Prealgebra Textbook

Second Edition

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Contents

Chapter 1

The Whole Numbers

Welcome to the study of prealgebra. In this first chapter of study, we will introduce the set of natural numbers, then follow with the set of whole numbers. We will then follow with a quick review of addition, subtraction, multiplication, and division skills involving whole numbers that are prerequisite for success in the study of prealgebra. Along the way we will introduce a number of properties of the whole numbers and show how that can be used to evaluate expressions involving whole number operations.

We will also define what is meant by *prime* and *composite* numbers, discuss a number of divisibility tests, then show how any composite number can be written uniquely as a product of prime numbers. This will lay the foundation for requisite skills with fractional numbers in later chapters.

Finally, we will introduce the concept of a *variable*, then introduce equations and technique required for their solution. We will use equations to model and solve a number of real-world applications along the way.

Let's begin the journey.

1.1 An Introduction to the Whole Numbers

A *set* is a collection of objects. If the set is finite, we can describe the set completely by simply listing all the objects in the set and enclosing the list in curly braces. For example, the set

$$S = \{\texttt{dog}, \texttt{cat}, \texttt{parakeet}\}$$

is the set whose members are "dog", "cat", and "parakeet." If the set is infinite, then we need to be more clever with our description. For example, the set of *natural numbers* (or *counting numbers*) is the set

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}.$$

Because this set is infinite (there are an infinite number of natural numbers), we can't list all of them. Instead, we list the first few then follow with "three dots," which essentially mean "etcetera." The implication is that the reader sees the intended pattern and can then intuit the remaining numbers in the set. Can you see that the next few numbers are 6, 7, 8, 9, etc.?

If we add the number zero to the set of natural numbers, then we have a set of numbers that are called the *whole numbers*.

The Whole Numbers. The set

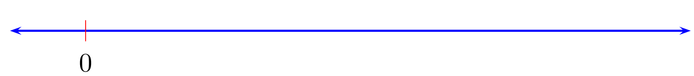
$$\mathbb{W} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

is called the set of *whole numbers*.

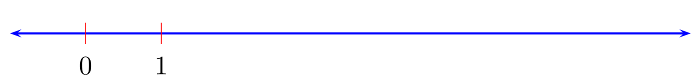
The whole numbers will be our focus in the remainder of this chapter.

Graphing numbers on the number line

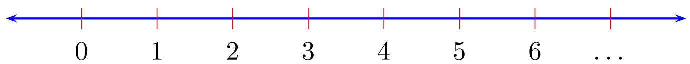
It is a simple matter to set up a correspondence between the whole numbers and points on a number line. First, draw a number line, then set a tick mark at zero.



The next step is to declare a unit length.



The remainder of the whole numbers now fall easily in place on the number line.



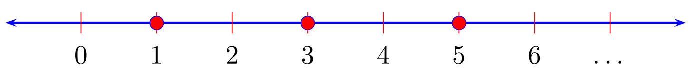
When asked to graph a whole number on a number line, shade in a solid dot at the position on the number line that corresponds to the given whole number.

You Try It!

EXAMPLE 1. Graph the whole numbers 1, 3, and 5 on the number line.

Graph the whole numbers 3, 4, and 6 on the number line.

Solution: Shade the numbers 1, 3, and 5 on the number line as solid dots.

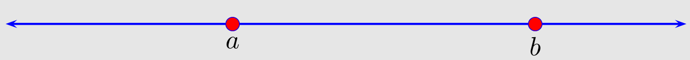


□

Ordering the whole numbers

Now that we have a correspondence between the whole numbers and points on the number line, we can *order* the whole numbers in a natural way. Note that as you move to the left along the number line, the numbers get smaller; as you move to the right, the numbers get bigger. This inspires the following definition.

Ordering the Whole Numbers. Suppose that a and b are whole numbers located on the number line so that the point representing the whole number a lies to the left of the point representing the whole number b.



Then the whole number a is "less than" the whole number b and write

$$a < b.$$

Alternatively, we can also say that the whole number b is"greater than" the whole number a and write

$$b > a.$$

Comparison Property: When comparing two whole numbers a and b, only one of three possibilities is true:

$$a < b \qquad \text{or} \qquad a = b \qquad \text{or} \qquad a > b.$$

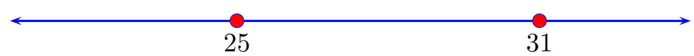
You Try It!

Compare the whole numbers 18 and 12.

Answer: $18 > 12$

EXAMPLE 2. Compare the whole numbers 25 and 31.

Solution: On the number line, 25 is located to the left of 31.



Therefore, 25 is less than 31 and write $25 < 31$. Alternatively, we could also note that 31 is located to the right of 25. Therefore, 31 is greater than 25 and write $31 > 25$.

□

Expanded notation

The whole numbers

$$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

are called *digits* and are used to construct larger whole numbers. For example, consider the whole number 222 (pronounced "two hundred twenty two"). It is made up of three twos, but the position of each two describes a different meaning or value.

2	2	2
hundreds	tens	ones

- The first two is in the "hundreds" position and represents two hundreds or 200.
- The second two is in the "tens" position and represents two tens or 20.
- The third two is in the "ones" position and represents two ones or 2.

Consider the larger number 123,456,789. The following table shows the place value of each digit.

1	2	3	4	5	6	7	8	9
hundred millions	ten millions	millions	hundred thousands	ten thousands	thousands	hundreds	tens	ones
millions			thousands			ones		

In "expanded notation," we would write

$$\begin{aligned}&1 \text{ hundred million} + 2 \text{ ten millions} + 3 \text{ millions} + 4 \text{ hundred thousands}\\ &\quad + 5 \text{ ten thousands} + 6 \text{ thousands} + 7 \text{ hundreds} + 8 \text{ tens} + 9 \text{ ones.}\end{aligned}$$

We read the numeral 123,456,789 as "one hundred twenty three million, four hundred fifty six thousand, seven hundred eighty nine."

Let's look at another example.

You Try It!

Write the number 54,615 in expanded notation. Pronounce the result.

EXAMPLE 3. Write the number 23,712 in expanded notation, then pronounce the result.

Solution: In expanded notation, 23,712 becomes

$$2 \text{ ten thousands} + 3 \text{ thousands} + 7 \text{ hundreds} + 1 \text{ ten} + 2 \text{ ones.}$$

This is pronounced "twenty three thousand, seven hundred twelve."

□

You Try It!

Write the number $430,705$ in expanded notation. Pronounce the result.

EXAMPLE 4. Write the number 203,405 in expanded notation, then pronounce the result.

Solution: In expanded notation, 203,405 becomes

$$\begin{aligned}&2 \text{ hundred thousands} + 0 \text{ ten thousands} + 3 \text{ thousands}\\ &\quad + 4 \text{ hundreds} + 0 \text{ tens} + 5 \text{ ones.}\end{aligned}$$

Since 0 ten thousands is zero and 0 tens is also zero, this can also be written

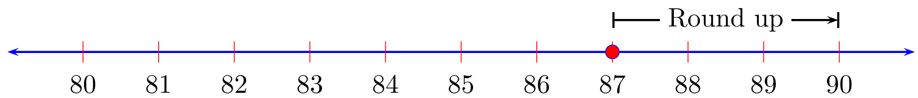
$$2 \text{ hundred thousands} + 3 \text{ thousands} + 4 \text{ hundreds} + 5 \text{ ones.}$$

This is pronounced "two hundred three thousand, four hundred five."

□

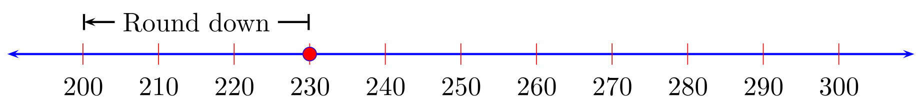
Rounding whole numbers

When less precision is needed, we round numbers to a particular place. For example, suppose a store owner needs approximately 87 boxes of ten-penny nails, but they can only be ordered in cartons containing ten boxes.



Note that 87 is located closer to 9 tens (or 90) than it is to 8 tens (or 80). Thus, rounded to the nearest ten, $87 \approx 90$ (87 approximately equals 90). The store owner decides that 90 boxes is probably a better fit for his needs.

On the other hand, the same store owner estimates that he will need 230 bags of peatmoss for his garden section.



Note that 230 is closer to 2 hundreds (or 200) than it is to 3 hundreds (or 300). The store owner worries that might have overestimated his need, so he rounds down to the nearest hundred, $230 \approx 200$ (230 approximately equals 200).

There is a simple set of rules to follow when rounding.

Rules for Rounding. To round a number to a particular place, follow these steps:

1. Mark the place you wish to round to. This is called the *rounding digit*.
2. Check the next digit to the right of your digit marked in step 1. This is called the *test digit*.
   a) If the test digit is greater than or equal to 5, add 1 to the rounding digit and replace all digits to the right of the rounding digit with zeros.
   b) If the test digit is less than 5, keep the rounding digit the same and replace all digits to the right of rounding digit with zeros.

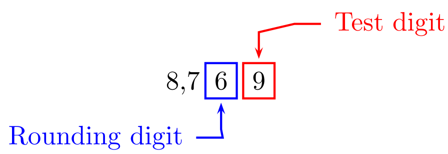
Let's try these rules with an example or two.

You Try It!

Round the number 9,443 to the nearest ten.

EXAMPLE 5. Round the number 8,769 to the nearest ten.

Solution: Mark the rounding and test digits.



The test digit is greater than 5. The "Rules for Rounding" require that we add 1 to the rounding digit, then make all digits to the right of the rounding digit zeros. Thus, rounded to the nearest ten,

$$8,769 \approx 8,770.$$

That is, 8,769 is *approximately equal* to 8,770.

Answer: 9,440

Mathematical Notation. The symbol

$$\approx$$

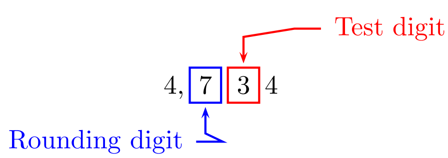
means *approximately equal*.

You Try It!

EXAMPLE 6. Round the number 4,734 to the nearest hundred.

Round the number 6,656 to the nearest hundred.

Solution: Mark the rounding and test digits.



The test digit is less than 5. The "Rules for Rounding" require that we keep the rounding digit the same, then make all digits to the right of the rounding digit zeros. Thus, rounded to the nearest hundred,

$$4,734 \approx 4,700.$$

Answer: $6,700$

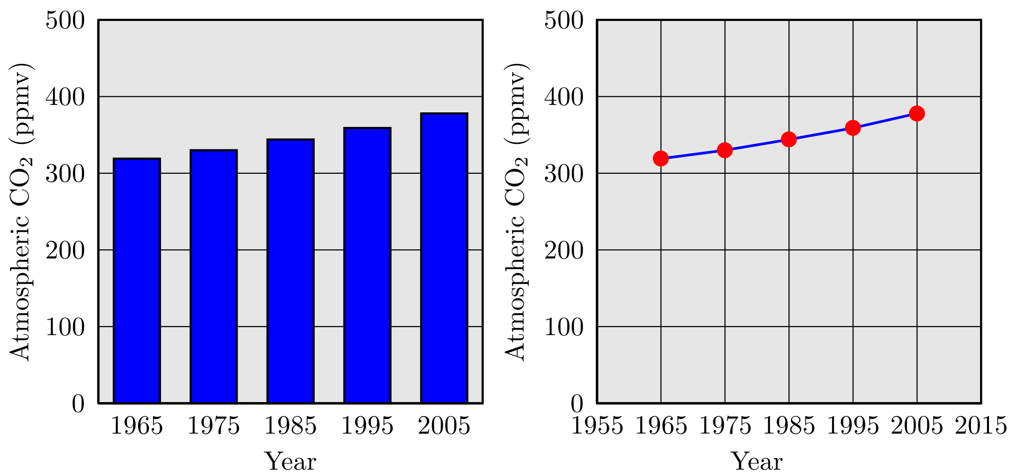
Year	1965	1975	1985	1995	2005
Atmospheric CO_2	319	330	344	359	378

Table 1.1: Atmospheric CO_2 values (ppmv) derived from in situ air samples collected at Mauna Loa, Hawaii, USA.

Tables and graphs

Reading data in graphical form is an important skill. The data in **Table 1.1** provides measures of the carbon dioxide content (CO_2) in the atmosphere, gathered in the month of January at the observatory atop Mauna Loa in Hawaii.

In **Figure 1.1(a)**, a *bar graph* is used to display the carbon dioxide measurements. The year the measurement was taken is placed on the horizontal axis, and the height of each bar equals the amount of carbon dioxide in the atmosphere during that year.



(a) Bar graph. (b) Line graph.

Figure 1.1: Using graphs to examine carbon dioxide data.

In **Figure 1.1(b)**, a *line graph* is used to display the carbon dioxide measurements. Again, the dates of measurement are placed on the horizontal axis, and the amount of carbon dioxide in the atmosphere is placed on the vertical axis. Instead of using the height of a bar to represent the carbon dioxide measurement, we place a dot at a height that represents the carbon monoxide content. Once each data point is plotted, we connect consecutive data points with line segments.

Exercises

In Exercises 1-12, sketch the given whole numbers on a number line, then arrange them in order, from smallest to largest.

1. 2, 8, and 4

2. 2, 7, and 4

3. 1, 8, and 2

4. 0, 4, and 3

5. 0, 4, and 1

6. 3, 6, and 5

7. 4, 9, and 6

8. 2, 4, and 3

9. 0, 7, and 4

10. 2, 8, and 6

11. 1, 6, and 5

12. 0, 9, and 5

In Exercises 13-24, create a number line diagram to determine which of the two given statements is true.

13. $3 < 8$ or $3 > 8$

14. $44 < 80$ or $44 > 80$

15. $59 < 24$ or $59 > 24$

16. $15 < 11$ or $15 > 11$

17. $0 < 74$ or $0 > 74$

18. $11 < 18$ or $11 > 18$

19. $1 < 81$ or $1 > 81$

20. $65 < 83$ or $65 > 83$

21. $43 < 1$ or $43 > 1$

22. $62 < 2$ or $62 > 2$

23. $43 < 28$ or $43 > 28$

24. $73 < 21$ or $73 > 21$

---

25. Which digit is in the thousands column of the number 2,054,867,372?

26. Which digit is in the hundreds column of the number 2,318,999,087?

27. Which digit is in the hundred thousands column of the number 8,311,900,272?

28. Which digit is in the tens column of the number 1,143,676,212?

29. Which digit is in the hundred millions column of the number 9,482,616,000?

30. Which digit is in the hundreds column of the number 375,518,067?

31. Which digit is in the ten millions column of the number 5,840,596,473?

32. Which digit is in the hundred thousands column of the number 6,125,412,255?

33. Which digit is in the hundred millions column of the number 5,577,422,501?

34. Which digit is in the thousands column of the number 8,884,966,835?

35. Which digit is in the tens column of the number 2,461,717,362?

36. Which digit is in the ten millions column of the number 9,672,482,548?

37. Round the number 93, 857 to the nearest thousand.

38. Round the number 56, 872 to the nearest thousand.

39. Round the number 9, 725 to the nearest ten.

40. Round the number 6, 815 to the nearest ten.

41. Round the number 58, 739 to the nearest hundred.

42. Round the number 93, 146 to the nearest hundred.

43. Round the number 2, 358 to the nearest ten.

44. Round the number 8, 957 to the nearest ten.

45. Round the number 39, 756 to the nearest thousand.

46. Round the number 24, 965 to the nearest thousand.

47. Round the number 5, 894 to the nearest ten.

48. Round the number 3, 281 to the nearest ten.

49. Round the number 56, 123 to the nearest hundred.

50. Round the number 49, 635 to the nearest hundred.

51. Round the number 5, 483 to the nearest ten.

52. Round the number 9, 862 to the nearest ten.

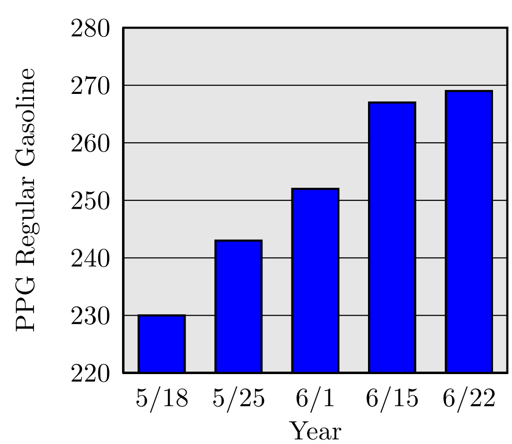
53. According to the U.S. Census Bureau, the estimated population of the US is 304,059,724 as of July 2008. Round to the nearest hundred thousand.

54. According to the U.S. Census Bureau, the estimated population of California is 36,756,666 as of July 2008. Round to the nearest hundred thousand.

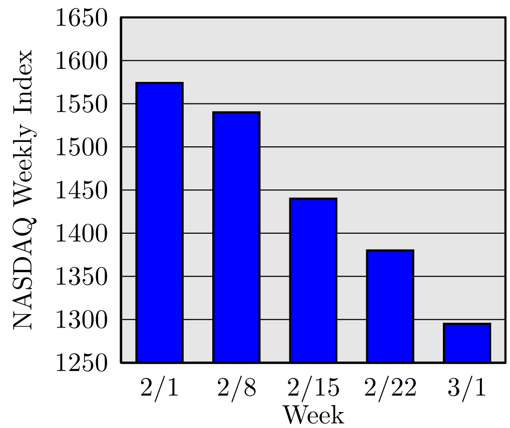
55. According to the U.S. Census Bureau, the estimated population of Humboldt County is 129,000 as of July 2008. Round to the nearest ten thousand.

56. According to the U.S. Census Bureau, the estimated population of the state of Alasks was 686,293 as of July 2008. Round to the nearest ten thousand.

57. The following bar chart shows the average price (in cents) of one gallon of regular gasoline in the United States over five consecutive weeks in 2009, running from May 18 (5/18) through June 22 (6/22). What was the price (in cents) of one gallon of regular gasoline on June 1, 2009?



58. The following bar chart shows the average weekly NASDAQ index for five consecutive weeks in 2009, beginning with week starting February 1 (2/1) and ending with the week starting March 1 (3/1). What was the average NASDAQ index for the week starting February 8, 2009?



59. The population of Humboldt County is broken into age brackets in the following table. *Source: WolframAlpha.*

Age in years	Number
under 5	7,322
5-18	26,672
18-65	78,142
over 65	16,194

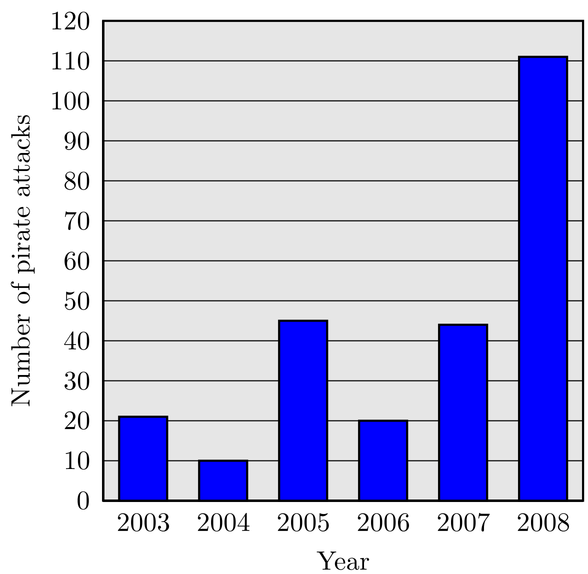
Create a bar chart for this data set with one bar for each age category.

60. The five cities with the largest number of reported violent crimes in the year 2007 are reported in the following table. *Source: Wikipedia.*

City	Violent Crimes
Detroit	2,289
St. Louis	2,196
Memphis	1,951
Oakland	1,918
Baltimore	1,631

Create a bar chart for this data set with one bar for each city.

61. The following bar chart tracks pirate attacks off the coast of Somalia.

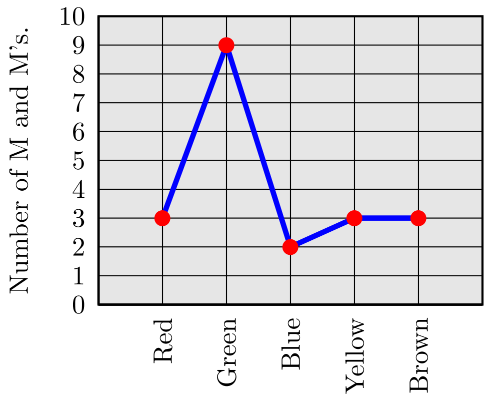


Source: ICC International Maritime Bureau, AP Times-Standard, 4/15/2009

a) How many pirate attacks were there in 2003?

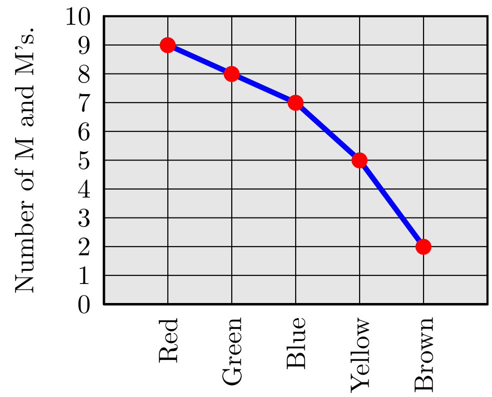
b) How many pirate attacks were there in 2008?

62. A team of students separated a small bowl of M and M's into five piles by color. The following line plot indicates the number of M and M's of each color.



How many red M and M's were in the bowl?

63. A team of students separated a small bowl of M and M's into five piles by color. The following line plot indicates the number of M and M's of each color.



How many red M and M's were in the bowl?

64. A team of students separated a small bowl of M and M's into five piles by color. The following table indicates the number of M and M's of each color.

Color	Number
Red	5
Green	9
Blue	7
Yellow	2
Brown	3

Create a lineplot for the M and M data. On the horizontal axis, arrange the colors in the same order as presented in the table above.

65. A team of students separated a small bowl of M and M's into five piles by color. The following table indicates the number of M and M's of each color.

Color	Number
Red	3
Green	7
Blue	2
Yellow	4
Brown	9

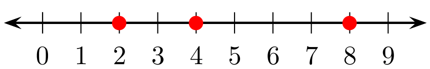
Create a lineplot for the M and M data. On the horizontal axis, arrange the colors in the same order as presented in the table above.

66. Salmon count. The table shows the number of adult coho salmon returning to the Shasta River over the past four years. Round the salmon count for each year to the nearest ten. *Times-Standard Shasta River coho rescue underway.*

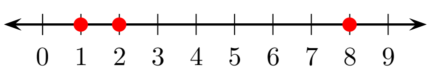
Year	Salmon count
2007	300
2008	31
2009	9
2010	4

Answers

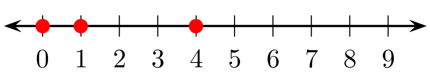
1. Smallest to largest: 2, 4, and 8.



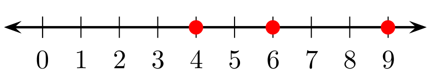
3. Smallest to largest: 1, 2, and 8.



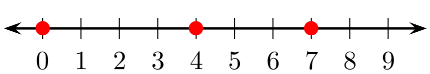
5. Smallest to largest: 0, 1, and 4.



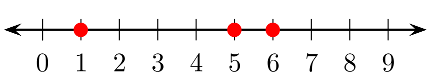
7. Smallest to largest: 4, 6, and 9.



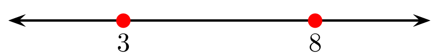
9. Smallest to largest: 0, 4, and 7.



11. Smallest to largest: 1, 5, and 6.

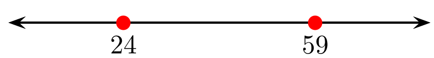


13.



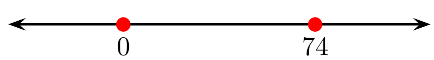
Therefore, $3 < 8$.

15.



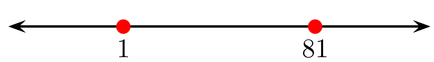
Therefore, $59 > 24$.

17.



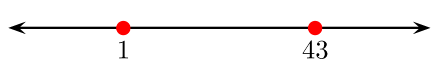
Therefore, $0 < 74$.

19.



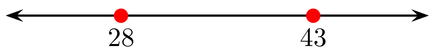
Therefore, $1 < 81$.

21.



Therefore, $43 > 1$.

23.



Therefore, $43 > 28$.

25. 7

27. 9

29. 4

31. 4

33. 5

35. 6

37. 94000

39. 9730

41. 58700

43. 2360

45. 40000

47. 5890

49. 56100

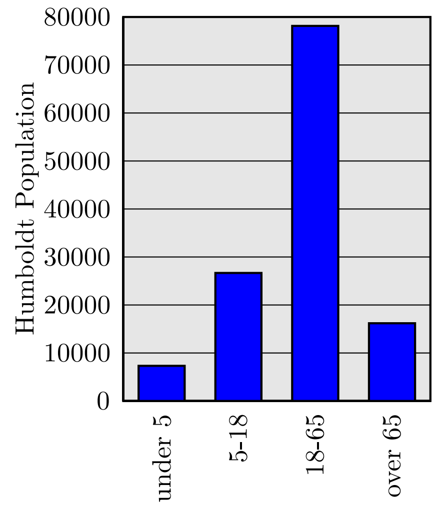
51. 5480

53. 304,100,000

55. 130,000

57. Approximately 252 cents

59.

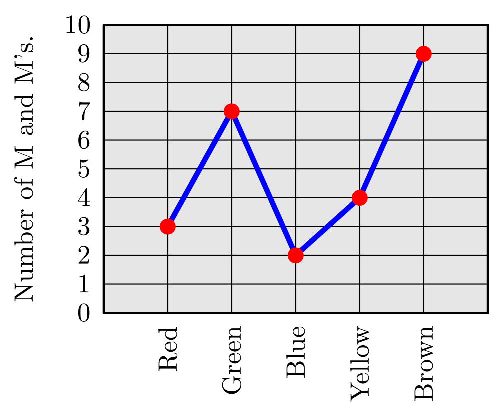


61. a) Approximately 21

b) Approximately 111

63. 9

65.



1.2 Adding and Subtracting Whole Numbers

In the expression $3+4$, which shows the sum of two whole numbers, the whole numbers 3 and 4 are called *addends* or *terms*. We can use a visual approach to find the sum of 3 and 4. First, construct a number line as shown in Figure 1.2.

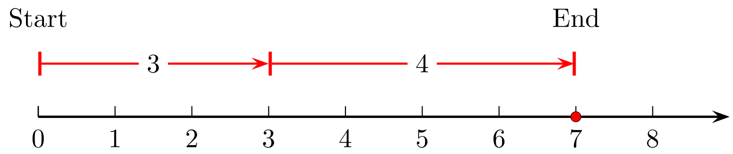


Figure 1.2: Adding whole numbers on a number line.

To add 3 and 4, proceed as follows.

1. Start at the number 0, then draw an arrow 3 units to the right, as shown in Figure 1.2. This arrow has magnitude (length) three and represents the whole number 3.
2. Draw a second arrow of length four, starting at the end of the first arrow representing the number 3. This arrow has magnitude (length) four and represents the whole number 4.
3. The sum of 3 and 4 could be represented by an arrow that starts at the number 0 and ends at the number 7. However, we prefer to mark this sum on the number line as a solid dot at the whole number 7. This number represents the sum of the whole numbers 3 and 4.

The Commutative Property of Addition

Let's change the order in which we add the whole numbers 3 and 4. That is, let's find the sum $4+3$ instead.

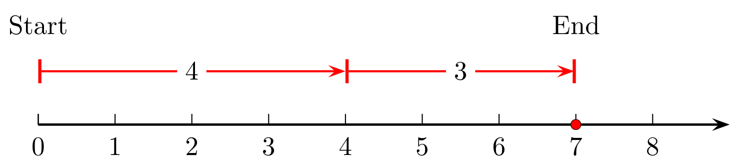


Figure 1.3: Addition is commutative; i.e., order doesn't matter.

As you can see in Figure 1.3, we start at zero then draw an arrow of length four, followed by an arrow of length three. However, the result is the same; i.e., $4+3=7$.

Thus, the order in which we add three and four does not matter; that is,

$$3+4=4+3.$$

This property of addition of whole numbers is known as the *commutative property* of addition.

The Commutative Property of Addition. Let a and b represent two whole numbers. Then,

$$a + b = b + a.$$

Grouping Symbols

In mathematics, we use *grouping symbols* to affect the order in which an expression is evaluated. Whether we use parentheses, brackets, or curly braces, the expression inside any pair of grouping symbols must be evaluated first. For example, note how we first evaluate the sum in the parentheses in the following calculation.

$$\begin{aligned}(3+4)+5 &= 7+5\\ &= 12\end{aligned}$$

The rule is simple: Whatever is inside the parentheses is evaluated first.

Writing Mathematics. When writing mathematical statements, follow the mantra:

One equal sign per line.

We can use brackets instead of parentheses.

$$\begin{aligned}5+[7+9] &= 5+16\\ &= 21\end{aligned}$$

Again, note how the expression inside the brackets is evaluated first.

We can also use curly braces instead of parentheses or brackets.

$$\begin{aligned}\{2+3\}+4 &= 5+4\\ &= 9\end{aligned}$$

Again, note how the expression inside the curly braces is evaluated first.

If grouping symbols are *nested*, we evaluate the innermost parentheses first. For example,

$$\begin{aligned}2+[3+(4+5)] &= 2+[3+9]\\ &= 2+12\\ &= 14.\end{aligned}$$

Grouping Symbols. Use parentheses, brackets, or curly braces to delimit the part of an expression you want evaluated first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.

The Associative Property of Addition

Consider the evaluation of the expression (2+3)+4. We evaluate the expression in parentheses first.

$$\begin{aligned}(2+3)+4 &= 5+4\\ &= 9\end{aligned}$$

Now, suppose we change the order of addition to $2+(3+4)$. Then,

$$\begin{aligned}2+(3+4) &= 2+7\\ &= 9.\end{aligned}$$

Although the grouping has changed, the result is the same. That is,

$$(2+3)+4 = 2+(3+4).$$

This property of addition of whole numbers is called the *associate property* of addition.

Associate Property of Addition. Let a, b, and c represent whole numbers. Then,

$$(a+b)+c = a+(b+c).$$

Because of the associate property of addition, when presented with a sum of three numbers, whether you start by adding the first two numbers or the last two numbers, the resulting sum is the same.

The Additive Identity

Imagine a number line visualization of the sum of four and zero; i.e., $4+0$.

In Figure 1.4, we start at zero, then draw an arrow of magnitude (length) four pointing to the right. Now, at the end of this arrow, attach a second arrow of length zero. Of course, that means that we remain right where we are, at 4. Hence the shaded dot at 4 is the sum. That is, $4+0=4$.

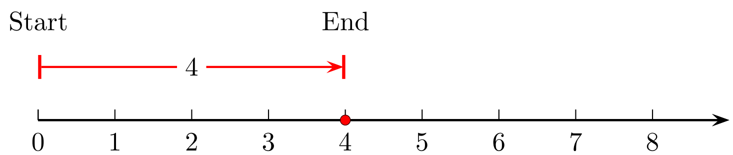


Figure 1.4: Adding zero to four.

The Additive Identity Property. The whole number zero is called the *additive identity.* If a is any whole number, then

$$a + 0 = a.$$

The number zero is called the additive identity because if you add zero to any number, you get the identical number back.

Adding Larger Whole Numbers

For completeness, we include two examples of adding larger whole numbers. Hopefully, the algorithm is familiar from previous coursework.

You Try It!

Simplify: $1,286 + 349$

EXAMPLE 1. Simplify: $1,234 + 498$.

Solution. Align the numbers vertically, then add, starting at the furthest column to the right. Add the digits in the ones column, $4 + 8 = 12$. Write the 2, then carry a 1 to the tens column. Next, add the digits in the tens column, $3 + 9 = 12$, add the carry to get 13, then write the 3 and carry a 1 to the hundreds column. Continue in this manner, working from right to left.

$$\begin{array}{cccccc} & & & 1 & 1 & \\ & & 1 & 2 & 3 & 4 \\ + & & & 4 & 9 & 8 \\ \hline & & 1 & 7 & 3 & 2 \end{array}$$

Therefore, $1,234 + 498 = 1,732$.

Answer: 1,635

□

Add three or more numbers in the same manner.

You Try It!

Simplify: $256 + 342 + 283$

EXAMPLE 2. Simplify: $256 + 322 + 418$.

Solution. Align the numbers vertically, then add, starting at the furthest column to the right. Add the digits in the ones column, $6 + 2 + 8 = 16$. Write the 6, then carry a 1 to the tens column. Continue in this manner, working from right to left.

$$\begin{array}{r} {\scriptstyle 1} \\ 2\,5\,6 \\ 3\,2\,2 \\ +\ 4\,1\,8 \\ \hline 9\,9\,6 \end{array}$$

Therefore, $256 + 322 + 418 = 996$. Answer: 881

□

Subtraction of Whole Numbers

The key idea is this: *Subtraction is the opposite of addition.* For example, consider the difference $7 - 4$ depicted on the number line in Figure 1.5.

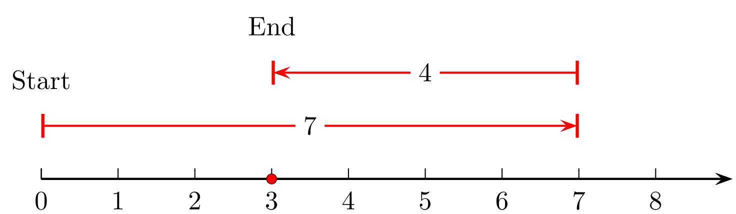


Figure 1.5: Subtraction means *add the opposite*.

If we were adding 7 and 4, we first draw an arrow starting at zero pointing to the right with magnitude (length) seven. Then, to add 4, we would draw a second arrow of magnitude (length) 4, attached to the end of the first arrow and pointing to the right.

However, because subtraction is the *opposite* of addition, in Figure 1.5 we attach an arrow of magnitude (length) four to the end of the first arrow, but *pointing in the opposite direction* (to the left). Note that this last arrow ends at the answer, which is a shaded dot on the number line at 3. That is, $7 - 4 = 3$.

Note that subtraction is **not commutative**; that is, it make no sense to say that $7 - 5$ is the same as $5 - 7$.

Subtraction is **not associative**. It is not the case that $(9 - 5) - 2$ is the same as $9 - (5 - 2)$. On the one hand,

$$\begin{aligned} (9 - 5) - 2 &= 4 - 2 \\ &= 2, \end{aligned}$$

but

$$\begin{aligned} 9 - (5 - 2) &= 9 - 3 \\ &= 6. \end{aligned}$$

Subtracting Larger Whole Numbers

Much as we did with adding larger whole numbers, to subtract two large whole numbers, align them vertically then subtract, working from right to left. You may have to "borrow" to complete the subtraction at any step.

You Try It!

Simplify: $5,635 - 288$.

EXAMPLE 3. Simplify: $1,755 - 328$.

Solution. Align the numbers vertically, then subtract, starting at the ones column, then working right to left. At the ones column, we cannot subtract 8 from 5, so we borrow from the previous column. Now, 8 from 15 is 7. Continue in this manner, working from right to left.

$$\begin{array}{rcccc} & & & 4 & \\ & 1 & 7 & \not{5} & {}^{1}5 \\ - & & 3 & 2 & 8 \\ \hline & 1 & 4 & 2 & 7 \end{array}$$

Answer: 5,347

Therefore, $1,755 - 328 = 1,427$.

Order of Operations

In the absence of grouping symbols, it is important to understand that addition holds no precedence over subtraction, and vice-versa.

> Perform all additions and subtractions in the order presented, moving left to right.

Let's look at an example.

You Try It!

Simplify: $25 - 10 + 8$.

EXAMPLE 4. Simplify the expression $15 - 8 + 4$.

Solution. This example can be trickier than it seems. However, if we follow the rule (perform all additions and subtractions in the order presented, moving left to right), we should have no trouble. First comes fifteen minus eight, which is seven. Then seven plus four is eleven.

$$\begin{aligned} 15 - 8 + 4 &= 7 + 4 \\ &= 11. \end{aligned}$$

Answer: 23

Caution! Incorrect answer ahead! Note that it is possible to arrive at a different (but incorrect) answer if we favor addition over subtraction in Example 4. If we first add eight and four, then $15 - 8 + 4$ becomes $15 - 12$, which is 3. However, note that **this is incorrect**, because it violates the rule "perform all additions and subtractions in the order presented, moving left to right."

Applications — Geometry

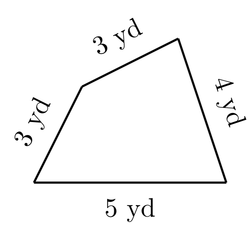
There are any number of applications that require a sum or difference of whole numbers. Let's examine a few from the world of geometry.

Perimeter of a Polygon. In geometry a polygon is a plane figure made up of a closed path of a finite sequence of segments. The segments are called the *edges* or *sides* of the polygon and the points where two edges meet are called the *vertices* of the polygon. The *perimeter* of any polygon is the sum of the lengths of its sides.

You Try It!

EXAMPLE 5. A quadrilateral is a polygon with four sides. Find the perimeter of the quadrilateral shown below, where the sides are measured in yards.

A quadrilateral has sides that measure 4 in., 3 in., 5 in., and 5 in. Find the perimeter.



Solution. To find the perimeter of the quadrilateral, find the sum of the lengths of the sides.

$$\text{Perimeter} = 3 + 3 + 4 + 5 = 15$$

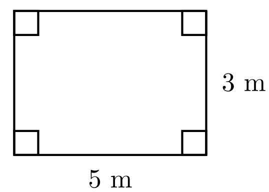
Hence, the perimeter of the quadrilateral is 15 yards.

Answer: 17 inches

You Try It!

A rectangle has length 12 meters and width 8 meters. Find its perimeter.

EXAMPLE 6. A quadrilateral (four sides) is a *rectangle* if all four of its angles are right angles. It can be shown that the opposite sides of a rectangle must be equal. Find the perimeter of the rectangle shown below, where the sides of the rectangle are measured in meters.



Solution. To find the perimeter of the rectangle, find the sum of the four sides. Because opposite sides have the same length, we have two sides of length 5 meters and two sides of length 3 meters. Hence,

$$\text{Perimeter} = 5 + 3 + 5 + 3 = 16.$$

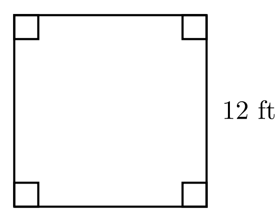
Thus, the perimeter of the rectangle is 16 meters. □

Answer: 40 meters

You Try It!

A square has a side that measures 18 centimeters. Find its perimeter.

EXAMPLE 7. A quadrilateral (four sides) is a *square* if all four of its sides are equal and all four of its angles are right angles. Pictured below is a square having a side of length 12 feet. Find the perimeter of the square.



Solution. Because the quadrilateral is a square, all four sides have the same length, namely 12 feet. To find the perimeter of the square, find the sum of the four sides.

$$\text{Perimeter} = 12 + 12 + 12 + 12 = 48$$

Hence, the perimeter of the square is 48 feet. □

Answer: 72 centimeters

Application — Alternative Fuels

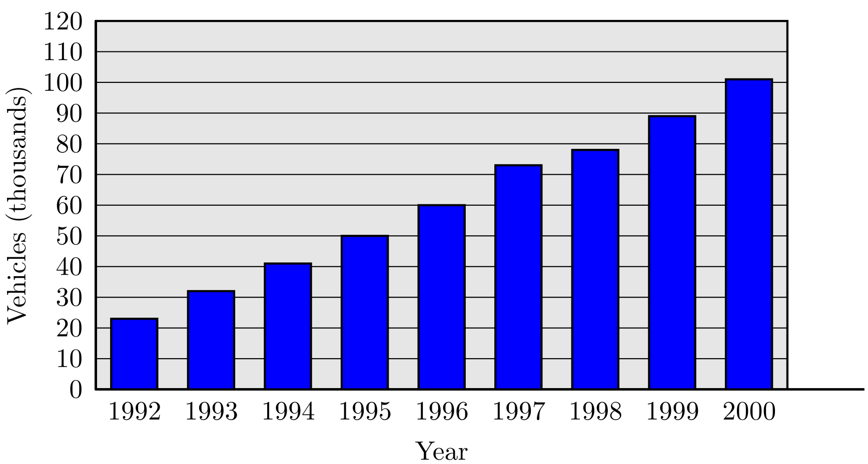
Automobiles that run on alternative fuels (other than gasoline) have increased in the United States over the years.

EXAMPLE 8. Table 1.2 show the number of cars (in thousands) running on compressed natural gas versus the year. Create a bar chart showing the number of cars running on compressed natural gas versus the year.

Year	1992	1993	1994	1995	1996	1997	1998	1999	2000
Number	23	32	41	50	60	73	78	89	101

Table 1.2: Number of vehicles (in thousands) running on compressed natural gas.

Solution. Place the years on the horizontal axis. At each year, sketch a bar having height equal to the number of cars in that year that are running on compressed natural gas. Scale the vertical axis in thousands.



□

You Try It!

The following table shows the number of hybrid cars (in thousands) by country.

Country	Number
U.S.	279
Japan	77
Canada	17
U.K.	14
Netherlands	11

Create a bar chart showing the number of cars versus the country of use.

You Try It!

The following table show Alphonso's percentage scores on his examinations in mathematics.

Exam	Percentage
Exam #1	52
Exam #2	45
Exam #3	72
Exam #4	889
Exam #5	76

Construct a line graph of Alphonso's exam scores versus exam number.

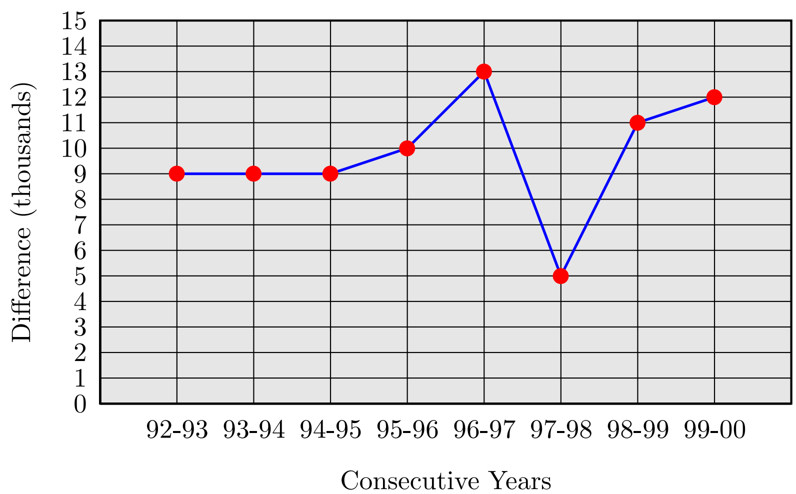
EXAMPLE 9. Using the data in Table 1.2, create a table that shows the differences in consecutive years, then create a line plot of the result. In what consecutive years did the United States see the greatest increase in cars powered by compressed natural gas?

Solution. Table 1.3 shows the differences in consecutive years.

Years	92-93	93-94	94-95	95-96	96-97	97-98	98-99	99-00
Difference	9	9	9	10	13	5	11	12

Table 1.3: Showing the differences in vehicles in consecutive years.

Next, craft a line graph. Place consecutive years on the horizontal axis. At each consecutive year pair, plot a point at a height equal to the difference in alternative fuel vehicles. Connect the points with straight line segments.



Note how the line graph makes it completely clear that the greatest increase in vehicles powered by compressed natural gas occurred in the consecutive years 1996-1997, an increase of 13,000 vehicles.

❧ ❧ ❧ Exercises ☙ ☙ ☙

1. Sketch a number line diagram depicting the sum $3 + 2$, as shown in Figure 1.2 in the narrative of this section.

2. Sketch a number line diagram depicting the sum $3 + 5$, as shown in Figure 1.2 in the narrative of this section.

3. Sketch a number line diagram depicting the sum $3 + 4$, as shown in Figure 1.2 in the narrative of this section.

4. Sketch a number line diagram depicting the sum $2 + 4$, as shown in Figure 1.2 in the narrative of this section.

5. Sketch a number line diagram depicting the sum $4 + 2$, as shown in Figure 1.2 in the narrative of this section.

6. Sketch a number line diagram depicting the sum $4 + 3$, as shown in Figure 1.2 in the narrative of this section.

7. Sketch a number line diagram depicting the sum $2 + 5$, as shown in Figure 1.2 in the narrative of this section.

8. Sketch a number line diagram depicting the sum $4 + 5$, as shown in Figure 1.2 in the narrative of this section.

9. Sketch a number line diagram depicting the sum $4 + 4$, as shown in Figure 1.2 in the narrative of this section.

10. Sketch a number line diagram depicting the sum $3 + 3$, as shown in Figure 1.2 in the narrative of this section.

In Exercises 11-28, determine which property of addition is depicted by the given identity.

11. $28 + 0 = 28$

12. $53 + 0 = 53$

13. $24 + 0 = 24$

14. $93 + 0 = 93$

15. $(51 + 66) + 88 = 51 + (66 + 88)$

16. $(90 + 96) + 4 = 90 + (96 + 4)$

17. $64 + 39 = 39 + 64$

18. $68 + 73 = 73 + 68$

19. $(70 + 27) + 52 = 70 + (27 + 52)$

20. $(8 + 53) + 81 = 8 + (53 + 81)$

21. $79 + 0 = 79$

22. $42 + 0 = 42$

23. $10 + 94 = 94 + 10$

24. $55 + 86 = 86 + 55$

25. $47 + 26 = 26 + 47$

26. $62 + 26 = 26 + 62$

27. $(61 + 53) + 29 = 61 + (53 + 29)$

28. $(29 + 96) + 61 = 29 + (96 + 61)$

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29. Sketch a number line diagram depicting the difference $8-2$, as shown in Figure 1.5 in the narrative of this section.

30. Sketch a number line diagram depicting the difference $8-4$, as shown in Figure 1.5 in the narrative of this section.

31. Sketch a number line diagram depicting the difference $7-2$, as shown in Figure 1.5 in the narrative of this section.

32. Sketch a number line diagram depicting the difference $9-5$, as shown in Figure 1.5 in the narrative of this section.

33. Sketch a number line diagram depicting the difference $7-4$, as shown in Figure 1.5 in the narrative of this section.

34. Sketch a number line diagram depicting the difference $6-4$, as shown in Figure 1.5 in the narrative of this section.

35. Sketch a number line diagram depicting the difference $9-4$, as shown in Figure 1.5 in the narrative of this section.

36. Sketch a number line diagram depicting the difference $6-5$, as shown in Figure 1.5 in the narrative of this section.

37. Sketch a number line diagram depicting the difference $8-5$, as shown in Figure 1.5 in the narrative of this section.

38. Sketch a number line diagram depicting the difference $9-3$, as shown in Figure 1.5 in the narrative of this section.

In Exercises 39-50, simplify the given expression.

39. $16 - 8 + 2$

40. $17 - 3 + 5$

41. $20 - 5 + 14$

42. $14 - 5 + 6$

43. $15 - 2 + 5$

44. $13 - 4 + 2$

45. $12 - 5 + 4$

46. $19 - 4 + 13$

47. $12 - 6 + 4$

48. $13 - 4 + 18$

49. $15 - 5 + 8$

50. $13 - 3 + 11$

In Exercises 51-58, the width W and length L of a rectangle are given. Find the perimeter P of the rectangle.

