \pi - X + \Theta I_\lambda^*(X))],$$

which concludes the proof. \square

A.5 PROOF OF PROPOSITION 4.2

Denote by $I_\lambda^*(x) = \max(\min(y_{\lambda,x}, x), 0)$. To prove Proposition 4.2, we need two lemmas.

Lemma A.1

For $\lambda > 0$, when Θ and X are independent, the function $x \mapsto y_{\lambda,x}$ is nondecreasing.

PROOF

Let $x_1 < x_2$. Denote by $y_1 = y_{\lambda,x_1}$ and $y_2 = y_{\lambda,x_2}$. The function $h_1 : x \mapsto \mathbb{E}[\Theta U'(W - \pi - x + \Theta y_1) - \lambda \Theta]$ is increasing. Therefore $h_1(x_1) < h_1(x_2)$. Define also $h_2 : y \mapsto \mathbb{E}[\Theta U'(W - \pi - x_2 + \Theta y) - \lambda \Theta]$

(where the expectation sign refers to the expectation with respect to the random variable Θ). It is clear that h_2 is decreasing and $h_2(y_2) = 0$ (by definition of y_2) and $h_1(x_2) = h_2(y_1)$. Then

$$\forall y \geq y_2, \quad h_2(y) \leq h_2(y_2) = 0.$$

Since $h_2(y_1) = h_1(x_2) > h_1(x_1) = 0$, then $y_1 < y_2$. □

Lemma A.2

For $x \in [0, \bar{x}]$, when Θ and X are independent,

$$I_\lambda^*(x) = 0 \Leftrightarrow \lambda \geq \frac{\mathbb{E}[\Theta U'(W - \pi - x)]}{\mathbb{E}[\Theta]}, \quad (\text{A.3})$$

$$I_\lambda^*(x) = x \Leftrightarrow \lambda \leq \frac{\mathbb{E}[\Theta U'(W - \pi - (1 - \Theta)x)]}{\mathbb{E}[\Theta]}. \quad (\text{A.4})$$

PROOF

First, we note that the function $\lambda \mapsto y_{\lambda,x}$ is nonincreasing. Let $\lambda_1 < \lambda_2$. It is clear that $y_{\lambda_1,x} > y_{\lambda_2,x}$ because of the inequality

$$\lambda_1 = \frac{\mathbb{E}[\Theta U'(W - \pi - x + \Theta y_{\lambda_1,x})]}{\mathbb{E}[\Theta]} < \frac{\mathbb{E}[\Theta U'(W - \pi - x + \Theta y_{\lambda_2,x})]}{\mathbb{E}[\Theta]} = \lambda_2,$$

and the fact that U' is decreasing.

Note that $y_{\lambda,x} = 0$ if and only if $\lambda = \mathbb{E}[\Theta U'(W - \pi - x)]/\mathbb{E}[\Theta]$. By monotonicity of $\lambda \mapsto y_{\lambda,x}$, and because $I_\lambda^*(x) = \max(\min(y_{\lambda,x}, x), 0)$ we obtain (28).

Note that $y_{\lambda,x} = x$ if and only if $\lambda = \mathbb{E}[\Theta U'(W - \pi - (1 - \Theta)x)]/\mathbb{E}[\Theta]$. By monotonicity of $\lambda \mapsto y_{\lambda,x}$, and because $I_\lambda^*(x) = \max(\min(y_{\lambda,x}, x), 0)$, we obtain (29). □

In addition to these two lemmas, the proof of Proposition 4.2 uses the “nondecreasing rearrangement” of a random variable. We recall here its definition (see definition 3.3 of Dana and Scarsini [2007, p. 157] and further key properties in Carlier and Dana [2003]).

DEFINITION A.1

For a given measurable function $f : [0, \bar{x}] \mapsto [0, \bar{x}]$, there exists a unique nondecreasing function \tilde{f} such that

$$\forall x \in [0, \bar{x}], \quad \Pr\{f \leq x\} = \Pr\{\tilde{f} \leq x\}.$$

\tilde{f} is called the nondecreasing rearrangement of f .