51. $W = 7$ in, $L = 9$ in

52. $W = 4$ in, $L = 6$ in

53. $W = 8$ in, $L = 9$ in

54. $W = 5$ in, $L = 9$ in

55. $W = 4$ cm, $L = 6$ cm

56. $W = 5$ in, $L = 8$ in

57. $W = 4$ cm, $L = 7$ cm

58. $W = 4$ in, $L = 9$ in

In Exercises 59-66, the length s of a side of a square is given. Find the perimeter P of the square.

59. $s = 25$ cm

60. $s = 21$ in

61. $s = 16$ cm

62. $s = 10$ in

63. $s = 18$ in

64. $s = 7$ in

65. $s = 3$ in

66. $s = 20$ in

In Exercises 67-86, find the sum.

67. $3005 + 5217$

68. $1870 + 5021$

69. $575 + 354 + 759$

70. $140 + 962 + 817$

71. $472 + (520 + 575)$

72. $318 + (397 + 437)$

73. $274 + (764 + 690)$

74. $638 + (310 + 447)$

75. $8583 + 592$

76. $5357 + 9936$

77. $899 + 528 + 116$

78. $841 + 368 + 919$

79. $(466 + 744) + 517$

80. $(899 + 996) + 295$

81. $563 + 298 + 611 + 828$

82. $789 + 328 + 887 + 729$

83. $607 + 29 + 270 + 245$

84. $738 + 471 + 876 + 469$

85. $(86 + 557) + 80$

86. $(435 + 124) + 132$

In Exercises 87-104, find the difference.

87. $3493 - 2034 - 227$

88. $3950 - 1530 - 2363$

89. $8338 - 7366$

90. $2157 - 1224$

91. $2974 - 2374$

92. $881 - 606$

93. $3838 - (777 - 241)$

94. $8695 - (6290 - 4233)$

95. $5846 - 541 - 4577$

96. $5738 - 280 - 4280$

97. $3084 - (2882 - 614)$

98. $1841 - (217 - 28)$

99. $2103 - (1265 - 251)$

100. $1471 - (640 - 50)$

101. $9764 - 4837 - 150$

102. $9626 - 8363 - 1052$

103. $7095 - 226$

104. $4826 - 1199$

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105. **Water Subsidies.** Since the drought began in 2007, California farms have received \$79 million in water subsidies. California cotton and rice farmers received an additional \$439 million. How much total water subsidies have farmers received? *Associated Press Times-Standard 4/15/09*

106. **War Budget.** The 2010 Federal budget allocates $534 billion for the Department of Defense base programs and an additional $130 billion for the nation's two wars. How much will the Department of Defense receive altogether? *Associated Press Times-Standard 5/8/09*

107. **Sun Frost.** Arcata, CA is home to Sun Frost, a manufacturer of highly efficient refrigerators and freezers. The AC model RF12 refrigerator/freezer costs $2,279 while an R16 model refrigerator/freezer costs $3,017. How much more does the R16 model cost? *Source: www.sunfrost.com/retail_pricelist.html*

108. **Shuttle Orbit.** The space shuttle usually orbits at 250 miles above the surface of the earth. To service the Hubble Space Telescope, the shuttle had to go to 350 miles above the surface. How much higher did the shuttle have to orbit?

109. **Earth's Orbit.** Earth orbits the sun in an ellipse. When earth is at its closest to the sun, called *perihelion*, earth is about 147 million kilometers. When earth is at its furthest point from the sun, called *aphelion*, earth is about 152 million kilometers from the sun. What's the difference in millions of kilometers between aphelion and perihelion?

110. **Pluto's Orbit.** Pluto's orbit is highly eccentric. Find the difference between Pluto's closest approach to the sun and Pluto's furthest distance from the sun if Pluto's perihelion (closest point on its orbit about the sun) is about 7 billion kilometers and its aphelion (furthest point on its orbit about the sun) is about 30 billion kilometers.

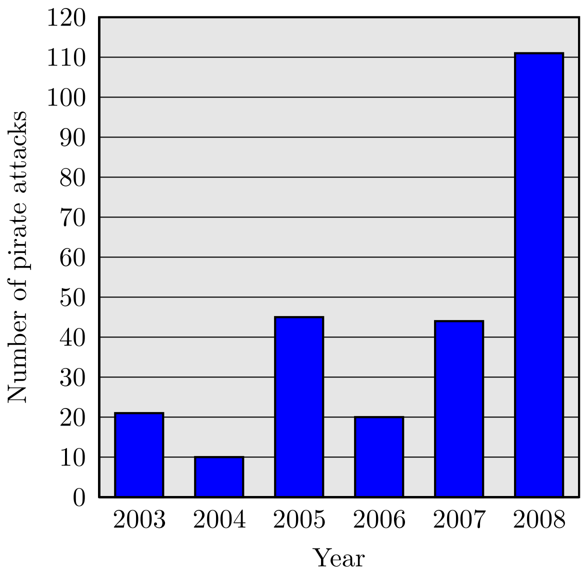
111. **Sunspot Temperature.** The surface of the sun is about 10,000 degrees Fahrenheit. Sunspots are darker regions on the surface of the sun that have a relatively cooler temperature of 6,300 degrees Fahrenheit. How many degrees cooler are sunspots?

112. **Jobs.** The Times-Standard reports that over the next year, the credit- and debit-card processing business Humboldt Merchant Services expects to cut 36 of its 80 jobs, but then turn around and hire another 21. How many people will be working for the company then? *Times-Standard 5/6/09*

113. **Wild tigers.** The chart shows the estimated wild tiger population, by region. According to this chart, what is the total wild tiger population worldwide? *Associated Press-Times-Standard 01/24/10 Pressure mounts to save the tiger.*

Region	Tiger population
India, Nepal and Bhutan	1650
China and Russia	450
Bangladesh	250
Sumatra (Indonesia)	400
Malaysia	500
other SE Asia	350

114. Pirate Attacks. The following bar chart tracks pirate attacks off the coast of Somalia.



Source: ICC International Maritime Bureau, AP Times-Standard, 4/15/2009

a) How many pirate attacks were there in 2003, 2004, and 2005 combined?

b) How many pirate attacks were there in 2006, 2007, and 2008 combined?

c) How many more pirate attacks were there in 2008 than in 2007?

115. Emily shows improvement on each successive examination throughout the term. Her exam scores are recorded in the following table.

Exam	Score
Exam #1	48
Exam #2	51
Exam #3	54
Exam #4	59
Exam #5	67
Exam #6	70

a) Create a bar plot for Emily's examination scores. Place the examination numbers on the horizontal axis in the same order shown in the table above.

b) Create a table that shows successive differences in examination scores. Make a line plot of these differences. Between which two exams did Emily show the greatest improvement?

116. Jason shows improvement on each successive examination throughout the term. His exam scores are recorded in the following table.

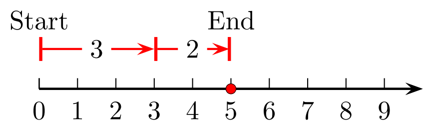
Exam	Score
Exam #1	34
Exam #2	42
Exam #3	45
Exam #4	50
Exam #5	57
Exam #6	62

a) Create a bar plot for Jason's examination scores. Place the examination numbers on the horizontal axis in the same order shown in the table above.

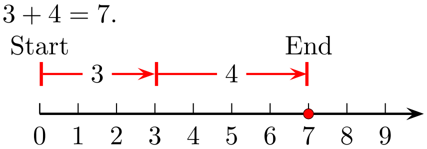
b) Create a table that shows successive differences in examination scores. Make a line plot of these differences. Between which two exams did Jason show the greatest improvement?

Answers

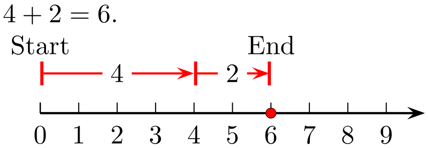
1. $3 + 2 = 5$.



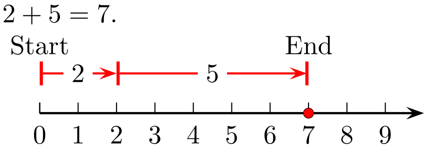
3. $3 + 4 = 7$.



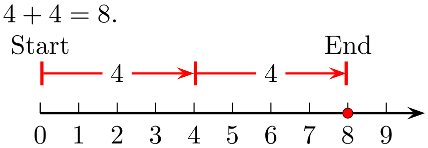
5. $4 + 2 = 6$.



7. $2 + 5 = 7$.



9. $4 + 4 = 8$.



11. Additive identity property of addition.

13. Additive identity property of addition.

15. Associative property of addition

17. Commutative property of addition

19. Associative property of addition

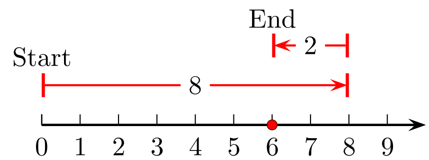
21. Additive identity property of addition.

23. Commutative property of addition

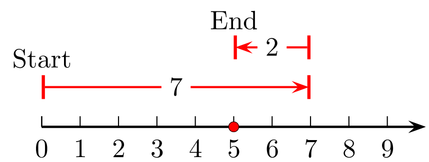
25. Commutative property of addition

27. Associative property of addition

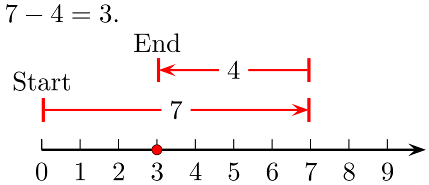
29. $8 - 2 = 6$.



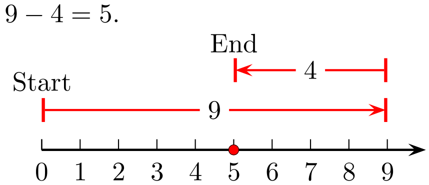
31. $7 - 2 = 5$.



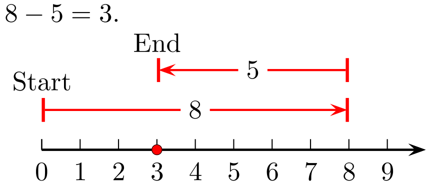
33. $7 - 4 = 3$.



35. $9 - 4 = 5$.



37. $8 - 5 = 3$.



39. 10

41. 29

43. 18

45. 11

47. 10

49. 18

51. $P = 32$ in

53. $P = 34$ in

55. $P = 20$ cm

57. $P = 22$ cm

59. $P = 100$ cm

61. $P = 64$ cm

63. $P = 72$ in

65. $P = 12$ in

67. 8222

69. 1688

71. 1567

73. 1728

75. 9175

77. 1543

79. 1727

81. 2300

83. 1151

85. 723

87. 1232

89. 972

91. 600

93. 3302

95. 728

97. 816

99. 1089

101. 4777

103. 6869

105. $518 million

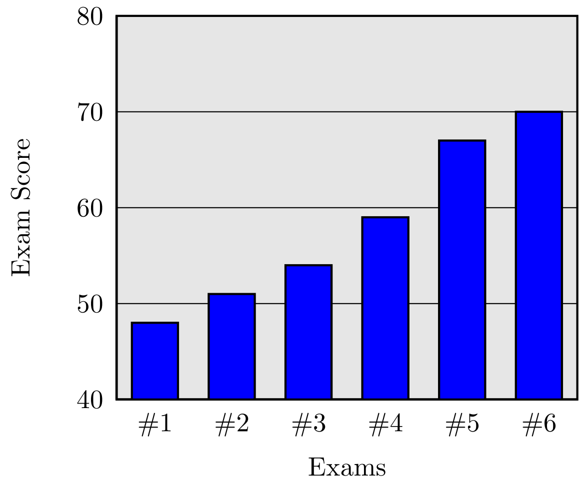
107. $738

109. 5 million kilometers

111. 3,700 degrees Fahrenheit

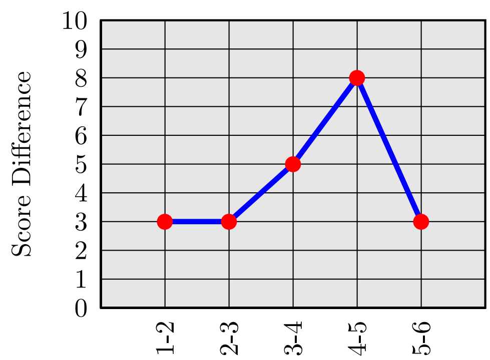
113. 3600

115. a) Bar chart.



b) Line plot of consecutive differences.

The line plot of consecutive examination score differences.



The largest improvement was between Exam #4 and Exam #5, where Emily improved by 8 points.

1.3 Multiplication and Division of Whole Numbers

We begin this section by discussing multiplication of whole numbers. The first order of business is to introduce the various symbols used to indicate multiplication of two whole numbers.

Mathematical symbols that indicate multiplication.

Symbol		Example
$\times$	times symbol	3×4
$\cdot$	dot	$3 \cdot 4$
( )	parentheses	(3)(4) or 3(4) of (3)4

Products and Factors. In the expression $3 \cdot 4$, the whole numbers 3 and 4 are called the **factors** and $3 \cdot 4$ is called the **product**.

The key to understanding multiplication is held in the following statement.

Multiplication is equivalent to repeated addition.

Suppose, for example, that we would like to evaluate the product $3 \cdot 4$. Because multiplication is equivalent to repeated addition, $3 \cdot 4$ is equivalent to adding three fours. That is,

$$3 \cdot 4 = \underbrace{4 + 4 + 4}_{\text{three fours}}$$

Thus, $3 \cdot 4 = 12$. You can visualize the product $3 \cdot 4$ as the sum of three fours on a number line, as shown in Figure 1.6.

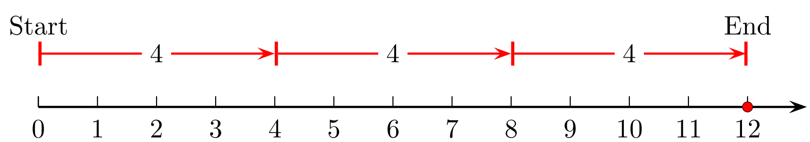


Figure 1.6: Note that $3 \cdot 4 = 4 + 4 + 4$. That is, $3 \cdot 4 = 12$.

Like addition, the order of the factors does not matter.

$$4 \cdot 3 = \underbrace{3 + 3 + 3 + 3}_{\text{four threes}}$$

Thus, $4 \cdot 3 = 12$. Consider the visualization of $4 \cdot 3$ in Figure 1.7.

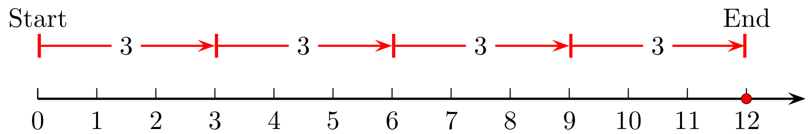


Figure 1.7: Note that $4 \cdot 3 = 3 + 3 + 3 + 3$. That is, $4 \cdot 3 = 12$.

The evidence in Figure 1.6 and Figure 1.7 show us that multiplication is *commutative*. That is,

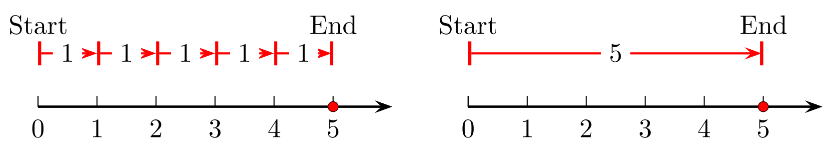
$$3 \cdot 4 = 4 \cdot 3.$$

Commutative Property of Multiplication. If a and b are any whole numbers, then

$$a \cdot b = b \cdot a.$$

The Multiplicative Identity

In Figure 1.8(a), note that five ones equals 5; that is, $5 \cdot 1 = 5$. On the other hand, in Figure 1.8(b), we see that one five equals five; that is, $1 \cdot 5 = 5$.



(a) Note that $5 \cdot 1 = 1 + 1 + 1 + 1 + 1$.

(b) Note that $1 \cdot 5 = 5$.

Figure 1.8: Note that $5 \cdot 1 = 5$ and $1 \cdot 5 = 5$.

Because multiplying a whole number by 1 equals that identical number, the whole number 1 is called the *multiplicative identity*.

The Multiplicative Identity Property. If a is any whole number, then

$$a \cdot 1 = a \qquad \text{and} \qquad 1 \cdot a = a.$$

Multiplication by Zero

Because $3 \cdot 4 = 4 + 4 + 4$, we can say that the product $3 \cdot 4$ represents "3 sets of 4," as depicted in Figure 1.9, where three groups of four boxes are each enveloped in an oval.

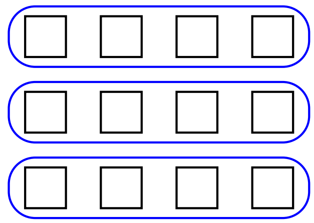


Figure 1.9: Three sets of four: $3 \cdot 4 = 12$.

Therefore, $0 \cdot 4$ would mean zero sets of four. Of course, zero sets of four is zero.

Multiplication by Zero. If a represents any whole number, then

$$a \cdot 0 = 0 \qquad \text{and} \qquad 0 \cdot a = 0.$$

The Associative Property of Multiplication

Like addition, multiplication of whole numbers is associative. Indeed,

$$\begin{aligned} 2 \cdot (3 \cdot 4) &= 2 \cdot 12 \\ &= 24, \end{aligned}$$

and

$$\begin{aligned} (2 \cdot 3) \cdot 4 &= 6 \cdot 4 \\ &= 24. \end{aligned}$$

The Associative Property of Multiplication. If a, b, and c are any whole numbers, then

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

Multiplying Larger Whole Numbers

Much like addition and subtraction of large whole numbers, we will also need to multiply large whole numbers. Again, we hope the algorithm is familiar from previous coursework.

You Try It!

Simplify: $56 \cdot 335$.

EXAMPLE 1. Simplify: $35 \cdot 127$.

Solution. Align the numbers vertically. The order of multiplication does not matter, but we'll put the larger of the two numbers on top of the smaller number. The first step is to multiply 5 times 127. Again, we proceed from right to left. So, 5 times 7 is 35. We write the 5, then carry the 3 to the tens column. Next, 5 times 2 is 10. Add the carry digit 3 to get 13. Write the 3 and carry the 1 to the hundreds column. Finally, 5 times 1 is 5. Add the carry digit to get 6.

$$\begin{array}{rrr} {\scriptstyle 1} & {\scriptstyle 3} & \\ 1 & 2 & 7 \\ \times & 3 & 5 \\ \hline 6 & 3 & 5 \end{array}$$

The next step is to multiply 3 times 127. However, because 3 is in the tens place, its value is 30, so we actually multiply 30 times 126. This is the same as multiplying 127 by 3 and placing a 0 at the end of the result.

$$\begin{array}{rrrr} & & {\scriptstyle 2} & \\ & 1 & 2 & 7 \\ & \times & 3 & 5 \\ \hline & 6 & 3 & 5 \\ 3 & 8 & 1 & 0 \\ \hline \end{array}$$

After adding the 0, 3 times 7 is 21. We write the 1 and carry the 2 above the 2 in the tens column. Then, 3 times 2 is 6. Add the carry digit 2 to get 8. Finally, 3 times 1 is 1.

All that is left to do is to add the results.

$$\begin{array}{rrrr} & 1 & 2 & 7 \\ & \times & 3 & 5 \\ \hline & 6 & 3 & 5 \\ 3 & 8 & 1 & 0 \\ \hline 4 & 4 & 4 & 5 \end{array}$$

Thus, $35 \cdot 127 = 4,445$.

Alternate Format. It does not hurt to omit the trailing zero in the second step of the multiplication, where we multiply 3 times 127. The result would look like this:

$$\begin{array}{rrrr} & 1 & 2 & 7 \\ & \times & 3 & 5 \\ \hline & 6 & 3 & 5 \\ 3 & 8 & 1 & \\ \hline 4 & 4 & 4 & 5 \end{array}$$

In this format, the zero is understood, so it is not necessary to have it physically present. The idea is that with each multiplication by a new digit, we indent the product one space from the right.

Answer: $18,760$

Division of Whole Numbers

We now turn to the topic of division of whole numbers. We first introduce the various symbols used to indicate division of whole numbers.

Mathematical symbols that indicate division.

Symbol		Example
$\div$	division symbol	$12 \div 4$
$-$	fraction bar	$\frac{12}{4}$
$\overline{)}$	division bar	$4\overline{)12}$

Note that each of the following say the same thing; that is, "12 divided by 4 is 3."

$$12 \div 4 = 3 \qquad \text{or} \qquad \frac{12}{4} = 3 \qquad \text{or} \qquad \begin{array}{r} 3 \\ 4\overline{)12} \end{array}$$

Quotients, Dividends, and Divisors. In the statement

$$\begin{array}{r} 3 \\ 4\overline{)12} \end{array}$$

the whole number 12 is called the *dividend*, the whole number 4 is called the *divisor*, and the whole number 3 is called the *quotient*. Note that this division bar notation is equivalent to

$$12 \div 4 = 3 \qquad \text{and} \qquad \frac{12}{4} = 3.$$

The expression a/b means "a divided by b," but this construct is also called a *fraction*.

Fraction. The expression

$$\frac{a}{b}$$

is called a *fraction*. The number a on top is called the *numerator* of the fraction; the number b on the bottom is called the *denominator* of the fraction.

The key to understanding division of whole numbers is contained in the following statement.

Division is equivalent to repeated subtraction.

Suppose for example, that we would like to divide the whole number 12 by the whole number 4. This is equivalent to asking the question "how many fours can we subtract from 12?" This can be visualized in a number line diagram, such as the one in Figure 1.10.

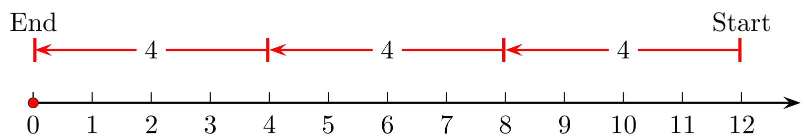


Figure 1.10: Division is repeated subtraction.

In Figure 1.10, note that we if we subtract three fours from twelve, the result is zero. In symbols,

$$12 - \underbrace{4 - 4 - 4}_{\text{three fours}} = 0.$$

Equivalently, we can also ask "How many groups of four are there in 12," and arrange our work as shown in Figure 1.11, where we can see that in an array of twelve objects, we can circle three groups of four ; i.e., $12 \div 4 = 3$.

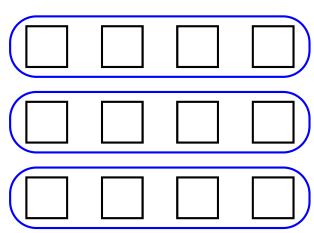


Figure 1.11: There are three groups of four in twelve.

In Figure 1.10 and Figure 1.11, note that the division (repeated subtraction) leaves no remainder. This is not always the case.

You Try It!

Use both the number line approach and the array of boxes approach to divide 12 by 5.

EXAMPLE 2. Divide 7 by 3.

Solution. In Figure 1.12, we see that we can subtract two threes from seven, leaving a remainder of one.

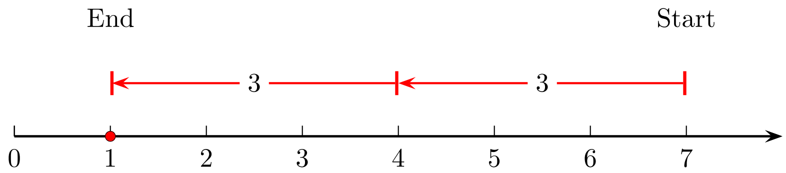


Figure 1.12: Division with a remainder.

Alternatively, in an array of seven objects, we can circle two groups of three, leaving a remainder of one.

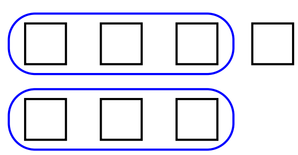


Figure 1.13: Dividing seven by three leaves a remainder of one.

Both Figure 1.12 and Figure1.13 show that there are two groups of three in seven, with one left over. We say "Seven divided by three is two, with a remainder of one."

□

Division is not Commutative

When dividing whole numbers, the order matters. For example,

$$12 \div 4 = 3,$$

but $4 \div 12$ is not even a whole number. Thus, if a and b are whole numbers, then $a \div b$ does **not** have to be the same as $b \div a$.

Division is not Associative

When you divide three numbers, the order in which they are grouped will usually affect the answer. For example,

$$\begin{aligned}(48 \div 8) \div 2 &= 6 \div 2 \\ &= 3,\end{aligned}$$

but

$$\begin{aligned} 48 \div (8 \div 2) &= 48 \div 4 \\ &= 12. \end{aligned}$$

Thus, if a, b, and c are whole numbers, $(a \div b) \div c$ does **not** have to be the same as $a \div (b \div c)$.

Division by Zero is Undefined

Suppose that we are asked to divide six by zero; that is, we are asked to calculate $6 \div 0$. In Figure 1.14, we have an array of six objects.

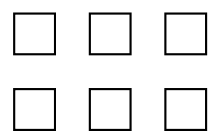


Figure 1.14: How many groups of zero do you see?

Now, to divide six by zero, we must answer the question "How many groups of zero can we circle in Figure 1.14?" Some thought will provide the answer: This is a meaningless request! It makes absolutely no sense to ask how many groups of zero can be circled in the array of six objects in Figure 1.14.

Division by Zero. Division by zero is **undefined**. Each of the expressions

$$6 \div 0 \qquad \text{and} \qquad \frac{6}{0} \qquad \text{and} \qquad 0\overline{)6}$$

is **undefined**.

On the other hand, it make sense to ask "What is zero divided by six?" If we create an array of zero objects, then ask how many groups of six we can circle, the answer is "zero groups of six." That is, zero divided by six is zero.

$$0 \div 6 = 0 \qquad \text{and} \qquad \frac{0}{6} = 0 \qquad \text{and} \qquad \begin{array}{r} 0 \\ 6\overline{)0}. \end{array}$$

Dividing Larger Whole Numbers

We'll now provide a quick review of division of larger whole numbers, using an algorithm that is commonly called *long division*. This is not meant to be a thorough discussion, but a cursory one. We're counting on the fact that our readers have encountered this algorithm in previous courses and are familiar with the process.

You Try It!

Divide: 980/35

EXAMPLE 3. Simplify: 575/23.

Solution. We begin by estimating how many times 23 will divide into 57, guessing 1. We put the 1 in the quotient above the 7, multiply 1 times 23, place the answer underneath 57, then subtract.

$$\begin{array}{r} 1 \\ 23\overline{)575} \\ \underline{23} \\ 34 \end{array}$$

Because the remainder is larger than the divisor, our estimate is too small. We try again with an estimate of 2.

$$\begin{array}{r} 2 \\ 23\overline{)575} \\ \underline{46} \\ 11 \end{array}$$

That's the algorithm. Divide, multiply, then subtract. You may continue only when the remainder is smaller than the divisor.

To continue, bring down the 5, estimate that 115 divided by 23 is 5, then multiply 5 times the divisor and subtract.

$$\begin{array}{r} 25 \\ 23\overline{)575} \\ \underline{46} \\ 115 \\ \underline{115} \\ 0 \end{array}$$

Because the remainder is zero, $575/23 = 25$.

Answer: 28

Application — Counting Rectangular Arrays

Consider the rectangular array of stars in Figure 1.15.

To count the number of stars in the array, we could use brute force, counting each star in the array one at a time, for a total of 20 stars. However, as we have four rows of five stars each, it is much faster to multiply: $4 \cdot 5 = 20$ stars.

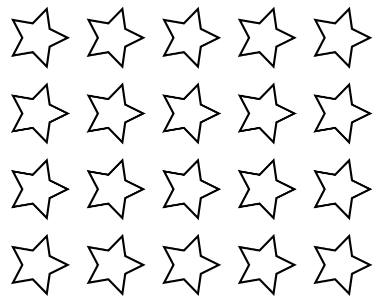
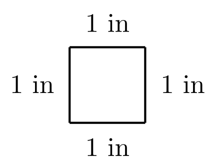
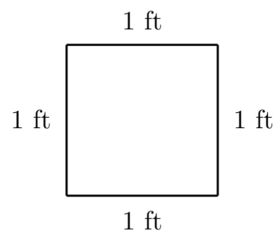


Figure 1.15: Four rows and five columns.



(a) One square inch.



(b) One square foot.

Figure 1.16: Measures of area are in square units.

Application — Area

In Figure 1.16(a), pictured is one square inch ($1\,\text{in}^2$), a square with one inch on each side. In Figure 1.16(b), pictured is one square foot ($1\,\text{ft}^2$), a square with one foot on each side. Both of these squares are *measures of area*.

Now, consider the rectangle shown in Figure 1.17. The length of this rectangle is four inches (4 in) and the width is three inches (3 in).

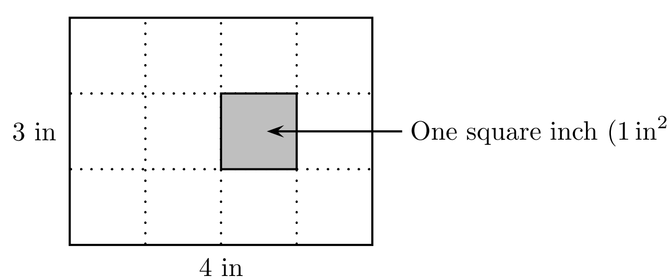
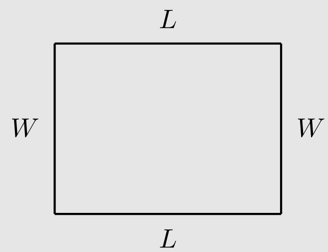


Figure 1.17: A rectangle with length 4 inches and width 3 inches.

To find the area of the figure, we can count the individual units of area that make up the area of the rectangle, twelve square inches ($12\,\text{in}^2$) in all. However, as we did in counting the stars in the array in Figure 1.15, it is much faster to note that we have three rows of four square inches. Hence, it is much faster to multiply the number of squares in each row by the number of squares in each column: $4 \cdot 3 = 12$ square inches.

The argument presented above leads to the following rule for finding the area of a rectangle.

Area of a Rectangle. Let L and W represent the length and width of a rectangle, respectively.



To find the area of the rectangle, calculate the product of the length and width. That is, if A represents the area of the rectangle, then the area of the rectangle is given by the formula

$$A = LW.$$

You Try It!

EXAMPLE 4. A rectangle has width 5 feet and length 12 feet. Find the area of the rectangle.

Solution. Substitute $L = 12\,\text{ft}$ and $W = 5\,\text{ft}$ into the area formula.

$$\begin{aligned} A &= LW \\ &= (12\,\text{ft})(5\,\text{ft}) \\ &= 60\,\text{ft}^2 \end{aligned}$$

Hence, the area of the rectangle is 60 square feet. □

A rectangle has width 17 inches and length 33 inches. Find the area of the rectangle.

Answer: 561 square inches.

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-4 use number line diagrams as shown in Figure 1.6 to depict the multiplication.

1. $2 \cdot 4$.

2. $3 \cdot 4$.

3. $4 \cdot 2$.

4. $4 \cdot 3$.

In Exercises 5-16, state the property of multiplication depicted by the given identity.

5. $9 \cdot 8 = 8 \cdot 9$

6. $5 \cdot 8 = 8 \cdot 5$

7. $8 \cdot (5 \cdot 6) = (8 \cdot 5) \cdot 6$

8. $4 \cdot (6 \cdot 5) = (4 \cdot 6) \cdot 5$

9. $6 \cdot 2 = 2 \cdot 6$

10. $8 \cdot 7 = 7 \cdot 8$

11. $3 \cdot (5 \cdot 9) = (3 \cdot 5) \cdot 9$

12. $8 \cdot (6 \cdot 4) = (8 \cdot 6) \cdot 4$

13. $21 \cdot 1 = 21$

14. $39 \cdot 1 = 39$

15. $13 \cdot 1 = 13$

16. $44 \cdot 1 = 44$

In Exercises 17-28, multiply the given numbers.

17. $78 \cdot 3$

18. $58 \cdot 7$

19. $907 \cdot 6$

20. $434 \cdot 80$

21. $128 \cdot 30$

22. $454 \cdot 90$

23. $799 \cdot 60$

24. $907 \cdot 20$

25. $14 \cdot 70$

26. $94 \cdot 90$

27. $34 \cdot 90$

28. $87 \cdot 20$

In Exercises 29-40, multiply the given numbers.

29. $237 \cdot 54$

30. $893 \cdot 94$

31. $691 \cdot 12$

32. $823 \cdot 77$

33. $955 \cdot 89$

34. $714 \cdot 41$

35. $266 \cdot 61$

36. $366 \cdot 31$

37. $365 \cdot 73$

38. $291 \cdot 47$

39. $955 \cdot 57$

40. $199 \cdot 33$

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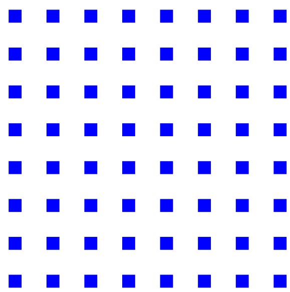
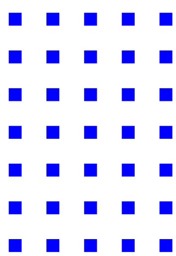
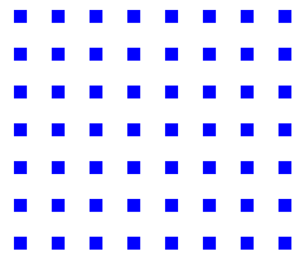
41. Count the number of objects in the array.



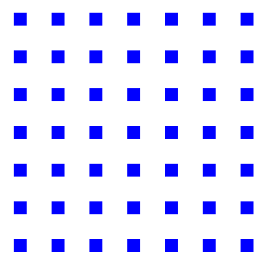
42. Count the number of objects in the array.



43. Count the number of objects in the array.



44. Count the number of objects in the array.



In Exercises 45-48, find the area of the rectangle having the given length and width.

45. $L = 50$ in, $W = 25$ in

46. $L = 48$ in, $W = 24$ in

47. $L = 47$ in, $W = 13$ in

48. $L = 19$ in, $W = 10$ in

In Exercises 49-52, find the perimeter of the rectangle having the given length and width.

49. $L = 25$ in, $W = 16$ in

50. $L = 34$ in, $W = 18$ in

51. $L = 30$ in, $W = 28$ in

52. $L = 41$ in, $W = 25$ in

---

53. A set of beads costs 50 cents per dozen. What is the cost (in dollars) of 19 dozen sets of beads?

54. A set of beads costs 60 cents per dozen. What is the cost (in dollars) of 7 dozen sets of beads?

55. If a math tutor worked for 47 hours and was paid \$15 each hour, how much money would she have made?

56. If a math tutor worked for 46 hours and was paid \$11 each hour, how much money would he have made?

57. There are 12 eggs in one dozen, and 12 dozen in one gross. How many eggs are in a shipment of 24 gross?

58. There are 12 eggs in one dozen, and 12 dozen in one gross. How many eggs are in a shipment of 11 gross?

59. If bricks weigh 4 kilograms each, what is the weight (in kilograms) of 5000 bricks?

60. If bricks weigh 4 pounds each, what is the weight (in pounds) of 2000 bricks?

In Exercises 61-68, which of the following four expressions differs from the remaining three?

61. $\dfrac{30}{5}$, $30 \div 5$, $5\overline{)30}$, $5 \div 30$

62. $\dfrac{12}{2}$, $12 \div 2$, $2\overline{)12}$, $2 \div 12$

63. $\dfrac{8}{2}$, $8 \div 2$, $2\overline{)8}$, $8\overline{)2}$

64. $\dfrac{8}{4}$, $8 \div 4$, $4\overline{)8}$, $8\overline{)4}$

65. $2\overline{)14}$, $14\overline{)2}$, $\dfrac{14}{2}$, $14 \div 2$

66. $9\overline{)54}$, $54\overline{)9}$, $\dfrac{54}{9}$, $54 \div 9$

67. $3\overline{)24}$, $3 \div 24$, $\dfrac{24}{3}$, $24 \div 3$

68. $3\overline{)15}$, $3 \div 15$, $\dfrac{15}{3}$, $15 \div 3$

In Exercises 69-82, simplify the given expression. If the answer doesn't exist or is undefined, write "undefined".

69. $0 \div 11$

70. $0 \div 5$

71. $17 \div 0$

72. $24 \div 0$

73. $10 \cdot 0$

74. $20 \cdot 0$

75. $\dfrac{7}{0}$

76. $\dfrac{23}{0}$

77. $16\overline{)0}$

78. $25\overline{)0}$

79. $\dfrac{0}{24}$

80. $\dfrac{0}{22}$

81. $0\overline{)0}$

82. $0 \div 0$

In Exercises 83-94, divide the given numbers.

83. $\frac{2816}{44}$

84. $\frac{1998}{37}$

85. $\frac{2241}{83}$

86. $\frac{2716}{97}$

87. $\frac{3212}{73}$

88. $\frac{1326}{17}$

89. $\frac{8722}{98}$

90. $\frac{1547}{91}$

91. $\frac{1440}{96}$

92. $\frac{2079}{27}$

93. $\frac{8075}{85}$

94. $\frac{1587}{23}$

In Exercises 95-106, divide the given numbers.

95. $\frac{17756}{92}$

96. $\frac{46904}{82}$

97. $\frac{11951}{19}$

98. $\frac{22304}{41}$

99. $\frac{18048}{32}$

100. $\frac{59986}{89}$

101. $\frac{29047}{31}$

102. $\frac{33264}{36}$

103. $\frac{22578}{53}$

104. $\frac{18952}{46}$

105. $\frac{12894}{14}$

106. $\frac{18830}{35}$

---

107. A concrete sidewalk is laid in square blocks that measure 6 feet on each side. How many blocks will there be in a walk that is 132 feet long?

108. A concrete sidewalk is laid in square blocks that measure 5 feet on each side. How many blocks will there be in a walk that is 180 feet long?

109. One boat to the island can take 5 people. How many trips will the boat have to take in order to ferry 38 people to the island? (Hint: Round up your answer.)

110. One boat to the island can take 4 people. How many trips will the boat have to take in order to ferry 46 people to the island? (Hint: Round up your answer.)

111. If street lights are placed at most 145 feet apart, how many street lights will be needed for a street that is 4 miles long, assuming that there are lights at each end of the street? (Note: 1 mile = 5280 feet.)

112. If street lights are placed at most 70 feet apart, how many street lights will be needed for a street that is 3 miles long, assuming that there are lights at each end of the street? (Note: 1 mile = 5280 feet.)

113. A concrete sidewalk is laid in square blocks that measure 4 feet on each side. How many blocks will there be in a walk that is 292 feet long?

114. A concrete sidewalk is laid in square blocks that measure 5 feet on each side. How many blocks will there be in a walk that is 445 feet long?

115. One boat to the island can take 3 people. How many trips will the boat have to take in order to ferry 32 people to the island? (Hint: Round up your answer.)

116. One boat to the island can take 4 people. How many trips will the boat have to take in order to ferry 37 people to the island? (Hint: Round up your answer.)

117. If street lights are placed at most 105 feet apart, how many street lights will be needed for a street that is 2 miles long, assuming that there are lights at each end of the street? (Note: 1 mile = 5280 feet.)

118. If street lights are placed at most 105 feet apart, how many street lights will be needed for a street that is 3 miles long, assuming that there are lights at each end of the street? (Note: 1 mile = 5280 feet.)

---

119. **Writing articles.** Eli writes an average of 4 articles a day, five days a week, to support product sales. How many articles does Eli write in one week?

120. **Machine gun.** A 0.50-caliber anti-aircraft machine gun can fire 800 rounds each minute. How many rounds could fire in three minutes? *Associated Press Times-Standard 4/15/09*

121. **Laps.** The swimming pool at CalCourts is 25 yards long. If one lap is up and back again, how many yards has Wendell swam doing 27 laps?

122. **Refrigerator wattage.** A conventional refrigerator will run about 12 hours each day can use 150 Watts of power each hour. How many Watts of power will a refrigerator use over the day?

123. **Horse hay.** A full-grown horse should eat a minimum of 12 pounds of hay each day and may eat much more depending on their weight. How many pounds minimum would a horse eat over a year?

124. **College costs.** After a $662 hike in fees, Califormia residents who want to attend the University of California as an undergraduate should expect to pay $8,700 in for the upcoming academic year 2009-2010. If the cost were to remain the same for the next several years, how much should a student expect to pay for a four-year degree program at a UC school?

125. **Non-resident costs.** Nonresident undergraduates who want to attend a University of California college should expect to pay about $22,000 for the upcoming academic year. Assuming costs remain the same, what can a four-year degree cost?

126. **Student tax.** The mayer of Providence, Rhode Island wants to tax its 25,000 Brown University students $150 each to contribute to tax receipts saying students should pay for the resources they use just like the town residents. How many dollars would the mayer generate?

127. **New iceberg.** A new iceberg, shaved off a glacier after a collision with another iceberg, measures about 48 miles long and 28 miles wide. What's the approximate area of the new iceberg? *Associated Press-Times-Standard 02/27/10 2 Huge icebergs set loose off Antarctica's coast.*

128. **Solar panels.** One of the solar panels on the International Space Station is 34 meters long and 11 meters wide. If there are eight of these, what's the total area for solar collection?

129. **Sidewalk.** A concrete sidewalk is to be 80 foot long and 4 foot wide. How much will it cost to lay the sidewalk at $8 per square foot?

130. **Hay bales.** An average bale of hay weighs about 60 pounds. If a horse eats 12 pounds of hay a day, how many days will one bale feed a horse?

131. **Sunspots.** Sunspots, where the sun's magnetic field is much higher, usually occur in pairs. If the total count of sunspots is 72, how many pairs of sunspots are there?

Answers

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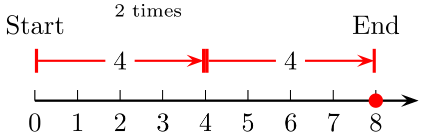
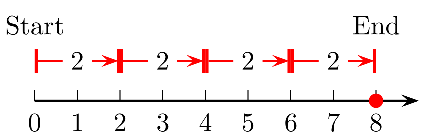
1. $2 \cdot 4 = \underbrace{4 + 4}_{\text{2 times}} = 8$



3. $4 \cdot 2 = \underbrace{2 + 2 + 2 + 2}_{\text{4 times}} = 8$



5. Commutative property of multiplication

7. Associative property of multiplication

9. Commutative property of multiplication

11. Associative property of multiplication

13. Multiplicative identity property

15. Multiplicative identity property

17. 234

19. 5442

21. 3840

23. 47940

25. 980

27. 3060

29. 12798

31. 8292

33. 84995

35. 16226

37. 26645

39. 54435

41. 64

43. 56

45. $1250\,\text{in}^2$

47. $611\,\text{in}^2$

49. 82 in

51. 116 in

53. 9.50

55. 705

57. 3456

59. 20000

61. $5 \div 30$

63. $8\overline{)2}$

65. $14\overline{)2}$

67. $3 \div 24$

69. 0

71. Undefined

73. 0

75. Undefined

77. 0

79. 0

81. Undefined

83. 64

85. 27

87. 44

89. 89

91. 15

93. 95

95. 193

97. 629

99. 564

101. 937

103. 426

105. 921

107. 22

109. 8

111. 147

113. 73

115. 11

117. 102

119. 20 articles

121. 1350 yards

123. 4380 pounds of hay

125. $88,000

127. $1344\,\text{mi}^2$

129. $2,560

131. 36

1.4 Prime Factorization

In the statement $3 \cdot 4 = 12$, the number 12 is called the *product*, while 3 and 4 are called *factors*.

EXAMPLE 1. Find all whole number factors of 18.

Solution. We need to find all whole number pairs whose product equals 18. The following pairs come to mind.

$$1 \cdot 18 = 18 \quad \text{and} \quad 2 \cdot 9 = 18 \quad \text{and} \quad 3 \cdot 6 = 18.$$

Hence, the factors of 18 are (in order) 1, 2, 3, 6, 9, and 18. □

You Try It!

Find all whole number factors of 21.

Answer: 1, 3, 7, and 21.

Divisibility

In Example 1, we saw $3 \cdot 6 = 18$, making 3 and 6 factors of 18. Because division is the inverse of multiplication, that is, divison by a number undoes the multiplication of that number, this immediately provides

$$18 \div 6 = 3 \quad \text{and} \quad 18 \div 3 = 6.$$

That is, 18 is divisible by 3 and 18 is divisible by 6. When we say that 18 is divisible by 3, we mean that when 18 is divided by 3, there is a zero remainder.

Divisible. Let a and b be whole numbers. Then a is *divisible* by b if and only if the remainder is **zero** when a is divided by b. In this case, we say that "b is a *divisor* of a."

EXAMPLE 2. Find all whole number divisors of 18.

Solution. In Example 1, we saw that $3 \cdot 6 = 18$. Therefore, 18 is divisible by both 3 and 6 ($18 \div 3 = 6$ and $18 \div 6 = 3$). Hence, when 18 is divided by 3 or 6, the remainder is zero. Therefore, 3 and 6 are divisors of 18. Noting the other products in Example 1, the complete list of divisors of 18 is 1, 2, 3, 6, 9, and 18. □

You Try It!

Find all whole number divisors of 21.

Answer: 1, 3, 7, and 21.

Example 1 and Example 2 show that when working with whole numbers, the words *factor* and *divisor* are interchangeable.

Factors and Divisors. If

$$c = a \cdot b,$$

then a and b are called *factors* of c. Both a and b are also called *divisors* of c.

Divisibility Tests

There are a number of very useful divisibility tests.

Divisible by 2. If a whole number ends in 0, 2, 4, 6, or 8, then the number is called an **even** number and is divisible by 2. Examples of even numbers are 238 and 1,246 ($238 \div 2 = 119$ and $1,246 \div 2 = 623$). A number that is **not** even is called an **odd** number. Examples of odd numbers are 113 and 2,339.

Divisible by 3. If the sum of the digits of a whole number is divisible by 3, then the number itself is divisible by 3. An example is 141. The sum of the digits is $1 + 4 + 1 = 6$, which is divisible by 3. Therefore, 141 is also divisible by 3 ($141 \div 3 = 47$).

Divisible by 4. If the number represented by the last two digits of a whole number is divisible by 4, then the number itself is divisible by 4. An example is 11,524. The last two digits represent 24, which is divisible by 4 ($24 \div 4 = 6$). Therefore, 11,524 is divisible by 4 ($11,524 \div 4 = 2,881$).

Divisible by 5. If a whole number ends in a zero or a 5, then the number is divisible by 5. Examples are 715 and 120 ($715 \div 5 = 143$ and $120 \div 5 = 24$).

Divisible by 6. If a whole number is divisible by 2 and by 3, then it is divisible by 6. An example is 738. First, 738 is even and divisible by 2. Second, 7+3+8=18, which is divisible by 3. Hence, 738 is divisible by 3. Because 738 is divisible by both 2 and 3, it is divisible by 6 ($738 \div 6 = 123$).

Divisible by 8. If the number represented by the last three digits of a whole number is divisible by 8, then the number itself is divisible by 8. An example is 73,024. The last three digits represent the number 24, which is divisible by 8 ($24 \div 8 = 3$). Thus, 73,024 is also divisible by 8 ($73,024 \div 8 = 9,128$).

Divisible by 9. If the sum of the digits of a whole number is divisible by 9, then the number itself is divisible by 9. An example is 117. The sum of the digits is $1 + 1 + 7 = 9$, which is divisible by 9. Hence, 117 is divisible by 9 ($117 \div 9 = 13$).

Prime Numbers

We begin with the definition of a *prime number*.

Prime Number. A whole number (other than 1) is a *prime number* if its only factors (divisors) are 1 and itself. Equivalently, a number is prime if and only if it has exactly two factors (divisors).

You Try It!

Which of the whole numbers 15, 23, 51, and 59 are prime numbers?

EXAMPLE 3. Which of the whole numbers 12, 13, 21, and 37 are prime numbers?

Solution.

- The factors (divisors) of 12 are 1, 2, 3, 4, 6, and 12. Hence, 12 is **not** a prime number.
- The factors (divisors) of 13 are 1 and 13. Because its only divisors are 1 and itself, 13 **is** a prime number.
- The factors (divisors) of 21 are 1, 3, 7, and 21. Hence, 21 is **not** a prime number.
- The factors (divisors) of 37 are 1 and 37. Because its only divisors are 1 and itself, 37 **is** a prime number.

Answer: 23 and 59.

□

You Try It!

List all the prime numbers less than 100.

EXAMPLE 4. List all the prime numbers less than 20.

Solution. The prime numbers less than 20 are 2, 3, 5, 7, 11, 13, 17, and 19.

□

Composite Numbers. If a whole number is not a prime number, then it is called a *composite number*.

You Try It!

Is the whole number 2,571 prime or composite?

Answer: Composite.

EXAMPLE 5. Is the whole number 1,179 prime or composite?

Solution. Note that $1 + 1 + 7 + 9 = 18$, which is divisible by both 3 and 9. Hence, 3 and 9 are both divisors of 1,179. Therefore, 1,179 is a composite number.

□

Factor Trees

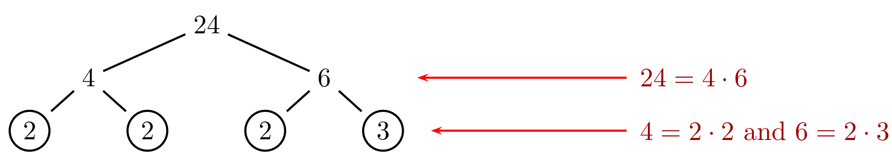
We will now learn how to express a composite number as a unique product of prime numbers. The most popular device for accomplishing this goal is the *factor tree.*

You Try It!

Express 36 as a product of prime factors.

EXAMPLE 6. Express 24 as a product of prime factors.

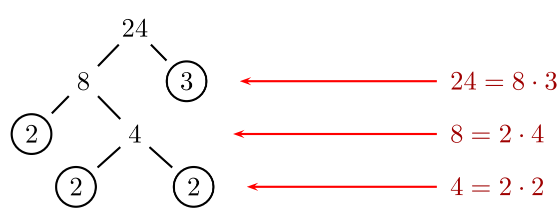
Solution. We use a factor tree to break 24 down into a product of primes.



At each level of the tree, break the current number into a product of two factors. The process is complete when all of the "circled leaves" at the bottom of the tree are prime numbers. Arranging the factors in the "circled leaves" in order,

$$24 = 2 \cdot 2 \cdot 2 \cdot 3.$$

The final answer does not depend on product choices made at each level of the tree. Here is another approach.



The final answer is found by including all of the factors from the "circled leaves" at the end of each branch of the tree, which yields the same result, namely $24 = 2 \cdot 2 \cdot 2 \cdot 3$.

Alternate Approach. Some favor repeatedly dividing by 2 until the result is no longer divisible by 2. Then try repeatedly dividing by the next prime until the result is no longer divisible by that prime. The process terminates when the last resulting quotient is equal to the number 1.

2	24	
2	12	← $24 \div 2 = 12$
2	6	← $12 \div 2 = 6$
3	3	← $6 \div 2 = 3$
	1	← $3 \div 3 = 1$

The first column reveals the prime factorization; i.e., $24 = 2 \cdot 2 \cdot 2 \cdot 3$.

Answer: $2 \cdot 2 \cdot 3 \cdot 3$.

□

The fact that the alternate approach in Example 6 yielded the same result is significant.

Unique Factorization Theorem. Every whole number can be **uniquely** factored as a product of primes.

This result guarantees that if the prime factors are ordered from smallest to largest, everyone will get the same result when breaking a number into a product of prime factors.

Exponents

We begin with the definition of an exponential expression.

Exponents. The expression a^m is defined to mean

$$a^m = \underbrace{a \cdot a \cdot \ldots \cdot a}_{m \text{ times}}.$$

The number a is called the *base* of the exponential expression and the number m is called the *exponent*. The exponent m tells us to repeat the base a as a factor m times.

You Try It!

Evaluate: 3^5.

EXAMPLE 7. Evaluate 2^5, 3^3 and 5^2.

Movie

Solution.

- In the case of 2^5, we have

$$\begin{aligned} 2^5 &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \\ &= 32. \end{aligned}$$

- In the case of 3^3, we have

$$\begin{aligned} 3^3 &= 3 \cdot 3 \cdot 3 \\ &= 27. \end{aligned}$$

- In the case of 5^2, we have

$$\begin{aligned} 5^2 &= 5 \cdot 5 \\ &= 25. \end{aligned}$$

Answer: 243.

□

You Try It!

Prime factor 54.

EXAMPLE 8. Express the solution to Example 6 in compact form using exponents.

Solution. In Example 6, we determined the prime factorization of 24.

$$24 = 2 \cdot 2 \cdot 2 \cdot 3$$

Because $2 \cdot 2 \cdot 2 = 2^3$, we can write this more compactly.

$$24 = 2^3 \cdot 3$$

Answer: $2 \cdot 3 \cdot 3 \cdot 3$.

□

You Try It!

Evaluate: $3^3 \cdot 5^2$.

EXAMPLE 9. Evaluate the expression $2^3 \cdot 3^2 \cdot 5^2$.

Solution. First raise each factor to the given exponent, then perform the multiplication in order (left to right).

$$\begin{aligned} 2^3 \cdot 3^2 \cdot 5^2 &= 8 \cdot 9 \cdot 25 \\ &= 72 \cdot 25 \\ &= 1800 \end{aligned}$$

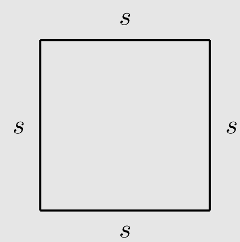
Answer: 675

□

Application

A square is a rectangle with four equal sides.

Area of a Square. Let s represent the length of each side of a square.



Because a square is also a rectangle, we can find the area of the square by multiplying its length and width. However, in this case, the length and width both equal s, so $A = (s)(s) = s^2$. Hence, the formula for the area of a square is

$$A = s^2.$$

You Try It!

EXAMPLE 10. The edge of a square is 13 centimeters. Find the area of the square.

Solution. Substitute $s = 13\,\text{cm}$ into the area formula.

$$\begin{aligned} A &= s^2 \\ &= (13\,\text{cm})^2 \\ &= (13\,\text{cm})(13\,\text{cm}) \\ &= 169\,\text{cm}^2 \end{aligned}$$

Hence, the area of the square is $169\,\text{cm}^2$; i.e., 169 square centimeters. □

The edge of a square is 15 meters. Find the area of the square.

Answer: 225 square meters.

Exercises

In Exercises 1-12, find all divisors of the given number.

1. 30

2. 19

3. 83

4. 51

5. 91

6. 49

7. 75

8. 67

9. 64

10. 87

11. 14

12. 89

In Exercises 13-20, which of the following numbers is **not** divisible by 2?

13. 117, 120, 342, 230

14. 310, 157, 462, 160

15. 30, 22, 16, 13

16. 382, 570, 193, 196

17. 105, 206, 108, 306

18. 60, 26, 23, 42

19. 84, 34, 31, 58

20. 66, 122, 180, 63

In Exercises 21-28, which of the following numbers is **not** divisible by 3?

21. 561, 364, 846, 564

22. 711, 850, 633, 717

23. 186, 804, 315, 550

24. 783, 909, 504, 895

25. 789, 820, 414, 663

26. 325, 501, 945, 381

27. 600, 150, 330, 493

28. 396, 181, 351, 606

In Exercises 29-36, which of the following numbers is **not** divisible by 4?

29. 3797, 7648, 9944, 4048

30. 1012, 9928, 7177, 1592

31. 9336, 9701, 4184, 2460

32. 2716, 1685, 2260, 9788

33. 9816, 7517, 8332, 7408

34. 1788, 8157, 7368, 4900

35. 1916, 1244, 7312, 7033

36. 7740, 5844, 2545, 9368

In Exercises 37-44, which of the following numbers is **not** divisible by 5?

37. 8920, 4120, 5285, 9896

38. 3525, 7040, 2185, 2442

39. 8758, 3005, 8915, 3695

40. 3340, 1540, 2485, 2543

41. 2363, 5235, 4145, 4240

42. 9030, 8000, 5445, 1238

43. 1269, 5550, 4065, 5165

44. 7871, 9595, 3745, 4480

In Exercises 45-52, which of the following numbers is **not** divisible by 6?

45. 328, 372, 990, 528

46. 720, 288, 148, 966

47. 744, 174, 924, 538

48. 858, 964, 930, 330

49. 586, 234, 636, 474

50. 618, 372, 262, 558

51. 702, 168, 678, 658

52. 780, 336, 742, 312

In Exercises 53-60, which of the following numbers is **not** divisible by 8?

53. 1792, 8216, 2640, 5418

54. 2168, 2826, 1104, 2816

55. 8506, 3208, 9016, 2208

56. 2626, 5016, 1392, 1736

57. 4712, 3192, 2594, 7640

58. 9050, 9808, 8408, 7280

59. 9808, 1232, 7850, 7912

60. 3312, 1736, 9338, 3912

In Exercises 61-68, which of the following numbers is **not** divisible by 9?

61. 477, 297, 216, 991

62. 153, 981, 909, 919

63. 153, 234, 937, 675

64. 343, 756, 927, 891

65. 216, 783, 594, 928

66. 504, 279, 307, 432

67. 423, 801, 676, 936

68. 396, 684, 567, 388

In Exercises 69-80, identify the given number as prime, composite, or neither.

69. 19

70. 95

71. 41

72. 88

73. 27

74. 61

75. 91

76. 72

77. 21

78. 65

79. 23

80. 36

In Exercises 81-98, find the prime factorization of the natural number.

81. 224

82. 320

83. 108

84. 96

85. 243

86. 324

87. 160

88. 252

89. 32

90. 128

91. 360

92. 72

93. 144

94. 64

95. 48

96. 200

97. 216

98. 392

In Exercises 99-110, compute the exact value of the given exponential expression.

99. $5^2 \cdot 4^1$

100. $2^3 \cdot 4^1$

101. 0^1

102. 1^3

103. $3^3 \cdot 0^2$

104. $3^3 \cdot 2^2$

105. 4^1

106. 5^2

107. 4^3

108. 4^2

109. $3^3 \cdot 1^2$

110. $5^2 \cdot 2^3$

In Exercises 111-114, find the area of the square with the given side.

111. 28 inches

112. 31 inches

113. 22 inches

114. 13 inches

Create factor trees for each number in Exercises 115-122. Write the prime factorization for each number in compact form, using exponents.

115. 12

116. 18

117. 105

118. 70

119. 56

120. 56

121. 72

122. 270

---

123. **Sieve of Eratosthenes.** This exercise introduces the *Sieve of Eratosthenes*, an ancient algorithm for finding the primes less than a certain number n, first created by the Greek mathematician Eratosthenes. Consider the grid of integers from 2 through 100.

2	3	4	5	6	7	8	9	10	11
12	13	14	15	16	17	18	19	20	21
22	23	24	25	26	27	28	29	30	31
32	33	34	35	36	37	38	39	40	41
42	43	44	45	46	47	48	49	50	51
52	53	54	55	56	57	58	59	60	61
62	63	64	65	66	67	68	69	70	71
72	73	74	75	76	77	78	79	80	81
82	83	84	85	86	87	88	89	90	91
92	93	94	95	96	97	98	99	100	

To find the primes less than 100, proceed as follows.

i) Strike out all multiples of 2 (4, 6, 8, etc.)

ii) The list's next number that has not been struck out is a prime number.

iii) Strike out from the list all multiples of the number you identified in step (ii).

iv) Repeat steps (ii) and (iii) until you can no longer strike any more multiples.

v) All unstruck numbers in the list are primes.

❧ ❧ ❧ Answers ☙ ☙ ☙

1. 1, 2, 3, 5, 6, 10, 15, 30

3. 1, 83

5. 1, 7, 13, 91

7. 1, 3, 5, 15, 25, 75

9. 1, 2, 4, 8, 16, 32, 64

11. 1, 2, 7, 14

13. 117

15. 13

17. 105

19. 31

21. 364

23. 550

25. 820

27. 493

29. 3797

31. 9701

33. 7517

35. 7033

37. 9896

39. 8758

41. 2363

43. 1269

45. 328

47. 538

49. 586

51. 658

53. 5418

55. 8506

57. 2594

59. 7850

61. 991

63. 937

65. 928

67. 676

69. prime

71. prime

73. composite

75. composite

77. composite

79. prime

81. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 7$

83. $2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$

85. $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

87. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$

89. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

91. $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$

93. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$

95. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$

97. $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$

99. 100

101. 0

103. 0

105. 4

107. 64

109. 27

111. $784\,\text{in}^2$

113. $484\,\text{in}^2$

115. $12 = 2^2 \cdot 3$

117. $105 = 3 \cdot 5 \cdot 7$

119. $56 = 2^3 \cdot 7$

121. $72 = 2^3 \cdot 3^2$

123. Unstruck numbers are primes: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

1.5 Order of Operations

The order in which we evaluate expressions can be ambiguous. Take for example, the expression $4 + 3 \cdot 2$. If we do the addition first, then

$$\begin{aligned} 4 + 3 \cdot 2 &= 7 \cdot 2 \\ &= 14. \end{aligned}$$

On the other hand, if we do the multiplication first, then

$$\begin{aligned} 4 + 3 \cdot 2 &= 4 + 6 \\ &= 10. \end{aligned}$$

So, what are we to do?

Of course, grouping symbols can remove the ambiguity.

Grouping Symbols. Parentheses, brackets, or curly braces can be used to group parts of an expression. Each of the following are equivalent:

$$(4 + 3) \cdot 2 \qquad \text{or} \qquad [4 + 3] \cdot 2 \qquad \text{or} \qquad \{4 + 3\} \cdot 2$$

In each case, the rule is "evaluate the expression inside the grouping symbols first." If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.

Thus, for example,

$$\begin{aligned} (4 + 3) \cdot 2 &= 7 \cdot 2 \\ &= 14. \end{aligned}$$

Note how the expression contained in the parentheses was evaluated first.

Another way to avoid ambiguities in evaluating expressions is to establish an order in which operations should be performed. The following guidelines should always be strictly enforced when evaluating expressions.

Rules Guiding Order of Operations. When evaluating expressions, proceed in the following order.

1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
2. Evaluate all exponents that appear in the expression.
3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.

4. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

You Try It!

EXAMPLE 1. Evaluate $4 + 3 \cdot 2$.

Simplify: $8 + 2 \cdot 5$.

Solution. Because of the established *Rules Guiding Order of Operations*, this expression is no longer ambiguous. There are no grouping symbols or exponents, so we immediately go to rule three, evaluate all multiplications and divisions in the order that they appear, moving left to right. After that we invoke rule four, performing all additions and subtractions in the order that they appear, moving left to right.

$$\begin{aligned} 4 + 3 \cdot 2 &= 4 + 6 \\ &= 10 \end{aligned}$$

Thus, $4 + 3 \cdot 2 = 10$.

Answer: 18

You Try It!

EXAMPLE 2. Evaluate $18 - 2 + 3$.

Simplify: $17 - 8 + 2$.

Solution. Follow the *Rules Guiding Order of Operations*. Addition has no precedence over subtraction, nor does subtraction have precedence over addition. We are to perform additions and subtractions as they occur, moving left to right.

$$\begin{aligned} 18 - 2 + 3 &= 16 + 3 && \text{Subtract: } 18 - 2 = 16. \\ &= 19 && \text{Add: } 16 + 3 = 19. \end{aligned}$$

Thus, $18 - 2 + 3 = 19$.

Answer: 11

You Try It!

EXAMPLE 3. Evaluate $54 \div 9 \cdot 2$.

Simplify: $72 \div 9 \cdot 2$.

Solution. Follow the *Rules Guiding Order of Operations*. Division has no precedence over multiplication, nor does multiplication have precedence over division. We are to perform divisions and multiplications as they occur, moving left to right.

$$\begin{aligned} 54 \div 9 \cdot 2 &= 6 \cdot 2 && \text{Divide: } 54 \div 9 = 6. \\ &= 12 && \text{Multiply: } 6 \cdot 2 = 12. \end{aligned}$$

Answer: 16

Thus, $54 \div 9 \cdot 2 = 12$. □

You Try It!

Simplify: $14 + 3 \cdot 4^2$.

EXAMPLE 4. Evaluate $2 \cdot 3^2 - 12$.

Solution. Follow the *Rules Guiding Order of Operations*, exponents first, then multiplication, then subtraction.

$$\begin{aligned} 2 \cdot 3^2 - 12 &= 2 \cdot 9 - 12 && \text{Evaluate the exponent: } 3^2 = 9. \\ &= 18 - 12 && \text{Perform the multiplication: } 2 \cdot 9 = 18. \\ &= 6 && \text{Perform the subtraction: } 18 - 12 = 6. \end{aligned}$$

Answer: 62

Thus, $2 \cdot 3^2 - 12 = 6$. □

You Try It!

Simplify: $3(2 + 3 \cdot 4)^2 - 11$.

EXAMPLE 5. Evaluate $12 + 2(3 + 2 \cdot 5)^2$.

Solution. Follow the *Rules Guiding Order of Operations*, evaluate the expression inside the parentheses first, then exponents, then multiplication, then addition.

$$\begin{aligned} 12 + 2(3 + 2 \cdot 5)^2 &= 12 + 2(3 + 10)^2 && \text{Multiply inside parentheses: } 2 \cdot 5 = 10. \\ &= 12 + 2(13)^2 && \text{Add inside parentheses: } 3 + 10 = 13. \\ &= 12 + 2(169) && \text{Exponents are next: } (13)^2 = 169. \\ &= 12 + 338 && \text{Multiplication is next: } 2(169) = 338. \\ &= 350 && \text{Time to add: } 12 + 338 = 350. \end{aligned}$$

Answer: 577

Thus, $12 + 2(3 + 2 \cdot 5)^2 = 350$. □

You Try It!

EXAMPLE 6. Evaluate $2\{2 + 2[2 + 2]\}$.

Simplify: $2\{3 + 2[3 + 2]\}$.

Solution. When grouping symbols are nested, evaluate the expression between the pair of innermost grouping symbols first.

$$\begin{aligned} 2\{2+2[2+2]\} &= 2\{2+2[4]\} && \text{Innermost grouping first: } 2+2=4. \\ &= 2\{2+8\} && \text{Multiply next: } 2[4]=8. \\ &= 2\{10\} && \text{Add inside braces: } 2+8=10. \\ &= 20 && \text{Multiply: } 2\{10\}=20 \end{aligned}$$

Thus, $2\{2 + 2[2 + 2]\} = 20$.

Answer: 26

□

Fraction Bars

Consider the expression

$$\frac{6^2+8^2}{(2+3)^2}.$$

Because a fraction bar means division, the above expression is equivalent to

$$(6^2+8^2) \div (2+3)^2.$$

The position of the grouping symbols signals how we should proceed. We should simplify the numerator, then the denominator, then divide.

Fractional Expressions. If a fractional expression is present, evaluate the numerator and denominator first, then divide.

You Try It!

EXAMPLE 7. Evaluate the expression

$$\frac{6^2+8^2}{(2+3)^2}.$$

Simplify: $\dfrac{12 + 3 \cdot 2}{6}$.

Solution. Simplify the numerator and denominator first, then divide.

$$\begin{aligned} \frac{6^2+8^2}{(2+3)^2} &= \frac{6^2+8^2}{(5)^2} && \text{Parentheses in denominator first: } 2+3=5. \\ &= \frac{36+64}{25} && \text{Exponents are next: } 6^2=36,\ 8^2=64,\ 5^2=25. \\ &= \frac{100}{25} && \text{Add in numerator: } 36+64=100. \\ &= 4 && \text{Divide: } 100 \div 25 = 4. \end{aligned}$$

Thus, $\dfrac{6^2 + 8^2}{(2+3)^2} = 4.$

Answer: 3

□

The Distributive Property

Consider the expression $2 \cdot (3 + 4)$. If we follow the "Rules Guiding Order of Operations," we would evaluate the expression inside the parentheses first.

$$\begin{aligned} 2 \cdot (3+4) &= 2 \cdot 7 && \text{Parentheses first: } 3+4=7. \\ &= 14 && \text{Multiply: } 2 \cdot 7 = 14. \end{aligned}$$

However, we could also choose to "distribute" the 2, first multiplying 2 times each addend in the parentheses.

$$\begin{aligned} 2 \cdot (3+4) &= 2 \cdot 3 + 2 \cdot 4 && \text{Multiply 2 times both 3 and 4.} \\ &= 6 + 8 && \text{Multiply: } 2 \cdot 3 = 6 \text{ and } 2 \cdot 4 = 8. \\ &= 14 && \text{Add: } 6 + 8 = 14. \end{aligned}$$

The fact that we get the same answer in the second approach is an illustration of an important property of whole numbers.[1]

The Distributive Property. Let a, b, and c be any whole numbers. Then,

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

We say that "multiplication is distributive with respect to addition."

Multiplication is distributive with respect to addition. If you are not computing the product of a number and a sum of numbers, the distributive property does not apply.

Caution! Wrong Answer Ahead! If you are calculating the product of a number and the product of two numbers, the distributive property must not be used. For example, here is a *common misapplication* of the distributive property.

$$\begin{aligned} 2 \cdot (3 \cdot 4) &= (2 \cdot 3) \cdot (2 \cdot 4) \\ &= 6 \cdot 8 \\ &= 48 \end{aligned}$$

[1]Later, we'll see that this property applies to all numbers, not just whole numbers.

This result is quite distant from the correct answer, which is found by computing the product within the parentheses first.

$$\begin{aligned} 2 \cdot (3 \cdot 4) &= 2 \cdot 12 \\ &= 24. \end{aligned}$$

In order to apply the distributive property, you must be multiplying times a sum.

You Try It!

EXAMPLE 8. Use the distributive property to calculate $4 \cdot (5 + 11)$.

Distribute: $5 \cdot (11 + 8)$.

Solution. This is the product of a number and a sum, so the distributive property may be applied.

$$\begin{aligned} 4 \cdot (5 + 11) &= 4 \cdot 5 + 4 \cdot 11 && \text{Distribute the 4 times each addend in the sum.} \\ &= 20 + 44 && \text{Multiply: } 4 \cdot 5 = 20 \text{ and } 4 \cdot 11 = 44. \\ &= 64 && \text{Add: } 20 + 44 = 64. \end{aligned}$$

Readers should check that the same answer is found by computing the sum within the parentheses first.

Answer: 95

The distributive property is the underpinning of the multiplication algorithm learned in our childhood years.

You Try It!

EXAMPLE 9. Multiply: $6 \cdot 43$.

Use the distributive property to evaluate $8 \cdot 92$.

Solution. We'll express 43 as sum, then use the distributive property.

$$\begin{aligned} 6 \cdot 43 &= 6 \cdot (40 + 3) && \text{Express 43 as a sum: } 43 = 40 + 3. \\ &= 6 \cdot 40 + 6 \cdot 3 && \text{Distribute the 6.} \\ &= 240 + 18 && \text{Multiply: } 6 \cdot 40 = 240 \text{ and } 6 \cdot 3 = 18. \\ &= 258 && \text{Add: } 240 + 18 = 258. \end{aligned}$$

Readers should be able to see this application of the distributive property in the more familiar algorithmic form:

$$\begin{array}{r} 43 \\ \times 6 \\ \hline 18 \\ 240 \\ \hline 258 \end{array}$$

Or in the even more condensed form with "carrying:"

$$\begin{array}{r} {}^{1}43 \\ \times 6 \\ \hline 2\,5\,8 \end{array}$$

Answer: 736

□

Multiplication is also distributive with respect to subtraction.

The Distributive Property (Subtraction). Let a, b, and c be any whole numbers. Then,

$$a \cdot (b - c) = a \cdot b - a \cdot c.$$

We say the multiplication is "distributive with respect to subtraction."

You Try It!

Distribute: $8 \cdot (9 - 2)$.

EXAMPLE 10. Use the distributive property to simplify: $3 \cdot (12 - 8)$.

Solution. This is the product of a number and a difference, so the distributive property may be applied.

$$\begin{aligned} 3 \cdot (12 - 8) &= 3 \cdot 12 - 3 \cdot 8 && \text{Distribute the 3 times each term in the difference.} \\ &= 36 - 24 && \text{Multiply: } 3 \cdot 12 = 36 \text{ and } 3 \cdot 8 = 24. \\ &= 12 && \text{Subtract: } 36 - 24 = 12. \end{aligned}$$

Alternate solution. Note what happens if we use the usual "order of operations" to evaluate the expression.

$$\begin{aligned} 3 \cdot (12 - 8) &= 3 \cdot 4 && \text{Parentheses first: } 12 - 8 = 4. \\ &= 12 && \text{Multiply: } 3 \cdot 4 = 12. \end{aligned}$$

Answer: 56

Same answer.

□

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-12, simplify the given expression.

1. $5 + 2 \cdot 2$

2. $5 + 2 \cdot 8$

3. $23 - 7 \cdot 2$

4. $37 - 3 \cdot 7$

5. $4 \cdot 3 + 2 \cdot 5$

6. $2 \cdot 5 + 9 \cdot 7$

7. $6 \cdot 5 + 4 \cdot 3$

8. $5 \cdot 2 + 9 \cdot 8$

9. $9 + 2 \cdot 3$

10. $3 + 6 \cdot 6$

11. $32 - 8 \cdot 2$

12. $24 - 2 \cdot 5$

In Exercises 13-28, simplify the given expression.

13. $45 \div 3 \cdot 5$

14. $20 \div 1 \cdot 4$

15. $2 \cdot 9 \div 3 \cdot 18$

16. $19 \cdot 20 \div 4 \cdot 16$

17. $30 \div 2 \cdot 3$

18. $27 \div 3 \cdot 3$

19. $8 - 6 + 1$

20. $15 - 5 + 10$

21. $14 \cdot 16 \div 16 \cdot 19$

22. $20 \cdot 17 \div 17 \cdot 14$

23. $15 \cdot 17 + 10 \div 10 - 12 \cdot 4$

24. $14 \cdot 18 + 9 \div 3 - 7 \cdot 13$

25. $22 - 10 + 7$

26. $29 - 11 + 1$

27. $20 \cdot 10 + 15 \div 5 - 7 \cdot 6$

28. $18 \cdot 19 + 18 \div 18 - 6 \cdot 7$

In Exercises 29-40, simplify the given expression.

29. $9 + 8 \div \{4 + 4\}$

30. $10 + 20 \div \{2 + 2\}$

31. $7 \cdot [8 - 5] - 10$

32. $11 \cdot [12 - 4] - 10$

33. $(18 + 10) \div (2 + 2)$

34. $(14 + 7) \div (2 + 5)$

35. $9 \cdot (10 + 7) - 3 \cdot (4 + 10)$

36. $9 \cdot (7 + 7) - 8 \cdot (3 + 8)$

37. $2 \cdot \{8 + 12\} \div 4$

38. $4 \cdot \{8 + 7\} \div 3$

39. $9 + 6 \cdot (12 + 3)$

40. $3 + 5 \cdot (10 + 12)$

In Exercises 41-56, simplify the given expression.

41. $2 + 9 \cdot [7 + 3 \cdot (9 + 5)]$

42. $6 + 3 \cdot [4 + 4 \cdot (5 + 8)]$

43. $7 + 3 \cdot [8 + 8 \cdot (5 + 9)]$

44. $4 + 9 \cdot [7 + 6 \cdot (3 + 3)]$

45. $6 - 5[11 - (2 + 8)]$

46. $15 - 1[19 - (7 + 3)]$

47. $11 - 1[19 - (2 + 15)]$

48. $9 - 8[6 - (2 + 3)]$

49. $4\{7[9 + 3] - 2[3 + 2]\}$

50. $4\{8[3 + 9] - 4[6 + 2]\}$

51. $9 \cdot [3 + 4 \cdot (5 + 2)]$

52. $3 \cdot [4 + 9 \cdot (8 + 5)]$

53. $3\{8[6 + 5] - 8[7 + 3]\}$

54. $2\{4[6 + 9] - 2[3 + 4]\}$

55. $3 \cdot [2 + 4 \cdot (9 + 6)]$

56. $8 \cdot [3 + 9 \cdot (5 + 2)]$

In Exercises 57-68, simplfiy the given expression.

57. $(5 - 2)^2$

58. $(5 - 3)^4$

59. $(4 + 2)^2$

60. $(3 + 5)^2$

61. $2^3 + 3^3$

62. $5^4 + 2^4$

63. $2^3 - 1^3$

64. $3^2 - 1^2$

65. $12 \cdot 5^2 + 8 \cdot 9 + 4$

66. $6 \cdot 3^2 + 7 \cdot 5 + 12$

67. $9 - 3 \cdot 2 + 12 \cdot 10^2$

68. $11 - 2 \cdot 3 + 12 \cdot 4^2$

In Exercises 69-80, simplify the given expression.

69. $4^2 - (13 + 2)$

70. $3^3 - (7 + 6)$

71. $3^3 - (7 + 12)$

72. $4^3 - (6 + 5)$

73. $19 + 3[12 - (2^3 + 1)]$

74. $13 + 12[14 - (2^2 + 1)]$

75. $17 + 7[13 - (2^2 + 6)]$

76. $10 + 1[16 - (2^2 + 9)]$

77. $4^3 - (12 + 1)$

78. $5^3 - (17 + 15)$

79. $5 + 7[11 - (2^2 + 1)]$

80. $10 + 11[20 - (2^2 + 1)]$

In Exercises 81-92, simplify the given expression.

81. $\dfrac{13+35}{3(4)}$

82. $\dfrac{35+28}{7(3)}$

83. $\dfrac{64-(8\cdot 6-3)}{4\cdot 7-9}$

84. $\dfrac{19-(4\cdot 3-2)}{6\cdot 3-9}$

85. $\dfrac{2+13}{4-1}$

86. $\dfrac{7+1}{8-4}$

87. $\dfrac{17+14}{9-8}$

88. $\dfrac{16+2}{13-11}$

89. $\dfrac{37+27}{8(2)}$

90. $\dfrac{16+38}{6(3)}$

91. $\dfrac{40-(3\cdot 7-9)}{8\cdot 2-2}$

92. $\dfrac{60-(8\cdot 6-3)}{5\cdot 4-5}$

In Exercises 93-100, use the distributive property to evaluate the given expression.

93. $5\cdot(8+4)$

94. $8\cdot(4+2)$

95. $7\cdot(8-3)$

96. $8\cdot(9-7)$

97. $6\cdot(7-2)$

98. $4\cdot(8-6)$

99. $4\cdot(3+2)$

100. $4\cdot(9+6)$

In Exercises 101-104, use the distributive property to evaluate the given expression using the technique shown in Example 9.

101. $9\cdot 62$

102. $3\cdot 76$

103. $3\cdot 58$

104. $7\cdot 57$

❧ ❧ ❧ Answers ☙ ☙ ☙

1. 9

3. 9

5. 22

7. 42

9. 15

11. 16

13. 75

15. 108

17. 45

19. 3

21. 266

23. 208

25. 19

27. 161

29. 10

31. 11

33. 7

35. 111

37. 10

39. 99

41. 443

43. 367

45. 1

47. 9

49. 296

51. 279

53. 24

55. 186

57. 9

59. 36

61. 35

63. 7

65. 376

67. 1203

69. 1

71. 8

73. 28

75. 38

77. 51

79. 47

81. 4

83. 1

85. 5

87. 31

89. 4

91. 2

93. 60

95. 35

97. 30

99. 20

101. 558

103. 174

1.6 Solving Equations by Addition and Subtraction

Let's start with the definition of a variable.

Variable. A *variable* is a symbol (usually a letter) that stands for a value that may vary.

Next we follow with the definition of an equation.

Equation. An *equation* is a mathematical statement that equates two mathematical expressions.

The key difference between a mathematical expression and an equation is the presence of an an equals sign. So, for example,

$$2 + 3[5 - 4 \cdot 2], \qquad x^2 + 2x - 3, \qquad \text{and} \qquad x + 2y + 3$$

are mathematical expressions (two of which contain variables), while

$$3 + 2(7 - 3) = 11, \qquad x + 3 = 4, \qquad \text{and} \qquad 3x = 9$$

are equations. Note that each of the equations contain an equals sign, but the expressions do not.

Next we have the definition of a solution of an equation.

What it Means to be a Solution. A *solution* of an equation is a numerical value that satisfies the equation. That is, when the variable in the equation is replaced by the solution, a true statement results.

You Try It!

Show that 27 is a solution of the equation $x - 12 = 15$.

EXAMPLE 1. Show that 3 is a solution of the equation $x + 8 = 11$.

Movie

Solution. Substitute 3 for x in the given equation and simplify.

$x + 8 = 11$	The given equation.
$3 + 8 = 11$	Substitute 3 for x.
$11 = 11$	Simplify both sides.

Since the left- and right-hand sides of the last line are equal, this shows that when 3 is substituted for x in the equation a true statement results. Therefore, 3 is a solution of the equation.

□

You Try It!

Is 8 a solution of $5 = 12 - y$?

EXAMPLE 2. Is 23 a solution of the equation $4 = y - 11$?

Solution. Substitute 23 for y in the given equation and simplify.

$4 = y - 11$	The given equation.
$4 = 23 - 11$	Substitute 23 for y.
$4 = 12$	Simplify both sides.

Since the left- and right-hand sides of the last line are **not** equal, this shows that when 23 is substituted for y in the equation a false statement results. Therefore, 23 is **not** a solution of the equation.

Answer: No.

Equivalent Equations

We start with the definition of equivalent equations.

Equivalent Equations. Two equations are equivalent if they have the same solution set.

You Try It!

Are the equations $x = 4$ and $x + 8 = 3$ equivalent?

EXAMPLE 3. Are the equations $x + 2 = 9$ and $x = 7$ equivalent?

Solution. The number 7 is the only solution of the equation $x + 2 = 9$. Similarly, 7 is the only solution of the equation $x = 7$. Therefore $x + 2 = 9$ and $x = 7$ have the same solution sets and are equivalent.

Answer: No.

You Try It!

Are the equations $x = 2$ and $x^2 = 2x$ equivalent?

EXAMPLE 4. Are the equations $x^2 = x$ and $x = 1$ equivalent?

Solution. By inspection, the equation $x^2 = x$ has two solutions, 0 and 1. On the other hand, the equation $x = 1$ has a single solution, namely 1. Hence, the equations $x^2 = x$ and $x = 1$ do not have the same solution sets and are **not** equivalent.

Answer: No.

Operations that Produce Equivalent Equations

There are many operations that will produce equivalent operations. In this section we look at two: addition and subtraction.

Adding the Same Quantity to Both Sides of an Equation. Adding the same quantity to both sides of an equation does not change the solution set. That is, if

$$a = b,$$

then adding c to both sides of the equation produces the equivalent equation

$$a + c = b + c.$$

Let's see if this works as advertised. Consider the equaton

$$x - 4 = 3.$$

By inspection, 7 is the only solution of the equation. Now, let's add 4 to both sides of the equation to see if the resulting equation is equivalent to $x - 4 = 3$.

$x - 4 = 3$	The given equation.
$x - 4 + 4 = 3 + 4$	Add 4 to both sides of the equation.
$x = 7$	Simplify both sides of the equation.

The number 7 is the only solution of the equation $x = 7$. Thus, the equation $x = 7$ is equivalent to the original equation $x - 4 = 3$ (they have the same solutions).

Important Point. Adding the same amount to both sides of an equation does not change its solutions.

It is also a fact that subtracting the same quantity from both sides of an equation produces an equivalent equation.

Subtracting the Same Quantity from Both Sides of an Equation. Subtracting the same quantity from both sides of an equation does not change the solution set. That is, if

$$a = b,$$

then subtracting c from both sides of the equation produces the equivalent equation

$$a - c = b - c.$$

Let's also see if this works as advertised. Consider the equation

$$x + 4 = 9.$$

By inspection, 5 is the only solution of the equation. Now, let's subtract 4 from both sides of the equation to see if the resulting equation is equivalent to $x + 4 = 9$.

$x + 4 = 9$	The given equation.
$x + 4 - 4 = 9 - 4$	Subtract 4 from both sides of the equation.
$x = 5$	Simplify both sides of the equation.

The number 5 is the only solution of the equation $x = 5$. Thus, the equation $x = 5$ is equivalent to the original equation $x + 4 = 9$ (they have the same solutions).

Important Point. Subtracting the same amount from both sides of an equation does not change its solutions.

Writing Mathematics. When solving equations, observe the following rules to neatly arrange your work:

1. **One equation per line.** This means that you should not arrange your work like this:

   $$x + 3 = 7 \qquad x + 3 - 3 = 7 - 3 \qquad x = 4$$

   That's three equations on a line. Rather, arrange your work one equation per line like this:

   $$\begin{aligned} x + 3 &= 7 \\ x + 3 - 3 &= 7 - 3 \\ x &= 4 \end{aligned}$$

2. **Add and subtract inline.** Don't do this:

   $$\begin{array}{r} x - 7 = \ 12 \\ +7 \quad +7 \\ \hline x = \ 19 \end{array}$$

   Instead, add 7 to both sides of the equation "inline."

   $$\begin{aligned} x - 7 &= 12 \\ x - 7 + 7 &= 12 + 7 \\ x &= 19 \end{aligned}$$

Wrap and Unwrap

Suppose that you are wrapping a gift for your cousin. You perform the following steps in order.

1. Put the gift paper on.
2. Put the tape on.
3. Put the decorative bow on.

When we give the wrapped gift to our cousin, he politely unwraps the present, "undoing" each of our three steps in inverse order.

1. Take off the decorative bow.
2. Take off the tape.
3. Take off the gift paper.

This seemingly frivolous wrapping and unwrapping of a gift contains some deeply powerful mathematical ideas.

Consider the mathematical expression $x+4$. To evaluate this expression at a particular value of x, we would start with the given value of x, then

1. Add 4.

Suppose we started with the number 7. If we add 4, we arrive at the following result: 11.

Now, how would we "unwrap" this result to return to our original number? We would start with our result, then

1. Subtract 4.

That is, we would take our result from above, 11, then subtract 4, which returns us to our original number, namely 7.

Addition and Subtraction as Inverse Operations. Two extremely important observations:

The inverse of addition is subtraction. If we start with a number x and add a number a, then subtracting a from the result will return us to the original number x. In symbols,

$$x + a - a = x.$$

The inverse of subtraction is addition. If we start with a number x and subtract a number a, then adding a to the result will return us to the original number x. In symbols,

$$x - a + a = x.$$

You Try It!

Solve $x + 5 = 12$ for x.

EXAMPLE 5. Solve $x - 8 = 10$ for x.

Solution. To undo the effects of subtracting 8, we add 8 to both sides of the equation.

$x - 8 = 10$	Original equation.
$x - 8 + 8 = 10 + 8$	Add 8 to both sides of the equation.
$x = 18$	On the left, adding 8 "undoes" the effect of subtracting 8 and returns x. On the right, 10+8=18.

Therefore, the solution of the equation is 18.

Check. To check, substitute the solution 18 into the original equation.

$x - 8 = 10$	Original equation.
$18 - 8 = 10$	Substitute 18 for x.
$10 = 10$	Simplify both sides.

The fact that the last line of our check is a true statement guarantees that 18 is a solution of $x - 8 = 10$.

Answer: $x = 7$.

□

You Try It!

Solve $y - 8 = 11$ for y.

EXAMPLE 6. Solve $11 = y + 5$ for y.

Solution. To undo the effects of adding 5, we subtract 5 from both sides of the equation.

$11 = y + 5$	Original equation.
$11 - 5 = y + 5 - 5$	Subtract 5 from both sides of the equation.
$6 = y$	On the right, subtracting "undoes" the effect of adding 5 and returns y. On the left, 11-5=6.

Therefore, the solution of the equation is 6.

Check. To check, substitute the solution 6 into the original equation.

$$\begin{aligned} 11 &= y + 5 && \text{Original equation.} \\ 11 &= 6 + 5 && \text{Substitute 6 for } y. \\ 11 &= 11 && \text{Simplify both sides.} \end{aligned}$$

The fact that the last line of our check is a true statement guarantees that 6 is a solution of $11 = y + 5$.

Answer: $y = 19$.

Word Problems

The solution of a word problem must incorporate each of the following steps.

Requirements for Word Problem Solutions.

1. **Set up a Variable Dictionary.** You must let your readers know what each variable in your problem represents. This can be accomplished in a number of ways:
   - Statements such as "Let P represent the perimeter of the rectangle."
   - Labeling unknown values with variables in a table.
   - Labeling unknown quantities in a sketch or diagram.
2. **Set up an Equation.** Every solution to a word problem must include a carefully crafted equation that accurately describes the constraints in the problem statement.
3. **Solve the Equation.** You must always solve the equation set up in the previous step.
4. **Answer the Question.** This step is easily overlooked. For example, the problem might ask for Jane's age, but your equation's solution gives the age of Jane's sister Liz. Make sure you answer the original question asked in the problem. Your solution should be written in a sentence with appropriate units.
5. **Look Back.** It is important to note that this step does not imply that you should simply check your solution in your equation. After all, it's possible that your equation incorrectly models the problem's situation, so you could have a valid solution to an incorrect equation. The important question is: "Does your answer make sense based on the words in the original problem statement."

Let's give these requirements a test drive.

You Try It!

12 more than a certain number is 19. Find the number.

EXAMPLE 7. Four more than a certain number is 12. Find the number.

Solution. In our solution, we will carefully address each step of the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* We can satisfy this requirement by simply stating "Let x represent a certain number."
2. *Set up an Equation.* "Four more than a certain number is 12" becomes

4	more than	a certain number	is	12
4	$+$	x	$=$	12

3. *Solve the Equation.* To "undo" the addition, subtract 4 from both sides of the equation.

$4 + x = 12$	Original equation.
$4 + x - 4 = 12 - 4$	Subtract 4 from both sides of the equation.
$x = 8$	On the left, subtracting 4 "undoes" the effect of adding 4 and returns x. On the right, $12 - 4 = 8$.

4. *Answer the Question.* The number is 8.
5. *Look Back.* Does the solution 8 satisfy the words in the original problem? We were told that "four more than a certain number is 12." Well, four more than 8 is 12, so our solution is correct.

Answer: 7

□

You Try It!

Fred withdraws \$230 from his account, lowering his balance to \$3,500. What was his original balance?

EXAMPLE 8. Amelie withdraws \$125 from her savings account. Because of the withdrawal, the current balance in her account is now \$1,200. What was the original balance in the account before the withdrawal?

Solution. In our solution, we will carefully address each step of the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* We can satisfy this requirement by simply stating "Let B represent the original balance in Amelie's account."

2. *Set up an Equation.* We can describe the situation in words and symbols.

Original Balance	minus	Amelie's Withdrawal	is	Current Balance
B	$-$	125	$=$	1200

3. *Solve the Equation.* To "undo" the subtraction, add 125 to both sides of the equation.

$$B - 125 = 1200 \quad \text{Original equation.}$$

$$B - 125 + 125 = 1200 + 125 \quad \text{Add 125 to both sides of the equation.}$$

$$B = 1325 \quad \text{On the left, adding 125 ``undoes'' the effect of subtracting 125 and returns } B\text{. On the right, } 1200+125=1325.$$

4. *Answer the Question.* The original balance was $1,325.

5. *Look Back.* Does the solution $1,325 satisfy the words in the original problem? Note that if Amelie withdraws $125 from this balance, the new balance will be $1,200. Hence, the solution is correct.

Answer: $3,730.

You Try It!

EXAMPLE 9. The perimeter of a triangle is 114 feet. Two of the sides of the triangle measure 30 feet and 40 feet, respectively. Find the measure of the third side of the triangle.

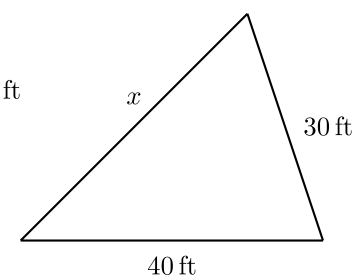
The perimeter of a quadrilater is 200 meters. If three of the sides measure 20, 40, and 60 meters, what is the length of the fourth side?

Movie

Solution. In our solution, we will carefully address each step of the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* When geometry is involved, we can create our variable dictionary by labeling a carefully constructed diagram. With this thought in mind, we draw a triangle, then label its known and unknown sides and its perimeter.

Perimeter = 114 ft



The figure makes it clear that x represents the length of the unknown side of the triangle. The figure also summarizes information needed for the solution.

2. *Set up an equation.* We know that the perimeter of a triangle is found by finding the sum of its three sides; in words and symbols,

Perimeter	is	First Side	plus	Second Side	plus	Third Side
114	=	x	+	30	+	40

Simplify the right-hand side by adding 30 and 40; i.e., $30 + 40 = 70$.

$$114 = x + 70$$

3. *Solve the Equation.* To "undo" adding 70, subtract 70 from both sides of the equation.

$114 = x + 70$	Our equation.
$114 - 70 = x + 70 - 70$	Subtract 70 from both sides.
$44 = x$	On the right, subtracting 70 "undoes" the effect of adding 70 and returns to x. On the left, $114 - 70 = 44$.

4. *Answer the Question.* The unknown side of the triangle is 44 feet.

5. *Look Back.* Does the solution 44 feet satisfy the words of the original problem? We were told that the perimeter is 114 feet and two of the sides have length 30 feet and 40 feet respectively. We found that the third side has length 44 feet. Now, adding the three sides, $30+40+44 = 114$, which equals the given perimeter of 114 feet. The answer works!

Answer: 80 meters.

□

Exercises

In Exercises 1-12, which of the numbers following the given equation are solutions of the given equation? Support your response with work similar to that shown in Examples 1 and 2.

1. $x-4=6$; 10, 17, 13, 11

2. $x-9=7$; 17, 23, 19, 16

3. $x+2=6$; 5, 11, 7, 4

4. $x+3=9$; 6, 9, 7, 13

5. $x+2=3$; 8, 1, 4, 2

6. $x+2=5$; 10, 3, 6, 4

7. $x-4=7$; 12, 11, 18, 14

8. $x-6=7$; 13, 16, 20, 14

9. $x+3=4$; 8, 4, 2, 1

10. $x+5=9$; 5, 11, 7, 4

11. $x-6=8$; 17, 21, 14, 15

12. $x-2=9$; 11, 14, 12, 18

In Exercises 13-52, solve the given equation for x.

13. $x+5=6$

14. $x+6=19$

15. $5=4+x$

16. $10=8+x$

17. $13+x=17$

18. $7+x=15$

19. $9+x=10$

20. $14+x=17$

21. $19=x-3$

22. $2=x-11$

23. $x-18=1$

24. $x-20=8$

25. $x-3=11$

26. $x-17=18$

27. $2+x=4$

28. $1+x=16$

29. $x-14=12$

30. $x-1=17$

31. $x+2=8$

32. $x+11=14$

33. $11+x=17$

34. $11+x=18$

35. $x+13=17$

36. $x+1=16$

37. $20=3+x$

38. $9=3+x$

39. $20=8+x$

40. $10=3+x$

41. $3=x-20$

42. $13=x-15$

43. $x+16=17$

44. $x+6=12$

45. $5=x-6$

46. $10=x-7$

47. $18=x-6$

48. $14=x-4$

49. $18=13+x$

50. $17=5+x$

51. $x-9=15$

52. $x-11=17$

53. 12 less than a certain number is 19. Find the number.