We will make use of the following important property (which is a variant of the Hardy Littlewood inequality).

Lemma A.3 (3.6 of Dana and Scarsini 2007)

If $L : [0, \bar{x}]^2 \mapsto \mathbb{R}$ is continuously differentiable such that for all $t \in [0, \bar{x}]$, the application $x \mapsto \partial L / \partial t(x, t)$ is increasing (supermodularity condition), then

$$\mathbb{E}[L(x, \tilde{f}(x))] \geq \mathbb{E}[L(x, f(x))], \quad (\text{A.5})$$

and the inequality is strict unless $f(x)$ is nondecreasing.

PROOF OF PROPOSITION 4.2

Under the assumption of independence between X and Θ , the auxiliary problem (13) involves conditional expectations. It can now be simplified as the following optimization problem:

$$\max_{0 \leq y \leq x} \{ \mathbb{E}[U(W - \pi - x + \Theta y)] - \lambda y \mathbb{E}[\Theta] \}. \quad (\text{A.6})$$

Denote by $I_\lambda^*(x)$ the optimal solution, then $I_\lambda^*(x) = \max(\min(y_{\lambda,x}, x), 0)$, where $y_{\lambda,x}$ is the unique solution to $\mathbb{E}[U'(W - \pi - x + \Theta y_{\lambda,x})] - \lambda \mathbb{E}[\Theta] = 0$. It is the pointwise optimum. For $K \in (0, \mathbb{E}[X])$, there exists $\lambda > 0$ such that $\mathbb{E}[I_\lambda^*(X)] = K$. It is clear that $I_\lambda^*(x)$ is continuous with respect to λ . Moreover due to lemma A.2 one has

$$\lim_{\lambda \rightarrow 0^+} \mathbb{E}[I_\lambda^*(X)] = \mathbb{E}[X]$$

and

$$\lim_{\lambda \rightarrow +\infty} \mathbb{E}[I_\lambda^*(X)] = 0^+.$$

We have proved the existence of an optimal solution thanks to Proposition 4.1. Lemma A.1 proves that this solution is nondecreasing.

The uniqueness comes from the unique rearrangement of $I(X)$ defined in definition A.1, which is nondecreasing with respect to X (also called comonotonic rearrangement), and of the supermodularity of $\Psi(x, y) = \mathbb{E}[U(W - \pi - x - \Theta y)]$. For any solution I , define \tilde{I} as the nondecreasing rearrangement of I with respect to X ; then

$$\mathbb{E}[U(W - \pi - X + \Theta \tilde{I}(X))] \geq \mathbb{E}[U(W - \pi - X + \Theta I(X))],$$

and the inequality is strict unless $I(\cdot)$ is nondecreasing (because of Lemma A.3).

When $y_{\lambda,x} \in (0, x)$ its derivative with respect to x verifies

$$\frac{\partial y_{\lambda,x}}{\partial x} = \frac{\mathbb{E}[\Theta U''(W - \pi - x + \Theta y_{\lambda,x})]}{\mathbb{E}[\Theta^2 U''(W - \pi - x + \Theta y_{\lambda,x})]}.$$

We know that U'' is negative and that $0 \leq \Theta^2 \leq \Theta \leq 1$; therefore $\partial y_{\lambda,x} / \partial x \geq 1$. This ends the proof of Proposition 4.2. \square

A.6 PROOF OF PROPOSITION 4.3**PROOF**

Starting with

$$\mathbb{E}[U'(W - \pi - x + \Theta y_{\lambda,x}) | X = x] - \lambda \mathbb{E}[\Theta | X = x] = 0, \quad (\text{A.7})$$

one can rewrite this equality as

$$p(x)U'(W - \pi - x + y_{\lambda,x}) + (1 - p(x))\theta_0 U'(W - \pi - x + \theta_0 y_{\lambda,x}) - \lambda(p(x) + \theta_0(1 - p(x))) = 0. \quad (\text{A.8})$$

If $\theta_0 = 0$, the equality (A.8) becomes $p(x)U'(W - \pi - x + y_{\lambda,x}) - \lambda p(x) = 0$. Since $p(x) \neq 0$, $y_{\lambda,x} = x - W + \pi + [U']^{-1}(\lambda)$.