54. 19 less than a certain number is 1. Find the number.

55. A triangle has a perimeter of 65 feet. It also has two sides measuring 19 feet and 17 feet, respectively. Find the length of the third side of the triangle.

56. A triangle has a perimeter of 55 feet. It also has two sides measuring 14 feet and 13 feet, respectively. Find the length of the third side of the triangle.

57. Burt makes a deposit to an account having a balance of $1900. After the deposit, the new balance in the account is $8050. Find the amount of the deposit.

58. Dave makes a deposit to an account having a balance of $3500. After the deposit, the new balance in the account is $4600. Find the amount of the deposit.

59. 8 more than a certain number is 18. Find the number.

60. 3 more than a certain number is 19. Find the number.

61. Michelle withdraws a $120 from her bank account. As a result, the new account balance is $1000. Find the account balance before the withdrawal.

62. Mercy withdraws a $430 from her bank account. As a result, the new account balance is $1200. Find the account balance before the withdrawal.

63. **Foreclosures.** Between January and March last year, 650,000 homes received a foreclosure notice. Between the first three months of this year, there were 804,000 foreclosure notices. What was the increase in home foreclosure notices? *Associated Press Times-Standard 4/22/09*

64. **Home Price.** According to the Humboldt State University Economics Department's Humboldt Economic Index, the median home price in the US fell $1500 over the last month to $265,000. What was the median home price before the price drop?

65. **Unmanned Areal Vehicle.** Northrup Grumman's Global Hawk unmanned drone can fly at 65,000 feet, 40,000 feet higher than NASA's Ikhana unmanned aircraft. How high can the Ikhana fly?

66. **Tribal Land.** The Yurok Tribe has the option to purchase 47,000 acres in order to increase its ancestral territory. The first phase would include 22,500 acres in the Cappel and Pecman watersheds. The second phase plans for acreage in the Blue Creek area. How many acres could be purchased in the second phase? *Times-Standard 4/15/09*

❧ ❧ ❧ Answers ☙ ☙ ☙

1. 10

3. 4

5. 1

7. 11

9. 1

11. 14

13. 1

15. 1

17. 4

19. 1

21. 22

23. 19

25. 14

27. 2

29. 26

31. 6

33. 6

35. 4

37. 17

39. 12

41. 23

43. 1

45. 11

47. 24

49. 5

51. 24

53. 31

55. 29

57. $6150

59. 10

61. $1120

63. 154,000

65. 25,000 feet

1.7 Solving Equations by Multiplication and Division

In Section 1.6, we stated that two equations that have the same solutions are *equivalent*. Furthermore, we saw that adding the same number to both sides of an equation produced an equivalent equation. Similarly, subtracting the same the number from both sides of an equation also produces an equivalent equation. We can make similar statements for multiplication and division.

Multiplying both Sides of an Equation by the Same Quantity. Multiplying both sides of an equation by the same quantity does not change the solution set. That is, if

$$a = b,$$

then multiplying both sides of the equation by c produces the equivalent equation

$$a \cdot c = b \cdot c,$$

provided $c \neq 0$.

A similar statement can be made about division.

Dividing both Sides of an Equation by the Same Quantity. Dividing both sides of an equation by the same quantity does not change the solution set. That is, if

$$a = b,$$

then dividing both sides of the equation by c produces the equivalent equation

$$\frac{a}{c} = \frac{b}{c},$$

provided $c \neq 0$.

In Section 1.6, we saw that addition and subtraction were inverse operations. If you start with a number, add 4 and subtract 4, you are back to the original number. This concept also works for multiplication and division.

Multiplication and Division as Inverse Operations. Two extremely important observations:

The inverse of multiplication is division. If we start with a number x and multiply by a number a, then dividing the result by the number a returns us to the original number x. In symbols,

$$\frac{a \cdot x}{a} = x.$$

The inverse of division is multiplication. If we start with a number x and divide by a number a, then multiplying the result by the number a returns us to the original number x. In symbols,

$$a \cdot \frac{x}{a} = x.$$

Let's put these ideas to work.

You Try It!

Solve for x: $5x = 120$

EXAMPLE 1. Solve the equation $3x = 24$ for x.

Solution. To undo the effects of multiplying by 3, we divide both sides of the equation by 3.

$3x = 24$	Original equation.
$\dfrac{3x}{3} = \dfrac{24}{3}$	Divide both sides of the equation by 3.
$x = 8$	On the left, dividing by 3 "undoes" the effect of multiplying by 3 and returns to x. On the right, $24/3 = 8$.

Solution. To check, substitute the solution 8 into the original equation.

$3x = 24$	Original equation.
$3(8) = 24$	Substitute 8 for x.
$24 = 24$	Simplify both sides.

That fact that the last line of our check is a true statement guarantees that 8 is a solution of $3x = 24$.

Answer: 24

□

You Try It!

Solve for x: $x/2 = 19$

EXAMPLE 2. Solve the following equation for x.

$$\frac{x}{7} = 12$$

Solution. To undo the effects of dividing by 7, we multiply both sides of the equation by 7.

$$\frac{x}{7} = 12$$ Original equation.

$$7 \cdot \frac{x}{7} = 7 \cdot 12$$ Multiply both sides of the equation by 7.

$$x = 84$$ On the left, multiplying by 7 "undoes" the effect of dividing by 7 and returns to x. On the right, $7 \cdot 12 = 84$.

Solution. To check, substitute the solution 84 into the original equation.

$$\frac{x}{7} = 12$$ Original equation.

$$\frac{84}{7} = 12$$ Substitute 84 for x.

$$12 = 12$$ Simplify both sides.

That fact that the last line of our check is a true statement guarantees that 84 is a solution of $x/7 = 12$.

Answer: 38

Word Problems

In Section 1.6 we introduced *Requirements for Word Problem Solutions.* Those requirements will be strictly adhered to in this section.

You Try It!

Seven times a certain number is one hundred five. Find the unknown number.

EXAMPLE 3. Fifteen times a certain number is 45. Find the unknown number.

Solution. In our solution, we will carefully address each step of the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* We can satisfy this requirement by simply stating "Let x represent a certain number."

2. *Set up an equation.* "Fifteen times a certain number is 45" becomes

15	times	a certain number	is	45
15	$\cdot$	x	$=$	45

3. *Solve the Equation.* To "undo" the multiplication by 15, divide both sides of the equation by 15.

$15x = 45$	Original equation. Write $15 \cdot x$ as $15x$
$\dfrac{15x}{15} = \dfrac{45}{15}$	Divide both sides of the equation by 15.
$x = 3$	On the left, dividing by 15 "undoes" the effect of multiplying by 15 and returns to x. On the right, $45/15 = 3$.

4. *Answer the Question.* The unknown number is 3.

5. *Look Back.* Does the solution 3 satisfy the words of the original problem? We were told that "15 times a certain number is 45." Well, 15 times 3 is 45, so our solution is correct.

Answer: 15

□

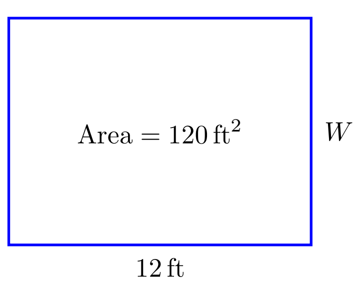
You Try It!

The area of a rectangle is 3,500 square meters. If the width is 50 meters, find the length.

EXAMPLE 4. The area of a rectangle is 120 square feet. If the length of the rectangle is 12 feet, find the width of the rectangle.

Solution. In our solution, we will carefully address each step of the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* When geometry is involved, we can create our variable dictionary by labeling a carefully constructed diagram. With this thought in mind, we draw a rectangle, then label its length, width, and area.



The figure makes it clear that W represents the width of the rectangle. The figure also summarizes information needed for the solution.

2. *Set up an equation.* We know that the area of a rectangle is found by multiplying its length and width; in symbols,

$$A = LW. \tag{1.1}$$

We're given the area is $A = 120\,\mathrm{ft}^2$ and the length is $L = 12\,\mathrm{ft}$. Substitute these numbers into the area formula (1.1) to get

$$120 = 12W.$$

3. *Solve the Equation.* To "undo" the multiplication by 12, divide both sides of the equation by 12.

$120 = 12W$	Our equation.
$\dfrac{120}{12} = \dfrac{12W}{12}$	Divide both sides of the equation by 12.
$10 = W$	On the right, dividing by 12 "undoes" the effect of multiplying by 12 and returns to W. On the left, $120/12 = 10$.

4. *Answer the Question.* The width is 10 feet.

5. *Look Back.* Does the found width satisfy the words of the original problem? We were told that the area is 120 square feet and the length is 12 feet. The area is found by multiplying the length and width, which gives us 12 feet times 10 feet, or 120 square feet. The answer works!

Answer: 70 meters

□

You Try It!

A class of 30 students averaged 75 points on an exam. How many total points were accumulated by the class as a whole?

EXAMPLE 5. A class of 23 students averaged 76 points on an exam.How many total points were accumulated by the class as a whole?

Movie

Solution. In our solution, we will carefully address each step of the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* We can set up our variable dictionary by simply stating "Let T represent the total points accumulated by the class."

2. *Set up an equation.* To find the average score on the exam, take the total points accumulated by the class, then divide by the number of students in the class. In words and symbols,

Total Points	divided by	Number of Students	equals	Average Score
T	$\div$	23	$=$	76

An equivalent representation is

$$\frac{T}{23} = 76.$$

3. *Solve the Equation.* To "undo" the division by 23, multiply both sides of the equation by 23.

$$\begin{aligned} \frac{T}{23} &= 76 && \text{Our equation.} \\ 23 \cdot \frac{T}{23} &= 76 \cdot 23 && \text{Multiply both sides of the equation by 23.} \\ T &= 1748 && \text{On the left, multiplying by 23 "undoes" the effect of dividing by 23 and returns to } T\text{. On the right, } 76 \cdot 23 = 1748. \end{aligned}$$

4. *Answer the Question.* The total points accumulated by the class on the exam is 1,748.

5. *Look Back.* Does the solution 1,748 satisfy the words of the original problem? To find the average on the exam, divide the total points 1,748 by 23, the number of students in the class. Note that this gives an average score of $1748 \div 23 = 76$. The answer works!

Answer: 2,250

□

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-12, which of the numbers following the given equation are solutions of the given equation?

1. $\frac{x}{6} = 4$; 24, 25, 27, 31

2. $\frac{x}{7} = 6$; 49, 42, 43, 45

3. $\frac{x}{2} = 3$; 6, 9, 13, 7

4. $\frac{x}{9} = 5$; 45, 46, 48, 52

5. $5x = 10$; 9, 2, 3, 5

6. $4x = 36$; 12, 16, 9, 10

7. $5x = 25$; 5, 6, 8, 12

8. $3x = 3$; 1, 8, 4, 2

9. $2x = 2$; 4, 8, 1, 2

10. $3x = 6$; 2, 9, 5, 3

11. $\frac{x}{8} = 7$; 57, 59, 63, 56

12. $\frac{x}{3} = 7$; 24, 21, 28, 22

In Exercises 13-36, solve the given equation for x.

13. $\frac{x}{6} = 7$

14. $\frac{x}{8} = 6$

15. $2x = 16$

16. $2x = 10$

17. $2x = 18$

18. $2x = 0$

19. $4x = 24$

20. $2x = 4$

21. $\frac{x}{4} = 9$

22. $\frac{x}{5} = 6$

23. $5x = 5$

24. $3x = 15$

25. $5x = 30$

26. $4x = 28$

27. $\frac{x}{3} = 4$

28. $\frac{x}{9} = 4$

29. $\frac{x}{8} = 9$

30. $\frac{x}{8} = 2$

31. $\frac{x}{7} = 8$

32. $\frac{x}{4} = 6$

33. $2x = 8$

34. $3x = 9$

35. $\frac{x}{8} = 5$

36. $\frac{x}{5} = 4$

37. The price of one bookcase is $370. A charitable organization purchases an unknown number of bookcases and the total price of the purchase is $4,810. Find the number of bookcases purchased.

38. The price of one computer is $330. A charitable organization purchases an unknown number of computers and the total price of the purchase is $3,300. Find the number of computers purchased.

39. When an unknown number is divided by 3, the result is 2. Find the unknown number.

40. When an unknown number is divided by 8, the result is 3. Find the unknown number.

41. A class of 29 students averaged 80 points on an exam. How many total points were accumulated by the class as a whole?

42. A class of 44 students averaged 87 points on an exam. How many total points were accumulated by the class as a whole?

43. When an unknown number is divided by 9, the result is 5. Find the unknown number.

44. When an unknown number is divided by 9, the result is 2. Find the unknown number.

45. The area of a rectangle is 16 square cm. If the length of the rectangle is 2 cm, find the width of the rectangle.

46. The area of a rectangle is 77 square ft. If the length of the rectangle is 7 ft, find the width of the rectangle.

47. The area of a rectangle is 56 square cm. If the length of the rectangle is 8 cm, find the width of the rectangle.

48. The area of a rectangle is 55 square cm. If the length of the rectangle is 5 cm, find the width of the rectangle.

49. The price of one stereo is $430. A charitable organization purchases an unknown number of stereos and the total price of the purchase is $6,020. Find the number of stereos purchased.

50. The price of one computer is $490. A charitable organization purchases an unknown number of computers and the total price of the purchase is $5,880. Find the number of computers purchased.

51. A class of 35 students averaged 74 points on an exam. How many total points were accumulated by the class as a whole?

52. A class of 44 students averaged 88 points on an exam. How many total points were accumulated by the class as a whole?

53. 5 times an unknown number is 20. Find the unknown number.

54. 5 times an unknown number is 35. Find the unknown number.

55. 3 times an unknown number is 21. Find the unknown number.

56. 2 times an unknown number is 10. Find the unknown number.

❧ ❧ ❧ Answers ☙ ☙ ☙

1. 24

3. 6

5. 2

7. 5

9. 1

11. 56

13. 42

15. 8

17. 9

19. 6

21. 36

23. 1

25. 6

27. 12

29. 72

31. 56

33. 4

35. 40

37. 13

39. 6

41. 2,320

43. 45

45. 8 cm

47. 7 cm

49. 14

51. 2,590

53. 4

55. 7

Chapter 2

The Integers

Today, much as we take for granted the fact that there exists a number zero, denoted by 0, such that $a+0=a$ for any whole number a, we similarly take for granted that for any whole number a there exists a unique number $-a$, called the "negative" or "opposite" of a, so that $a+(-a)=0$.

In a natural way, or so it seems to modern-day mathematicians, this easily introduces the concept of a *negative* number. However, history teaches us that the concept of negative numbers was not embraced wholeheartedly by mathematicians until somewhere around the 17th century.

In his work *Arithmetica* (c. 250 AD), the Greek mathematician Diophantus (c. 200-284 AD), who some call the "Father of Algebra," described the equation $4=4x+20$ as "absurd," for how could one talk about an answer less than nothing? Girolamo Cardano (1501-1576), in his seminal work *Ars Magna* (c. 1545 AD) referred to negative numbers as "numeri ficti," while the German mathematician Michael Stifel (1487-1567) referred to them as "numeri absurdi." John Napier (1550-1617) (the creator of logarithms) called negative numbers "defectivi," and Rene Descartes (1596-1650) (the creator of analytic geometry) labeled negative solutions of algebraic equations as "false roots."

On the other hand, there were mathematicians whose treatment of negative numbers resembled somewhat our modern notions of the properties held by negative numbers. The Indian mathematician Brahmagupta described arithmetical rules in terms of fortunes (positive number) and debts (negative numbers). Indeed, in his work *Brahmasphutasiddhanta*, he writes "a fortune subtracted from zero is a debt," which in modern notation would resemble $0-4=-4$.

If you find the study of the integers somewhat difficult, do not be discouraged, as centuries of mathematicians have struggled mightily with the topic. With this thought it mind, let's begin the study of the integers.

2.1 An Introduction to the Integers

As we saw in the introduction to the chapter, negative numbers have a rich and storied history. One of the earliest applications of negative numbers had to do with credits and debits. For example, if \$5 represents a credit or profit, then −\$5 represents a debit or loss. Of course, the ancients had a different monetary system than ours, but you get the idea. Note that if a vendor experiences a profit of \$5 on a sale, then a loss of −\$5 on a second sale, the vendor *breaks even*. That is, the sum of \$5 and −\$5 is zero.

In much the same way, every whole number has an *opposite* or *negative* counterpart.

The Opposite or Negative of a Whole Number. For every whole number a, there is a unique number $-a$, called the **opposite** or **negative** of a, such that $a + (-a) = 0$.

The opposite or negative of any whole number is easily located on the number line.

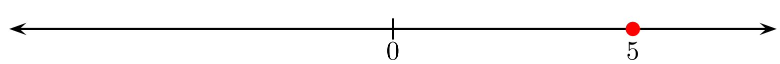
Number Line Location. To locate the opposite (or negative) of any whole number, first locate the whole number on the number line. The opposite is the reflection of the whole number through the origin (zero).

You Try It!

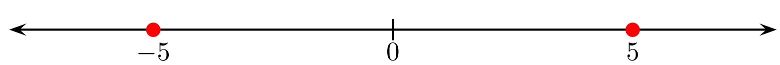
Locate the number −7 and its opposite on the number line.

EXAMPLE 1. Locate the whole number 5 and its opposite (negative) on the number line.

Solution. Draw a number line, then plot the whole number 5 on the line as a shaded dot.



To find its opposite, reflect the number 5 through the origin. This will be the location of the opposite (negative) of the whole number 5, which we indicate by the symbol -5.



Note the symmetry. The whole number 5 is located five units to the right of zero. Its negative is located five units to the left of zero.

□

Important Pronunciation. The symbol -5 is pronounced in one of two ways: (1) "negative five," or (2) "the opposite of five."

In similar fashion, we can locate the opposite or negative of any whole number by reflecting the whole number through the origin (zero), which leads to the image shown in Figure 2.1.

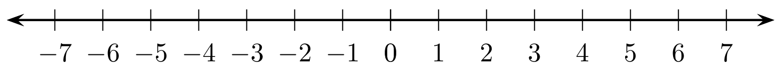


Figure 2.1: The opposite (negative) of any whole number is a reflection of that number through the origin (zero).

The Integers

The collection of numbers arranged on the number line in Figure 2.1 extend indefinitely to the right, and because the numbers on the left are reflections through the origin, the numbers also extend indefinitely to the left. This collection of numbers is called the set of *integers*.

The Integers. The infinite collection of numbers

$$\{\ldots, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, \ldots\}$$

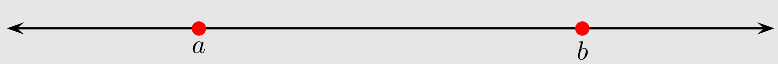
is called the **set of integers**.

The ellipsis ... at each end of this infinite collection means "etcetera," as the integers continue indefinitely to the right and left. Thus, for example, both 23,456 and $-117,191$ are elements of this set and are therefore integers.

Ordering the Integers

As we saw with the whole numbers, as you move to the right on the number line, the numbers get larger; as you move to the left, the numbers get smaller.

Order on the Number Line. Let a and b be integers located on the number line so that the point representing the integer a lies to the left of the point representing the integer b.



Then the integer a is "less than" the integer b and we write

$$a < b$$

Alternatively, we can also say that the integer b is "larger than" the integer a and write

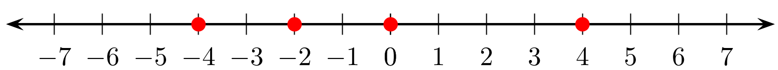
$$b > a.$$

You Try It!

Arrange the numbers −3, −5, 2, and −8 in order, from smallest to largest.

EXAMPLE 2. Arrange the integers 4, 0, −4, and −2 in order, from smallest to largest.

Solution. Place each of the numbers 4, 0, −4, and −2 as shaded dots on the number line.



Thus, −4 is the smallest integer, −2 is the next largest, followed by 0, then 4. Arranging these numbers in order, from smallest to largest, we have

$$-4,\ -2,\ 0,\ 4.$$

Answer: −8, −5, −3, 2

□

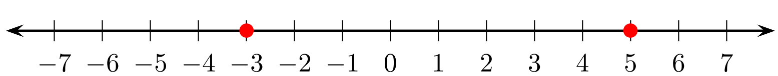
You Try It!

Compare −12 and −11.

EXAMPLE 3. Replace each shaded box with < (less than) or > (greater than) so the resulting inequality is a true statement.

$$-3 \quad \square \quad 5 \qquad\qquad -2 \quad \square \quad -4$$

Solution. For the first case, locate −3 and 5 on the number line as shaded dots.

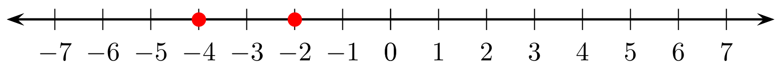


Note that −3 lies to the left of 5, so:

$$-3 \quad \boxed{<} \quad 5$$

That is, -3 is "less than" 5.

In the second case, locate -2 and -4 as shaded dots on the number line.



Note that -2 lies to the right of -4, so:

$$-2 \;\boxed{>}\; -4$$

That is, -2 is "greater than" -4.

Answer: $-12 < -11$

> **Important Observation**. In Example 3, note that the "pointy end" of the inequality symbol always points towards the smaller number.

Opposites of Opposites

We stated earlier that every integer has a unique number called its "opposite" or "negative." Thus, the integer -5 is the opposite (negative) of the integer 5. Thus, we can say that the pair -5 and 5 are opposites. Each is the opposite of the other. Logically, this leads us to the conclusion that the opposite of -5 is 5. In symbols, we would write

$$-(-5) = 5.$$

> **Opposites of Opposites.** Let a be an integer. Then the "opposite of the opposite of a is a." In symbols, we write
>
> $$-(-a) = a.$$
>
> We can also state that the "negative of a negative a is a."

You Try It!

EXAMPLE 4. Simplify $-(-13)$ and $-(-119)$.

Simplify: $-(-50)$.

Solution. The opposite of the opposite of a number returns the original number. That is,

$$-(-13) = 13 \qquad \text{and} \qquad -(-119) = 119.$$

Answer: 50

Positive and Negative

We now define the terms *positive integer* and *negative integer*.

Positive Integer. If a is an integer that lies to the right of zero (the origin) on the number line, then a is a **positive** integer. This means that a is a positive integer if and only if $a > 0$.

Thus, 2, 5, and 117 are positive integers.

Negative Integer. If a is an integer that lies to the left of zero (the origin) on the number line, then a is a **negative** integer. This means that a is a negative integer if and only if $a < 0$.

Thus, -4, -8, and $-1,123$ are negative integers.

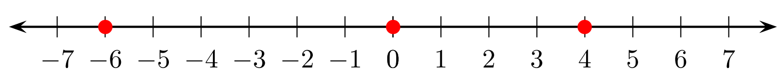
Zero. The integer zero is neither positive nor negative.

You Try It!

Classify -11 as positive, negative, or neither

EXAMPLE 5. Classify each of the following numbers as negative, positive, or neither: 4, -6, and 0.

Solution. Locate 4, -6, and 0 on the number line.



Thus:

- 4 lies to the right of zero. That is, $4 > 0$, making 4 a positive integer.
- -6 lies to the left of zero. That is, $-6 < 0$, making -6 a negative integer.
- The number 0 is neutral. It is neither negative nor positive.

Answer: Negative

□

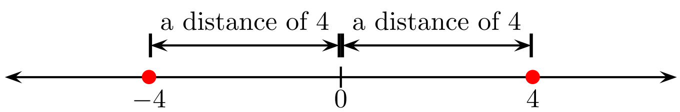
Absolute Value

We define the absolute value of an integer.

Absolute Value. The absolute value of an integer is defined as its distance from the origin (zero).

It is important to note that distance is always a nonnegative quantity (not negative); i.e., distance is either positive or zero.

As an example, we've shaded the integers -4 and 4 on a number line.



The number line above shows two cases:

- The integer -4 is 4 units from zero. Because absolute value measures the distance from zero, $|-4| = 4$.
- The integer 4 is also 4 units from zero. Again, absolute value measures the distance from zero, so $|4| = 4$.

Let's look at another example.

You Try It!

EXAMPLE 6. Determine the value of each expression: a) $|-7|$, b) $|3|$, and c) $|0|$.

Simplify: $|-33|$.

Solution. The absolute value of any integer is equal to the distance that number is from the origin (zero) on the number line. Thus:

a) The integer -7 is 7 units from the origin; hence, $|-7| = 7$.

b) The integer 3 is 3 units from the origin; hence, $|3| = 3$.

c) The integer 0 is 0 units from the origin; hence, $|0| = 0$.

Answer: 33

□

You Try It!

EXAMPLE 7. Determine the value of each expression: a) $-(-8)$ and b) $-|-8|$

Simplify: $-|-50|$.

Solution. These are distinctly different problems.

a) The opposite of -8 is 8. That is, $-(-8) = 8$.

b) However, in this case, we take the absolute value of -8 first, which is 8, then the opposite of that result to get -8. That is,

$$\begin{aligned} -|-8| &= -(8) && \text{First: } |-8| = 8. \\ &= -8 && \text{Second: The opposite of 8 is } -8. \end{aligned}$$

Answer: 50

Applications

There are a number of applications that benefit from the use of integers.

You Try It!

The following table contains record low temperatures (degrees Fahrenheit) for Jackson Hole, Wyoming for the indicated months.

Month	Temp
Sept.	14
Oct.	2
Nov.	-27
Dec.	-49
Jan	-50

Create a bar graph of temperature versus months.

EXAMPLE 8. Profits and losses for the first six months of the fiscal year for a small business are shown in Table 2.1. Profits and losses are measured in thousands of dollars. A positive number represents a profit, while a negative number represents a loss. Create a bar graph representing the profits and losses for this small business for each month of the first half of the fiscal year.

Month	Jan	Feb	Mar	Apr	May	Jun
Profit/Loss	10	12	7	-2	-4	5

Table 2.1: Profit and loss are measured in thousands of dollars.

Solution. Start by labeling the horizontal axis with the months of the fiscal year. Once you've completed that task, scale the vertical axis to accommodate the Profit/Loss values recorded in **Table 2.1**. Finally, starting at the 0 level on the horizontal axis, sketch bars having heights equal to the profit and loss for each month.

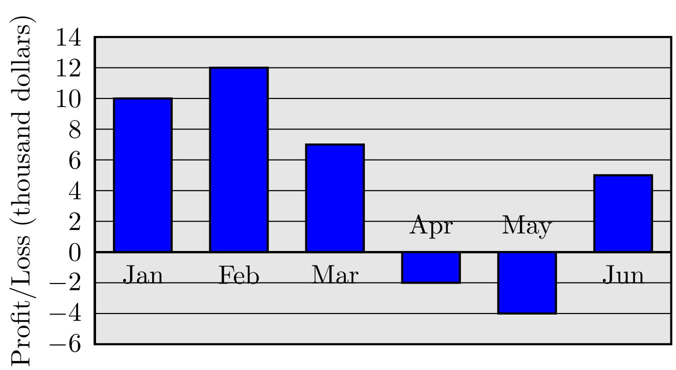


Figure 2.2: Profit and loss bar graph.

Note that the bars in Figure 2.2 for the months January, February, March, and June have heights greater than zero, representing a profit in each of those months. The bars for the months April and May have heights that are less than zero, representing a loss for each of those months.

□

You Try It!

A man stands on the roof of a multistory building and throws a baseball vertically upward. The height (in feet) of the ball above the edge of the roof at measured times (in seconds) is given in the following table.

Time	Height
0	0
1	24
2	16
3	-24
4	-96

Create a line graph of the height of the ball versus time in the air.

EXAMPLE 9. Table 2.2 contains the low temperature recordings (degrees Fahrenheit) on five consecutive days in Fairbanks, Alaska, 1995. Create a line graph for the data in Table 2.2.

Date	Jan 21	Jan 22	Jan 23	Jan 24	Jan 25
Low Temp	1	10	5	-20	-28

Table 2.2: Temperature readings are in degrees Fahrenheit.

Solution. Start by labeling the horizontal axis with the days in January that the temperatures occurred. Scale the vertical axis to accommodate the temperatures given in Table 2.2. Finally, plot points on each day at a height that equals the temperature for that given day. Connect consecutive pairs of points with line segments to produce the line graph shown in Figure 2.3.

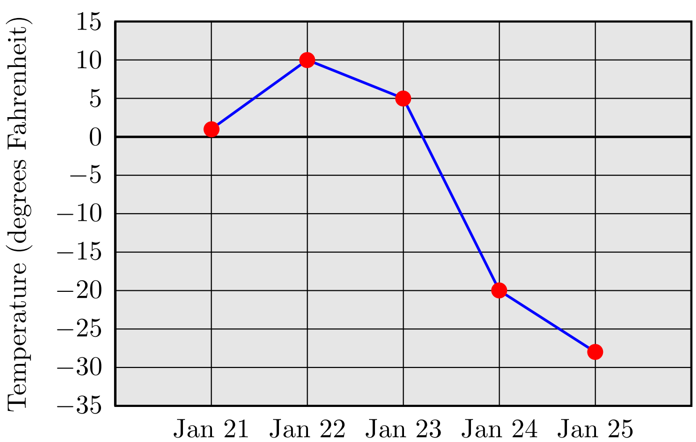


Figure 2.3: Plottting daily low temps versus the date they occurred.

Note that the points in Figure 2.3 have heights greater than zero for January 21-23, representing temperatures greater than zero. The points representing January 24-45 have negative heights corresponding to the negative temperatures of Table 2.2.

□

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-12, for each of the following questions, provide a number line sketch with your answer.

1. What number lies three units to the left of 4 on the number line?

2. What number lies three units to the left of 1 on the number line?

3. What number lies three units to the left of 6 on the number line?

4. What number lies four units to the left of -2 on the number line?

5. What number lies two units to the right of 0 on the number line?

6. What number lies four units to the right of -2 on the number line?

7. What number lies two units to the right of 1 on the number line?

8. What number lies two units to the right of -4 on the number line?

9. What number lies four units to the left of 6 on the number line?

10. What number lies two units to the left of 0 on the number line?

11. What number lies two units to the right of -5 on the number line?

12. What number lies three units to the right of -6 on the number line?

In Exercises 13-24, for each of the following sets of integers, perform the following tasks:

i) Plot each of the integers on a numberline.

ii) List the numbers in order, from smallest to largest.

13. 6, 1, -3, and -5

14. 5, -3, -5, and 2

15. 5, -6, 0, and 2

16. 4, 2, 6, and -4

17. -3, -5, 3, and 5

18. -4, 5, 2, and -6

19. -5, 4, 2, and -3

20. 6, 1, -3, and -1

21. 3, 5, -5, and -1

22. -4, 6, -2, and 3

23. -2, -4, 3, and -6

24. 2, -6, -4, and 5

In Exercises 25-36, in each of the following exercises, enter the inequality symbol $<$ or the symbol $>$ in the shaded box in order that the resulting inequality is a true statement.

25. -4 ☐ 0

26. -4 ☐ 3

27. -2 ☐ -1

28. 3 ☐ 0

29. -3 ☐ -1

30. 6 ☐ 5

31. $3 \ \square \ 6$

32. $-4 \ \square \ -2$

33. $-3 \ \square \ -6$

34. $0 \ \square \ -3$

35. $-1 \ \square \ 4$

36. $1 \ \square \ -4$

In Exercises 37-52, simplify each of the following expressions.

37. $-(-4)$.

38. $-(-6)$.

39. $|7|$.

40. $|1|$.

41. $|5|$.

42. $|3|$.

43. $-|-11|$.

44. $-|-1|$.

45. $|-5|$.

46. $|-1|$.

47. $-|-20|$.

48. $-|-8|$.

49. $|-4|$.

50. $|-3|$.

51. $-(-2)$.

52. $-(-17)$.

In Exercises 53-64, for each of the following exercises, provide a number line sketch with your answer.

53. Find two integers on the number line that are 2 units away from the integer 2.

54. Find two integers on the number line that are 2 units away from the integer -3.

55. Find two integers on the number line that are 4 units away from the integer -3.

56. Find two integers on the number line that are 2 units away from the integer -2.

57. Find two integers on the number line that are 3 units away from the integer -2.

58. Find two integers on the number line that are 4 units away from the integer 1.

59. Find two integers on the number line that are 2 units away from the integer 3.

60. Find two integers on the number line that are 3 units away from the integer 3.

61. Find two integers on the number line that are 3 units away from the integer 0.

62. Find two integers on the number line that are 4 units away from the integer 2.

63. Find two integers on the number line that are 2 units away from the integer 0.

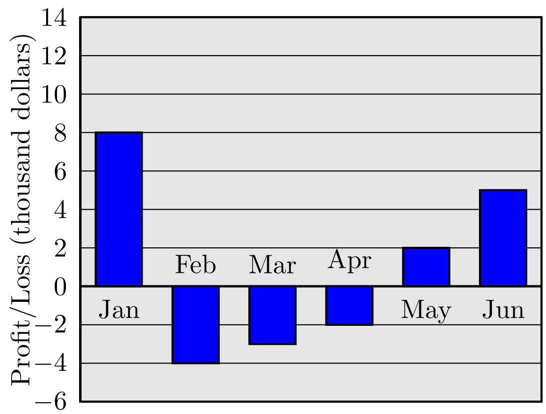
64. Find two integers on the number line that are 3 units away from the integer 1.

---

65. **Dam.** Utah's lowest point of elevation is 2,350 feet above sea level and occurs at Beaver Dam Wash. Express the height as an integer.

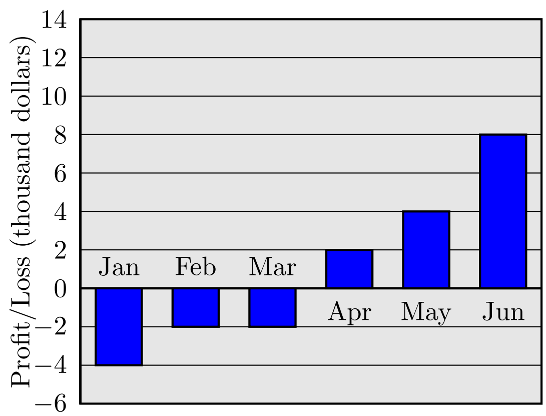
66. **Underwater Glider.** The National Oceanic and Atmospheric Administration's underwater glider samples the bottom of the Atlantic Ocean at 660 feet below sea level. Express that depth as an integer. *Associated Press Times-Standard 4/19/09*

67. Profits and losses for the first six months of the fiscal year for a small business are shown in the following bar chart.



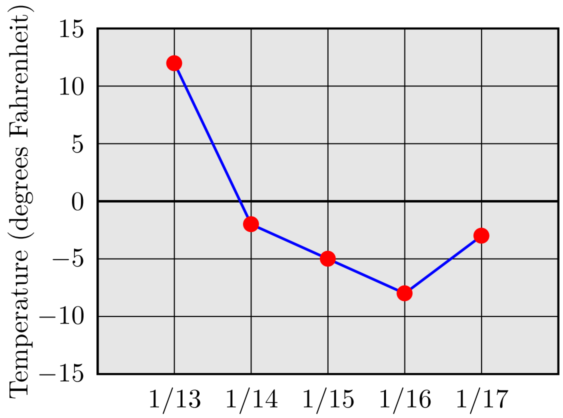
Create a table showing the profit and loss for each month. Use positive integers for the profit and negative integers for the loss. Create a line graph using the data in your table.

68. Profits and losses for the first six months of the fiscal year for a small business are shown in the following bar chart.

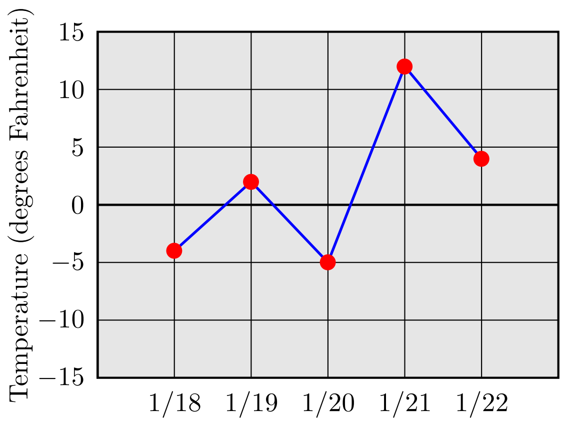


Create a table showing the profit and loss for each month. Use positive integers for the profit and negative integers for the loss. Create a line graph using the data in your table.

69. The following line graph displays the low temperature recordings (degrees Fahrenheit) on five consecutive days in Big Bear, California. What was the lowest temperature reading for the week and on what date did it occur?



70. The following line graph displays the low temperature recordings (degrees Fahrenheit) on five consecutive days in Ogden, Utah. What was the lowest temperature reading for the week and on what date did it occur?



71. The following table contains the low temperature recordings (degrees Fahrenheit) on five consecutive days in Littletown, Ohio. Create a line graph for the data.

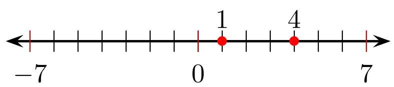
Date	Feb 11	Feb 12	Feb 13	Feb 14	Feb 15
Low Temp	10	−2	−5	−12	8

72. The following table contains the low temperature recordings (degrees Fahrenheit) on five consecutive days in MyTown, Ottawa. Create a line graph for the data.

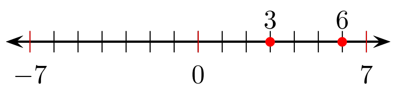
Date	Apr 20	Apr 21	Apr 22	Apr 23	Apr 24
Low Temp	−10	−2	8	5	−5

Answers

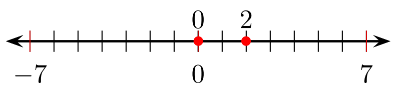
1.



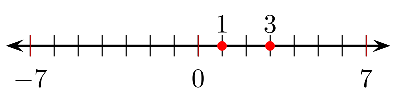
3.



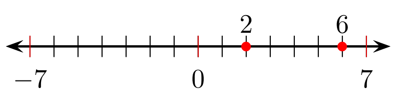
5.



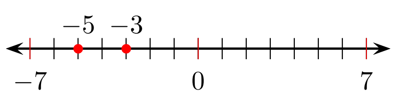
7.



9.

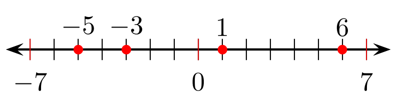


11.



13.

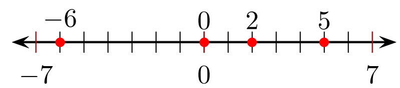
i) Arrange the integers 6, 1, −3, and −5 on a number line.



ii) −5, −3, 1, 6

15.

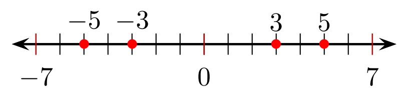
i) Arrange the integers 5, −6, 0, and 2 on a number line.



ii) −6, 0, 2, 5

17.

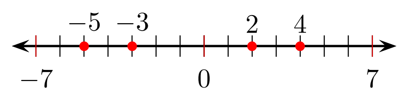
i) Arrange the integers −3, −5, 3, and 5 on a number line.



ii) −5, −3, 3, 5

19.

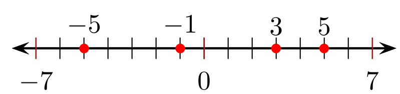
i) Arrange the integers −5, 4, 2, and −3 on a number line.



ii) −5, −3, 2, 4

21.

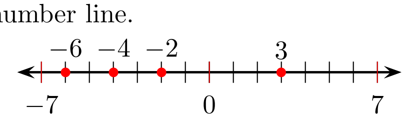
i) Arrange the integers 3, 5, −5, and −1 on a number line.



ii) −5, −1, 3, 5

23.

i) Arrange the integers −2, −4, 3, and −6 on a number line.



ii) −6, −4, −2, 3

25. $-4 < 0$

27. $-2 < -1$

29. $-3 < -1$

31. $3 < 6$

33. $-3 > -6$

35. $-1 < 4$

37. 4

39. 7

41. 5

43. -11

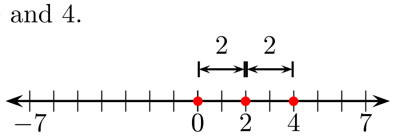
45. 5

47. -20

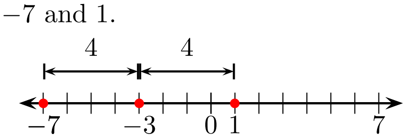
49. 4

51. 2

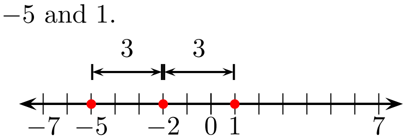
53. 0 and 4.



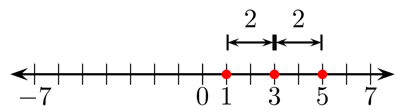
55. -7 and 1.



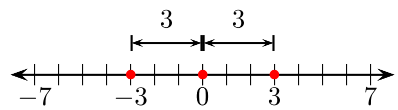
57. -5 and 1.



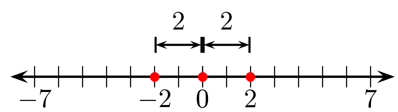
59. 1 and 5.



61. -3 and 3.



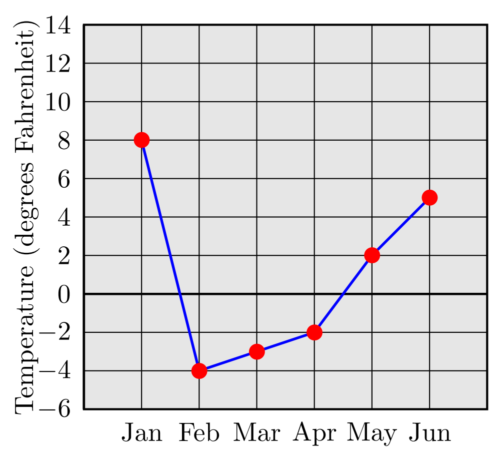
63. -2 and 2.



65. 2,350 feet

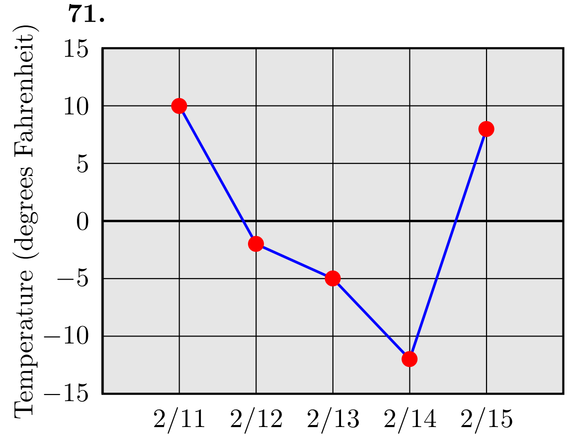
67.

Month	Profit/Loss
Jan	8
Feb	-4
Mar	-3
Apr	-2
May	2
Jun	5



69. Approximately -8° Fahrenheit on January 16.

71.



2.2 Adding Integers

Like our work with the whole numbers, addition of integers is best explained through the use of number line diagrams. However, before we start, let's take a moment to discuss the concept of a vector.

Vectors. A *vector* is a mathematical object that possesses two import qualities: (1) magnitude or length, and (2) directon.

Vectors are a fundamental problem solving tool in mathematics, science, and engineering. In physics, vectors are used to represent forces, position, velocity, and acceleration, while engineers use vectors to represent both internal and external forces on structures, such as bridges and buildings. In this course, and in this particular section, we will concentrate on the use of vectors to help explain addition of integers.

Vectors on the Number Line

Consider the number line in Figure 2.4.

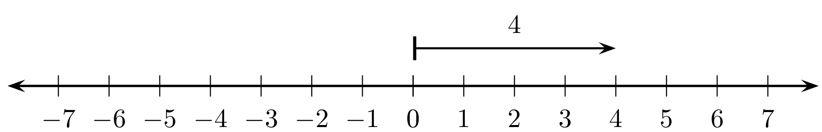


Figure 2.4: A vector representing positive four.

Above the line we've drawn a vector with tail starting at the integer 0 and arrowhead finishing at the integer 4. There are two important things to note about this vector:

1. The magnitude (length) of the vector in Figure 2.4 is four.
2. The vector in Figure 2.4 points to the right.

We will agree that the vector in Figure 2.4 represents positive four.

It is not important that the vector start at the origin. Consider, for example, the vector pictured in Figure 2.5.

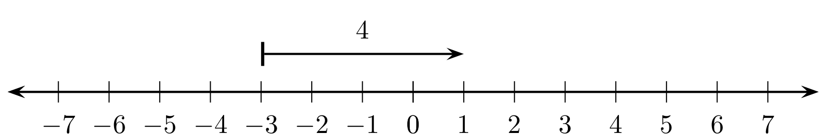


Figure 2.5: A vector representing positive four.

Again, there are two important observations to be made:

1. The magnitude (length) of the vector in Figure 2.5 is four.

2. The vector in Figure 2.5 points to the right.

Hopefully, you have the idea. Any vector that has length 4 and points to the right will represent positive four, regardless of its starting or finishing point.

Conversely, consider the vector in Figure 2.6, which starts at the integer 4 and finishes at the integer -3.

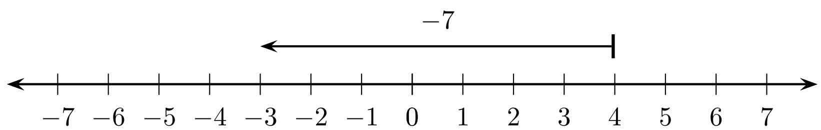


Figure 2.6: A vector representing negative seven.

Two observations:

1. The magnitude (length) of the vector in Figure 2.6 is seven.

2. The vector in Figure 2.6 points to the left.

We will agree that the vector in Figure 2.6 represents negative seven. We could select different starting and finishing points for our vector, but as long as the vector has length seven and points to the left, it represents the integer -7.

Important Observation. A vector pointing to the right represents a **positive** number. A vector pointing to the left represents a **negative** number.

Magnitude and Absolute Value

In Figure 2.4 and Figure 2.5, the vectors pictured represent the integer positive four. Note that the absolute value of four is four; that is, $|4| = 4$. Note also that this absolute value is the magnitude or length of the vectors representing the integer positive four in Figure 2.4 and Figure 2.5.

In Figure 2.6, the vector pictured represents the integer -7. Note that $|-7| = 7$. This shows that the absolute value represents the magnitude or length of the vector representing -7.

Magnitude and Absolute Value. If a is an integer, then $|a|$ gives the magnitude or length of the vector that represents the integer a.

Adding Integers with Like Signs

Because the positive integers are also whole numbers, we've already seen how to add to them in Section 1.2.

EXAMPLE 1. Find the sum $3 + 4$.

Solution. To add the positive integers 3 and 4, proceed as follows.

1. Start at the integer 0, then draw a vector 3 units in length pointing to the right, as shown in Figure 2.7. This arrow has magnitude (length) three and represents the positive integer 3.