In the general case $0 < \theta_0 < 1$. Note that when $x = W + \pi - [U']^{-1}(\lambda)$ then $y_{\lambda,x} = 0$. In fact, we have the following implications:

$$\begin{aligned}
x &\leq W + \pi - [U']^{-1}(\lambda) \Leftrightarrow U'(W - \pi - x) \geq \lambda \\
&\Leftrightarrow \Theta U'(W - \pi - x) \geq \lambda \Theta a.s. \\
&\Leftrightarrow \mathbb{E}[\Theta U'(W - \pi - x)|X = x] \geq \lambda \mathbb{E}[\Theta|X = x] \\
&\Leftrightarrow \mathbb{E}[\Theta U'(W - \pi - x)|X = x] \geq \mathbb{E}[\Theta U'(W - \pi - x + \Theta y_{\lambda,x})|X = x] \\
&\Rightarrow y_{\lambda,x} \leq 0 \\
&\Rightarrow I_{\lambda}^*(x) = 0.
\end{aligned}$$

The above relationships show that the optimal contract has a deductible level at $W + \pi - [U']^{-1}(\lambda)$, so that the expression (16) of the optimal indemnity is proved.

In the general case $0 < \theta_0 < 1$, from the expression (A.7) we also obtain for all $x \geq 0$,

$$\begin{aligned}
p(x)[U'(W - \pi - x + y_{\lambda,x}) - \theta_0 U'(W - \pi - x + \theta_0 y_{\lambda,x}) - \lambda + \lambda \theta_0] \\
+ \theta_0(-\lambda + U'(W - \pi - x + \theta_0 y_{\lambda,x})) = 0. \quad (\text{A.9})
\end{aligned}$$

For typographical convenience in the proof below, we now denote by $\phi(x) \triangleq y_{\lambda,x}$ and by $\phi'(x) = \partial y_{\lambda,x} / \partial x$ and differentiate the above equality to obtain

$$\begin{aligned}
p'(x)[U'(W - \pi - x + \phi(x)) - \theta_0 U'(W - \pi - x + \theta_0 \phi(x)) - \lambda + \lambda \theta_0] \\
+ p(x)[U''(W - \pi - x + \phi(x))(\phi'(x) - 1) - \theta_0 U''(W - \pi - x + \theta_0 \phi(x))(\theta_0 \phi'(x) - 1)] \\
+ \theta_0 U''(W - \pi - x + \theta_0 \phi(x))(\theta_0 \phi'(x) - 1) = 0. \quad (\text{A.10})
\end{aligned}$$

Using (A.9) in the first line, and after simplifying, (17) follows, and we obtain the expression of $A(x)\phi'(x)$ given in Proposition 4.3.

Evaluating this expression at $x^* = W - \pi - [U']^{-1}(\lambda)$, we obtain $1 + 1 - p(x^*)/p(x^*)\theta_0$. Moreover

$$A(x^*) = 1 + \frac{\theta_0^2(1 - p(x^*))}{p(x^*)}.$$

Therefore

$$\phi'(x^*) = \frac{p(x^*) + (1 - p(x^*))\theta_0}{p(x^*) + (1 - p(x^*))\theta_0^2} > 1$$

because $\theta_0^2 < \theta_0$. □

A.7 PROOF OF COROLLARY 4.1

PROOF

We continue to use $\phi(x) = y_{\lambda,x}$. An alternative representation from (A.10) is

$$\begin{aligned}
\{p(x)U''(W - \pi - x + \phi(x)) + \theta_0^2(1 - p(x))U''(W - \pi - x + \theta_0 \phi(x))\}\phi'(x) \\
= -p'(x)U'(W - \pi - x + \phi(x)) + p(x)U''(W - \pi - x + \phi(x)) \\
+ (1 - p(x))\theta_0 U''(W - \pi - x + \phi(x)) + \theta_0 p'(x)U'(W - \pi - x + \theta_0 \phi(x)) + \lambda(1 - \theta_0)p'(x). \quad (\text{A.11})
\end{aligned}$$

This formula does not require that $p(x) \neq 0$. It can be written as $B(x)\phi'(x) = C(x)$. In (A.11) we see that the coefficient $B(x)$ on the left-hand side is always negative, while on the right-hand side all the terms in $C(x)$ except the first one are negative again. Therefore, unless $p'(x)U'(W - \pi - x + \phi(x))$ is large, we expect that $\phi' > 0$ and the optimal contract is increasing.

If at x , $p'(x) = 0$, then immediately $\phi'(x) \geq 0$.

To understand the structure of (A.11) better, we may combine the terms on the right-hand side in two ways. First, consider $-p'(x)U'(W - \pi - x + \phi(x)) + p(x)U''(W - \pi - x + \phi(x))$. Rearranging, this term is negative if and only if

$$-\frac{U''(W - \pi - x + \phi(x))}{U'(W - \pi - x + \phi(x))} > -\frac{p'(x)}{p(x)},$$

which depends on the absolute risk aversion of the utility function compared to the sensitivity of the probability of default. This proves the local monotonicity of the optimum when the second condition (19) holds.

Finally, consider

$$-p'(x)U'(W - \pi - x + \phi(x)) + \theta_0 p'(x)U'(W - \pi - x + \theta_0 \phi(x)).$$

The sign of this term depends on whether the function $t \mapsto \varpi(t) \triangleq tU'(W - \pi - x + t\phi(x))$ is increasing or decreasing. Indeed, the above is $p'(x)(\varpi(\theta_0) - \varpi(1))$, and a sufficient condition for it to be negative is $\varpi'(t) < 0$ on $[\theta_0, 1]$. This proves the monotonicity of the optimum when the condition (20) holds. \square

A.8 PROOF OF PROPOSITION 4.4

PROOF

As for the proof of Proposition 4.2 we make use of Lemma A.3 given in Appendix A.5 (which was originally proved by Dana and Scarsini 2007). We differentiate $\Phi(x, y) := \mathbb{E}[U(W - \pi - x + \Theta y) | X = x] = (1 - p(x))U(W - \pi - x + \theta_0 y) + p(x)U(W - \pi - x + y)$ with respect to y . The condition in Proposition 4.4 ensures that its derivative with respect to y is nondecreasing in x and thus that we can apply Lemma A.3 to obtain

$$\mathbb{E}[\Phi(X, I(X))] \leq \mathbb{E}[\Phi(X, \tilde{I}(X))]. \quad (\text{A.12})$$

Equality $I(X) = \tilde{I}(X)$ a.s. holds if and only if $I(X)$ is almost surely nondecreasing. Therefore, the optimal indemnity of Problem A is then nondecreasing because of (A.12) and because the constraints of Problem A are also verified. Note that $\Psi(x, y) = -\mathbb{E}[\Theta y | X = x]$ also satisfies the assumption of Lemma A.3; thus $\mathbb{E}[\Theta \tilde{Y}] \leq \mathbb{E}[\Theta Y] \leq K$. \square

A.9 PROOF OF PROPOSITION 4.5

PROOF

Recall that the optimum $y_{\lambda, x}$ is implicitly defined by (15). This equation becomes

$$\theta_0 U'(W - \pi - x + \theta_0 y_{\lambda, x})(1 - p(x)) + U'(W - \pi - x + y_{\lambda, x})p(x) - \lambda(\theta_0(1 - p(x)) + p(x)) = 0. \quad (\text{A.13})$$

When $\theta_0 = 0$, in both cases for $x \geq x_0(p(x) = p_0 > 0)$ or for $x < x_0(p(x) = 1)$, this simplifies to

$$U'(W - \pi - x + y_{\lambda, x}) - \lambda = 0,$$

and therefore $y_{\lambda, x} = x - W + \pi + [U']^{-1}(\lambda)$ is a stop-loss contract.