2. Draw a second vector of length four that points to the right, starting at the end of the first vector representing the positive integer 3. This arrow has magnitude (length) four and represents the positive integer 4.

3. The sum of the positive integers 3 and 4 could be represented by a vector that starts at the integer 0 and ends at the positive integer 7. However, we prefer to mark this sum on the number line as a solid dot at the positive integer 7. This integer represents the sum of the positive integers 3 and 4.

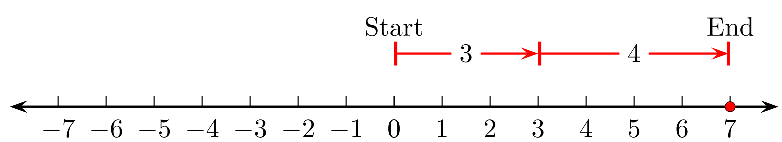


Figure 2.7: Adding two positive integers on the number line.

Thus, $3 + 4 = 7$.

You Try It!

Use a number line diagram to show the sum $5 + 7$.

Answer: 12.

Negative integers are added in a similar fashion.

EXAMPLE 2. Find the sum $-3 + (-4)$.

Solution. To add the negative integers -3 and -4, proceed as follows.

1. Start at the integer 0, then draw a vector 3 units in length pointing to the left, as shown in Figure 2.8. This arrow has magnitude (length) three and represents the negative integer -3.

2. Draw a second vector of length four that points to the left, starting at the end of the first vector representing the negative integer -3. This arrow has magnitude (length) four and represents the negative integer -4.

You Try It!

Use a number line diagram to show the sum $-7 + (-3)$.

3. The sum of the negative integers −3 and −4 could be represented by a vector that starts at the integer 0 and ends at the negative integer −7. However, we prefer to mark this sum on the number line as a solid dot at the negative integer −7. This integer represents the sum of the negative integers −3 and −4.

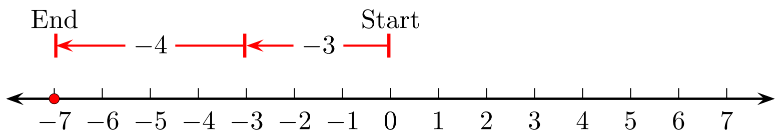


Figure 2.8: Adding two positive integers on the number line.

Thus, $-3 + (-4) = -7$.

Drawing on Physical Intuition. Imagine that you are "walking the number line" in Figure 2.8. You start at the origin (zero) and take 3 paces *to the left*. Next, you walk an additional four paces *to the left*, landing at the number −7.

Answer: −10.

□

It should come as no surprise that the procedure used to add two negative integers comprises two steps.

Adding Two Negative Integers. To add two negative integers, proceed as follows:

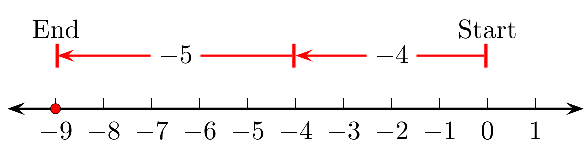
1. Add the magnitudes of the integers.
2. Prefix the common negative sign.

You Try It!

Find the sum: $-5 + (-9)$.

EXAMPLE 3. Find the sums: (a) $-4 + (-5)$, (b) $-12 + (-9)$, and (c) $-2 + (-16)$. **Solution.** We'll examine three separate but equivalent approaches, as discussed in the narrative above.

a) The number line schematic



shows that $(-4) + (-5) = -9$.

b) Drawing on physical intuition, start at zero, walk 12 units to the left, then an additional 9 units to the left. You should find yourself 21 units to the left of the origin (zero). Hence, $-12 + (-9) = -21$.

c) Following the algorithm above in “Adding Two Negative Integers,” first add the magnitudes of -2 and -16; that is, $2+16 = 18$. Now prefix the common sign. Hence, $-2 + (-16) = -18$.

Answer: -14.

Adding Integers with Unlike Signs

Adding integers with unlike signs is no harder than adding integers with like signs.

You Try It!

Use a number line diagram to show the sum $-9 + 2$.

EXAMPLE 4. Find the sum $-8 + 4$.

Solution. To find the sum $-8 + 4$, proceed as follows:

1. Start at the integer 0, then draw a vector eight units in length pointing to the left, as shown in Figure 2.9. This arrow has magnitude (length) eight and represents the negative integer -8.

2. Draw a second vector of length four that points to the right, starting at the end of the first vector representing the negative integer -8. This arrow (also shown in Figure 2.9) has magnitude (length) four and represents the positive integer 4.

3. The sum of the negative integers -8 and 4 could be represented by a vector that starts at the integer 0 and ends at the negative integer -4. However, we prefer to mark this sum on the number line as a solid dot at the negative integer -4. This integer represents the sum of the integers -8 and 4.

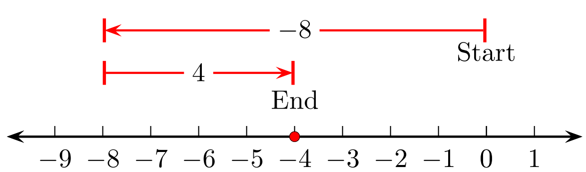


Figure 2.9: Adding -8 and 4 on the number line.

Thus, $-8 + 4 = -4$.

Drawing on Physical Intuition. Imagine you are "walking the number line in Figure 2.9. You start at the origin (zero) and walk eight paces *to the left*. Next, turn around and walk four paces *to the right*, landing on the number -4.

Answer: -7.

Note that adding integers with unlike signs is a *subtractive process*. This is due to the reversal of direction experienced in drawing Figure 2.9 in Example 4.

Adding Two Integers with Unlike Signs. To add two integers with unlike signs, proceed as follows:

1. Subtract the smaller magnitude from the larger magnitude.
2. Prefix the sign of the number with the larger magnitude.

For example, to find the sum $-8+4$ of Example 4, we would note that the integers -8 and 4 have magnitudes 8 and 4, respectively. We would then apply the process outlined in "Adding Two Integers with Unlike Signs."

1. Subtract the smaller magnitude from the larger magnitude; that is, $8-4=4$.
2. Prefix the sign of the number with the larger magnitude. Because -8 has the larger magnitude and its sign is negative, we prefix a negative sign to the difference of the magnitudes. Thus, $-8+4=-4$.

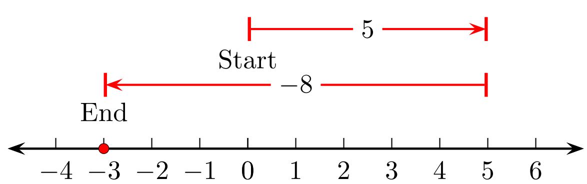
You Try It!

Use a number line diagram to show the sum $5+(-11)$.

EXAMPLE 5. Find the sums: (a) $5+(-8)$, (b) $-12+16$, and (c) $-117+115$.

Solution. We'll examine three separate but equivalent approaches, as discussed in the narrative above.

a) The number line schematic



shows that $5+(-8)=-3$.

b) Drawing on physical intuition, start at zero, walk 12 units to the left, then turn around and walk 16 units to the right. You should find yourself 4 units to the right of the origin (zero). Hence, $-12+16=4$.

c) Following the algorithm in "Adding Two Integers with Unlike Signs," subtract the smallar magnitude from the larger magnitude, thus $117 - 115 = 2$. Because -117 has the larger magnitude and its sign is negative, we prefix a negative sign to the difference of the magnitudes. Thus, $-117 + 115 = -2$.

Answer: -6.

Properties of Addition of Integers

You will be pleased to learn that the properties of addition for whole numbers also apply to addition of integers.

The Commutative Property of Addition. Let a and b represent two integers. Then,

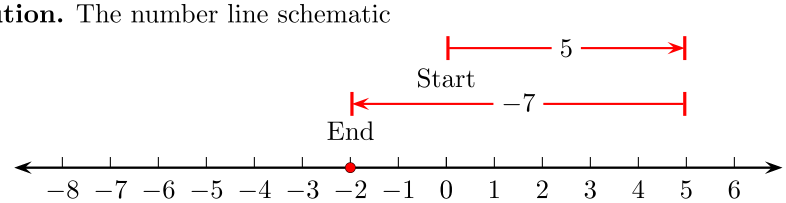
$$a + b = b + a.$$

You Try It!

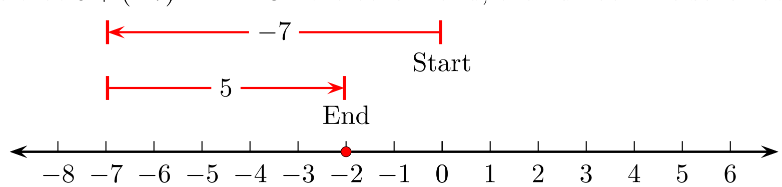
Use a number line diagram to show that $-8 + 6$ is the same as $6 + (-8)$.

EXAMPLE 6. Show that $5 + (-7) = -7 + 5$.

Solution. The number line schematic



shows that $5 + (-7) = -2$. On the other hand, the number line schematic



shows that $-7 + 5 = -2$. Therefore, $5 + (-7) = -7 + 5$.

Addition of integers is also associative.

The Associative Property of Addition. Let a, b, and c represent integers. Then,

$$(a + b) + c = a + (b + c).$$

You Try It!

Show that the expression $(-8+5)+3$ is the same as $-8+(5+3)$ by simplifying each of the two expressions independently.

EXAMPLE 7. Show that $(-9+6)+2 = -9+(6+2)$.

Solution. On the left, the grouping symbols demand that we add -9 and 6 first. Thus,

$$\begin{aligned}(-9+6)+2 &= -3+2 \\ &= -1.\end{aligned}$$

On the right, the grouping symbols demand that we add 6 and 2 first. Thus,

$$\begin{aligned}-9+(6+2) &= -9+8 \\ &= -1.\end{aligned}$$

Both sides simplify to -1. Therefore, $(-9+6)+2 = -9+(6+2)$.

The Additive Identity Property. The integer zero is called the *additive identity*. If a is any integer, then

$$a+0=a \qquad \text{and} \qquad 0+a=a.$$

Thus, for example, $-8+0=-8$ and $0+(-113)=-113$.

Finally, every integer has a unique opposite, called its *additive inverse*.

The Additive Inverse Property. Let a represent any integer. Then there is a unique integer $-a$, called the **opposite** or **additive inverse** of a, such that

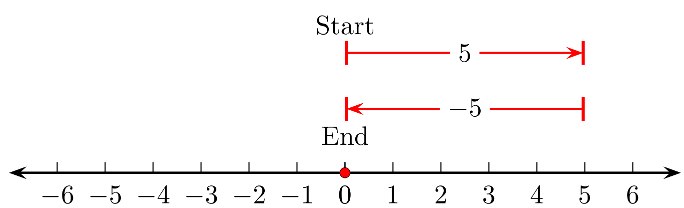
$$a+(-a)=0 \qquad \text{and} \qquad -a+a=0.$$

You Try It!

Use a number line diagram to show that $9+(-9)=0$.

EXAMPLE 8. Show that $5+(-5)=0$.

Solution. The number line schematic



clearly shows that $5+(-5)=0$.

Important Observation. We have used several equivalent phrases to pronounce the integer $-a$. We've used "the opposite of a," "negative a," and "the additive inverse of a." All are equivalent pronunciations.

Grouping for Efficiency

Order of operations require that we perform all additions as they occur, working from left to right.

You Try It!

EXAMPLE 9. Simplify $-7+8+(-9)+12$.

Simplify: $-8+9+(-4)+2$.

Solution. We perform the additions as they occur, working left to right.

$$\begin{aligned} -7+8+(-9)+12 &= 1+(-9)+12 && \text{Working left to right, } -7+8=1. \\ &= -8+12 && \text{Working left to right, } 1+(-9)=-8. \\ &= 4 && -8+12=4 \end{aligned}$$

Thus, $-7+8+(-9)+12=4$.

Answer: -1

The commutative property of addition tells us that changing the order of addition does not change the answer. The associative property of addition tells us that a sum is not affected by regrouping. Let's work Example 9 again, first grouping positive and negative numbers together.

You Try It!

EXAMPLE 10. Simplify $-7+8+(-9)+12$.

Simplify: $-11+7+(-12)+3$.

Solution. The commutative and associative properties allows us to change the order of addition and regroup.

$$\begin{aligned} -7+8+(-9)+12 &= -7+(-9)+8+12 && \text{Use the commutative property to change the order.} \\ &= [-7+(-9)]+[8+12] && \text{Use the associative property to regroup.} \\ &= -16+20 && \text{Add the negatives. Add the positives.} \\ &= 4 && \text{One final addition.} \end{aligned}$$

Thus, $-7+8+(-9)+12=4$.

Answer: -13

At first glance, there seems to be no advantage in using the technique in Example 10 over the technique used in Example 9. However, the technique in Example 10 is much quicker in practice, particularly if you eliminate some of the explanatory steps.

Efficient Grouping. When asked to find the sum of a number of integers, it is most efficient to first add all the positive integers, then add the negatives, then add the results.

You Try It!

Simplify: $-11 + 3 + (-2) + 7$.

EXAMPLE 11. Simplify $-7 + 8 + (-9) + 12$.

Solution. Add the positive integers first, then the negatives, then add the results.

$$\begin{aligned} -7 + 8 + (-9) + 12 &= 20 + (-16) && \text{Add the positives: } 8 + 12 = 20. \\ & && \text{Add the negatives: } -7 + (-9) = -16. \\ &= 4 && \text{Add the results: } 20 + (-16) = 4. \end{aligned}$$

Thus, $-7 + 8 + (-9) + 12 = 4$.

Answer: -7.

□

Using Correct Notation. Never write $+-$! That is, the notation

$$9 + -4 \qquad \text{and} \qquad -8 + -6$$

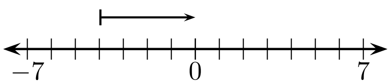
should not be used. Instead, use grouping symbols as follows:

$$9 + (-4) \qquad \text{and} \qquad -8 + (-6)$$

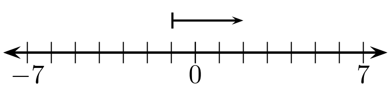
Exercises

In Exercises 1-12, what integer is represented by the given vector?

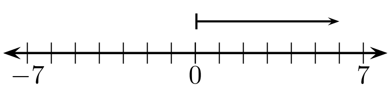
1.



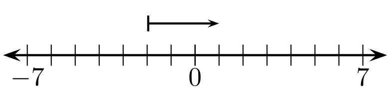
2.



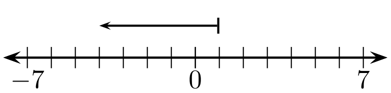
3.



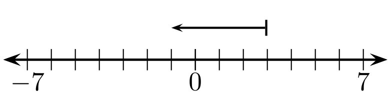
4.



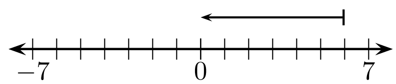
5.



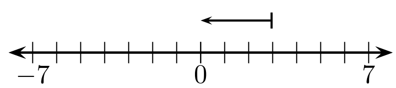
6.



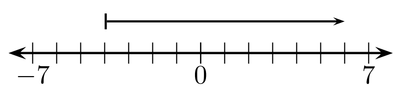
7.



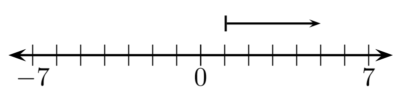
8.



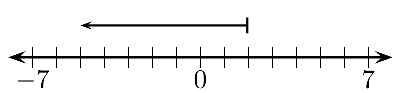
9.



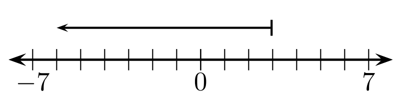
10.



11.



12.



In Exercises 13-36, find the sum of the given integers.

13. $-15+1$

14. $-1+18$

15. $18+(-10)$

16. $2+(-19)$

17. $-10+(-12)$

18. $-1+(-7)$

19. $5+10$

20. $1+12$

21. $2+5$

22. $14+1$

23. $19+(-15)$

24. $20+(-17)$

25. $-2+(-7)$

26. $-14+(-6)$

27. $-6+16$

28. $-2+14$

29. $-11+(-6)$

30. $-7+(-8)$

31. $14+(-9)$

32. $5+(-15)$

33. $10+11$

34. $14+11$

35. $-13+1$

36. $-8+2$

In Exercises 37-52, state the property of addition depicted by the given identity.

37. $-1+(3+(-8))=(-1+3)+(-8)$

38. $-4+(6+(-5))=(-4+6)+(-5)$

39. $7+(-7)=0$

40. $14+(-14)=0$

41. $15+(-18)=-18+15$

42. $14+(-8)=-8+14$

43. $-15+0=-15$

44. $-11+0=-11$

45. $-7+(1+(-6))=(-7+1)+(-6)$

46. $-4+(8+(-1))=(-4+8)+(-1)$

47. $17+(-2)=-2+17$

48. $5+(-13)=-13+5$

49. $-4+0=-4$

50. $-7+0=-7$

51. $19+(-19)=0$

52. $5+(-5)=0$

In Exercises 53-64, state the additive inverse of the given integer.

53. 18

54. 10

55. 12

56. 15

57. -16

58. -4

59. 11

60. 13

61. -15

62. -19

63. -18

64. -9

In Exercises 65-80, find the sum of the given integers.

65. $6 + (-1) + 3 + (-4)$

66. $6 + (-3) + 2 + (-7)$

67. $15 + (-1) + 2$

68. $11 + (-16) + 16$

69. $-17 + 12 + 3$

70. $-5 + (-3) + 2$

71. $7 + 20 + 19$

72. $14 + (-14) + (-20)$

73. $4 + (-8) + 2 + (-5)$

74. $6 + (-3) + 7 + (-2)$

75. $7 + (-8) + 2 + (-1)$

76. $8 + (-9) + 5 + (-3)$

77. $9 + (-3) + 4 + (-1)$

78. $1 + (-9) + 7 + (-6)$

79. $9 + 10 + 2$

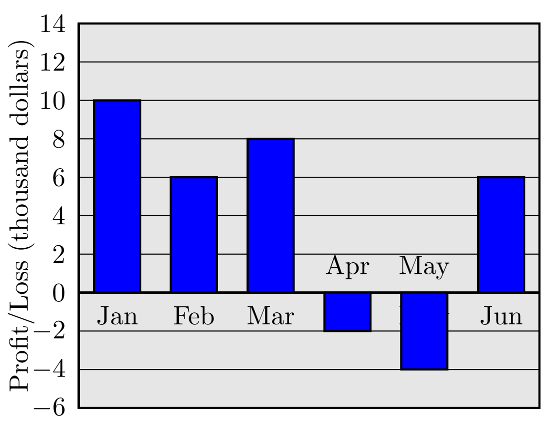
80. $-6 + 15 + (-18)$

---

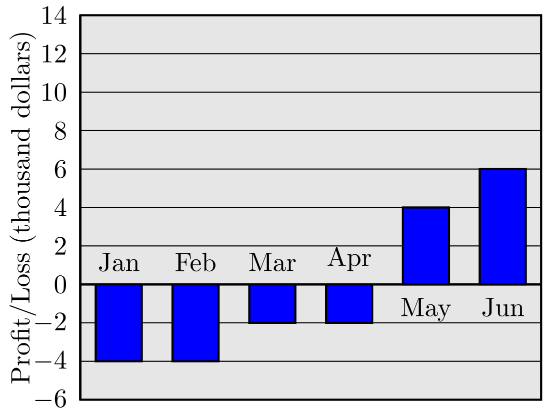
81. **Bank Account.** Gerry opened a new bank account, depositing a check for \$215. He then made several withdraws of \$40, \$75, and \$20 before depositing another check for \$185. How much is in Gerry's account now?

82. **Dead Sea Sinking.** Due to tectonic plate movement, the Dead Sea is sinking about 1 meter each year. If it's currently -418 meters now, what will Dead Sea elevation be in 5 years? Write an expression that models this situation and compute the result.

83. **Profit and Loss.** Profits and losses for the first six months of the fiscal year for a small business are shown in the following bar chart. Sum the profits and losses from each month. Was there a net profit or loss over the six-month period? How much?



84. **Profit and Loss.** Profits and losses for the first six months of the fiscal year for a small business are shown in the following bar chart. Sum the profits and losses from each month. Was there a net profit or loss over the six-month period? How much?



Answers

1. 4

3. 6

5. −5

7. −6

9. 10

11. −7

13. −14

15. 8

17. −22

19. 15

21. 7

23. 4

25. −9

27. 10

29. −17

31. 5

33. 21

35. −12

37. Associative property of addition

39. Additive inverse property

41. Commutative property of addition

43. Additive identity property

45. Associative property of addition

47. Commutative property of addition

49. Additive identity property

51. Additive inverse property

53. -18

55. -12

57. 16

59. -11

61. 15

63. 18

65. 4

67. 16

69. -2

71. 46

73. -7

75. 0

77. 9

79. 21

81. $265

83. Net Profit: $24,000

2.3 Subtracting Integers

In Section 1.2, we stated that "Subtraction is the opposite of addition." Thus, to subtract 4 from 7, we walked seven units to the right on the number line, but then walked 4 units in the opposite direction (to the left), as shown in Figure 2.10.

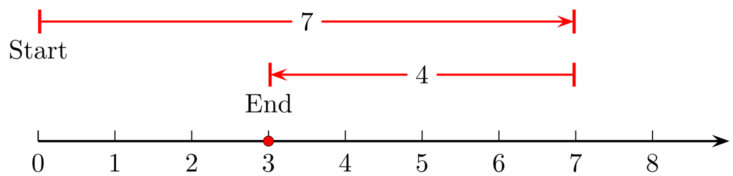


Figure 2.10: Subtraction requires that we reverse direction.

Thus, $7 - 4 = 3$.

The key phrase is "add the opposite." Thus, the subtraction $7 - 4$ becomes the addition $7 + (-4)$, which we would picture on the number line as shown in Figure 2.11.

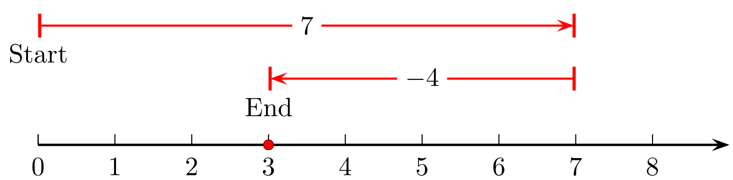


Figure 2.11: Subtraction means *add the opposite*.

Figure 2.10 and Figure 2.11 provide ample evidence that the subtraction $7-4$ is identical to the addition $7+(-4)$. Again, subtraction means "add the opposite." That is, $7 - 4 = 7 + (-4)$.

Defining Subtraction. Subtraction means "add the opposite." That is, if a and b are any integers, then

$$a - b = a + (-b).$$

Thus, for example, $-123-150 = -123+(-150)$ and $-57-(-91) = -57+91$. In each case, subtraction means "add the opposite." In the first case, subtracting 150 is the same as adding -150. In the second case, subtracting -91 is the same as adding 91.

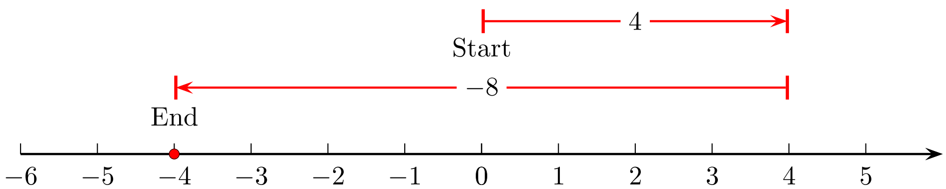
You Try It!

Use each of the techniques in parts (a), (b), and (c) of Example 1 to evaluate the difference $-11-(-9)$.

EXAMPLE 1. Find the differences: (a) $4-8$, (b) $-15-13$, and (c) $-117-(-115)$.

Solution. In each case, subtraction means "add the opposite."

a) Change the subtraction to addition with the phrase "subtraction means add the opposite." That is, $4-8=4+(-8)$. We can now perform this addition on the number line.



Thus, $4-8=4+(-8)=-4$.

b) First change the subtraction into addition by "adding the opposite." That is, $-15-13=-15+(-13)$. We can now use physical intuition to perform the addition. Start at the origin (zero), walk 15 units to the left, then an additional 13 units to the left, arriving at the answer -28. That is,

$$\begin{aligned}-15-13&=-15+(-13)\\&=-28.\end{aligned}$$

c) First change the subtraction into addition by "adding the opposite." That is, $-117-(-115)=-117+115$. Using "Adding Two Integers with Unlike Signs" from Section 2.2, first subtract the smaller magnitude from the larger magnitude; that is, $117-115=2$. Because -117 has the larger magnitude and its sign is negative, prefix a negative sign to the difference in magnitudes. Thus,

$$\begin{aligned}-117-(-115)&=-117+115\\&=-2.\end{aligned}$$

Answer: -2

□

Order of Operations

We will now apply the "Rules Guiding Order of Operations" from Section 1.5 to a number of example exercises.

You Try It!

Simplify: $-3-(-9)-11$.

EXAMPLE 2. Simplify $-5-(-8)-7$.

Solution. We work from left to right, changing each subtraction by "adding the opposite."

$-5-(-8)-7 = -5+8+(-7)$	Add the opposite of -8, which is 8. Add the opposite of 7, which is -7.
$= 3+(-7)$	Working left to right, $-5+8=3$.
$= -4$	$3+(-7)=-4$.

Answer: -5

□

Grouping symbols say "do me first."

You Try It!

Simplify: $-3-(-3-3)$.

EXAMPLE 3. Simplify $-2-(-2-4)$.

Solution. Parenthetical expressions must be evaluated first.

$-2-(-2-4) = -2-(-2+(-4))$	Simplify the parenthetical expression first. Add the opposite of 4, which is -4.
$= -2-(-6)$	Inside the parentheses, $-2+(-4)=-6$.
$= -2+6$	Subtracting a -6 is the same as adding a 6.
$= 4$	Add: $-2+6=4$.

Answer: 3

□

Change as a Difference

Suppose that when I leave my house in the early morning, the temperature outside is 40° Fahrenheit. Later in the day, the temperature measures 60° Fahrenheit. How do I measure the change in the temperature?

The Change in a Quantity. To measure the change in a quantity, always subtract the *former* measurement from the *latter* measurement. That is:

$$\text{Change in a Quantity} = \text{Latter Measurement} - \text{Former Measurement}$$

Thus, to measure the change in temperature, I perform a subtraction as follows:

Change in Temperature	=	Latter Measurement	−	Former Measurement
	=	60° F	−	40° F
	=	20° F		

Note that the positive answer is in accord with the fact that the temperature has increased.

You Try It!

EXAMPLE 4. Suppose that in the afternoon, the temperature measures 65° Fahrenheit, then late evening the temperature drops to 44° Fahrenheit. Find the change in temperature.

Marianne awakes to a morning temperature of 54° Fahrenheit. A storm hits, dropping the temperature to 43° Fahrenheit.Find the change in temperature.

Solution. To measure the change in temperature, we must subtract the former measurement from the latter measurement.

Change in Temperature	=	Latter Measurement	−	Former Measurement
	=	44° F	−	65° F
	=	−11° F		

Note that the negative answer is in accord with the fact that the temperature has decreased. There has been a "change" of −11° Fahrenheit.

Answer: −11° Fahrenheit

□

You Try It!

EXAMPLE 5. Sometimes a bar graph is not the most appropriate visualization for your data. For example, consider the bar graph in Figure 2.12 depicting the Dow Industrial Average for seven consecutive days in March of 2009. Because the bars are of almost equal height, it is difficult to detect fluctuation or change in the Dow Industrial Average.

Let's determine the change in the Dow Industrial average on a day-to-day basis. Remember to subtract the latter measurement minus the former (current day minus former day). This gives us the following changes.

Consecutive Days	Change in Dow Industrial Average
Sun-Mon	$6900 - 7000 = -100$
Mon-Tues	$6800 - 6900 = -100$
Tues-Wed	$6800 - 6800 = 0$
Wed-Thu	$7000 - 6800 = 200$
Thu-Fri	$7100 - 7000 = 100$
Fri-Sat	$7200 - 7100 = 100$

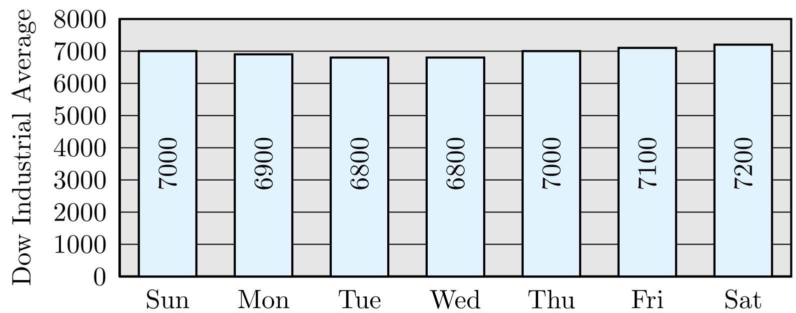


Figure 2.12: Profit and loss bar graph.

We will use the data in the table to construct a line graph. On the horizontal axis, we place the pairs of consecutive days (see Figure 2.13). On the vertical axis we place the Change in the Industrial Dow Average. At each pair of days we plot a point at a height equal to the change in Dow Industrial Average as calculated in our table.

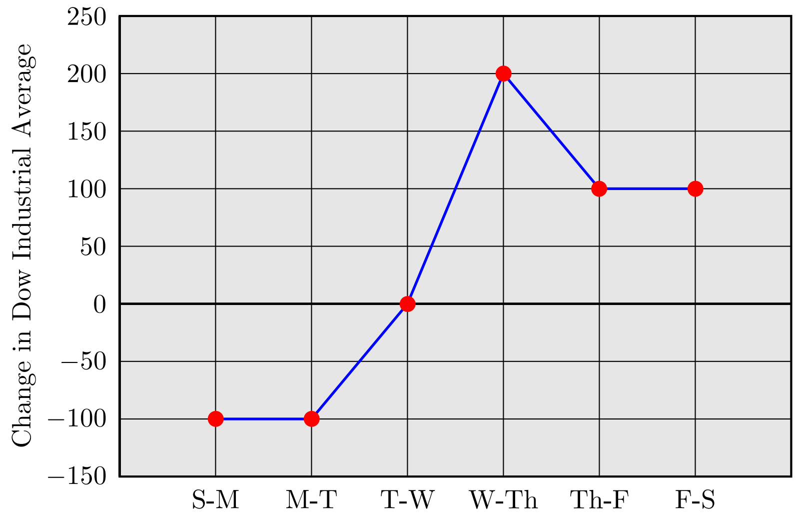


Figure 2.13: Plotting change in Dow versus consecutive days.

Note that the data as displayed by Figure 2.13 more readily shows the changes in the Dow Industrial Average on a day-to-day basis. For example, it is now easy to pick the day that saw the greatest increase in the Dow (from Wednesday to Thursday, the Dow rose 200 points).

□

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-24, find the difference.

1. $16 - 20$

2. $17 - 2$

3. $10 - 12$

4. $16 - 8$

5. $14 - 11$

6. $5 - 8$

7. $7 - (-16)$

8. $20 - (-10)$

9. $-4 - (-9)$

10. $-13 - (-3)$

11. $8 - (-3)$

12. $14 - (-20)$

13. $2 - 11$

14. $16 - 2$

15. $-8 - (-10)$

16. $-14 - (-2)$

17. $13 - (-1)$

18. $12 - (-13)$

19. $-4 - (-2)$

20. $-6 - (-8)$

21. $7 - (-8)$

22. $13 - (-14)$

23. $-3 - (-10)$

24. $-13 - (-9)$

In Exercises 25-34, simplify the given expression.

25. $14 - 12 - 2$

26. $-19 - (-7) - 11$

27. $-20 - 11 - 18$

28. $7 - (-13) - (-1)$

29. $5 - (-10) - 20$

30. $-19 - 12 - (-8)$

31. $-14 - 12 - 19$

32. $-15 - 4 - (-6)$

33. $-11 - (-7) - (-6)$

34. $5 - (-5) - (-14)$

In Exercises 35-50, simplify the given expression.

35. $-2 - (-6 - (-5))$

36. $6 - (-14 - 9)$

37. $(-5 - (-8)) - (-3 - (-2))$

38. $(-6 - (-8)) - (-9 - 3)$

39. $(6 - (-9)) - (3 - (-6))$

40. $(-2 - (-3)) - (3 - (-6))$

41. $-1 - (10 - (-9))$

42. $7 - (14 - (-8))$

43. $3 - (-8 - 17)$

44. $1 - (-1 - 4)$

45. $13-(16-(-1))$

46. $-7-(-3-(-8))$

47. $(7-(-8))-(5-(-2))$

48. $(6-5)-(7-3)$

49. $(6-4)-(-8-2)$

50. $(2-(-6))-(-9-(-3))$

---

51. The first recorded temperature is 42° F. Four hours later, the second temperature is 65° F. What is the change in temperature?

52. The first recorded temperature is 79° F. Four hours later, the second temperature is 46° F. What is the change in temperature?

53. The first recorded temperature is 30° F. Four hours later, the second temperature is 51° F. What is the change in temperature?

54. The first recorded temperature is 109° F. Four hours later, the second temperature is 58° F. What is the change in temperature?

55. Typical temperatures in Fairbanks, Alaska in January are -2 degrees Fahrenheit in the daytime and -19 degrees Fahrenheit at night. What is the change in temperature from day to night?

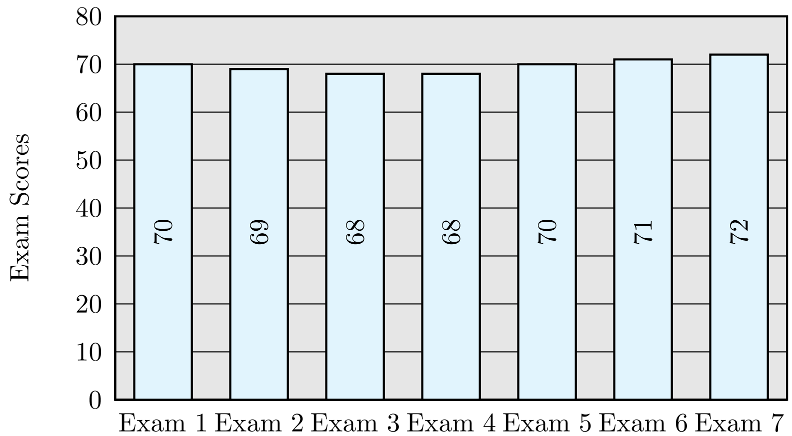
56. Typical summertime temperatures in Fairbanks, Alaska in July are 79 degrees Fahrenheit in the daytime and 53 degrees Fahrenheit at night. What is the change in temperature from day to night?

57. **Communication.** A submarine 1600 feet below sea level communicates with a pilot flying 22,500 feet in the air directly above the submarine. How far is the communique traveling?

58. **Highest to Lowest.** The highest spot on earth is on Mount Everest in Nepal-Tibet at 8,848 meters. The lowest point on the earth's crust is the Mariana's Trench in the North Pacific Ocean at 10,923 meters below sea level. What is the distance between the highest and the lowest points on earth? *Wikipedia http://en.wikipedia.org/wiki/Extremes_on_Earth*

59. **Lowest Elevation.** The lowest point in North America is Death Valley, California at -282 feet. The lowest point on the entire earth's landmass is on the shores of the Dead Sea along the Israel-Jordan border with an elevation of -1,371 feet. How much lower is the Dead Sea shore from Death Valley?

60. Exam Scores. Freida's scores on her first seven mathematics exams are shown in the following bar chart. Calculate the differences between consecutive exams, then create a line graph of the differences on each pair of consecutive exams. Between which two pairs of consecutive exams did Freida show the most improvement?



❧ ❧ ❧ Answers ☙ ☙ ☙

1. -4

3. -2

5. 3

7. 23

9. 5

11. 11

13. -9

15. 2

17. 14

19. -2

21. 15

23. 7

25. 0

27. -49

29. -5

31. -45

33. 2

35. -1

37. 4

39. 6

41. -20

43. 28

45. -4

47. 8

49. 12

51. 23° F

53. 21° F

55. -17 degrees Fahrenheit

57. 24,100 feet

59. 1,089 feet lower

2.4 Multiplication and Division of Integers

Before we begin, let it be known that the integers satisfy the same properties of multiplication as do the whole numbers.

Integer Properties of Multiplication.

Commutative Property. If a and b are integers, then their product commutes. That is,

$$a \cdot b = b \cdot a.$$

Associative Property. If a, b, and c are integers, then their product is associative. That is,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

Multiplicative Identity Property. If a is any integer, then

$$a \cdot 1 = a \qquad \text{and} \qquad 1 \cdot a = a.$$

Because multiplying any integer by 1 returns the identical integer, the integer 1 is called the **multiplicative identity**.

In Section 1.3, we learned that multiplication is equivalent to *repeated addition.* For example,

$$3 \cdot 4 = \underbrace{4 + 4 + 4}_{\text{three fours}}.$$

On the number line, three sets of four is equivalent to walking three sets of four units to the right, starting from zero, as shown in Figure 2.14.

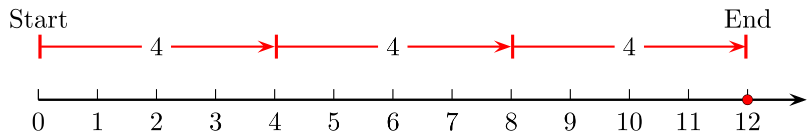


Figure 2.14: Note that $3 \cdot 4 = 4 + 4 + 4$. That is, $3 \cdot 4 = 12$.

This example and a little thought should convince readers that the product of two positive integers will always be a positive integer.

The Product of Two Positive Integers. If a and b are two positive integers, then their product ab is also a positive integer.

For example, $2 \cdot 3 = 6$ and $13 \cdot 117 = 1521$. In each case, the product of two positive integers is a positive integer.

The Product of a Positive Integer and a Negative Integer

If we continue with the idea that multiplication is equivalent to repeated addition, then it must be that

$$3 \cdot (-4) = \underbrace{-4 + (-4) + (-4)}_{\text{three negative fours}}.$$

Pictured on the number line, $3 \cdot (-4)$ would then be equivalent to walking three sets of negative four units (to the left), starting from zero, as shown in Figure 2.15.

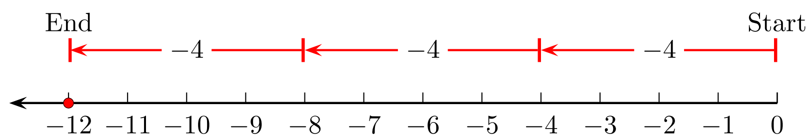


Figure 2.15: Note that $3 \cdot (-4) = -4 + (-4) + (-4)$. That is, $3 \cdot (-4) = -12$.

Note, at least in this particular case, that the product of a positive integer and a negative integer is a negative integer.

We've shown that $3 \cdot (-4) = -12$. However, integer multiplication is commutative, so it must also be true that $-4 \cdot 3 = -12$. That is, the product of a negative integer and a positive integer is also a negative integer. Although not a proof, this argument motivates the following fact about integer multiplication.

The Product of a Positive Integer and a Negative Integer. Two facts are true:

1. If a is a positive integer and b is a negative integer, then the product ab is a negative integer.

2. If a is a negative integer and b is a positive integer, then the product ab is a negative integer.

Thus, for example, $5 \cdot (-12) = -60$ and $-13 \cdot 2 = -26$. In each case the answer is negative because we are taking a product where one of the factors is positive and the other is negative.

The Distributive Property

The integers satisfy the *distributive property.*

The Distributive Property. Let a, b, and c be integers. Then,

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

We say that "multiplication is distributive with respect to addition."

Note how the a is "distributed." The a is multiplied times each term in the parentheses.

For example, consider the expression $3 \cdot (4 + 5)$. We can evaluate this expression according to the order of operations, simplifying the expression inside the parentheses first.

$$\begin{aligned} 3 \cdot (4 + 5) &= 3 \cdot 9 && (2.1) \\ &= 27 && (2.2) \end{aligned}$$

But we can also use the distributive property, multiplying each term inside the parentheses by three, then simplifying the result.

$$\begin{aligned} 3 \cdot (4 + 5) &= 3 \cdot 4 + 3 \cdot 5 && \text{Distribute the 3.} \\ &= 12 + 15 && \text{Perform multiplications first:} \\ & && 3 \cdot 4 = 12 \text{ and } 3 \cdot 5 = 15. \\ &= 27 && \text{Add: } 12 + 15 = 27. \end{aligned}$$

Note that evaluating $3 \cdot (4 + 5)$ using the distributive property provides the same result as the evaluation (2.1) using the order of operations.

The Multiplicative Property of Zero

The distributive property can be used to provide proofs of a number of important properties of integers. One important property is the fact that if you multiply an integer by zero, the product is zero. Here is a proof of that fact that uses the distributive property.

Let a be any integer. Then,

$$\begin{aligned} a \cdot 0 &= a \cdot (0 + 0) && \text{Additive Identity Property: } 0 + 0 = 0. \\ a \cdot 0 &= a \cdot 0 + a \cdot 0 && \text{Distribute } a \text{ times each zero in the parentheses.} \end{aligned}$$

Next, to "undo" the effect of adding $a \cdot 0$, subtract $a \cdot 0$ from both sides of the equation.

$$\begin{aligned} a \cdot 0 - a \cdot 0 &= a \cdot 0 + a \cdot 0 - a \cdot 0 && \text{Subtract } a \cdot 0 \text{ from both sides.} \\ 0 &= a \cdot 0 && a \cdot 0 - a \cdot 0 = 0 \text{ on each side.} \end{aligned}$$

Multiplicative Property of Zero. Let a represent any integer. Then

$$a \cdot 0 = 0 \qquad \text{and} \qquad 0 \cdot a = 0.$$

Thus, for example, $-18 \cdot 0 = 0$ and $0 \cdot 122 = 0$.

Multiplying by Minus One

Here is another useful application of the distributive property.

$$\begin{aligned}
(-1)a + a &= (-1)a + 1a && \text{Replace } a \text{ with } 1a.\\
&= (-1+1)a && \text{Use the distributive property to factor out } a.\\
&= 0a && \text{Replace } -1+1 \text{ with } 0.\\
&= 0 && \text{Replace } 0a \text{ with } 0.
\end{aligned}$$

Thus, $(-1)a + a = 0$. That is, if you add $(-1)a$ to a you get zero. However, the Additive Inverse Property says that $-a$ is the *unique* number that you add to a to get zero. The conclusion must be that $(-1)a = -a$.

Multiplying by Minus One. If a is any integer, then

$$(-1)a = -a.$$

Thus, for example, $-1(4) = -4$ and $-1(-4) = -(-4) = 4$.

This property is rather important, as we will see in future work. Not only does it tell us that $(-1)a = -a$, but it also tells us that if we see $-a$, then it can be interpreted to mean $(-1)a$.

The Product of Two Negative Integers

We can employ the multiplicative property of -1, that is, $(-1)a = -a$ to find the product of two negative numbers.

$$\begin{aligned}
(-4)(-3) &= [(-1)(4)](-3) && \text{Replace } -4 \text{ with } (-1)(4).\\
&= (-1)[(4)(-3)] && \text{Use the associative property to regroup.}\\
&= (-1)(-12) && \text{We know: } (4)(-3) = -12.\\
&= -(-12) && (-1)a = -a. \text{ Here } (-1)(-12) = -(-12).\\
&= 12 && -(-a) = a. \text{ Here } -(-12) = 12.
\end{aligned}$$

Thus, at least in the case of $(-4)(-3)$, the product of two negative integers is a positive integer. This is true in general.

The Product of Two Negative Integers. If both a and b are negative integers, then their product ab is a positive integer.

Thus, for example, $(-5)(-7) = 35$ and $(-112)(-6) = 672$. In each case the answer is positive, because the product of two negative integers is a positive integer.

Memory Device

Here's a simple memory device to help remember the rules for finding the product of two integers.

Like and Unlike Signs. There are two cases:

Unlike Signs. The product of two integers with *unlike* signs is negative. That is:

$$(+)(-) = -$$
$$(-)(+) = -$$

Like Signs. The product of two integers with *like* signs is positive. That is:

$$(+)(+) = +$$
$$(-)(-) = +$$

You Try It!

EXAMPLE 1. Simplify: (a) $(-3)(-2)$, (b) $(4)(-10)$, and (c) $(12)(-3)$.

Simplify: (a) $(-12)(4)$ and (b) $(-3)(-11)$.

Solution. In each example, we use the "like" and "unlike" signs approach.

a) Like signs gives a positive result. Hence, $(-3)(-2) = 6$.

b) Unlike signs gives a negative result. Hence, $(4)(-10) = -40$.

c) Unlike signs gives a negative result. Hence, $(12)(-3) = -36$.

Answer: (a) -48, (b) 33

□

You Try It!

Simplify: $(-2)(-3)(4)(-1)$.

EXAMPLE 2. Simplify $(-3)(2)(-4)(-2)$.

Solution. Order of operations demands that we work from left to right.

$$\begin{aligned}
(-3)(2)(-4)(-2) &= (-6)(-4)(-2) && \text{Work left to right: } (-3)(2) = -6.\\
&= (24)(-2) && \text{Work left to right: } (-6)(-4) = 24.\\
&= -48 && \text{Multiply: } (24)(-2) = -48.
\end{aligned}$$

Answer: -24.

Hence, $(-3)(2)(-4)(-2) = -48$.

□

You Try It!

Simplify: (a) $(-2)^2$ and (b) -2^2.

EXAMPLE 3. Simplify: (a) $(-2)^3$ and (c) $(-3)^4$.

Solution. In each example, use

$$a^m = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ times}},$$

then work left to right with the multiplication.

a) Use the definition of an exponent, then order of operations.

$$\begin{aligned}
(-2)^3 &= (-2)(-2)(-2) && \text{Write } -2 \text{ as a factor three times.}\\
&= 4(-2) && \text{Work left to right: } (-2)(-2) = 4.\\
&= -8
\end{aligned}$$

b) Use the definition of an exponent, then order of operations.

$$\begin{aligned}
(-3)^4 &= (-3)(-3)(-3)(-3) && \text{Write } -3 \text{ as a factor four times.}\\
&= 9(-3)(-3) && \text{Work left to right: } (-3)(-3) = 9.\\
&= -27(-3) && \text{Work left to right: } 9(-3) = -27.\\
&= 81
\end{aligned}$$

Answer: (a) 4 and (b) -4.

□

Example 3 motivates the following fact.

Even and Odd. Two facts are apparent.

1. If a product contains an odd number of negative factors, then the product is negative.
2. If a product contains an even number of negative factors, then the product is positive.

Thus, for example,

$$(-2)^5 = (-2)(-2)(-2)(-2)(-2) = -32$$

quickly evaluates as -32 as it has an odd number of negative factors. On the other hand,

$$(-2)^6 = (-2)(-2)(-2)(-2)(-2)(-2) = 64$$

quickly evaluates as 64 as it has an even number of negative factors.

Division of Integers

Consider that

$$\frac{12}{3} = 4 \text{ because } 3(4) = 12 \qquad \text{and} \qquad \frac{-12}{-3} = 4 \text{ because } -3(4) = -12.$$

In like manner,

$$\frac{12}{-3} = -4 \text{ because } -3(-4) = 12 \qquad \text{and} \qquad \frac{-12}{3} = -4 \text{ because } 3(-4) = -12.$$

Thus, the rules for dividing integers are the same as the rules for multiplying integers.