When $\theta_0 = 1$, it is the standard optimal insurance problem (with no counterparty risk and $p_0 = 1$) so the stop-loss contract is optimal.

When $\theta_0 \in (0, 1)$, we discuss two cases.

For $x < x_0$, $p(x) = 1$, and one has $U'(W - \pi - x + \theta_0 y_{\lambda, x}) = \lambda$ (from eq. [A.13]). Therefore $y_{\lambda, x} = x - W + \pi + [U']^{-1}(\lambda)$ and $\partial y_{\lambda, x} / \partial x = 1$.

In the case when $x > x_0$, we can follow the proof of Proposition 4.3 and show that although $p(\cdot)$ is not differentiable everywhere on $(0, \bar{x})$, equations (17) and (18) still hold (where for $x > x_0$, $p'(x) = 0$ and $p(x) = p_0$). Denote by $D(x) = \theta_0(1 - p_0)U''(W - \pi - x + \theta_0\phi(x))/p_0U''(W - \pi - x + \phi(x))$. It follows that $A(x)$ in (18) is given by $A(x) = 1 + \theta_0 D(x)$ when $x > x_0$. The slope of the contract given in (17) can then be simplified as

$$\frac{\partial y_{\lambda,x}}{\partial x} = \frac{1 + D(x)}{1 + \theta_0 D(x)} > 1,$$

because $D(x) > 0$ and $\theta_0 < 1$.

If x_0 is larger than the stop-loss level $W - \pi - [U']^{-1}(\lambda)$, then the optimal indemnity is first a linear stop-loss with slope 1, and when x_0 is reached, the marginal demand for insurance becomes bigger. \square

A.10 PROOF OF PROPOSITION 5.1

PROOF

Finding the optimum of Problem AS is very similar to finding the optimum of Problem A. We first solve for a pointwise optimum $z_{\lambda,x}$ ignoring the constraints on the indemnity. We then add the constraints such that the pointwise optimum will be equal to

$$\max(0, \min(z_{\lambda,x}, x)).$$

The Lagrange multiplier should then be chosen such that $\mathbb{E}[I^*(X)] = K$.

For all $x \geq 0$, the pointwise optimum $z_{\lambda,x}$ verifies $\mathbb{E}[\Theta U'(W - \pi - x + \Theta z_{\lambda,x}) | X = x] - \lambda = 0$. One can rewrite this equality as

$$p(x)U'(W - \pi - x + z_{\lambda,x}) + (1 - p(x))\theta_0 U'(W - \pi - x + \theta_0 z_{\lambda,x}) - \lambda = 0. \quad (\text{A.14})$$

If $\theta_0 = 0$, from (A.14), $z_{\lambda,x} = x - W + \pi + [U']^{-1}(\lambda/p(x))$.

In the general case $0 < \theta_0 < 1$, one obtains for all $x \geq 0$,

$$p(x)[U'(W - \pi - x + z_{\lambda,x}) - \theta_0 U'(W - \pi - x + \theta_0 z_{\lambda,x})] + \theta_0 U'(W - \pi - x + \theta_0 z_{\lambda,x}) - \lambda = 0. \quad (\text{A.15})$$

For simplicity of notation, we now denote by $\phi(x) \triangleq z_{\lambda,x}$ and differentiating the above equality find

$$\begin{aligned} p'(x)[U'(W - \pi - x + \phi(x)) - \theta_0 U'(W - \pi - x + \theta_0 \phi(x))] \\ + p(x)[U''(W - \pi - x + \phi(x))(\phi'(x) - 1) - \theta_0 U''(W - \pi - x + \theta_0 \phi(x))(\theta_0 \phi'(x) - 1)] \\ + \theta_0 U''(W - \pi - x + \theta_0 \phi(x))(\theta_0 \phi'(x) - 1) = 0. \end{aligned} \quad (\text{A.16})$$

Using (A.14) in the first line of (A.16), one obtains (25). \square

7. ACKNOWLEDGMENTS

The authors acknowledge support from the Society of Actuaries, Committee on Knowledge, Education and Research, and the Actuarial Education Research Fund. Bernard is partially supported by the Natural Sciences and Engineering Research Council of Canada. The authors thank the anonymous referees, whose initial feedback improved our presentation and proofs.