Like and Unlike Signs. There are two cases:

Unlike Signs. The quotient of two integers with *unlike* signs is negative. That is,

$$\frac{(+)}{(-)} = -$$
$$\frac{(-)}{(+)} = -$$

Like Signs. The quotient of two integers with *like* signs is positive. That is,

$$\frac{(+)}{(+)} = +$$
$$\frac{(-)}{(-)} = +$$

Thus, for example, $12/(-6) = -2$ and $-44/(-4) = 11$. In the first case, unlike signs gives a negative quotient. In the second case, like signs gives a positive quotient.

One final reminder.

Division by Zero is Undefined. If a is any integer, the quotient

$$\frac{a}{0}$$

is undefined. Division by zero is meaningless.

See the discussion in Section 1.3 for a discussion on division by zero.

You Try It!

Simplify: (a) $-24/4$ and (b) $-28/(-7)$.

EXAMPLE 4. Simplify: (a) $-12/(-4)$, (b) $6/(-3)$, and (c) $-15/0$.

Solution. In each example, we use the "like" and "unlike" signs approach.

a) Like signs gives a positive result. Hence,

$$\frac{-12}{-4} = 3.$$

b) Unlike signs gives a negative result. Hence,

$$\frac{6}{-3} = -2.$$

c) Division by zero is undefined. Hence,

$$\frac{-15}{0}$$

is undefined.

Answer: (a) -6, (b) 4

□

Exercises

In Exercises 1-16, state the property of multiplication depicted by the given identity.

1. $(-2)\big[(-16)(13)\big] = \big[(-2)(-16)\big](13)$

2. $(10)\big[(-15)(-6)\big] = \big[(10)(-15)\big](-6)$

3. $(-17)(-10) = (-10)(-17)$

4. $(-5)(3) = (3)(-5)$

5. $(4)(11) = (11)(4)$

6. $(-5)(-11) = (-11)(-5)$

7. $16 \cdot \big(8 + (-15)\big) = 16 \cdot 8 + 16 \cdot (-15)$

8. $1 \cdot \big(-16 + (-6)\big) = 1 \cdot (-16) + 1 \cdot (-6)$

9. $(17)\big[(20)(11)\big] = \big[(17)(20)\big](11)$

10. $(14)\big[(-20)(-18)\big] = \big[(14)(-20)\big](-18)$

11. $-19 \cdot 1 = -19$

12. $-17 \cdot 1 = -17$

13. $8 \cdot 1 = 8$

14. $-20 \cdot 1 = -20$

15. $14 \cdot \big(-12 + 7\big) = 14 \cdot (-12) + 14 \cdot 7$

16. $-14 \cdot \big(-3 + 6\big) = -14 \cdot (-3) + (-14) \cdot 6$

In Exercises 17-36, simplify each given expression.

17. $4 \cdot 7$

18. $4 \cdot 2$

19. $3 \cdot (-3)$

20. $7 \cdot (-9)$

21. $-1 \cdot 10$

22. $-1 \cdot 11$

23. $-1 \cdot 0$

24. $-8 \cdot 0$

25. $-1 \cdot (-14)$

26. $-1 \cdot (-13)$

27. $-1 \cdot (-19)$

28. $-1 \cdot (-17)$

29. $2 \cdot 0$

30. $-6 \cdot 0$

31. $-3 \cdot 8$

32. $7 \cdot (-3)$

33. $7 \cdot 9$

34. $6 \cdot 3$

35. $-1 \cdot 5$

36. $-1 \cdot 2$

In Exercises 37-48, simplify each given expression.

37. $(-7)(-1)(3)$

38. $(10)(6)(3)$

39. $(-7)(9)(10)(-10)$

40. $(-8)(-5)(7)(-9)$

41. $(6)(5)(8)$

42. $(7)(-1)(-9)$

43. $(-10)(4)(-3)(8)$

44. $(8)(-2)(-5)(2)$

45. $(6)(-3)(-8)$

46. $(-5)(-4)(1)$

47. $(2)(1)(3)(4)$

48. $(7)(5)(1)(4)$

In Exercises 49-60, compute the exact value.

49. $(-4)^4$

50. $(-3)^4$

51. $(-5)^4$

52. $(-2)^2$

53. $(-5)^2$

54. $(-3)^3$

55. $(-6)^2$

56. $(-6)^4$

57. $(-4)^5$

58. $(-4)^2$

59. $(-5)^3$

60. $(-3)^2$

In Exercises 61-84, simplify each given expression.

61. $-16 \div (-8)$

62. $-33 \div (-3)$

63. $\dfrac{-8}{1}$

64. $\dfrac{40}{-20}$

65. $\dfrac{-1}{0}$

66. $\dfrac{2}{0}$

67. $-3 \div 3$

68. $-58 \div 29$

69. $\dfrac{56}{-28}$

70. $\dfrac{60}{-12}$

71. $0 \div 15$

72. $0 \div (-4)$

73. $\dfrac{63}{21}$

74. $\dfrac{-6}{-1}$

75. $\dfrac{78}{13}$

76. $\dfrac{-84}{-14}$

77. $0 \div 5$

78. $0 \div (-16)$

79. $\dfrac{17}{0}$

80. $\dfrac{-20}{0}$

81. $-45 \div 15$

82. $-28 \div 28$

83. $12 \div 3$

84. $-22 \div (-22)$

---

85. **Scuba.** A diver goes down 25 feet. A second diver then dives down 5 times further than the first diver. Write the final depth of the second diver as an integer.

86. **Investing Loss.** An investing club of five friends has lost $4400 on a trade. If they share the loss equally, write each members' loss as an integer.

Answers

1. Associative property of multiplication

3. Commutative property of multiplication

5. Commutative property of multiplication

7. Distributive property

9. Associative property of multiplication

11. Multiplicative identity property

13. Multiplicative identity property

15. Distributive property

17. 28

19. -9

21. -10

23. 0

25. 14

27. 19

29. 0

31. -24

33. 63

35. -5

37. 21

39. 6300

41. 240

43. 960

45. 144

47. 24

49. 256

51. 625

53. 25

55. 36

57. -1024

59. -125

61. 2

63. -8

65. Division by zero is undefined.

67. -1

69. -2

71. 0

73. 3

75. 6

77. 0

79. Division by zero is undefined.

81. -3

83. 4

85. -125 feet

2.5 Order of Operations with Integers

For convenience, we repeat the "Rules Guiding Order of Operations" first introduced in Section 1.5.

Rules Guiding Order of Operations. When evaluating expressions, proceed in the following order.

1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
2. Evaluate all exponents that appear in the expression.
3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.
4. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

Let's look at a number of examples that require the use of these rules.

You Try It!

Simplify: -2^2.

EXAMPLE 1. Simplify: (a) $(-3)^2$ and (b) -3^2.

Solution. Recall that for any integer a, we have $(-1)a = -a$. Because negating is equivalent to multiplying by -1, the "Rules Guiding Order of Operations" require that we address grouping symbols and exponents before negation.

a) Because of the grouping symbols, we negate first, then square. That is,

$$\begin{aligned}(-3)^2 &= (-3)(-3)\\ &= 9.\end{aligned}$$

b) There are no grouping symbols in this example. Thus, we must square first, then negate. That is,

$$\begin{aligned}-3^2 &= -(3 \cdot 3)\\ &= -9.\end{aligned}$$

Answer: -4

□

You Try It!

EXAMPLE 2. Simplify: $-2-3(5-7)$.

Simplify: $-3-2(6-8)$.

Solution. Grouping symbols first, then multiplication, then subtraction.

$$\begin{aligned} -2-3(5-7) &= -2-3(-2) && \text{Perform subtraction within parentheses.} \\ &= -2-(-6) && \text{Multiply: } 3(-2)=-6. \\ &= -2+6 && \text{Add the opposite.} \\ &= 4 \end{aligned}$$

Answer: 1

You Try It!

EXAMPLE 3. Simplify: $-2(2-4)^2-3(3-5)^3$.

Simplify:

$$-2(5-6)^3-3(5-7)^2$$

.

Solution. Grouping symbols first, then multiplication, and subtraction, in that order.

$$\begin{aligned} -2(2-4)^2-3(3-5)^3 &= -2(-2)^2-3(-2)^3 && \text{Perform subtraction within parentheses first.} \\ &= -2(4)-3(-8) && \text{Exponents are next.} \\ &= -8-(-24) && \text{Multiplications are next.} \\ &= -8+24 && \text{Add the opposite.} \\ &= 16 \end{aligned}$$

Answer: -10

You Try It!

EXAMPLE 4. Simplify: $-24 \div 8(-3)$.

Simplify: $-48 \div 6(-2)$.

Solution. Division has no preference over multiplication, or vice versa. Divisions and multiplications must be performed in the order that they occur, moving left to right.

$$\begin{aligned} -24 \div 8(-3) &= -3(-3) && \text{Division first: } -24 \div 8 = -3. \\ &= 9 \end{aligned}$$

Note that if you multiply first, which would be incorrect, you would get a completely different answer.

Answer: 16

You Try It!

Simplify: $(-4)(-2)^2(-1)^3$.

EXAMPLE 5. Simplify: $(-2)(-3)(-2)^3$.

Solution. Exponents first, then multiplication in the order that it occurs, moving left to right.

$$\begin{aligned}
(-2)(-3)(-2)^3 &= (-2)(-3)(-8) && \text{Exponent first: } (-2)^3 = -8.\\
&= 6(-8) && \text{Multiply from left to right: } (-2)(-3) = 6.\\
&= -48
\end{aligned}$$

Answer: 16

□

Evaluating Fractions

If a fraction bar is present, evaluate the numerator and denominator separately according to the "Rules Guiding Order of Operations," then perform the division in the final step.

You Try It!

Simplify:

$$\frac{6-2(-6)}{-2-(-2)^2}$$

EXAMPLE 6. Simplify:

$$\frac{-5-5(2-4)^3}{-22-3(-5)}$$

Solution. Evaluate numerator and denominator separately, then divide.

$$\begin{aligned}
\frac{-5-5(2-4)^3}{-22-3(-5)} &= \frac{-5-5(-2)^3}{-22-(-15)} && \text{Numerator: parentheses first. Denominator: multiply } 3(-5) = -15.\\
&= \frac{-5-5(-8)}{-22+15} && \text{Numerator: exponent } (-2)^3 = -8. \text{ Denominator: add the opposite.}\\
&= \frac{-5-(-40)}{-7} && \text{Numerator: multiply } 5(-8) = -40. \text{ Denominator: add } -22+15 = -7.\\
&= \frac{-5+40}{-7} && \text{Numerator: add the opposite.}\\
&= \frac{35}{-7} && \text{Numerator: } -5+40 = 35.\\
&= -5 && \text{Divide: } 35/-7 = -5
\end{aligned}$$

Answer: -3

□

Absolute Value

Absolute value calculates the magnitude of the vector associated with an integer, which is equal to the distance between the number and the origin (zero) on the number line. Thus, for example, $|4| = 4$ and $|-5| = 5$.

But absolute value bars also act as grouping symbols, and according to the "Rules Guiding Order of Operations," you should evaluate the expression inside a pair of grouping symbols first.

You Try It!

EXAMPLE 7. Simplify: (a) $-(-3)$ and (b) $-|-3|$.

Simplify: $-|-8|$.

Solution. There is a huge difference between simple grouping symbols and absolute value.

a) This is a case of $-(-a) = a$. Thus, $-(-3) = 3$.

b) This case is much different. The absolute value of -3 is 3, and then the negative of that is -3. In symbols,

$$-|-3| = -3.$$

Answer: -8

□

You Try It!

EXAMPLE 8. Simplify: $-3 - 2|5 - 7|$.

Simplify: $-2 - 4|6 - 8|$.

Solution. Evaluate the expression inside the absolute value bars first. Then multiply, then subtract.

$$\begin{aligned}
-3 - 2|5 - 7| &= -3 - 2|-2| && \text{Subtract inside absolute value bars.}\\
&= -3 - 2(2) && \text{Take the absolute value: } |-2| = 2.\\
&= -3 - 4 && \text{Multiply: } 2(2) = 4.\\
&= -7 && \text{Subtract.}
\end{aligned}$$

Answer: -10

□

Exercises

In Exercises 1-40, compute the exact value of the given expression.

1. $7 - \frac{-14}{2}$

2. $-2 - \frac{-16}{4}$

3. $-7 - \frac{-18}{9}$

4. $-6 - \frac{-7}{7}$

5. -5^4

6. -3^3

7. $9 - 1(-7)$

8. $85 - 8(9)$

9. -6^3

10. -3^5

11. $3 + 9(4)$

12. $6 + 7(-1)$

13. $10 - 72 \div 6 \cdot 3 + 8$

14. $8 - 120 \div 5 \cdot 6 + 7$

15. $6 + \frac{14}{2}$

16. $16 + \frac{8}{2}$

17. -3^4

18. -2^2

19. $3 - 24 \div 4 \cdot 3 + 4$

20. $4 - 40 \div 5 \cdot 4 + 9$

21. $64 \div 4 \cdot 4$

22. $18 \div 6 \cdot 1$

23. $-2 - 3(-5)$

24. $64 - 7(7)$

25. $15 \div 1 \cdot 3$

26. $30 \div 3 \cdot 5$

27. $8 + 12 \div 6 \cdot 1 - 5$

28. $9 + 16 \div 2 \cdot 4 - 9$

29. $32 \div 4 \cdot 4$

30. $72 \div 4 \cdot 6$

31. $-11 + \frac{16}{16}$

32. $4 + \frac{-20}{4}$

33. -5^2

34. -4^3

35. $10 + 12(-5)$

36. $4 + 12(4)$

37. $2 + 6 \div 1 \cdot 6 - 1$

38. $1 + 12 \div 2 \cdot 2 - 6$

39. $40 \div 5 \cdot 4$

40. $30 \div 6 \cdot 5$

In Exercises 41-80, simplify the given expression.

41. $-11 + |-1 - (-6)^2|$

42. $13 + |-21 - (-4)^2|$

43. $|0(-4)| - 4(-4)$

44. $|10(-3)| - 3(-1)$

45. $(2 + 3 \cdot 4) - 8$

46. $(11 + 5 \cdot 2) - 10$

47. $(8 - 1 \cdot 12) + 4$

48. $(9 - 6 \cdot 1) + 3$

49. $(6 + 10 \cdot 4) - 6$

50. $(8 + 7 \cdot 6) - 12$

51. $10 + (6 - 4)^3 - 3$

52. $5 + (12 - 7)^2 - 6$

53. $(6 - 8)^2 - (4 - 7)^3$

54. $(3 - 8)^2 - (4 - 9)^3$

55. $|0(-10)| + 4(-4)$

56. $|12(-5)| + 7(-5)$

57. $|8(-1)| - 8(-7)$

58. $|6(-11)| - 7(-1)$

59. $3 + (3 - 8)^2 - 7$

60. $9 + (8 - 3)^3 - 6$

61. $(4 - 2)^2 - (7 - 2)^3$

62. $(1 - 4)^2 - (3 - 6)^3$

63. $8 - |-25 - (-4)^2|$

64. $20 - |-22 - 4^2|$

65. $-4 - |30 - (-5)^2|$

66. $-8 - |-11 - (-6)^2|$

67. $(8 - 7)^2 - (2 - 6)^3$

68. $(2 - 7)^2 - (4 - 7)^3$

69. $4 - (3 - 6)^3 + 4$

70. $6 - (7 - 8)^3 + 2$

71. $-3 + |-22 - 5^2|$

72. $12 + |23 - (-6)^2|$

73. $(3 - 4 \cdot 1) + 6$

74. $(12 - 1 \cdot 6) + 4$

75. $1 - (1 - 5)^2 + 11$

76. $9 - (3 - 1)^3 + 10$

77. $(2 - 6)^2 - (8 - 6)^3$

78. $(2 - 7)^2 - (2 - 4)^3$

79. $|9(-3)| + 12(-2)$

80. $|0(-3)| + 9(-7)$

In Exercises 81-104, simplify the given expression.

81. $\dfrac{4(-10) - 5}{-9}$

82. $\dfrac{-4 \cdot 6 - (-8)}{-4}$

83. $\dfrac{10^2 - 4^2}{2 \cdot 6 - 10}$

84. $\dfrac{3^2 - 9^2}{2 \cdot 7 - 5}$

85. $\dfrac{3^2 + 6^2}{5 - 1 \cdot 8}$

86. $\dfrac{10^2 + 4^2}{1 - 6 \cdot 5}$

87. $\dfrac{-8 - 4}{7 - 13}$

88. $\dfrac{13 - 1}{8 - 4}$

89. $\dfrac{2^2+6^2}{11-4\cdot 4}$

90. $\dfrac{7^2+3^2}{10-8\cdot 1}$

91. $\dfrac{1^2-5^2}{9\cdot 1-5}$

92. $\dfrac{5^2-7^2}{2\cdot 2-12}$

93. $\dfrac{4^2-8^2}{6\cdot 3-2}$

94. $\dfrac{7^2-6^2}{6\cdot 3-5}$

95. $\dfrac{10^2+2^2}{7-2\cdot 10}$

96. $\dfrac{2^2+10^2}{10-2\cdot 7}$

97. $\dfrac{16-(-2)}{19-1}$

98. $\dfrac{-8-20}{-15-(-17)}$

99. $\dfrac{15-(-15)}{13-(-17)}$

100. $\dfrac{7-(-9)}{-1-1}$

101. $\dfrac{4\cdot 5-(-19)}{3}$

102. $\dfrac{10\cdot 7-(-11)}{-3}$

103. $\dfrac{-6\cdot 9-(-4)}{2}$

104. $\dfrac{-6\cdot 2-10}{-11}$

Answers

1. 14

3. −5

5. −625

7. 16

9. −216

11. 39

13. −18

15. 13

17. −81

19. −11

21. 64

23. 13

25. 45

27. 5

29. 32

31. −10

33. −25

35. −50

37. 37

39. 32

41. 26

43. 16

45. 6

47. 0

49. 40

51. 15

53. 31

55. −16

57. 64

59. 21

61. −121

63. −33

65. −9

67. 65

69. 35

71. 44

73. 5

75. −4

77. 8

79. 3

81. 5

83. 42

85. −15

87. 2

89. −8

91. −6

93. −3

95. −8

97. 1

99. 1

101. 13

103. −25

2.6 Solving Equations Involving Integers

Recall (see Section 1.6) that a *variable* is a symbol (usually a letter) that stands for a value that varies. If a variable in an equation is replaced by a number and a true statement results, then that number is called a *solution* of the equation.

You Try It!

Is -4 a solution of $8 - 2x = 5$?

EXAMPLE 1. Is -6 a solution of the equation $2x + 5 = -7$?

Solution. Substitute -6 for x in the equation.

$2x + 5 = -7$	Original equation.
$2(-6) + 5 = -7$	Substitute -6 for x.
$-12 + 5 = -7$	On the left, multiply first.
$-7 = -7$	On the left, add.

Because the last statement is a true statement, -6 is a solution of the equation. □

Answer: No.

Adding or Subtracting the Same Amount

Two equations having the same set of solutions are *equivalent*. For example, $2x+5 = -7$ and $x = -6$ have the same solutions. Therefore, they are equivalent equations. Certain algebraic operations produce equivalent equations.

Producing Equivalent Equations.

Adding the Same Quantity to Both Sides of an Equation. If we start with the equation

$$a = b,$$

then adding c to both sides of the equation produces the equivalent equation

$$a + c = b + c.$$

Subtracting the Same Quantity from Both Sides of an Equation. If we start with the equation

$$a = b,$$

then subtracting c from both sides of the equation produces the equivalent equation

$$a - c = b - c.$$

That is, adding or subtracting the same amount from both sides of an equation will not change the solutions of the equation.

You Try It!

EXAMPLE 2. Solve for x: $x + 3 = -7$.

Solve for x:

$$x + 9 = -11$$

Solution. To undo the effect of adding 3, subtract 3 from both sides of the equation.

$$\begin{aligned} x + 3 &= -7 && \text{Original equation.} \\ x + 3 - 3 &= -7 - 3 && \text{Subtract 3 from both sides.} \\ x &= -7 + (-3) && \text{Simplify the left hand side. On the right, express subtraction as adding the opposite.} \\ x &= -10 \end{aligned}$$

To check the solution, substitute -10 for x in the original equation and simplify.

$$\begin{aligned} x + 3 &= -7 && \text{Original equation.} \\ -10 + 3 &= -7 && \text{Substitute } -10 \text{ for } x. \\ -7 &= -7 && \text{Simplify both sides.} \end{aligned}$$

Since the last line of the check is a true statement, this confirms that -10 is a solution.

Answer: $x = -20$

□

You Try It!

EXAMPLE 3. Solve for x: $x - 8 = -11$.

Solve for x:

$$x - 2 = -7$$

Solution. To undo the effect of subtracting 8, add 8 to both sides of the equation.

$$\begin{aligned} x - 8 &= -11 && \text{Original equation.} \\ x - 8 + 8 &= -11 + 8 && \text{Add 8 to both sides.} \\ x &= -3 && \text{Simplify both sides.} \end{aligned}$$

To check the solution, substitute -3 for x in the original equation and simplify.

$$\begin{aligned} x - 8 &= -11 && \text{Original equation.} \\ -3 - 8 &= -11 && \text{Substitute } -3 \text{ for } x. \\ -11 &= -11 && \text{Simplify both sides.} \end{aligned}$$

Since the last line of the check is a true statement, this confirms that -3 is a solution.

Answer: $x = -5$

□

Sometimes a bit of simplification is in order before you start the solution process.

You Try It!

Solve for y:

$$y + 2(-4) = -8 + 6$$

EXAMPLE 4. Solve for y: $-8 + 2 = y - 11(-4)$.

Solution. First, simplify both sides of the equation.

$-8 + 2 = y - 11(-4)$	Original equation.
$-6 = y - (-44)$	Simplify. On the left, $-8 + 2 = -6$. On the right, $11(-4) = -44$.
$-6 = y + 44$	Express subtraction as adding the opposite.
$-6 - 44 = y + 44 - 44$	Subtract 44 from both sides of the equation.
$-6 + (-44) = y$	Express subtraction as addition. Simplify on the right.
$-50 = y$	

To check the solution, substitute -50 for y in the original equation and simplify.

$-8 + 2 = y - 11(-4)$	Original equation.
$-8 + 2 = -50 - 11(-4)$	Substitute -50 for y.
$-6 = -50 - (-44)$	On the left, add. On the right, multiply first: $11(-4) = -44$.
$-6 = -50 + 44$	Express subtraction on the right as addition.
$-6 = -6$	On the right, add: $-50 + 44 = -6$.

Since the last line of the check is a true statement, this confirms that -50 is a solution.

Answer: $y = 6$

□

Multiplying or Dividing by the Same Amount

Adding and subtracting is not the only way to produce an equivalent equation.

Producing Equivalent Equations.

Multiplying Both Sides of an Equation by the Same Quantity. If we start with the equation

$$a = b,$$

then multiplying both sides of the equation by c produces the equivalent equation

$$a \cdot c = b \cdot c, \qquad \text{or equivalently,} \qquad ac = bc,$$

provided $c \neq 0$.

Dividing Both Sides of an Equation by the Same Quantity. If we start with the equation

$$a = b,$$

then dividing both sides of the equation by c produces the equivalent equation

$$\frac{a}{c} = \frac{b}{c},$$

provided $c \neq 0$.

That is, multiplying or dividing both sides of an equation by the same amount will not change the solutions of the equation.

You Try It!

EXAMPLE 5. Solve for x: $-3x = 30$.

Solve for z:

$$-4z = -28$$

Solution. To undo the effect of multiplying by -3, divide both sides of the equation by -3.

$$\begin{aligned} -3x &= 30 && \text{Original equation.} \\ \frac{-3x}{-3} &= \frac{30}{-3} && \text{Divide both sides by } -3. \\ x &= -10 && \text{On the left, } -3 \text{ times } x \text{, divided by } -3 \text{ is } x. \\ & && \text{On the right, } 30/(-3) = -10. \end{aligned}$$

To check the solution, substitute -10 for x in the original equation and simplify.

$$\begin{aligned} -3x &= 30 && \text{Original equation.} \\ -3(-10) &= 30 && \text{Substitute } -10 \text{ for } x. \\ 30 &= 30 && \text{Simplify.} \end{aligned}$$

Because the last line of the check is a true statement, this confirms that -10 is a solution.

Answer: $z = 7$

□

You Try It!

EXAMPLE 6. Solve for x: $\dfrac{x}{-2} = -20$.

Solve for t:

$$\frac{t}{3} = -11$$

Solution. To undo the effect of dividing by -2, multiply both sides of the equation by -2.

$$\begin{aligned} \frac{x}{-2} &= -20 && \text{Original equation.} \\ -2\left(\frac{x}{-2}\right) &= -2(-20) && \text{Multiply both sides by } -2. \\ x &= 40 && \text{On the left, } x \text{ divided by } -2 \text{, multiplied by } -2 \text{,} \\ & && \text{the result is } x. \text{ On the right, } -2(-20) = 40. \end{aligned}$$

To check the solution, substitute 40 foro x in the original equation and simplify.

$$\begin{aligned} \frac{x}{-2} &= -20 && \text{Original equation.} \\ \frac{40}{-2} &= -20 && \text{Substitute 40 for } x. \\ -20 &= -20 && \text{Simplify both sides.} \end{aligned}$$

Because the last line of the check is a true statement, this confirms that 40 is a solution.

Answer: $t = -33$

□

Combining Operations

Recall the "Wrap" and "Unwrap" discussion from Section 1.6. To wrap a present we: (1) put the gift paper on, (2) put the tape on, and (3) put the decorative bow on. To unwrap the gift, we must "undo" each of these steps in inverse order. Hence, to unwrap the gift we: (1) take off the decorative bow, (2) take off the tape, and (3) take off the gift paper.

Now, imagine a machine that takes its input, then: (1) multiplies the input by 2, and (2) adds 3 to the result. This machine is pictured on the left in Figure 2.16.

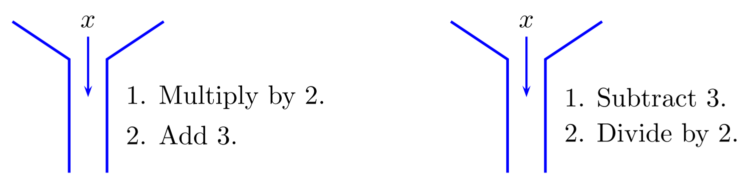


Figure 2.16: The second machine "unwraps" the first machine.

To "unwrap" the effect of the machine on the left, we need a machine that will "undo" each of the steps of the first machine, but in inverse order. The "unwrapping" machine is pictured on the right in Figure 2.16. It will first subtract three from its input, then divide the result by 2. Note that each of these operations "undoes" the corresponding operation of the first machine, but in inverse order.

For example, drop the integer 7 into the first machine on the left in Figure 2.16. First, we double 7, then add 3 to the result. The result is $2(7) + 3 = 17$.

Now, to "unwrap" this result, we drop 17 into the second machine. We first subtract 3, then divide by 2. The result is $(17 - 3)/2 = 7$, the original integer input into the first machine.

Now, consider the equation

$$2x + 3 = 7.$$

On the left, order of operations demands that we first multiply x by 2, then add 3. To solve this equation for x, we must "undo" each of these operations in inverse order. Thus, we will (1) subtract three from both sides of the equation, then (2) divide both sides of the resulting equation by 2.

$$\begin{aligned} 2x + 3 - 3 &= 7 - 3 && \text{Subtract 3 from both sides.} \\ 2x &= 4 && \text{Simplify both sides.} \\ \frac{2x}{2} &= \frac{4}{2} && \text{Divide both sides by 2.} \\ x &= 2 && \text{Simplify both sides.} \end{aligned}$$

Readers should check this solution in the original equation.

You Try It!

Solve for x:

$$\frac{x}{2} + 6 = 4$$

EXAMPLE 7. Solve for x: $\dfrac{x}{4} - 3 = -7$.

Solution. On the left, order of operations demands that we first divide x by 4, then subtract 3. To solve this equation for x, we must "undo" each of these operations in inverse order. Thus, we will (1) add 3 to both sides of the equation, then (2) multiply both sides of the resulting equation by 4.

$$\begin{aligned} \frac{x}{4} - 3 &= -7 && \text{Original equation.} \\ \frac{x}{4} - 3 + 3 &= -7 + 3 && \text{Add 3 to both sides.} \\ \frac{x}{4} &= -4 && \text{Simplify both sides.} \\ 4\left(\frac{x}{4}\right) &= 4(-4) && \text{Multiply both sides by 4.} \\ x &= -16 && \text{Simplify both sides.} \end{aligned}$$

Check. Substitute -16 for x in the original equation.

$$\begin{aligned} \frac{x}{4} - 3 &= -7 && \text{Original equation.} \\ \frac{-16}{4} - 3 &= -7 && \text{Substitute } -16 \text{ for } x. \\ -4 - 3 &= -7 && \text{Divide first: } -16/4 = -4. \\ -7 &= -7 && \text{Subtract: } -4 - 3 = -7. \end{aligned}$$

Because the last line of the check is a true statement, -16 is a solution of the original equation.

Answer: $x = -4$

□

You Try It!

Solve for r:

$$0 = 9 + 3r$$

Answer: $r = -3$

EXAMPLE 8. Solve for t: $0 = 8 - 2t$.

Solution. On the right, order of operations demands that we first multiply t by -2, then add 8. To solve this equation for t, we must "undo" each of these operations in inverse order. Thus, we will (1) subtract 8 from both sides of the equation, then (2) divide both sides of the resulting equation by -2.

$0 = 8 - 2t$	Original equation.
$0 - 8 = 8 - 2t - 8$	Subtract 8 from both sides.
$-8 = -2t$	Simplify both sides.
$\dfrac{-8}{-2} = \dfrac{-2t}{-2}$	Divide both sides by -2.
$4 = t$	Simplify both sides.

Check. Substitute 4 for t in the original equation.

$0 = 8 - 2t$	Original equation.
$0 = 8 - 2(4)$	Substitute 4 for t.
$0 = 8 - 8$	Multiply first: $2(4) = 8$.
$0 = 0$	Subtract: $8 - 8 = 0$.

Because the last line in the check is a true statement, 4 is a solution of the original equation.

□

You Try It!

Solve for q:

$$\frac{q}{-2} - 9 = -8 + 3$$

EXAMPLE 9. Solve for p: $-12 + 3 = -8 + 4 + \dfrac{p}{-3}$.

Solution. Always simplify when possible.

$-12 + 3 = -8 + 4 + \dfrac{p}{-3}$	Original equation.
$-9 = -4 + \dfrac{p}{-3}$	Simplify both sides.

On the right, order of operations demands that we first divide p by -3, then add -4. To solve this equation for p, we must "undo" each of these operations in inverse order. Thus, we will (1) add a positive 4 to both sides of the equation, then (2) multiply both sides of the resulting equation by −3.

$$\begin{aligned} -9+4 &= -4+\frac{p}{-3}+4 && \text{Add 4 to both sides.} \\ -5 &= \frac{p}{-3} && \text{Simplify both sides.} \\ -3(-5) &= -3\left(\frac{p}{-3}\right) && \text{Multiply both sides by } -3. \\ 15 &= p && \text{Simplify both sides.} \end{aligned}$$

Check. Substitute 15 for p in the original equation.

$$\begin{aligned} -12+3 &= -8+4+\frac{p}{-3} && \text{Original equation.} \\ -12+3 &= -8+4+\frac{15}{-3} && \text{Substitute 15 for } p. \\ -9 &= -8+4+(-5) && \text{On the left, add: } -12+3=-9. \text{ On the right, divide: } 15/(-3)=-5. \\ -9 &= -4+(-5) && \text{On the right, add: } -8+4=-4. \\ -9 &= -9 && \text{On the right, add: } -4+(-5)=-9. \end{aligned}$$

Because the last line in the check is a true statement, 15 is a solution of the original equation.

Answer: $q = -8$

□

Applications

Let's look at some applications of equations involving integers. First, we remind readers that a solution of a word problem must incorporate each of the following steps.

Requirements for Word Problem Solutions.

1. **Set up a Variable Dictionary.** You must let your readers know what each variable in your problem represents. This can be accomplished in a number of ways:
   - Statements such as "Let P represent the perimeter of the rectangle."
   - Labeling unknown values with variables in a table.
   - Labeling unknown quantities in a sketch or diagram.

2. Set up an Equation. Every solution to a word problem must include a carefully crafted equation that accurately describes the constraints in the problem statement.

3. Solve the Equation. You must always solve the equation set up in the previous step.

4. Answer the Question. This step is easily overlooked. For example, the problem might ask for Jane's age, but your equation's solution gives the age of Jane's sister Liz. Make sure you answer the original question asked in the problem. Your solution should be written in a sentence with appropriate units.

5. Look Back. It is important to note that this step does not imply that you should simply check your solution in your equation. After all, it's possible that your equation incorrectly models the problem's situation, so you could have a valid solution to an incorrect equation. The important question is: "Does your answer make sense based on the words in the original problem statement."

You Try It!

After withdrawing $125 from his account, Allen finds that he is overdrawn by $15. What was his account balance before his withdrawal?

EXAMPLE 10. A student's bank account is overdrawn. After making a deposit of $120, he finds that his account is still overdrawn by an amount of $75. What was his balance before he made his deposit?

Solution. In our solution, we address each step of the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* In this case, the unknown is the original balance in the student's account. Let B represent this original balance.

2. *Set up an Equation.* A positive integer represents a healthy balance, while a negative number represents an account that is overdrawn. After the student's deposit, the account is still overdrawn by $75. We will say that this balance is −$75. Thus,

Original Balance	plus	Student Deposit	equals	Current Balance
B	$+$	\$120	$=$	−\$75

3. *Solve the Equation.* To "undo" the addition, subtract 120 from both sides of the equation.

$$\begin{aligned} B + 120 &= -75 && \text{Original equation.} \\ B + 120 - 120 &= -75 - 120 && \text{Subtract 120 from both sides.} \\ B &= -195 && \text{Simplify both sides.} \end{aligned}$$

4. *Answer the Question.* The original balance was overdrawn to the tune of \$195.

5. *Look Back.* If the original balance was overdrawn by \$195, then we let −\$195 represent this balance. The student makes a deposit of \$120. Add this to the original balance to get −\$195 + \$120 = −\$75, the correct current balance.

Answer: \$110

□

You Try It!

Five less than twice a certain number is −7. Find the unknown number.

EXAMPLE 11. Three more than twice a certain number is −11. Find the unknown number.

Solution. In our solution, we address each step of the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* Let x represent the unknown number.

2. *Set up an Equation.* "Three more than twice a certain number" becomes:

Three	more than	Twice a Certain Number	is	−11
3	$+$	$2x$	$=$	-11

3. *Solve the Equation.* On the left, order of operations requires that we first multiply x by 2, then add 3. To solve this equation for x, we must "undo" each of these operations in inverse order. Thus, we will (1) subtract 3 from both sides of the equation, then (2) divide both sides of the resulting equation by 2.

$$\begin{aligned} 3 + 2x &= -11 && \text{Original equation.} \\ 3 + 2x - 3 &= -11 - 3 && \text{Subtract 3 from both sides.} \\ 2x &= -14 && \text{Simplify both sides.} \\ \frac{2x}{2} &= \frac{-14}{2} && \text{Divide both sides by 2.} \\ x &= -7 && \text{Simplify both sides.} \end{aligned}$$

4. *Answer the Question.* The unknown number is -7.

5. *Look Back.* Does the answer satisfy the problem constraints? Three more than twice -7 is three more than -14, or -11. So the solution is correct.

Answer: -1

□

Exercises

1. Is -11 a solution of $2x+3=-19$?

2. Is -8 a solution of $2x+7=-9$?

3. Is 6 a solution of $3x+1=19$?

4. Is -6 a solution of $2x+7=-5$?

5. Is 12 a solution of $4x+5=-8$?

6. Is -8 a solution of $-3x+8=18$?

7. Is 15 a solution of $2x+6=-9$?

8. Is 3 a solution of $-4x+1=-20$?

9. Is -15 a solution of $-3x+6=-17$?

10. Is -18 a solution of $-3x+9=-9$?

11. Is -6 a solution of $-2x+3=15$?

12. Is 7 a solution of $-3x+5=-16$?

In Exercises 13-28, solve the given equation for x.

13. $x-13=11$

14. $x-6=12$

15. $x-3=6$

16. $x-3=-19$

17. $x+10=17$

18. $x+3=9$

19. $x-6=1$

20. $x-10=12$

21. $x-15=-12$

22. $x-2=13$

23. $x+11=-19$

24. $x+3=17$

25. $x+2=1$

26. $x+2=-20$

27. $x+5=-5$

28. $x+14=-15$

In Exercises 29-44, solve the given equation for x.

29. $-x=-20$

30. $5x=-35$

31. $\dfrac{x}{-7}=10$

32. $\dfrac{x}{-6}=-20$

33. $\dfrac{x}{-10}=12$

34. $\dfrac{x}{2}=11$

35. $\dfrac{x}{9}=-16$

36. $\dfrac{x}{-3}=-7$

37. $-10x=20$

38. $-17x=-85$

39. $14x=84$

40. $-10x=-40$

41. $-2x=28$

42. $-14x=42$

43. $\dfrac{x}{-10}=15$

44. $\dfrac{x}{-8}=-1$

In Exercises 45-68, solve the given equation for x.

45. $-4x - 4 = 16$

46. $-6x - 14 = 4$

47. $4x - 4 = 76$

48. $-5x - 15 = 45$

49. $5x - 14 = -79$

50. $15x - 2 = 43$

51. $-10x - 16 = 24$

52. $2x - 7 = -11$

53. $9x + 5 = -85$

54. $8x + 8 = -16$

55. $7x + 15 = -55$

56. $2x + 2 = -38$

57. $-x + 8 = 13$

58. $-5x + 20 = -50$

59. $12x - 15 = -3$

60. $-19x - 17 = -36$

61. $4x - 12 = -56$

62. $7x - 16 = 40$

63. $19x + 18 = 113$

64. $-6x + 20 = -64$

65. $-14x + 12 = -2$

66. $-9x + 5 = 104$

67. $14x + 16 = 44$

68. $-14x + 10 = -60$

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69. Two less than eight times an unknown number is -74. Find the unknown number.

70. Six less than three times an unknown number is 21. Find the unknown number.

71. Eight more than two times an unknown number is 0. Find the unknown number.

72. Five more than eight times an unknown number is -35. Find the unknown number.

73. The number -6 is 2 more than an unknown number. Find the unknown number.

74. The number -4 is 7 more than an unknown number. Find the unknown number.

75. Three more than eight times an unknown number is -29. Find the unknown number.

76. Four more than nine times an unknown number is 85. Find the unknown number.

77. Alan's scores on his first three exams are 79, 61, and 54. What must Alan score on his next exam to average 71 for all four exams?

78. Benny's scores on his first three exams are 54, 68, and 54. What must Benny score on his next exam to average 61 for all four exams?

79. The quotient of two integers is 5. One of the integers is -2. Find the other integer.

80. The quotient of two integers is 3. One of the integers is -7. Find the other integer.

81. The quotient of two integers is 9. One of the integers is -8. Find the other integer.

82. The quotient of two integers is 9. One of the integers is -2. Find the other integer.

83. The number −5 is 8 more than an unknown number. Find the unknown number.

84. The number −6 is 8 more than an unknown number. Find the unknown number.

85. A student's bank account is overdrawn. After making a deposit of \$260, he finds that his account is still overdrawn by an amount of \$70. What was his balance before he made his deposit?

86. A student's bank account is overdrawn. After making a deposit of \$300, he finds that his account is still overdrawn by an amount of \$70. What was his balance before he made his deposit?

87. A student's bank account is overdrawn. After making a deposit of \$360, he finds that his account is still overdrawn by an amount of \$90. What was his balance before he made his deposit?

88. A student's bank account is overdrawn. After making a deposit of \$260, he finds that his account is still overdrawn by an amount of \$50. What was his balance before he made his deposit?

89. The number −10 is −5 times larger than an unknown number. Find the unknown number.

90. The number −3 is −3 times larger than an unknown number. Find the unknown number.

91. The number −15 is −5 times larger than an unknown number. Find the unknown number.

92. The number −16 is 4 times larger than an unknown number. Find the unknown number.

93. Two less than nine times an unknown number is 7. Find the unknown number.

94. Four less than two times an unknown number is 8. Find the unknown number.

95. Mark's scores on his first three exams are 79, 84, and 71. What must Mark score on his next exam to average 74 for all four exams?

96. Alan's scores on his first three exams are 85, 90, and 61. What must Alan score on his next exam to average 77 for all four exams?

❧ ❧ ❧ Answers ☙ ☙ ☙

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1. Yes

3. Yes

5. No

7. No

9. No

11. Yes

13. 24

15. 9

17. 7

19. 7

21. 3

23. −30

25. −1

27. −10

29. 20

31. −70

33. −120

35. −144

37. −2

39. 6

41. −14

43. −150

45. −5

47. 20

49. −13

51. −4

53. −10

55. −10

57. −5

59. 1

61. −11

63. 5

65. 1

67. 2

69. −9

71. −4

73. −8

75. −4

77. 90

79. −10

81. −72

83. −13

85. −$330

87. −$450

89. 2

91. 3

93. 1

95. 62

Chapter 3

The Fundamentals of Algebra

As his name portends, Abu Jafr Muhammad ibn Musa al-Khwarizmi was one of the greatest Arab mathematicians of his time. While living in Baghdad during the ninth century AD he became the Chief Librarian at the *House of Wisdom*, a library and major center of intellectual study. In the year 820AD, al-Khwarizmi wrote *Al-Kitab al-mukhtasar ti Hisab al-jabr w'al-muqabala*, translated to, *The Compendious Book on Calculation by Restoration and Reduction*, the first book to generalize solving equations using the principles of equality. In fact, the word algebra itself comes from *al-jabr*, meaning reunion or completion.

Many earlier cultures had employed what we might call algebra in the service of business, land management, inheritance, and trade. The Bablyonians were solving quadratic equations in 2000BC, but only specific equations, with specific numbers. Hindus on the Indian continent were also developing algebra along side their invention of the symbol for zero 0. But like al-Khwarizmi and the Moslem Arabs of the first millenium, to write equations these early cultures did not use symbols like x or y, or even equal signs $=$ that we use today. al-Khwarizmi wrote absolutely everything in words! 42 would be forty-two!

Early algebra written all with words is called rhetorical algebra, and a thousand years ago, each mathematician had their own way of expressing it. Algebra was a language with many different dialects, and communicating it from one mathematician to another was difficult as they had to explain themselves with words. It wasn't until well after the Gutenberg printing press was invented in 1436 and print became standardized, that Rene Descartes, a Frenchman began to develop a modern symbolic algebra.

In this section we'll learn how to manipulate symbols in order to al-muqabalah (combine like terms) and al-jabr (restore and balance equations). But we'll use modern notation to make it easier!

3.1 Mathematical Expressions

Recall the definition of a *variable* presented in Section 1.6.

Variable. A variable is a symbol (usually a letter) that stands for a value that may vary.

Let's add the definition of a *mathematical expression.*

Mathematical Expression. When we combine numbers and variables in a valid way, using operations such as addition, subtraction, multiplication, division, exponentiation, and other operations and functions as yet unlearned, the resulting combination of mathematical symbols is called a *mathematical expression.*

Thus,

$$2a, \qquad x+5, \qquad \text{and} \qquad y^2,$$

being formed by a combination of numbers, variables, and mathematical operators, are valid mathematical expressions.

A mathematical expression must be *well-formed*. For example,

$$2 + \div 5x$$

is *not a valid expression* because there is no term following the plus sign (it is not valid to write $+\div$ with nothing between these operators). Similarly,

$$2 + 3(2$$

is not well-formed because parentheses are not balanced.

Translating Words into Mathematical Expressions

In this section we turn our attention to translating word phrases into mathematical expressions. We begin with phrases that translate into *sums*. There is a wide variety of word phrases that translate into sums. Some common examples are given in Table 3.1(a), though the list is far from complete. In like manner, a number of phrases that translate into differences are shown in Table 3.1(b).

Let's look at some examples, some of which translate into expressions involving sums, and some of which translate into expressions involving differences.

You Try It!

Translate the following phrases into mathematical expressions: (a) "13 more than x", and (b) "12 fewer than y".

EXAMPLE 1. Translate the following phrases into mathematical expressions: (a) "12 larger than x," (b) "11 less than y," and (c) "r decreased by 9."

Solution. Here are the translations.

a) "12 larger than x" becomes $x + 12$.

b) "11 less than y" becomes $y - 11$.

c) "r decreased by 9" becomes $r - 9$.

Answers: (a) $x + 13$ and (b) $y - 12$

□

Phrase	Translates to:
sum of x and 12	$x + 12$
4 greater than b	$b + 4$
6 more than y	$y + 6$
44 plus r	$44 + r$
3 larger than z	$z + 3$

(a) Phrases that are sums.

Phrase	Translates to:
difference of x and 12	$x - 12$
4 less than b	$b - 4$
7 subtracted from y	$y - 7$
44 minus r	$44 - r$
3 smaller than z	$z - 3$

(b) Phrases that are differences.

Table 3.1: Translating words into symbols.

You Try It!

The width of a rectangle is 5 inches shorter than its length L. Express the width of the rectangle in terms of its length L.

EXAMPLE 2. Let W represent the width of the rectangle. The length of a rectangle is 4 feet longer than its width. Express the length of the rectangle in terms of its width W.

Solution. We know that the width of the rectangle is W. Because the length of the rectangle is 4 feet longer that the width, we must add 4 to the width to find the length.

Length	is	4	more than	the width
Length	=	4	+	W

Thus, the length of the rectangle, in terms of its width W, is $4 + W$.

Answer: $L - 5$

□

You Try It!

A string is cut into two pieces, the first of which measures 12 inches. Express the total length of the string as a function of x, where x represents the length of the second piece of string.

EXAMPLE 3. A string measures 15 inches is cut into two pieces. Let x represent the length of one of the resulting pieces. Express the length of the second piece in terms of the length x of the first piece.

Solution. The string has original length 15 inches. It is cut into two pieces and the first piece has length x. To find the length of the second piece, we must subtract the length of the first piece from the total length.

Length of the second piece	is	Total length	minus	the length of first piece
Length of the second piece	$=$	15	$-$	x

Thus, the length of the second piece, in terms of the length x of the first piece, is $15 - x$.

Answer: $12 + x$

There is also a wide variety of phrases that translate into products. Some examples are shown in Table 3.2(a), though again the list is far from complete. In like manner, a number of phrases translate into quotients, as shown in Table 3.2(b).

Phrase	Translates to:
product of x and 12	$12x$
4 times b	$4b$
twice r	$2r$

(a) Phrases that are products.

Phrase	Translates to:
quotient of x and 12	$x/12$
4 divided by b	$4/b$
the ratio of 44 to r	$44/r$

(b) Phrases that are differences.

Table 3.2: Translating words into symbols.

Let's look at some examples, some of which translate into expressions involving products, and some of which translate into expressions involving quotients.

You Try It!

Translate into mathematical symbols: (a) "the product of 5 and x" and (b) "12 divided by y."

EXAMPLE 4. Translate the following phrases into mathematical expressions: (a) "11 times x," (b) "quotient of y and 4," and (c) "twice a."

Solution. Here are the translations.

a) "11 times x" becomes $11x$.

b) "quotient of y and 4" becomes $y/4$, or equivalently, $\dfrac{y}{4}$.

c) "twice a" becomes $2a$.

Answer: (a) $5x$ and (b) $12/y$.

You Try It!

A carpenter cuts a board of unknown length L into three equal pieces. Express the length of each piece in terms of L.

EXAMPLE 5. A plumber has a pipe of unknown length x. He cuts it into 4 equal pieces. Find the length of each piece in terms of the unknown length x.

Solution. The total length is unknown and equal to x. The plumber divides it into 4 equal pieces. To find the length of each pieces, we must divide the total length by 4.

Length of each piece	is	Total length	divided by	4
Length of each piece	$=$	x	$\div$	4

Thus, the length of each piece, in terms of the unknown length x, is $x/4$, or equivalently, $\dfrac{x}{4}$.

Answer: $L/3$.

You Try It!

EXAMPLE 6. Mary invests A dollars in a savings account paying 2% interest per year. She invests five times this amount in a certificate of deposit paying 5% per year. How much does she invest in the certificate of deposit, in terms of the amount A in the savings account?

David invest K dollars in as savings account paying 3% per year. He invests half this amount in a mutual fund paying 4% per year. Express the amount invested in the mutual fund in terms of K, the amount invested in the savings account.

Solution. The amount in the savings account is A dollars. She invests five times this amount in a certificate of deposit.

Amount in CD	is	5	times	Amount in savings
Amount in CD	$=$	5	$\cdot$	A

Thus, the amount invested in the certificate of deposit, in terms of the amount A in the savings account, is $5A$.

Answer: $\frac{1}{2}K$

Combinations

Some phrases require combinations of the mathematical operations employed in previous examples.

You Try It!

EXAMPLE 7. Let the first number equal x. The second number is 3 more than twice the first number. Express the second number in terms of the first number x.

A second number is 4 less than 3 times a first number. Express the second number in terms of the first number y.

Solution. The first number is x. The second number is 3 more than twice the first number.

Second number	is	3	more than	twice the first number
Second number	$=$	3	$+$	$2x$

Answer: $3y - 4$

Therefore, the second number, in terms of the first number x, is $3 + 2x$. □

You Try It!

The width of a rectangle is W. The length is 7 inches longer than twice the width. Express the length of the rectangle in terms of its length L.

EXAMPLE 8. The length of a rectangle is L. The width is 15 feet less than 3 times the length. What is the width of the rectangle in terms of the length L?

Solution. The length of the rectangle is L. The width is 15 feet less than 3 times the length.

Width	is	3 times the length	less	15
Width	=	$3L$	−	15

Answer: $2W + 7$

Therefore, the width, in terms of the length L, is $3L - 15$. □

Exercises

In Exercises 1-20, translate the phrase into a mathematical expression involving the given variable.

1. "8 times the width n "

2. "2 times the length z "

3. "6 times the sum of the number n and 3"

4. "10 times the sum of the number n and 8"

5. "the demand b quadrupled"

6. "the supply y quadrupled"

7. "the speed y decreased by 33"

8. "the speed u decreased by 30"

9. "10 times the width n "

10. "10 times the length z "

11. "9 times the sum of the number z and 2"

12. "14 times the sum of the number n and 10"

13. "the supply y doubled"

14. "the demand n quadrupled"

15. "13 more than 15 times the number p "

16. "14 less than 5 times the number y "

17. "4 less than 11 times the number x "

18. "13 less than 5 times the number p "

19. "the speed u decreased by 10"

20. "the speed w increased by 32"

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21. **Representing Numbers.** Suppose n represents a whole number.

i) What does $n + 1$ represent?

ii) What does $n + 2$ represent?

iii) What does $n - 1$ represent?

22. Suppose $2n$ represents an even whole number. How could we represent the next even number after $2n$?

23. Suppose $2n + 1$ represents an odd whole number. How could we represent the next odd number after $2n + 1$?

24. There are b bags of mulch produced each month. How many bags of mulch are produced each year?

25. Steve sells twice as many products as Mike. Choose a variable and write an expression for each man's sales.

26. Find a mathematical expression to represent the values.

i) How many quarters are in d dollars?

ii) How many minutes are in h hours?

iii) How many hours are in d days?

iv) How many days are in y years?

v) How many months are in y years?

vi) How many inches are in f feet?

vii) How many feet are in y yards?

❧ ❧ ❧ Answers ☙ ☙ ☙

1. $8n$

3. $6(n+3)$

5. $4b$

7. $y-33$

9. $10n$

11. $9(z+2)$

13. $2y$

15. $15p+13$

17. $11x-4$

19. $u-10$

21.
i) $n+1$ represents the next whole number after n.

ii) $n+2$ represents the next whole number after $n+1$, or, two whole numbers after n.

iii) $n-1$ represents the whole number before n.

23. $2n+3$

25. Let Mike sell p products. Then Steve sells $2p$ products.

3.2 Evaluating Algebraic Expressions

In this section we will evaluate *algebraic expressions* for given values of the variables contained in the expressions. Here are some simple tips to help you be successful.

Tips for Evaluating Algebraic Expressions.

1. Replace all occurrences of variables in the expression with open parentheses. Leave room between the parentheses to substitute the given value of the variable.
2. Substitute the given values of variables in the open parentheses prepared in the first step.
3. Evaluate the resulting expression according to the *Rules Guiding Order of Operations*.

Let's begin with an example.

You Try It!

If $x = -2$ and $y = -1$, evaluate $x^3 - y^3$.

EXAMPLE 1. Evaluate the expression $x^2 - 2xy + y^2$ at $x = -3$ and $y = 2$.

Solution. Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression $x^2 - 2xy + y^2$ with open parentheses.

$$x^2 - 2xy + y^2 = (\quad)^2 - 2(\quad)(\quad) + (\quad)^2$$

Secondly, replace each variable with its given value, and thirdly, follow the "Rules Guiding Order of Operations" to evaluate the resulting expression.

$x^2 - 2xy + y^2$	Original expression.
$= (-3)^2 - 2(-3)(2) + (2)^2$	Substitute -3 for x and 2 for y.
$= 9 - 2(-3)(2) + 4$	Evaluate exponents first.
$= 9 - (-6)(2) + 4$	Left to right, multiply: $2(-3) = -6$.
$= 9 - (-12) + 4$	Left to right, multiply: $(-6)(2) = -12$.
$= 9 + 12 + 4$	Add the opposite.
$= 25$	Add.

Answer: -7

□

You Try It!

If $a = 3$ and $b = -5$, evaluate $a^2 - b^2$.

EXAMPLE 2. Evaluate the expression $(a-b)^2$ at $a = 3$ and $b = -5$.

Solution. Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression $(a - b)^2$ with open parentheses.

$$(a-b)^2 = ((\quad) - (\quad))^2$$

Secondly, replace each variable with its given value, and thirdly, follow the "Rules Guiding Order of Operations" to evaluate the resulting expression.

$$\begin{aligned}(a-b)^2 &= ((3) - (-5))^2 && \text{Substitute 3 for } a \text{ and } -5 \text{ for } b.\\ &= (3+5)^2 && \text{Add the opposite: } (3) - (-5) = 3 + 5\\ &= 8^2 && \text{Simplify inside parentheses: } 3 + 5 = 8\\ &= 64 && \text{Evaluate exponent: } 8^2 = 64\end{aligned}$$

Answer: -16

□

You Try It!

If $a = 5$ and $b = -7$, evaluate $2|a| - 3|b|$.

EXAMPLE 3. Evaluate the expression $|a| - |b|$ at $a = 5$ and $b = -7$.

Solution. Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression $|a| - |b|$ with open parentheses.

$$|a| - |b| = |(\quad)| - |(\quad)|$$

Secondly, replace each variable with its given value, and thirdly, follow the "Rules Guiding Order of Operations" to evaluate the resulting expression.

$$\begin{aligned}|a| - |b| &= |(5)| - |(-7)| && \text{Substitute 5 for } a \text{ and } -7 \text{ for } b.\\ &= 5 - 7 && \text{Absolute values first: } |(5)| = 5 \text{ and } |(-7)| = 7\\ &= 5 + (-7) && \text{Add the opposite: } 5 - 7 = 5 + (-7).\\ &= -2 && \text{Add: } 5 + (-7) = -2.\end{aligned}$$

Answer: -11

□

You Try It!

If $a = 5$ and $b = -7$, evaluate $|2a - 3b|$.

EXAMPLE 4. Evaluate the expression $|a - b|$ at $a = 5$ and $b = -7$.

Solution. Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression $|a - b|$ with open parentheses.

$$|a-b| = |(\quad) - (\quad)|$$

Secondly, replace each variable with its given value, and thirdly, follow the "Rules Guiding Order of Operations" to evaluate the resulting expression.

$$\begin{aligned} |a-b| &= |(5)-(-7)| && \text{Substitute 5 for } a \text{ and } -7 \text{ for } b. \\ &= |5+7| && \text{Add the opposite: } 5-(-7)=5+7. \\ &= |12| && \text{Add: } 5+7=12. \\ &= 12 && \text{Take the absolute value: } |12|=12. \end{aligned}$$

Answer: 31

□

You Try It!

If $a=-7$, $b=-3$, $c=-15$, and $d=-14$, evaluate:

$$\frac{a^2+b^2}{c+d}$$

Answer -2

EXAMPLE 5. Evaluate the expression

$$\frac{ad-bc}{a+b}$$

at $a=5$, $b=-3$, $c=2$, and $d=-4$.

Solution. Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression with open parentheses.

$$\frac{ad-bc}{a+b}=\frac{(\quad)(\quad)-(\quad)(\quad)}{(\quad)+(\quad)}$$

Secondly, replace each variable with its given value, and thirdly, follow the "Rules Guiding Order of Operations" to evaluate the resulting expression.

$$\begin{aligned} \frac{ad-bc}{a+b} &= \frac{(5)(-4)-(-3)(2)}{(5)+(-3)} && \text{Substitute: 5 for } a, -3 \text{ for } b, 2 \text{ for } c, -4 \text{ for } d. \\ &= \frac{-20-(-6)}{2} && \text{Numerator: } (5)(-4)=-20, (-3)(2)=-6. \\ & && \text{Denominator: } 5+(-3)=2. \\ &= \frac{-20+6}{2} && \text{Numerator: Add the opposite.} \\ &= \frac{-14}{2} && \text{Numerator: } -20+6=-14. \\ &= -7 && \text{Divide.} \end{aligned}$$

□

You Try It!

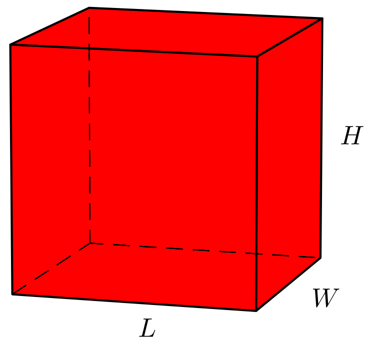
The surface area of the prism pictured in this example is given by the following formula:

$$S = 2(WH + LH + LW)$$

If $L = 12$, $W = 4$, and $H = 6$ feet, respectively, calculate the surface area.

Answer: 288 square feet.

EXAMPLE 6. Pictured below is a rectangular prism.



The volume of the rectangular prism is given by the formula

$$V = LWH,$$

where L is the length, W is the width, and H is the height of the rectangular prism. Find the volume of a rectangular prism having length 12 feet, width 4 feet, and height 6 feet.

Solution. Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of of L, W, and H in the formula

$$V = LWH$$

with open parentheses.

$$V = (\quad)(\quad)(\quad)$$

Next, substitute $12\,\text{ft}$ for L, $4\,\text{ft}$ for W, and $6\,\text{ft}$ for H and simplify.

$$\begin{aligned} V &= \big(12\,\text{ft}\big)\big(4\,\text{ft}\big)\big(6\,\text{ft}\big) \\ &= 288\,\text{ft}^3 \end{aligned}$$

Hence, the volume of the rectangular prism is 288 cubic feet.

□

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-12, evaluate the expression at the given value of x.

1. $-3x^2 - 6x + 3$ at $x = 7$

2. $7x^2 - 7x + 1$ at $x = -8$

3. $-6x - 6$ at $x = 3$

4. $6x - 1$ at $x = -10$

5. $5x^2 + 2x + 4$ at $x = -1$

6. $4x^2 - 9x + 4$ at $x = -3$

7. $-9x - 5$ at $x = -2$

8. $-9x + 12$ at $x = 5$

9. $4x^2 + 2x + 6$ at $x = -6$

10. $-3x^2 + 7x + 4$ at $x = -7$

11. $12x + 10$ at $x = -12$

12. $-6x + 7$ at $x = 11$

In Exercises 13-28, evaluate the expression at the given values of x and y.

13. $|x| - |y|$ at $x = -5$ and $y = 4$

14. $|x| - |y|$ at $x = -1$ and $y = -2$

15. $-5x^2 + 2y^2$ at $x = 4$ and $y = 2$

16. $-5x^2 - 4y^2$ at $x = -2$ and $y = -5$

17. $|x| - |y|$ at $x = 0$ and $y = 2$

18. $|x| - |y|$ at $x = -2$ and $y = 0$

19. $|x - y|$ at $x = 4$ and $y = 5$

20. $|x - y|$ at $x = -1$ and $y = -4$

21. $5x^2 - 4xy + 3y^2$ at $x = 1$ and $y = -4$

22. $3x^2 + 5xy + 3y^2$ at $x = 2$ and $y = -1$

23. $|x - y|$ at $x = 4$ and $y = 4$

24. $|x - y|$ at $x = 3$ and $y = -5$

25. $-5x^2 - 3xy + 5y^2$ at $x = -1$ and $y = -2$

26. $3x^2 - 2xy - 5y^2$ at $x = 2$ and $y = 5$

27. $5x^2 + 4y^2$ at $x = -2$ and $y = -2$

28. $-4x^2 + 2y^2$ at $x = 4$ and $y = -5$

In Exercises 29-40, evaluate the expression at the given value of x.

29. $\dfrac{9 + 9x}{-x}$ at $x = -3$

30. $\dfrac{9 - 2x}{-x}$ at $x = -1$

31. $\dfrac{-8x + 9}{-9 + x}$ at $x = 10$

32. $\dfrac{2x + 4}{1 + x}$ at $x = 0$

33. $\dfrac{-4 + 9x}{7x}$ at $x = 2$

34. $\dfrac{-1 - 9x}{x}$ at $x = -1$

35. $\dfrac{-12 - 7x}{x}$ at $x = -1$

36. $\dfrac{12 + 11x}{3x}$ at $x = -6$

37. $\dfrac{6x - 10}{5 + x}$ at $x = -6$

38. $\dfrac{11x + 11}{-4 + x}$ at $x = 5$

39. $\dfrac{10x+11}{5+x}$ at $x=-4$

40. $\dfrac{6x+12}{-3+x}$ at $x=2$

---

41. The formula

$$d = 16t^2$$

gives the distance (in feet) that an object falls from rest in terms of the time t that has elapsed since its release. Find the distance d (in feet) that an object falls in $t = 4$ seconds.

42. The formula

$$d = 16t^2$$

gives the distance (in feet) that an object falls from rest in terms of the time t that has elapsed since its release. Find the distance d (in feet) that an object falls in $t = 24$ seconds.

43. The formula

$$C = \frac{5(F-32)}{9}$$

gives the Celcius temperature C in terms of the Fahrenheit temperature F. Use the formula to find the Celsius temperature ($^\circ$ C) if the Fahrenheit temperature is $F = 230^\circ$ F.

44. The formula

$$C = \frac{5(F-32)}{9}$$

gives the Celcius temperature C in terms of the Fahrenheit temperature F. Use the formula to find the Celsius temperature ($^\circ$ C) if the Fahrenheit temperature is $F = 95^\circ$ F.

45. The Kelvin scale of temperature is used in chemistry and physics. Absolute zero occurs at 0° K, the temperature at which molecules have zero kinetic energy. Water freezes at 273° K and boils at $K = 373^\circ$ K. To change Kelvin temperature to Fahrenheit temperature, we use the formula

$$F = \frac{9(K-273)}{5} + 32.$$

Use the formula to change 28° K to Fahrenheit.

46. The Kelvin scale of temperature is used in chemistry and physics. Absolute zero occurs at 0° K, the temperature at which molecules have zero kinetic energy. Water freezes at 273° K and boils at $K = 373^\circ$ K. To change Kelvin temperature to Fahrenheit temperature, we use the formula

$$F = \frac{9(K-273)}{5} + 32.$$

Use the formula to change 248° K to Fahrenheit.

47. A ball is thrown vertically upward. Its velocity t seconds after its release is given by the formula

$$v = v_0 - gt,$$

where v_0 is its initial velocity, g is the acceleration due to gravity, and v is the velocity of the ball at time t. The acceleration due to gravity is $g = 32$ feet per second per second. If the initial velocity of the ball is $v_0 = 272$ feet per second, find the speed of the ball after $t = 6$ seconds.

48. A ball is thrown vertically upward. Its velocity t seconds after its release is given by the formula

$$v = v_0 - gt,$$

where v_0 is its initial velocity, g is the acceleration due to gravity, and v is the velocity of the ball at time t. The acceleration due to gravity is $g = 32$ feet per second per second. If the initial velocity of the ball is $v_0 = 470$ feet per second, find the speed of the ball after $t = 4$ seconds.

---

49. **Even numbers.** Evaluate the expression $2n$ for the following values:

i) $n = 1$

ii) $n = 2$

iii) $n = 3$

iv) $n = -4$

v) $n = -5$

vi) Is the result always an even number? Explain.

50. **Odd numbers.** Evaluate the expression $2n + 1$ for the following values:

i) $n = 1$

ii) $n = 2$

iii) $n = 3$

iv) $n = -4$

v) $n = -5$

vi) Is the result always an odd number? Explain.

❧ ❧ ❧ Answers ☙ ☙ ☙

1. -186

3. -24

5. 7

7. 13

9. 138

11. -134

13. 1

15. -72

17. -2

19. 1

21. 69

23. 0

25. 9

27. 36

29. −6

31. −71

33. 1

35. 5

37. 46

39. −29

41. 256 feet

43. 110 degrees

45. −409° F

47. 80 feet per second

49.
i) 2
ii) 4
iii) 6
iv) −8
v) −10
vi) Yes, the result will always be an even number because 2 will always be a factor of the product $2n$.

3.3 Simplifying Algebraic Expressions

Recall the commutative and associative properties of multiplication.

The Commutative Property of Multiplication. If a and b are any integers, then

$$a \cdot b = b \cdot a, \qquad \text{or equivalently,} \qquad ab = ba.$$

The Associative Property of Multiplication. If a, b, and c are any integers, then

$$(a \cdot b) \cdot c = a \cdot (b \cdot c), \qquad \text{or equivalently,} \qquad (ab)c = a(bc).$$

The commutative property allows us to change the order of multiplication without affecting the product or answer. The associative property allows us to regroup without affecting the product or answer.

You Try It!

Simplify: $-5(7y)$

EXAMPLE 1. Simplify: $2(3x)$.

Solution. Use the associative property to regroup, then simplify.

$$\begin{aligned} 2(3x) &= (2 \cdot 3)x && \text{Regrouping with the associative property.} \\ &= 6x && \text{Simplify: } 2 \cdot 3 = 6. \end{aligned}$$

Answer: $-35y$

□

The statement $2(3x) = 6x$ is an *identity*. That is, the left-hand side and right-hand side of $2(3x) = 6x$ are the same *for all values of* x. Although the derivation in Example 1 should be the proof of this statement, it helps the intuition to check the validity of the statement for one or two values of x.

If $x = 4$, then

$$\begin{aligned} 2(3x) &= 2(3(4)) \\ &= 2(12) \\ &= 24 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 6x &= 6(4) \\ &= 24 \end{aligned}$$

If $x = -5$, then

$$\begin{aligned} 2(3x) &= 2(3(-5)) \\ &= 2(-15) \\ &= -30 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 6x &= 6(-5) \\ &= -30 \end{aligned}$$

The above calculations show that $2(3x) = 6x$ for both $x = 4$ and $x = -5$. Indeed, the statement $2(3x) = 6x$ is true, regardless of what is substituted for x.

You Try It!

Simplify: $(-8a)(5)$

EXAMPLE 2. Simplify: $(-3t)(-5)$.

Solution. In essence, we are multiplying three numbers, -3, t, and -5, but the grouping symbols ask us to multiply the -3 and the t first. The associative and commutative properties allow us to change the order and regroup.

$$\begin{aligned} (-3t)(-5) &= ((-3)(-5))t && \text{Change the order and regroup.} \\ &= 15t && \text{Multiply: } (-3)(-5) = 15. \end{aligned}$$

Answer: $-40a$

□

You Try It!

Simplify: $(-4a)(5b)$

EXAMPLE 3. Simplify: $(-3x)(-2y)$.

Solution. In essence, we are multiplying four numbers, -3, x, -2, and y, but the grouping symbols specify a particular order. The associative and commutative properties allow us to change the order and regroup.

$$\begin{aligned} (-3x)(-2y) &= ((-3)(-2))(xy) && \text{Change the order and regroup.} \\ &= 6xy && \text{Multiply: } (-3)(-2) = 6. \end{aligned}$$

Answer: $-20ab$

□

Speeding Things Up

The meaning of the expression $2 \cdot 3 \cdot 4$ is clear. Parentheses and order of operations are really not needed, as the commutative and associative properties explain that it doesn't matter which of the three numbers you multiply together first.

- You can multiply 2 and 3 first:

$$\begin{aligned} 2 \cdot 3 \cdot 4 &= (2 \cdot 3) \cdot 4 \\ &= 6 \cdot 4 \\ &= 24. \end{aligned}$$

- Or you can multiply 3 and 4 first:

$$\begin{aligned} 2 \cdot 3 \cdot 4 &= 2 \cdot (3 \cdot 4) \\ &= 2 \cdot 12 \\ &= 24. \end{aligned}$$

- Or you can multiply 2 and 4 first:

$$\begin{aligned} 2 \cdot 3 \cdot 4 &= (2 \cdot 4) \cdot 3 \\ &= 8 \cdot 3 \\ &= 24. \end{aligned}$$

So, it doesn't matter which two factors you multiply first.

Of course, this would not be the case if there were a mixture of multiplication and other operators (division, addition, subtraction). Then we would have to strictly follow the "Rules Guiding Order of Operations." But if the only operator is multiplication, the order of multiplication is irrelevant.

Thus, when we see $2(3x)$, as in Example 1, we should think "It's all multiplication and it doesn't matter which two numbers I multiply together first, so I'll multiply the 2 and the 3 and get $2(3x) = 6x$."

Our comments apply equally well to a product of four or more factors. It simply doesn't matter how you group the multiplication. So, in the case of $(-3x)(-2y)$, as in Example 3, find the product of -2 and -3 and multiply the result by the product of x and y. That is, $(-3x)(-2y) = 6xy$.

You Try It!

EXAMPLE 4. Simplify: $(2a)(3b)(4c)$.

Simplify: $(-3x)(-2y)(-4z)$

Solution. The only operator is multiplication, so we can order and group as we please. So, we'll take the product of 2, 3, and 4, and multiply the result by the product of a, b, and c. That is,

$$(2a)(3b)(4c) = 24abc.$$

Answer: $-24xyz$

□

The Distributive Property

Multiplication is distributive with respect to addition.

The Distributive Property. If a, b, and c are any integers, then

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad \text{or equivalently,} \quad a(b + c) = ab + ac.$$

For example, if we follow the "Rules Guiding Order of Operations" and first evaluate the expression inside the parentheses, then

$$\begin{aligned} 3(4 + 5) &= 3(9) && \text{Parentheses first: } 4 + 5 = 9. \\ &= 27. && \text{Multiply: } 3(9) = 27. \end{aligned}$$

But if we "distribute" the 3, we get the same answer.

$$\begin{aligned} 3(4+5) &= 3(4+5) && \text{Each number in parentheses is multiplied by the number 3 outside the parentheses.} \\ &= 3(4)+3(5) \\ &= 12+15 && \text{Multiply first: } 3(4)=12,\ 3(5)=15. \\ &= 27 && \text{Add.} \end{aligned}$$

You Try It!

Use the distributive property to simplify: $2(5z+7)$

EXAMPLE 5. Use the distributive property to simplify: $3(4x+5)$.

Solution. Distribute the 3.

$$\begin{aligned} 3(4x+5) &= 3(4x)+3(5) && \text{Each number in parentheses is multiplied by the number 3 outside the parentheses.} \\ &= 12x+15 && \text{Multiply first: } 3(4x)=12x,\ 3(5)=15. \end{aligned}$$

Answer: $10z+14$

□

Multiplication is also distributive with respect to subtraction.

The Distributive Property. If a, b, and c are any integers, then

$$a\cdot(b-c)=a\cdot b-a\cdot c, \quad \text{or equivalently,} \quad a(b-c)=ab-ac.$$

The application of this form of the distributive property is identical to the first, the only difference being the subtraction symbol.

You Try It!

Use the distributive property to simplify: $7(4a-5)$

EXAMPLE 6. Use the distributive property to simplify: $5(3x-2)$.

Solution. Distribute the 5.

$$\begin{aligned} 5(3x-2) &= 5(3x)-5(2) && \text{Each number in parentheses is multiplied by the number 5 outside the parentheses.} \\ &= 15x-10 && \text{Multiply first: } 5(3x)=15x,\ 5(2)=10. \end{aligned}$$

Answer: $28a-35$

□

You Try It!

Remove parentheses: $-3(4t-11)$

EXAMPLE 7. Remove parentheses: (a) $-9(2t+7)$, and (b) $-5(4-3y)$.

Solution.

a) Use the distributive property.

$$\begin{aligned} -9(2t+7) &= -9(2t)+(-9)(7) && \text{Distribute multiplication by } -9.\\ &= -18t+(-63) && \text{Multiply: } -9(2t)=-18t \text{ and } -9(7)=-63.\\ &= -18t-63 && \text{Write the answer in simpler form. Adding } -63 \text{ is the same as subtracting } 63. \end{aligned}$$

b) Use the distributive property.

$$\begin{aligned} -5(4-3y) &= -5(4)-(-5)(3y) && \text{Distribute multiplication by } -5.\\ &= -20-(-15y) && \text{Multiply: } -5(4)=-20 \text{ and } (-5)(3y)=-15y.\\ &= -20+15y && \text{Write the answer in simpler form. Subtracting } -15y \text{ is the same as adding } 15y. \end{aligned}$$

Answer: $-12t+33$

□

Writing Mathematics. Example 7 stresses the importance of using as few symbols as possible to write your final answer. Hence, $-18t-63$ is favored over $-18t+(-63)$ and $-20+15y$ is favored over $-20-(-15y)$. You should always make these final simplifications.

Moving a Bit Quicker

Once you've applied the distributive property to a number of problems, showing all the work as in Example 7, you should try to eliminate some of the steps. For example, consider again Example 7(a). It's not difficult to apply the distributive property without writing down a single step, getting:

$$-9(2t+7)=-18t-63.$$

Here's the thinking behind this technique:

1. First, multiply -9 times $2t$, getting $-18t$.

2. Second, multiply -9 times $+7$, getting -63.

Note that this provides exactly the same solution found in Example 7(a).

Let try this same technique on Example 7(b).

$$-5(4-3y)=-20+15y$$

Here's the thinking behind this technique.

1. First, multiply -5 times 4, getting -20.
2. Second, multiply -5 times $-3y$, getting $+15y$.

Note that this provides exactly the same solution found in Example 7(b).

Extending the Distributive Property

Suppose that we add an extra term inside the parentheses.

Distributive Property. If a, b, c, and d are any integers, then

$$a(b+c+d)=ab+ac+ad.$$

Note that we "distributed" the a times each term inside the parentheses. Indeed, if we added still another term inside the parentheses, we would "distribute" a times that term as well.

You Try It!

Remove parentheses: $-3(4a-5b+7)$

EXAMPLE 8. Remove parentheses: $-5(2x-3y+8)$.

Solution. We will use the "quicker" technique, "distributing" -5 times each term in the parentheses mentally.

$$-5(2x-3y+8)=-10x+15y-40$$

Here is our thought process:

1. First, multiply -5 times $2x$, getting $-10x$.
2. Second, multiply -5 times $-3y$, getting $+15y$.
3. Third, multiply -5 times $+8$, getting -40.

Answer: $-12a+15b-21$

□

You Try It!

EXAMPLE 9. Remove parentheses: $-4(-3a + 4b - 5c + 12)$.

Remove parentheses: $-2(-2x + 4y - 5z - 11)$

Solution. We will use the "quicker" technique, "distributing" -4 times each term in the parentheses mentally.

$$-4(-3a + 4b - 5c + 12) = 12a - 16b + 20c - 48$$

Here is our thought process:

1. First, multiply -4 times $-3a$, getting $12a$.
2. Second, multiply -4 times $+4b$, getting $-16b$.
3. Third, multiply -4 times $-5c$, getting $+20c$.
4. Fourth, multiply -4 times $+12$, getting -48.

Answer: $4x - 8y + 10z + 22$

□

Distributing a Negative

It is helpful to recall that negating is equivalent to multiplying by -1.

Multiplying by -1. Let a be any integer, then

$$(-1)a = -a \qquad \text{and} \qquad -a = (-1)a.$$

We can use this fact, combined with the distributive property, to negate a sum.

You Try It!

EXAMPLE 10. Remove parentheses: $-(a + b)$.

Remove parentheses: $-(x + 2y)$

Solution. Change the negative symbol into multiplying by -1, then distribute the -1.

$$\begin{aligned} -(a+b) &= (-1)(a+b) && \text{Negating is equivalent to multiplying by } -1. \\ &= -a - b && \text{Distribute the } -1. \end{aligned}$$

We chose to use the "quicker" technique of "distributing" the -1. Here is our thinking:

1. Multiply -1 times a, getting $-a$.
2. Multiply -1 times $+b$, getting $-b$.

Answer: $-x-2y$

You Try It!

Remove parentheses: $-(4a-3c)$

EXAMPLE 11. Remove parentheses: $-(a-b)$.

Solution. Change the negative symbol into multiplying by -1, then distribute the -1.

$$\begin{aligned} -(a-b) &= (-1)(a-b) && \text{Negating is equivalent to multiplying by } -1. \\ &= -a+b && \text{Distribute the } -1. \end{aligned}$$

We chose to use the "quicker" technique of "distributing" the -1. Here is our thinking:

1. Multiply -1 times a, getting $-a$.
2. Multiply -1 times $-b$, getting $+b$.

Answer: $-4a+3c$

The results in Example 10 and Example 11 show us how to negate a sum: Simply negate each term of the sum. Positive terms change to negative, negative terms turn to positive.

Negating a Sum. To negate a sum, simply negate each term of the sum. For example, if a and b are integers, then

$$-(a+b) = -a-b \qquad \text{and} \qquad -(a-b) = -a+b.$$

You Try It!

Remove parentheses: $-(5-2x+4y-5z)$

EXAMPLE 12. Remove parentheses: $-(5-7u+3t)$.

Solution. Simply negate each term in the parentheses.

$$-(5-7u+3t) = -5+7u-3t.$$

Answer: $-5+2x-4y+5z$

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-20, use the associative and commutative properties of multiplication to simplify the expression.

1. $10(-4x)$

2. $7(-8x)$

3. $(-10x)(-3)$

4. $(-5x)(-8)$

5. $-5(3x)$

6. $9(6x)$

7. $(-4x)10$

8. $(-10x)(-6)$

9. $(5x)3$

10. $(3x)3$

11. $(5x)10$

12. $(-2x)(-10)$

13. $-9(-7x)$

14. $-10(5x)$

15. $6(2x)$

16. $3(-10x)$

17. $-8(-9x)$

18. $3(-3x)$

19. $(6x)7$

20. $(-8x)(-5)$

In Exercises 21-44, simplify the expression.

21. $8(7x+8)$

22. $-2(5x+5)$

23. $9(-2+10x)$

24. $-9(4+9x)$

25. $-(-2x+10y-6)$

26. $-(-6y+9x-7)$

27. $2(10+x)$

28. $2(10-6x)$

29. $3(3+4x)$

30. $3(4+6x)$

31. $-(-5-7x+2y)$

32. $-(4x-8-7y)$

33. $4(-6x+7)$

34. $6(4x+9)$

35. $4(8x-9)$

36. $10(-10x+1)$

37. $-(4-2x-10y)$

38. $-(-4x+6-8y)$

39. $-(-5x+1+9y)$

40. $-(-10-5x-4y)$

41. $-(6x+2-10y)$

42. $-(6x+4-10y)$

43. $-(-3y-4+4x)$

44. $-(-7-10x+7y)$

❧ ❧ ❧ Answers ☙ ☙ ☙

1. $-40x$

3. $30x$

5. $-15x$

7. $-40x$

9. $15x$

11. $50x$

13. $63x$

15. $12x$

17. $72x$

19. $42x$

21. $56x + 64$

23. $-18 + 90x$

25. $2x - 10y + 6$

27. $20 + 2x$

29. $9 + 12x$

31. $5 + 7x - 2y$

33. $-24x + 28$

35. $32x - 36$

37. $-4 + 2x + 10y$

39. $5x - 1 - 9y$

41. $-6x - 2 + 10y$

43. $3y + 4 - 4x$

3.4 Combining Like Terms

We begin our discussion with the definition of a *term*.

Term. A *term* is a single number or variable, or it can be the product of a number (called its *coefficient*) and one or more variables (called its *variable part*). The terms in an algebraic expression are separated by *addition* symbols.

You Try It!

How many terms are in the algebraic expression $3x^2 + 2xy - 3y^2$?

EXAMPLE 1. Identify the terms in the algebraic expression

$$3x^2 + 5xy + 9y^2 + 12.$$

For each term, identify its coefficient and variable part.

Solution. In tabular form, we list each term of the expression $3x^2 + 5xy + 9y^2 + 12$, its coefficient, and its variable part.

Term	Coefficient	Variable Part
$3x^2$	3	x^2
$5xy$	5	xy
$9y^2$	9	y^2
12	12	None

Answer: 3

□

You Try It!

How many terms are in the algebraic expression $11 - a^2 - 2ab + 3b^2$?

EXAMPLE 2. Identify the terms in the algebraic expression

$$a^3 - 3a^2b + 3ab^2 - b^3.$$

For each term, identify its coefficient and variable part.

Solution. The first step is to write each difference as a sum, because the terms of an expression are defined above to be those items separated by addition symbols.

$$a^3 + (-3a^2b) + 3ab^2 + (-b^3)$$

In tabular form, we list each term of the expression $a^3+(-3a^2b)+3ab^2+(-b^3)$, its coefficient, and its variable part.

Term	Coefficient	Variable Part
a^3	1	a^3
$-3a^2b$	-3	a^2b
$3ab^2$	3	ab^2
$-b^3$	-1	b^3

Answer: 4

□

Like Terms

We define what is meant by "like terms" and "unlike terms."

Like and Unlike Terms. The variable parts of two terms determine whether the terms are *like terms* or *unlike terms*.

Like Terms. Two terms are called *like terms* if they have identical variable parts, which means that the terms must contain the same variables raised to the same exponential powers.

Unlike Terms. Two terms are called *unlike terms* if their variable parts are different.

You Try It!

Are $-3xy$ and $11xy$ *like* or *unlike* terms?

EXAMPLE 3. Classify each of the following pairs as either *like terms* or *unlike terms*: (a) $3x$ and $-7x$, (b) $2y$ and $3y^2$, (c) $-3t$ and $5u$, and (d) $-4a^3$ and $3a^3$.

Solution. Like terms must have *identical* variable parts.

a) $3x$ and $-7x$ have identical variable parts. They are "like terms."

b) $2y$ and $3y^2$ do **not** have identical variable parts (the exponents differ). They are "unlike terms."

c) $-3t$ and $5u$ do **not** have identical variable parts (different variables). They are "unlike terms."

d) $-4a^3$ and $3a^3$ have identical variable parts. They are "like terms."

Answer: *Like* terms

□

Combining Like Terms

When using the distributive property, it makes no difference whether the multiplication is on the left or the right, one still distributes the multiplication times each term in the parentheses.

Distributive Property. If a, b, and c are integers, then

$$a(b+c) = ab + ac \qquad \text{and} \qquad (b+c)a = ba + ca.$$

In either case, you distribute a times each term of the sum.

"Like terms" can be combined and simplified. The tool used for combining like terms is the distributive property. For example, consider the expression $3y + 7y$, composed of two "like terms" with a common variable part. We can use the distributive property and write

$$3y + 7y = (3 + 7)y.$$

Note that we are using the distributive property in reverse, "factoring out" the common variable part of each term. *Checking our work*, note that if we redistribute the variable part y times each term in the parentheses, we are returned to the original expression $3y + 7y$.

You Try It!

Simplify: $-8z - 11z$

EXAMPLE 4. Use the distributive property to combine like terms (if possible) in each of the following expressions: (a) $-5x^2 - 9x^2$, (b) $-5ab + 7ab$, (c) $4y^3 - 7y^2$, and (d) $3xy^2 - 7xy^2$.

Solution. If the terms are "like terms," you can use the distributive property to "factor out" the common variable part.

a) Factor out the common variable part x^2.

$$\begin{aligned} -5x^2 - 9x^2 &= (-5 - 9)x^2 && \text{Use the distributive property.} \\ &= -14x^2 && \text{Simplify: } -5 - 9 = -5 + (-9) = -14. \end{aligned}$$

b) Factor out the common variable part ab.

$$\begin{aligned} -5ab + 7ab &= (-5 + 7)ab && \text{Use the distributive property.} \\ &= 2ab && \text{Simplify: } -5 + 7 = 2. \end{aligned}$$

c) The terms in the expression $4y^3 - 7y^2$ have different variable parts (the exponents are different). These are "unlike terms" and cannot be combined.

d) Factor out the common variable part xy^2.

$$\begin{aligned} 3xy^2 - 7xy^2 &= (3-7)xy^2 && \text{Use the distributive property.} \\ &= -4xy^2 && \text{Simplify: } 3-7 = 3+(-7) = -4. \end{aligned}$$

Answer: $-19z$

□

Speeding Things Up a Bit

Once you've written out all the steps for combining like terms, like those shown in Example 4, you can speed things up a bit by following this rule:

Combining Like Terms. To combine like terms, simply add their coefficients and keep the common variable part.

Thus for example, when presented with the sum of two like terms, such as in $5x + 8x$, simply add the coefficients and repeat the common variable part; that is, $5x + 8x = 13x$.

You Try It!

Combine: $-3x^2 - 4x^2$

EXAMPLE 5. Combine like terms: (a) $-9y - 8y$, (b) $-3y^5 + 4y^5$, and (c) $-3u^2 + 2u^2$.

Solution.

a) Add the coefficients and repeat the common variable part. Therefore,

$$-9y - 8y = -17y.$$

b) Add the coefficients and repeat the common variable part. Therefore,

$$-3y^5 + 4y^5 = 1y^5.$$

However, note that $1y^5 = y^5$. Following the rule that the final answer should use as few symbols as possible, a better answer is $-3y^5 + 4y^5 = y^5$.

c) Add the coefficients and repeat the common variable part. Therefore,

$$-3u^2 + 2u^2 = (-1)u^2.$$

However, note that $(-1)u^2 = -u^2$. Following the rule that the final answer should use as few symbols as possible, a better answer is $-3u^2 + 2u^2 = -u^2$.

Answer: $-7x^2$

□

Simplify

A frequently occurring instruction asks the reader to *simplify* an expression.

Simplify. The instruction *simplify* is a generic term that means "try to write the expression in its most compact form, using the fewest symbols possible."

One way you can accomplish this goal is by combining like terms when they are present.

You Try It!

EXAMPLE 6. Simplify: $2x + 3y - 5x + 8y$.

Simplify: $-3a + 4b - 7a - 9b$

Solution. Use the commutative property to reorder terms and the associative and distributive properties to regroup and combine like terms.

$$2x + 3y - 5x + 8y = (2x - 5x) + (3y + 8y)$$ Reorder and regroup.

$$= -3x + 11y$$ Combine like terms: $2x - 5x = -3x$ and $3y + 8y = 11y$.

Alternate solution. Of course, you do not need to show the regrouping step. If you are more comfortable combining like terms in your head, you are free to present your work as follows:

$$2x + 3y - 5x + 8y = -3x + 11y.$$

Answer: $-10a - 5b$

□

You Try It!

EXAMPLE 7. Simplify: $-2x - 3 - (3x + 4)$.

Simplify: $-9a - 4 - (4a - 8)$

Solution. First, distribute the negative sign.

$$-2x - 3 - (3x + 4) = -2x - 3 - 3x - 4$$ $-(3x + 4) = -3x - 4$.

Next, use the commutative property to reorder, then the associative property to regroup. Then combine like terms.

$$= (-2x - 3x) + (-3 - 4)$$ Reorder and regroup.

$$= -5x + (-7)$$ Combine like terms: $-2x - 3x = -5x$.

$$= -5x - 7$$ Simplify: $-5x + (-7) = -5x - 7$.

Alternate solution. You may skip the second step if you wish, simply combining like terms mentally. That is, it is entirely possible to order your work as follows:

$$\begin{aligned} -2x - 3 - (3x + 4) &= -2x - 3 - 3x - 4 && \text{Distribute negative sign.} \\ &= -5x - 7 && \text{Combine like terms.} \end{aligned}$$

Answer: $-13a + 4$

□

You Try It!

Simplify:

$-2(3a - 4) - 2(5 - a)$

EXAMPLE 8. Simplify: $2(5 - 3x) - 4(x + 3)$.

Solution. Use the distributive property to expand, then use the commutative and associative properties to group the like terms and combine them.

$$\begin{aligned} 2(5 - 3x) - 4(x + 3) &= 10 - 6x - 4x - 12 && \text{Use the distributive property.} \\ &= (-6x - 4x) + (10 - 12) && \text{Group like terms.} \\ &= -10x - 2 && \text{Combine like terms:} \\ & && -6x - 4x = -10x \text{ and} \\ & && 10 - 12 = -2. \end{aligned}$$

Alternate solution. You may skip the second step if you wish, simply combining like terms mentally. That is, it is entirely possible to order your work as follows:

$$\begin{aligned} 2(5 - 3x) - 4(x + 3) &= 10 - 6x - 4x - 12 && \text{Distribute.} \\ &= -10x - 2 && \text{Combine like terms.} \end{aligned}$$

Answer: $-4a - 2$

□

You Try It!

Simplify:
$(a^2 - 2ab) - 2(3ab + a^2)$

EXAMPLE 9. Simplify: $-8(3x^2y - 9xy) - 8(-7x^2y - 8xy)$.

Solution. We will proceed a bit quicker with this solution, using the distributive property to expand, then combining like terms mentally.

$$\begin{aligned} -8(3x^2y - 9xy) - 8(-7x^2y - 8xy) &= -24x^2y + 72xy + 56x^2y + 64xy \\ &= 32x^2y + 136xy \end{aligned}$$

Answer: $-a^2 - 8ab$

□

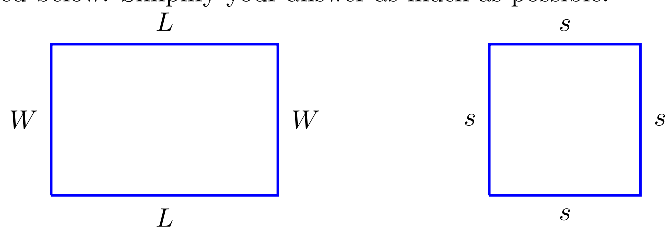
Applications

We can simplify a number of useful formulas by combining like terms.

You Try It!

A regular hexagon has six equal sides, each with length x. Find its perimeter in terms of x.

EXAMPLE 10. Find the perimeter P of the (a) rectangle and (b) square pictured below. Simplify your answer as much as possible.



Solution. The perimeter of any polygonal figure is the sum of the lengths of its sides.

a) To find the perimeter P of the rectangle, sum its four sides.

$$P = L + W + L + W.$$

Combine like terms.

$$P = 2L + 2W.$$

b) To find the perimeter P of the square, sum its four sides.

$$P = s + s + s + s.$$

Combine like terms.

$$P = 4s.$$

Answer: $P = 6x$

□

Sometimes it is useful to replace a variable with an expression containing another variable.

You Try It!

The length L of a rectangle is 5 meters longer than twice its width W. Find the perimeter P of the rectangle in terms of its width W.

EXAMPLE 11. The length of a rectangle is three feet longer than twice its width. Find the perimeter P of the rectangle in terms of its width alone.

Solution. From the previous problem, the perimeter of the rectangle is given by

$$P = 2L + 2W, \tag{3.1}$$

where L and W are the length and width of the rectangle, respectively. This equation gives the perimeter in terms of its length and width, but we're asked to get the perimeter in terms of the width alone.

However, we're also given the fact that the length is three feet longer than twice the width.

Length	is	Three Feet	longer than	Twice the Width
L	$=$	3	$+$	$2W$

Because $L = 3+2W$, we can replace L with $3+2W$ in the perimeter equation 3.1.

$$P = 2L + 2W$$
$$P = 2(3 + 2W) + 2W$$

Use the distributive property, then combine like terms.

$$P = 6 + 4W + 2W$$
$$P = 6 + 6W.$$

This last equation gives the perimeter P in terms of the width W alone.

Answer: $P = 6W + 10$

□

You Try It!

The width W of a rectangle is 5 feet less than twice its width L. Find the perimeter P of the rectangle in terms of its length L.

EXAMPLE 12. The width of a rectangle is two feet less than its length. Find the perimeter P of the rectangle in terms of its length alone.

Solution. Again, the perimeter of a rectangle is given by the equation

$$P = 2L + 2W, \tag{3.2}$$

where L and W are the length and width of the rectangle, respectively. This equation gives the perimeter in terms of its length and width, but we're asked to get the perimeter in terms of the length alone.

However, we're also given the fact that the width is two feet less than the length.

Width	is	Length	minus	Two feet
W	$=$	L	$-$	2

Because $W = L-2$, we can replace W with $L-2$ in the perimeter equation 3.2.

$$P = 2L + 2W$$
$$P = 2L + 2(L - 2)$$

Use the distributive property, then combine like terms.

$$P = 2L + 2L - 4$$
$$P = 4L - 4.$$

This last equation gives the perimeter P in terms of the length L alone.

Answer: $P = 6L - 10$

□

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-16, combine like terms by first using the distributive property to factor out the common variable part, and then simplifying.

1. $17xy^2 + 18xy^2 + 20xy^2$

2. $13xy - 3xy + xy$

3. $-8xy^2 - 3xy^2 - 10xy^2$

4. $-12xy - 2xy + 10xy$

5. $4xy - 20xy$

6. $-7y^3 + 15y^3$

7. $12r - 12r$

8. $16s - 5s$

9. $-11x - 13x + 8x$

10. $-9r - 10r + 3r$

11. $-5q + 7q$

12. $17n + 15n$

13. $r - 13r - 7r$

14. $19m + m + 15m$

15. $3x^3 - 18x^3$

16. $13x^2y + 2x^2y$

In Exercises 17-32, combine like terms by first rearranging the terms, then using the distributive property to factor out the common variable part, and then simplifying.

17. $-8 + 17n + 10 + 8n$

18. $11 + 16s - 14 - 6s$

19. $-2x^3 - 19x^2y - 15x^2y + 11x^3$

20. $-9x^2y - 10y^3 - 10y^3 + 17x^2y$

21. $-14xy - 2x^3 - 2x^3 - 4xy$

22. $-4x^3 + 12xy + 4xy - 12x^3$

23. $-13 + 16m + m + 16$

24. $9 - 11x - 8x + 15$

25. $-14x^2y - 2xy^2 + 8x^2y + 18xy^2$

26. $-19y^2 + 18y^3 - 5y^2 - 17y^3$

27. $-14x^3 + 16xy + 5x^3 + 8xy$

28. $-16xy + 16y^2 + 7xy + 17y^2$

29. $9n + 10 + 7 + 15n$

30. $-12r + 5 + 17 + 17r$

31. $3y + 1 + 6y + 3$

32. $19p + 6 + 8p + 13$

In Exercises 33-56, simplify the expression by first using the distributive property to expand the expression, and then rearranging and combining like terms mentally.