REFERENCES

- ARROW, K. J. 1963. Uncertainty and the Welfare Economics of Medical Care. *American Economic Review* 53: 941–973.
 ARROW, K. J. 1971. *Essays in the Theory of Risk Bearing*. Chicago: Markham.
 ARTZNER, P., F. DELBAEN, J.-M. EBER, AND D. HEATH. 1999. Coherent Measures of Risk. *Mathematical Finance* 9: 203–228.

- BERNARD, C., AND W. TIAN. 2010. Insurance Market Effects of Risk Management Metrics. *Geneva Risk and Insurance Review* 35: 47–80.
- BIFFIS, E., AND P. MILLOSOVICH. 2010. Optimal Insurance with Counterparty Default Risk. Working Paper. Available at <http://ssrn.com/abstract=1634883>.
- CAI, J., AND K. TAN. 2007. Optimal Retention for a Stop-Loss Reinsurance under the VaR and CTE Risk Measures. *ASTIN Bulletin* 37: 92–112.
- CAPÉRAA, P., AND J. LEFOLL. 1983. Aversion pour le risque croissante avec la richesse initiale aléatoire. *Econometrica* 53: 473–475.
- CARDENAS, V., AND O. MAHUL. 2006. A Note on Catastrophe Risk Financing with Unreliable Insurance. Paper presented at ARIA 2006 Conference.
- CARLIER, G., AND R.-A. DANA. 2003. Pareto Efficient Insurance Contracts When the Insurer's Cost Function Is Discontinuous. *Economic Theory* 21: 871–893.
- CUMMINS, J., AND O. MAHUL. 2003. Optimal Insurance with Divergent Beliefs about Insurer Total Default Risk. *Journal of Risk and Uncertainty* 27: 121–138.
- DANA, R.-A., AND M. SCARSINI. 2007. Optimal Risk Sharing with Background Risk. *Journal of Economic Theory* 133: 152–176.
- DOHERTY, N., AND H. SCHLESINGER. 1990. Rational Insurance Purchasing: Consideration of Contract Nonperformance. *Quarterly Journal of Economics* 105: 243–253.
- ECKHOUDT, L., C. GOLLIER, AND H. SCHLESINGER. 2005. *Economic and Financial Decisions under Risk*. Princeton: Princeton University Press.
- FRANKE, G., H. SCHLESINGER, AND R. C. STAPLETON. 2006. Multiplicative Background Risk. *Management Science* 52: 146–153.
- GOLLIER, C. 1987. The Design of Optimal Insurance Contracts without the Nonnegativity Constraint on Claims. *Journal of Risk and Insurance* 54: 314–324.
- MAHUL, O., AND B. WRIGHT. 2004. Implications of Incomplete Performance for Optimal Insurance. *Economica* 71: 661–670.
- MAHUL, O., AND B. WRIGHT. 2007. Optimal Coverage for Incompletely Reliable Insurance. *Economic Letters* 95: 456–461.
- MOSSIN, J. 1968. Aspects of Rational Insurance Purchasing. *Journal of Political Economy* 76: 553–568.
- RAVIV, A. 1979. The Design of an Optimal Insurance Policy. *American Economic Review* 69: 84–96.
- SCHLESINGER, H. 1981. The Optimal Level of Deductibility in Insurance Contracts. *Journal of Risk and Insurance* 48: 465–481.
- SCHLESINGER, H. 2000. The Theory of Insurance, in *Handbook of Insurance*, ed. G. Dionne, ch. 5. Boston: Kluwer.

Discussions on this paper can be submitted until July 1, 2012. The authors reserve the right to reply to any discussion. Please see the Submission Guidelines for Authors on the inside back cover for instructions on the submission of discussions.