33. $-4(9x^2y + 8) + 6(10x^2y - 6)$

34. $-4(-4xy + 5y^3) + 6(-5xy - 9y^3)$

35. $3(-4x^2 + 10y^2) + 10(4y^2 - x^2)$

36. $-7(-7x^3 + 6x^2) - 7(-10x^2 - 7x^3)$

37. $-s + 7 - (-1 - 3s)$

38. $10y - 6 - (-10 - 10y)$

39. $-10q - 10 - (-3q + 5)$

40. $-2n + 10 - (7n - 1)$

41. $7(8y + 7) - 6(8 - 7y)$

42. $-6(-5n - 4) - 9(3 + 4n)$

43. $7(10x^2 - 8xy^2) - 7(9xy^2 + 9x^2)$

44. $10(8x^2y - 10xy^2) + 3(8xy^2 + 2x^2y)$

45. $-2(6 + 4n) + 4(-n - 7)$

46. $-6(-2 - 6m) + 5(-9m + 7)$

47. $8 - (4 + 8y)$

48. $-1 - (8 + s)$

49. $-8(-n + 4) - 10(-4n + 3)$

50. $3(8r - 7) - 3(2r - 2)$

51. $-5 - (10p + 5)$

52. $-1 - (2p - 8)$

53. $7(1 + 7r) + 2(4 - 5r)$

54. $(5 - s) + 10(9 + 5s)$

55. $-2(-5 - 8x^2) - 6(6)$

56. $8(10y^2 + 3x^3) - 5(-7y^2 - 7x^3)$

---

57. The length L of a rectangle is 2 feet longer than 6 times its width W. Find the perimeter of the rectangle in terms of its width alone.

58. The length L of a rectangle is 7 feet longer than 6 times its width W. Find the perimeter of the rectangle in terms of its width alone.

59. The width W of a rectangle is 8 feet shorter than its length L. Find the perimeter of the rectangle in terms of its length alone.

60. The width W of a rectangle is 9 feet shorter than its length L. Find the perimeter of the rectangle in terms of its length alone.

61. The length L of a rectangle is 9 feet shorter than 4 times its width W. Find the perimeter of the rectangle in terms of its width alone.

62. The length L of a rectangle is 2 feet shorter than 6 times its width W. Find the perimeter of the rectangle in terms of its width alone.

❧ ❧ ❧ Answers ☙ ☙ ☙

1. $55xy^2$

3. $-21xy^2$

5. $-16xy$

7. 0

9. $-16x$

11. $2q$

13. $-19r$

15. $-15x^3$

17. $2 + 25n$

19. $9x^3 - 34x^2y$

21. $-18xy - 4x^3$

23. $3 + 17m$

25. $-6x^2y + 16xy^2$

27. $-9x^3 + 24xy$

29. $24n + 17$

31. $9y + 4$

33. $24x^2y - 68$

35. $-22x^2 + 70y^2$

37. $2s + 8$

39. $-7q - 15$

41. $98y + 1$

43. $7x^2 - 119xy^2$

45. $-40 - 12n$

47. $4 - 8y$

49. $48n - 62$

51. $-10 - 10p$

53. $15 + 39r$

55. $-26 + 16x^2$

57. $4 + 14W$

59. $4L - 16$

61. $10W - 18$

3.5 Solving Equations Involving Integers II

We return to solving equations involving integers, only this time the equations will be a bit more advanced, requiring the use of the distributive property and skill at combining like terms. Let's begin.

You Try It!

Solve for x:

$$-6x - 5x = 22$$

EXAMPLE 1. Solve for x: $7x - 11x = 12$.

Solution. Combine like terms.

$$\begin{aligned} 7x - 11x &= 12 && \text{Original equation.} \\ -4x &= 12 && \text{Combine like terms: } 7x - 11x = -4x. \end{aligned}$$

To undo the effect of multiplying by -4, divide both sides of the last equation by -4.

$$\begin{aligned} \frac{-4x}{-4} &= \frac{12}{-4} && \text{Divide both sides by } -4. \\ x &= -3 && \text{Simplify: } 12/(-4) = -3. \end{aligned}$$

Check. Substitute -3 for x in the original equation.

$$\begin{aligned} 7x - 11x &= 12 && \text{Original equation.} \\ 7(-3) - 11(-3) &= 12 && \text{Substitute } -3 \text{ for } x. \\ -21 + 33 &= 12 && \text{On the left, multiply first.} \\ 12 &= 12 && \text{On the left, add.} \end{aligned}$$

Because the last line of the check is a true statement, -3 is a solution of the original equation.

Answer: $x = -2$

□

You Try It!

Solve for x:

$$11 = 3x - (1 - x)$$

EXAMPLE 2. Solve for x: $12 = 5x - (4 + x)$.

Solution. To take the negative of a sum, negate each term in the sum (change each term to its opposite). Thus, $-(4 + x) = -4 - x$.

$$\begin{aligned} 12 &= 5x - (4 + x) && \text{Original equation.} \\ 12 &= 5x - 4 - x && -(4 + x) = -4 - x. \\ 12 &= 4x - 4 && \text{Combine like terms: } 5x - x = 4x. \end{aligned}$$

To undo the effect of subtracting 4, add 4 to both sides of the last equation.

$$\begin{aligned} 12 + 4 &= 4x - 4 + 4 && \text{Add 4 to both sides.} \\ 16 &= 4x && \text{Simplify both sides.} \end{aligned}$$

To undo the effect of multiplying by 4, divide both sides of the last equation by 4.

$$\frac{16}{4} = \frac{4x}{4} \quad \text{Divide both sides by 4.}$$
$$4 = x \quad \text{Simplify: } 16/4 = 4.$$

Check. Substitute 4 for x in the original equation.

$$12 = 5x - (4 + x) \quad \text{Original equation.}$$
$$12 = 5(4) - (4 + 4) \quad \text{Substitute 4 for } x.$$
$$12 = 20 - 8 \quad \text{On the right, } 5(4) = 20 \text{ and evaluate parentheses: } 4 + 4 = 8.$$
$$12 = 12 \quad \text{Simplify.}$$

Because the last line of the check is a true statement, 4 is a solution of the original equation.

Answer: $x = 3$

□

Variables on Both Sides

Variables can occur on both sides of the equation.

Goal. Isolate the terms containing the variable you are solving for on one side of the equation.

You Try It!

Solve for x:

$$4x - 3 = x$$

EXAMPLE 3. Solve for x: $5x = 3x - 18$.

Solution. To isolate the variables on one side of the equation, subtract $3x$ from both sides of the equation and simplify.

$$5x = 3x - 18 \quad \text{Original equation.}$$
$$5x - 3x = 3x - 18 - 3x \quad \text{Subtract } 3x \text{ from both sides.}$$
$$2x = -18 \quad \text{Combine like terms: } 5x - 3x = 2x \text{ and } 3x - 3x = 0.$$

Note that the variable is now isolated on the left-hand side of the equation. To undo the effect of multiplying by 2, divide both sides of the last equation by 2.

$$\frac{2x}{2} = \frac{-18}{2} \quad \text{Divide both sides by 2.}$$
$$x = -9 \quad \text{Simplify: } -18/2 = -9.$$

Check. Substitute -9 for x in the original equation.

$$\begin{aligned} 5x &= 3x - 18 && \text{Original equation.} \\ 5(-9) &= 3(-9) - 18 && \text{Substitute } -9 \text{ for } x. \\ -45 &= -27 - 18 && \text{Multiply first on both sides.} \\ -45 &= -45 && \text{Subtract on the right: } -27 - 18 = -45. \end{aligned}$$

Because the last line of the check is a true statement, -9 is a solution of the original equation.

Answer: $x = 1$

□

You Try It!

Solve for x:

$$7x = 18 + 9x$$

EXAMPLE 4. Solve for x: $5x = 3 + 6x$.

Solution. To isolate the variables on one side of the equation, subtract $6x$ from both sides of the equation and simplify.

$$\begin{aligned} 5x &= 3 + 6x && \text{Original equation.} \\ 5x - 6x &= 3 + 6x - 6x && \text{Subtract } 6x \text{ from both sides.} \\ -x &= 3 && \text{Combine like terms: } 5x - 6x = -x \\ & && \text{and } 6x - 6x = 0. \end{aligned}$$

Note that the variable is now isolated on the left-hand side of the equation.

There are a couple of ways we can finish this solution. Remember, $-x$ is the same as $(-1)x$, so we could undo the effects of multiplying by -1 by dividing both sides of the equation by -1. Multiplying both sides of the equation by -1 will work equally well. But perhaps the easiest way to proceed is to simply negate both sides of the equation.

$$\begin{aligned} -(-x) &= -3 && \text{Negate both sides.} \\ x &= -3 && \text{Simplify: } -(-x) = x. \end{aligned}$$

Check. Substitute -3 for x in the original equation.

$$\begin{aligned} 5x &= 3 + 6x && \text{Original equation.} \\ 5(-3) &= 3 + 6(-3) && \text{Substitute } -3 \text{ for } x. \\ -15 &= 3 - 18 && \text{Multiply first on both sides.} \\ -15 &= -15 && \text{Subtract on the right: } 3 - 18 = -15. \end{aligned}$$

Because the last line of the check is a true statement, -3 is a solution of the original equation.

Answer: $x = -9$

□

Dealing with −x. If your equation has the form

$$-x = c,$$

where c is some integer, note that this is equivalent to the equation $(-1)x = c$. Therefore, dividing both sides by -1 will produce a solution for x. Multiplying both sides by -1 works equally well. However, perhaps the easiest thing to do is negate each side, producing

$$-(-x) = -c, \quad \text{which is equivalent to} \quad x = -c.$$

You Try It!

Solve for x:

$$2x + 3 = 18 - 3x$$

EXAMPLE 5. Solve for x: $6x - 5 = 12x + 19$.

Solution. To isolate the variables on one side of the equation, subtract $12x$ from both sides of the equation and simplify.

$$\begin{aligned} 6x - 5 &= 12x + 19 && \text{Original equation.} \\ 6x - 5 - 12x &= 12x + 19 - 12x && \text{Subtract } 12x \text{ from both sides.} \\ -6x - 5 &= 19 && \text{Combine like terms: } 6x - 12x = -6x \\ & && \text{and } 12x - 12x = 0. \end{aligned}$$

Note that the variable is now isolated on the left-hand side of the equation. Next, to "undo" subtracting 5, add 5 to both sides of the equation.

$$\begin{aligned} -6x - 5 + 5 &= 19 + 5 && \text{Add 5 to both sides.} \\ -6x &= 24 && \text{Simplify: } -5 + 5 = 0 \text{ and } 19 + 5 = 24. \end{aligned}$$

Finally, to "undo" multiplying by -6, divide both sides of the equation by -6.

$$\begin{aligned} \frac{-6x}{-6} &= \frac{24}{-6} && \text{Divide both sides by } -6. \\ x &= -4 && \text{Simplify: } 24/(-6) = -4. \end{aligned}$$

Check. Substitute -4 for x in the original equation.

$$\begin{aligned} 6x - 5 &= 12x + 19 && \text{Original equation.} \\ 6(-4) - 5 &= 12(-4) + 19 && \text{Substitute } -4 \text{ for } x. \\ -24 - 5 &= -48 + 19 && \text{Multiply first on both sides.} \\ -29 &= -29 && \text{Add: } -24 - 5 = -29 \text{ and } -48 + 19 = -29. \end{aligned}$$

Because the last line of the check is a true statement, -4 is a solution of the original equation.

Answer: $x = 3$

□

You Try It!

Solve for x:

$3(2x-4)-2(5-x)=18$

EXAMPLE 6. Solve for x: $2(3x+2)-3(4-x)=x+8$.

Solution. Use the distributive property to remove parentheses on the left-hand side of the equation.

$$\begin{aligned} 2(3x+2)-3(4-x) &= x+8 && \text{Original equation.} \\ 6x+4-12+3x &= x+8 && \text{Use the distributive property.} \\ 9x-8 &= x+8 && \text{Combine like terms: } 6x+3x=9x \\ & && \text{and } 4-12=-8. \end{aligned}$$

Isolate the variables on the left by subtracting x from both sides of the equation.

$$\begin{aligned} 9x-8-x &= x+8-x && \text{Subtract } x \text{ from both sides.} \\ 8x-8 &= 8 && \text{Combine like terms: } 9x-x=8x \\ & && \text{and } x-x=0. \end{aligned}$$

Note that the variable is now isolated on the left-hand side of the equation. Next, to "undo" subtracting 8, add 8 to both sides of the equation.

$$\begin{aligned} 8x-8+8 &= 8+8 && \text{Add 8 to both sides.} \\ 8x &= 16 && \text{Simplify: } -8+8=0 \text{ and } 8+8=16. \end{aligned}$$

Finally, to "undo" multiplying by 8, divide both sides of the equation by 8.

$$\begin{aligned} \frac{8x}{8} &= \frac{16}{8} && \text{Divide both sides by 8.} \\ x &= 2 && \text{Simplify: } 16/8=2. \end{aligned}$$

Check. Substitute 2 for x in the original equation.

$$\begin{aligned} 2(3x+2)-3(4-x) &= x+8 && \text{Original equation.} \\ 2(3(2)+2)-3(4-2) &= 2+8 && \text{Substitute 2 for } x. \\ 2(6+2)-3(2) &= 10 && \text{Work parentheses on left, add on the right.} \\ 2(8)-3(2) &= 10 && \text{Add in parentheses on left.} \\ 16-6 &= 10 && \text{Multiply first on left.} \\ 10 &= 10 && \text{Subtract on left.} \end{aligned}$$

Because the last line of the check is a true statement, 2 is a solution of the original equation.

Answer: $x=5$

□

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-16, solve the equation.

1. $-9x + x = -8$

2. $4x - 5x = -3$

3. $-4 = 3x - 4x$

4. $-6 = -5x + 7x$

5. $27x + 51 = -84$

6. $-20x + 46 = 26$

7. $9 = 5x + 9 - 6x$

8. $-6 = x + 3 - 4x$

9. $0 = -18x + 18$

10. $0 = -x + 71$

11. $41 = 28x + 97$

12. $-65 = -x - 35$

13. $8x - 8 - 9x = -3$

14. $6x + 7 - 9x = 4$

15. $-85x + 85 = 0$

16. $17x - 17 = 0$

In Exercises 17-34, solve the equation.

17. $-6x = -5x - 9$

18. $-5x = -3x - 2$

19. $6x - 7 = 5x$

20. $3x + 8 = -5x$

21. $4x - 3 = 5x - 1$

22. $x - 2 = 9x - 2$

23. $-3x + 5 = 3x - 1$

24. $-5x + 9 = -4x - 3$

25. $-5x = -3x + 6$

26. $3x = 4x - 6$

27. $2x - 2 = 4x$

28. $6x - 4 = 2x$

29. $-6x + 8 = -2x$

30. $4x - 9 = 3x$

31. $6x = 4x - 4$

32. $-8x = -6x + 8$

33. $-8x + 2 = -6x + 6$

34. $-3x + 6 = -2x - 5$

In Exercises 35-52, solve the equation.

35. $1 - (x - 2) = -3$

36. $1 - 8(x - 8) = 17$

37. $-7x + 6(x + 8) = -2$

38. $-8x + 4(x + 7) = -12$

39. $8(-6x - 1) = -8$

40. $-7(-2x - 4) = -14$

41. $-7(-4x - 6) = -14$

42. $-2(2x + 8) = -8$

43. $2 - 9(x - 5) = -16$

44. $7 - 2(x + 4) = -1$

45. $7x + 2(x + 9) = -9$

46. $-8x + 7(x - 2) = -14$

47. $2(-x + 8) = 10$

48. $2(-x - 2) = 10$

49. $8 + 2(x - 5) = -4$

50. $-5 + 2(x + 5) = -5$

51. $9x - 2(x + 5) = -10$

52. $-8x - 5(x - 3) = 15$

In Exercises 53-68, solve the equation.

53. $4(-7x + 5) + 8 = 3(-9x - 1) - 2$

54. $-4(-x + 9) + 5 = -(-5x - 4) - 2$

55. $-8(-2x - 6) = 7(5x - 1) - 2$

56. $5(-4x - 8) = -9(-6x + 4) - 4$

57. $2(2x - 9) + 5 = -7(-x - 8)$

58. $-6(-4x - 9) + 4 = -2(-9x - 8)$

59. $6(-3x + 4) - 6 = -8(2x + 2) - 8$

60. $-5(5x - 9) - 3 = -4(2x + 5) - 6$

61. $2(-2x - 3) = 3(-x + 2)$

62. $-2(7x + 1) = -2(3x - 7)$

63. $-5(-9x + 7) + 7 = -(-9x - 8)$

64. $7(-2x - 6) + 1 = 9(-2x + 7)$

65. $5(5x - 2) = 4(8x + 1)$

66. $5(-x - 4) = -(-x + 8)$

67. $-7(9x - 6) = 7(5x + 7) - 7$

68. $-8(2x + 1) = 2(-9x + 8) - 2$

❧ ❧ ❧ Answers ☙ ☙ ☙

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1. 1

3. 4

5. -5

7. 0

9. 1

11. -2

13. -5

15. 1

17. 9

19. 7

21. -2

23. 1

25. -3

27. -1

29. 2

31. -2

33. -2

35. 6

37. 50

39. 0

41. -2

43. 7

45. -3

47. 3

49. -1

51. 0

53. 33

55. 3

57. -23

59. 21

61. -12

63. 1

65. -2

67. 0

3.6 Applications

Because we've increased our fundamental ability to simplify algebraic expressions, we're now able to tackle a number of more advanced applications. Before we begin, we remind readers of required steps that must accompany solutions of applications.

Requirements for Word Problem Solutions.

1. **Set up a Variable Dictionary.** You must let your readers know what each variable in your problem represents. This can be accomplished in a number of ways:
   - Statements such as "Let P represent the perimeter of the rectangle."
   - Labeling unknown values with variables in a table.
   - Labeling unknown quantities in a sketch or diagram.
2. **Set up an Equation.** Every solution to a word problem must include a carefully crafted equation that accurately describes the constraints in the problem statement.
3. **Solve the Equation.** You must always solve the equation set up in the previous step.
4. **Answer the Question.** This step is easily overlooked. For example, the problem might ask for Jane's age, but your equation's solution gives the age of Jane's sister Liz. Make sure you answer the original question asked in the problem.
5. **Look Back.** It is important to note that this step does not imply that you should simply check your solution in your equation. After all, it's possible that your equation incorrectly models the problem's situation, so you could have a valid solution to an incorrect equation. The important question is: "Does your answer make sense based on the words in the original problem statement."

Consecutive Integers

The integers are *consecutive*, in the sense that one follows right after another. For example, 5 and 6 are a pair of consecutive integers. The important relation to notice is the fact that the second integer of this pair is one larger than its predecessor. That is, $6 = 5 + 1$.

Consecutive Integers. Let k represent an integer. The next consecutive integer is the integer $k+1$.

Thus, if k is an integer, then $k+1$ is the next integer, $k+2$ is the next integer after that, and so on.

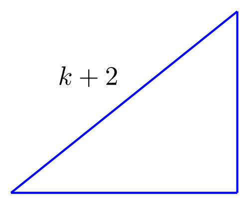
You Try It!

EXAMPLE 1. The three sides of a triangle are consecutive integers and the perimeter is 72 inches. Find the measure of each side of the triangle.

The three sides of a triangle are consecutive integers and the perimeter is 57 centimeters. Find the measure of each side of the triangle.

Solution. We follow the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* In this case, a carefully labeled diagram is the best way to indicate what the unknown variable represents.



In our schematic diagram, we've labeled the three sides of the triangle with expressions representing the consecutive integers k, $k+1$, and $k+2$.

2. *Set up an Equation.* To find the perimeter P of the triangle, sum the three sides.

$$P = k + (k+1) + (k+2)$$

However, we're given the fact that the perimeter is 72 inches. Thus,

$$72 = k + (k+1) + (k+2)$$

3. *Solve the Equation.* On the right, regroup and combine like terms.

$$72 = 3k + 3$$

Now, solve.

$$72 - 3 = 3k + 3 - 3 \quad \text{Subtract 3 from both sides.}$$
$$69 = 3k \quad \text{Simplify.}$$
$$\frac{69}{3} = \frac{3k}{3} \quad \text{Divide both sides by 3.}$$
$$23 = k \quad \text{Simplify.}$$

4. *Answer the Question.* We've only found one side, but the question asks for the measure of all three sides. However, the remaining two sides can be found by substituting 23 for k into the expressions $k+1$ and $k+2$.

$$\begin{aligned} k+1 &= 23+1 \\ &= 24 \end{aligned} \qquad \text{and} \qquad \begin{aligned} k+2 &= 23+2 \\ &= 25 \end{aligned}$$

Hence, the three sides measure 23 inches, 24 inches, and 25 inches.

5. *Look Back.* Does our solution make sense? Well, the three sides are certainly consecutive integers, and their sum is 23 inches + 24 inches + 25 inches = 72 inches, which was the given perimeter. Therefore, our solution is correct.

Answer: 18, 19, and 20 cm

□

Consecutive Odd Integers

The integer pair 19 and 21 are an example of a pair of *consecutive odd integers.* The important relation to notice is the fact that the second integer of this pair is two larger than its predecessor. That is, $21 = 19 + 2$.

Consecutive Odd Integers. Let k represent an *odd* integer. The next consecutive odd integer is $k+2$.

Thus, if k is an odd integer, then $k+2$ is the next odd integer, $k+4$ is the next odd integer after that, and so on.

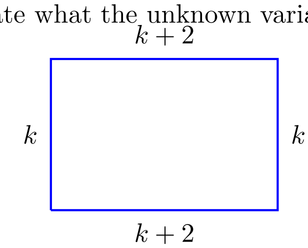
You Try It!

The length and width of a rectangle are consecutive odd integers and the perimeter is 120 meters. Find the length and width of the rectangle.

EXAMPLE 2. The length and width of a rectangle are consecutive odd integers and the perimeter is 168 centimeters. Find the length and width of the rectangle.

Solution. We follow the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* In this case, a carefully labeled diagram is the best way to indicate what the unknown variable represents.



In our schematic diagram, if the width k is an odd integer, then the length $k+2$ is the next consecutive odd integer.

2. *Set up an Equation.* To find the perimeter of the rectangle, sum the four sides.

$$P = k + (k+2) + k + (k+2)$$

However, we're given the fact that the perimeter is 168 centimeters. Thus,

$$168 = k + (k+2) + k + (k+2)$$

3. *Solve the Equation.* On the right, regroup and combine like terms.

$$168 = 4k + 4$$

Now, solve.

$$\begin{aligned} 168 - 4 &= 4k + 4 - 4 && \text{Subtract 4 from both sides.} \\ 164 &= 4k && \text{Simplify.} \\ \frac{164}{4} &= \frac{4k}{4} && \text{Divide both sides by 4.} \\ 41 &= k && \text{Simplify.} \end{aligned}$$

4. *Answer the Question.* We've only found the width, but the question asks for the measure of both the width and the length. However, the length can be found by substituting 41 for k into the expression $k + 2$.

$$\begin{aligned} k + 2 &= 41 + 2 \\ &= 43 \end{aligned}$$

Hence, the width is 41 centimeters and the length is 43 centimeters.

5. *Look Back.* Does our solution make sense? Well, the width is 41 cm and the length is 43 cm, certainly consecutive odd integers. Further, the perimeter would be 41 cm + 43 cm + 41 cm + 43 cm = 168 cm, so our solution is correct.

Answer: $W = 29$ cm, $L = 31$ cm

□

Tables

In the remaining applications in this section, we will strive to show how tables can be used to summarize information, define variables, and construct equations to help solve the application.

You Try It!

EXAMPLE 3. Hue inherits \$10,000 and decides to invest in two different types of accounts, a savings account paying 2% interest, and a certificate of deposit paying 4% interest. He decides to invest \$1,000 more in the certificate of deposit than in savings. Find the amount invested in each account.

Dylan invests a total of \$2,750 in two accounts, a savings account paying 3% interest, and a mutual fund paying 5% interest. He invests \$250 less in the mutual fund than in savings. Find the amount invested in each account.

Solution. We follow the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* We're going to use a table to summarize information and declare variables. In the table that follows, we let S represent the amount Hue invests in the savings account. *Using a variable letter that "sounds like" the quantity it represents is an excellent strategy. Thus, in this case, letting S represent the amount invested in savings is far better than letting x represent the amount invested in savings.*

Account Type	Amount Deposited
Savings Account (2%)	S
Certificate of Deposit (4%)	$S + 1000$
Totals	10000

   Because S represents the investment in savings, and we're told that the investment in the certificate of deposit (CD) is \$1,000 more than the investment in savings, the investment in the CD is therefore $S + 1000$, as indicated in the table.

2. *Set up an Equation.* The second column of the table reveals that the sum of the individual investments in the CD and savings totals \$10,000. Hence, the equation that models this application is

$$(S + 1000) + S = 10000.$$

3. *Solve the Equation.* On the left, regroup and combine like terms.

$$2S + 1000 = 10000$$

   Now, solve.

$$\begin{aligned} 2S + 1000 - 1000 &= 10000 - 1000 && \text{Subtract 1000 from both sides.} \\ 2S &= 9000 && \text{Simplify.} \\ \frac{2S}{2} &= \frac{9000}{2} && \text{Divide both sides by 2.} \\ S &= 4500 && \text{Simplify.} \end{aligned}$$

4. *Answer the Question.* We've only found the investment in savings, but the question also asks for the amount invested in the CD. However, the investment in the CD is easily found by substituting 4500 for S in the expression $S + 1000$.

$$\begin{aligned} S + 1000 &= 4500 + 1000 \\ &= 5500. \end{aligned}$$

   Hence, the investment in savings is \$4,500 and the investment in the CD is \$5,500.

5. *Look Back.* Does our solution make sense? Well, the amount invested in the CD is $5,500, which is certainly $1,000 more than the $4,500 invested in savings. Secondly, the two investments total $\$5,500 + \$4,500 = \$10,000$, so our solution is correct.

Answer: $1,500 in savings, $1,250 in the mutual fund

You Try It!

David keeps his change in a bowl made by his granddaughter. There is $1.95 in change in the bowl, all in dimes and quarters. There are two fewer quarters than dimes. How many dimes and quarters does he have in the bowl?

EXAMPLE 4. Jose cracks open his piggy bank and finds that he has $3.25 (325 cents), all in nickels and dimes. He has 10 more dimes than nickels. How many dimes and nickels does Jose have?

Solution. We follow the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* We're going to use a table to summarize information and declare variables. In the table that follows, we let N represent the number of nickels from the piggy bank. *Using a variable letter that "sounds like" the quantity it represents is an excellent strategy. Thus, in this case, letting N represent the number of nickels is far better than letting x represent the number of nickels.*

Coins	Number of Coins	Value (cents)
Nickels (5 cents apiece)	N	$5N$
Dimes (10 cents apiece)	$N+10$	$10(N+10)$
Totals	—	325

Because there are 10 more dimes than nickels, the number of dimes is $N + 10$, recorded in the second column. In the third column, N nickels, worth 5 cents apiece, have a value of $5N$ cents. Next, $N + 10$ dimes, worth 10 cents apiece, have a value of $10(N + 10)$ cents. The final entry in the column gives the total value of the coins as 325 cents.

2. *Set up an Equation.* The third column of the table reveals that the sum of the coin values is 325 cents. Hence, the equation that models this application is

$$5N + 10(N + 10) = 325,$$

which sums the value of the nickels and the value of the dimes to a total of 325 cents.

3. *Solve the Equation.* On the left, use the distributive property to remove parentheses.

$$5N + 10N + 100 = 325$$

Combine like terms.

$$15N + 100 = 325$$

Now, solve.

$$\begin{aligned} 15N + 100 - 100 &= 325 - 100 && \text{Subtract 100 from both sides.} \\ 15N &= 225 && \text{Simplify.} \\ \frac{15N}{15} &= \frac{225}{15} && \text{Divide both sides by 15.} \\ N &= 15 && \text{Simplify.} \end{aligned}$$

4. *Answer the Question.* We've only found the number of nickels, but the question also asks for the number of dimes. However, the number of dimes is easily found by substituting 15 for N in the expression $N + 10$.

$$\begin{aligned} N + 10 &= 15 + 10 \\ &= 25. \end{aligned}$$

Hence, Jose has 15 nickels and 25 dimes.

5. *Look Back.* Does our solution make sense? Well, the number of dimes is 25, which is certainly 10 more than 15 nickels. Also, the monetary value of 15 nickels is 75 cents and the monetary value of 25 dimes is 250 cents, a total of 325 cents, or \$3.25, so our solution is correct.

Answer: 7 dimes, 5 quarters

□

You Try It!

Emily purchase tickets to the IMAX theater for her family. An adult ticket cost \$12 and a child ticket costs \$4. She buys two more child tickets than adult tickets and the total cost is \$136. How many adult and child tickets did she buy?

EXAMPLE 5. A large children's organization purchases tickets to the circus. The organization has a strict rule that every five children must be accompanied by one adult guardian. Hence, the organization orders five times as many child tickets as it does adult tickets. Child tickets are three dollars and adult tickets are six dollars. If the total cost of tickets is \$4,200, how many child and adult tickets were purchased?

Solution. We follow the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* We're going to use a table to summarize information and declare variables. In the table that follows, we let A represent the number of adult tickets purchased. *Using a variable letter that "sounds like" the quantity it represents is an excellent strategy. Thus, in this case, letting A represent the number of adult tickets is far better than letting x represent the number of adult tickets.*

	Number of Tickets	Cost (dollars)
Adults (\$6 apiece)	A	$6A$
Children (\$3 apiece)	$5A$	$3(5A)$
Totals	—	4200

Because there are 5 times as many children's tickets purchased than adult tickets, the number of children's tickets purchased is $5A$, recorded in the second column. In the third column, $5A$ children's tickets at \$3 apiece will cost $3(5A)$ dollars, and A adult tickets at \$6 apiece will cost $6A$ dollars. The final entry in the column gives the total cost of all tickets as \$4,200.

2. *Set up an Equation.* The third column of the table reveals that the sum of the costs for both children and adult tickets is \$4,200. Hence, the equation that models this application is

$$6A + 3(5A) = 4200$$

which sums the cost of children and adult tickets at \$4,200.

3. *Solve the Equation.* On the left, use the associative property to remove parentheses.

$$6A + 15A = 4200$$

Combine like terms.

$$21A = 4200$$

Now, solve.

$$\begin{aligned} \frac{21A}{21} &= \frac{4200}{21} && \text{Divide both sides by 21.} \\ A &= 200 && \text{Simplify.} \end{aligned}$$

4. *Answer the Question.* We've only found the number of adult tickets, but the question also asks for the number of children's tickets. However, the number of children's tickets is easily found by substituting 200 for A in the expression $5A$.

$$\begin{aligned} 5A &= 5(200) \\ &= 1000. \end{aligned}$$

Hence, 1000 children's tickets and 200 adult tickets were purchased.

5. *Look Back.* Does our solution make sense? Well, the number of children's tickets purchased is 1000, which is certainly 5 times more than the 200 adult tickets purchased. Also, the monetary value of 1000 children's tickets at \$3 apiece is \$3,000, and the monetary value of 200 adult tickets at \$6 apiece is \$1,200, a total cost of \$4,200. Our solution is correct.

Answer: 8 adult and 10 child tickets

□

❧ ❧ ❧ Exercises ☙ ☙ ☙

1. The three sides of a triangle are consecutive odd integers. If the perimeter of the triangle is 39 inches, find the lengths of the sides of the triangle.

2. The three sides of a triangle are consecutive odd integers. If the perimeter of the triangle is 51 inches, find the lengths of the sides of the triangle.

3. The width and length of a rectangle are consecutive integers. If the perimeter of the rectangle is 142 inches, find the width and length of the rectangle.

4. The width and length of a rectangle are consecutive integers. If the perimeter of the rectangle is 166 inches, find the width and length of the rectangle.

5. The three sides of a triangle are consecutive even integers. If the perimeter of the triangle is 240 inches, find the lengths of the sides of the triangle.

6. The three sides of a triangle are consecutive even integers. If the perimeter of the triangle is 30 inches, find the lengths of the sides of the triangle.

7. The width and length of a rectangle are consecutive integers. If the perimeter of the rectangle is 374 inches, find the width and length of the rectangle.

8. The width and length of a rectangle are consecutive integers. If the perimeter of the rectangle is 318 inches, find the width and length of the rectangle.

9. The width and length of a rectangle are consecutive odd integers. If the perimeter of the rectangle is 208 inches, find the width and length of the rectangle.

10. The width and length of a rectangle are consecutive odd integers. If the perimeter of the rectangle is 152 inches, find the width and length of the rectangle.

11. The width and length of a rectangle are consecutive even integers. If the perimeter of the rectangle is 76 inches, find the width and length of the rectangle.

12. The width and length of a rectangle are consecutive even integers. If the perimeter of the rectangle is 300 inches, find the width and length of the rectangle.

13. The three sides of a triangle are consecutive even integers. If the perimeter of the triangle is 144 inches, find the lengths of the sides of the triangle.

14. The three sides of a triangle are consecutive even integers. If the perimeter of the triangle is 198 inches, find the lengths of the sides of the triangle.

15. The three sides of a triangle are consecutive integers. If the perimeter of the triangle is 228 inches, find the lengths of the sides of the triangle.

16. The three sides of a triangle are consecutive integers. If the perimeter of the triangle is 216 inches, find the lengths of the sides of the triangle.

17. The width and length of a rectangle are consecutive even integers. If the perimeter of the rectangle is 92 inches, find the width and length of the rectangle.

18. The width and length of a rectangle are consecutive even integers. If the perimeter of the rectangle is 228 inches, find the width and length of the rectangle.

19. The three sides of a triangle are consecutive integers. If the perimeter of the triangle is 105 inches, find the lengths of the sides of the triangle.

20. The three sides of a triangle are consecutive integers. If the perimeter of the triangle is 123 inches, find the lengths of the sides of the triangle.

21. The width and length of a rectangle are consecutive odd integers. If the perimeter of the rectangle is 288 inches, find the width and length of the rectangle.

22. The width and length of a rectangle are consecutive odd integers. If the perimeter of the rectangle is 352 inches, find the width and length of the rectangle.

23. The three sides of a triangle are consecutive odd integers. If the perimeter of the triangle is 165 inches, find the lengths of the sides of the triangle.

24. The three sides of a triangle are consecutive odd integers. If the perimeter of the triangle is 99 inches, find the lengths of the sides of the triangle.

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25. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 8 children must be accompanied by one adult guardian. Hence, the organization orders 8 times as many child tickets as it does adult tickets. Child tickets are $7 and adult tickets are $19. If the total cost of tickets is $975, how many adult tickets were purchased?

26. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 2 children must be accompanied by one adult guardian. Hence, the organization orders 2 times as many child tickets as it does adult tickets. Child tickets are $6 and adult tickets are $16. If the total cost of tickets is $532, how many adult tickets were purchased?

27. Judah cracks open a piggy bank and finds $3.30 (330 cents), all in nickels and dimes. There are 15 more dimes than nickels. How many nickels does Judah have?

28. Texas cracks open a piggy bank and finds $4.90 (490 cents), all in nickels and dimes. There are 13 more dimes than nickels. How many nickels does Texas have?

29. Steve cracks open a piggy bank and finds $4.00 (400 cents), all in nickels and dimes. There are 7 more dimes than nickels. How many nickels does Steve have?

30. Liz cracks open a piggy bank and finds $4.50 (450 cents), all in nickels and dimes. There are 15 more dimes than nickels. How many nickels does Liz have?

31. Jason inherits $20,300 and decides to invest in two different types of accounts, a savings account paying 2.5% interest, and a certificate of deposit paying 5% interest. He decides to invest $7,300 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

32. Trinity inherits $24,300 and decides to invest in two different types of accounts, a savings account paying 2% interest, and a certificate of deposit paying 5.75% interest. She decides to invest $8,500 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

33. Gina cracks open a piggy bank and finds $4.50 (450 cents), all in nickels and dimes. There are 15 more dimes than nickels. How many nickels does Gina have?

34. Dylan cracks open a piggy bank and finds \$4.05 (405 cents), all in nickels and dimes. There are 6 more dimes than nickels. How many nickels does Dylan have?

35. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 2 children must be accompanied by one adult guardian. Hence, the organization orders 2 times as many child tickets as it does adult tickets. Child tickets are \$4 and adult tickets are \$10. If the total cost of tickets is \$216, how many adult tickets were purchased?

36. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 2 children must be accompanied by one adult guardian. Hence, the organization orders 2 times as many child tickets as it does adult tickets. Child tickets are \$7 and adult tickets are \$11. If the total cost of tickets is \$375, how many adult tickets were purchased?

37. Connie cracks open a piggy bank and finds \$3.70 (370 cents), all in nickels and dimes. There are 7 more dimes than nickels. How many nickels does Connie have?

38. Don cracks open a piggy bank and finds \$3.15 (315 cents), all in nickels and dimes. There are 3 more dimes than nickels. How many nickels does Don have?

39. Mary inherits \$22,300 and decides to invest in two different types of accounts, a savings account paying 2% interest, and a certificate of deposit paying 4% interest. She decides to invest \$7,300 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

40. Amber inherits \$26,000 and decides to invest in two different types of accounts, a savings account paying 2.25% interest, and a certificate of deposit paying 4.25% interest. She decides to invest \$6,200 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

41. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 8 children must be accompanied by one adult guardian. Hence, the organization orders 8 times as many child tickets as it does adult tickets. Child tickets are \$6 and adult tickets are \$16. If the total cost of tickets is \$1024, how many adult tickets were purchased?

42. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 3 children must be accompanied by one adult guardian. Hence, the organization orders 3 times as many child tickets as it does adult tickets. Child tickets are \$3 and adult tickets are \$18. If the total cost of tickets is \$351, how many adult tickets were purchased?

43. Alan inherits \$25,600 and decides to invest in two different types of accounts, a savings account paying 3.5% interest, and a certificate of deposit paying 6% interest. He decides to invest \$6,400 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

44. Mercy inherits \$27,100 and decides to invest in two different types of accounts, a savings account paying 3% interest, and a certificate of deposit paying 4% interest. She decides to invest \$8,700 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

45. Tony inherits \$20,600 and decides to invest in two different types of accounts, a savings account paying 2% interest, and a certificate of deposit paying 4% interest. He decides to invest \$9,200 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

46. Connie inherits \$17,100 and decides to invest in two different types of accounts, a savings account paying 2% interest, and a certificate of deposit paying 5.5% interest. She decides to invest \$6,100 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

47. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 2 children must be accompanied by one adult guardian. Hence, the organization orders 2 times as many child tickets as it does adult tickets. Child tickets are \$2 and adult tickets are \$14. If the total cost of tickets is \$234, how many adult tickets were purchased?

48. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 8 children must be accompanied by one adult guardian. Hence, the organization orders 8 times as many child tickets as it does adult tickets. Child tickets are \$8 and adult tickets are \$13. If the total cost of tickets is \$1078, how many adult tickets were purchased?

Answers

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1. 11 in., 13 in., 15 in.

3. 35 in., 36 in.

5. 78 in., 80 in., 82 in.

7. 93 in., 94 in.

9. 51 in., 53 in.

11. 18 in., 20 in.

13. 46 in., 48 in., 50 in.

15. 75 in., 76 in., 77 in.

17. 22 in., 24 in.

19. 34 in., 35 in., 36 in.

21. 71 in., 73 in.

23. 53 in., 55 in., 57 in.

25. 13 adult tickets

27. 12 nickels

29. 22 nickels

31. \$6, 500

33. 20 nickels

35. 12 child tickets

37. 20 nickels

39. \$7, 500

41. 16 child tickets

43. \$9, 600

45. \$5, 700

47. 13 child tickets

Chapter 4

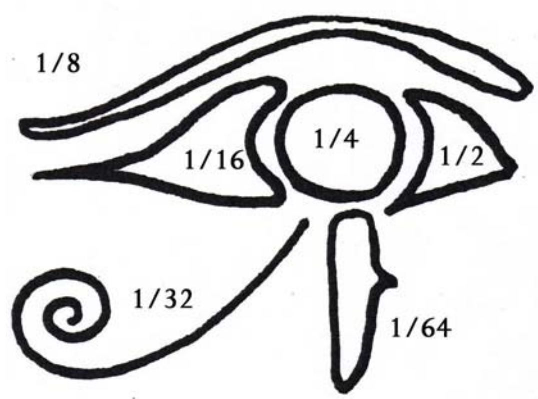
Fractions

Around 3000BC, Egyptians were carving hierglyphs into stone monuments to their kings and queens. Hierglyphs are pictures that represent objects and they were used for words and numbers.

Oddly, fractions were always written as sums of "unit fractions," fractions whose numerator is always 1. For instance, instead of writing 3/5 , they would write a sum of unit fractions.

$$\frac{3}{5} = \frac{1}{2} + \frac{1}{10}$$

Much of the ancient Egyptian math that we know of was in service to the agricultural and economic life of the people. in measuring dry goods such as grains, special glyphs were used to represent basic fractional amounts, glyphs that came together to represent the Eye of Horus.



Horus was a falcon-god whose father Osirus was murdered by his own brother Seth. When Horus attempted to avenge his father's death, Seth ripped out Horus' eye and cut it into six pieces, scattering them throughout Egypt.

Taking pity on Horus, Thot, the god of learning and magic, found the pieces and put them back together making Horus healthy and whole again.

Each piece of the Eye of Horus represents a different fraction of a hekat, or volume of grain. It was written that an apprentice scribe added the fractions one day and got

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}=\frac{63}{64}.$$

Asking where the missing 1/64 was, he was told that Thot would make up the difference to anyone "who sought and accepted is protection."

In this chapter, you'll learn how we use fractions.

4.1 Equivalent Fractions

In this section we deal with fractions, numbers or expressions of the form a/b.

Fractions. A number of the form

$$\frac{a}{b}$$

where a and b are numbers is called a *fraction*. The number a is called the *numerator* of the fraction, while the number b is called the *denominator* of the fraction.

Near the end of this section, we'll see that the numerator and denominator of a fraction can also be algebraic expressions, but for the moment we restrict our attention to fractions whose numerators and denominators are integers.

We start our study of fractions with the definition of *equivalent fractions*.

Equivalent Fractions. Two fractions are *equivalent* if they represent the same numerical value.

But how can we tell if two fractions represent the same number? Well, one technique involves some simple visualizations. Consider the image shown in Figure 4.1, where the shaded region represents $1/3$ of the total area of the figure (one of three equal regions is shaded).

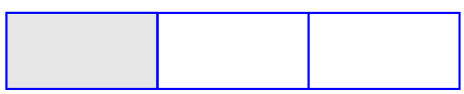


Figure 4.1: The shaded region is $1/3$ of the whole region.

In Figure 4.2, we've shaded $2/6$ of the entire region (two of six equal regions are shaded).

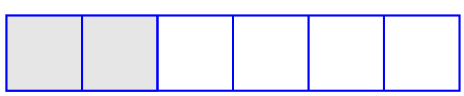


Figure 4.2: The shaded region is $2/6$ of the whole region.

In Figure 4.3, we've shaded 4/12 of the entire region (four of twelve equal regions are shaded).

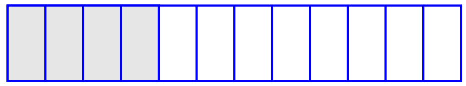


Figure 4.3: The shaded region is 4/12 of the whole region.

Let's take the diagrams from Figure 4.1, Figure 4.2, and Figure 4.3 and stack them one atop the other, as shown in Figure 4.4.

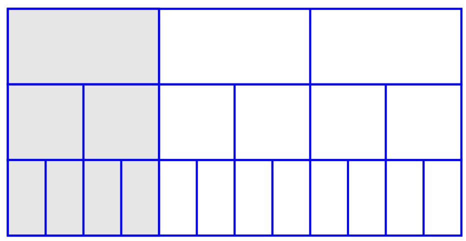


Figure 4.4: One of three equals two of six equals four of twelve.

Figure 4.4 provides solid visual evidence that the following fractions are equivalent.

$$\frac{1}{3} = \frac{2}{6} = \frac{4}{12}$$

Key Observations

1. If we start with the fraction 1/3, then multiply both numerator and denominator by 2, we get the following result.

$$\begin{aligned} \frac{1}{3} &= \frac{1 \cdot 2}{3 \cdot 2} && \text{Multiply numerator and denominator by 2.} \\ &= \frac{2}{6} && \text{Simplify numerator and denominator..} \end{aligned}$$

   This is precisely the same thing that happens going from Figure 4.1 to 4.2, where we double the number of available boxes (going from 3 available to 6 available) and double the number of shaded boxes (going from 1 shaded to 2 shaded).

2. If we start with the fraction 1/3, then multiply both numerator and denominator by 4, we get the following result.

$$\begin{aligned} \frac{1}{3} &= \frac{1 \cdot 4}{3 \cdot 4} && \text{Multiply numerator and denominator by 4.} \\ &= \frac{4}{12} && \text{Simplify numerator and denominator.} \end{aligned}$$

This is precisely the same thing that happens going from Figure 4.1 to 4.3, where we multiply the number of available boxes by 4 (going from 3 available to 12 available) and multiply the number of shaded boxes by 4 (going from 1 shaded to 4 shaded).

The above discussion motivates the following fundamental result.

Creating Equivalent Fractions. If you start with a fraction, then multiply both its numerator and denominator by the same number, the resulting fraction is equivalent (has the same numerical value) to the original fraction. In symbols,

$$\frac{a}{b} = \frac{a \cdot x}{b \cdot x}.$$

Arguing in Reverse. Reversing the above argument also holds true.

1. If we start with the fraction 2/6, then divide both numerator and denominator by 2, we get the following result.

$$\begin{aligned} \frac{2}{6} &= \frac{2 \div 2}{6 \div 2} && \text{Divide numerator and denominator by 2.} \\ &= \frac{1}{3} && \text{Simplify numerator and denominator.} \end{aligned}$$

   This is precisely the same thing that happens going backwards from Figure 4.2 to 4.1, where we divide the number of available boxes by 2 (going from 6 available to 3 available) and dividing the number of shaded boxes by 2 (going from 2 shaded to 1 shaded).

2. If we start with the fraction 4/12, then divide both numerator and denominator by 4, we get the following result.

$$\begin{aligned} \frac{4}{12} &= \frac{4 \div 4}{12 \div 4} && \text{Multiply numerator and denominator by 4.} \\ &= \frac{1}{3} && \text{Simplify numerator and denominator.} \end{aligned}$$

   This is precisely the same thing that happens going backwards from Figure 4.3 to 4.1, where we divide the number of available boxes by 4 (going from 12 available to 3 available) and divide the number alignof shaded boxes by 4 (going from 4 shaded to 1 shaded).

The above discussion motivates the following fundamental result.

Creating Equivalent Fractions. If you start with a fraction, then divide both its numerator and denominator by the same number, the resulting fraction is equivalent (has the same numerical value) to the original fraction. In symbols,

$$\frac{a}{b} = \frac{a \div x}{b \div x}.$$

The Greatest Common Divisor

We need a little more terminology.

Divisor. If d and a are natural numbers, we say that "d divides a" if and only if when a is divided by d, the remainder is zero. In this case, we say that "d is a divisor of a."

For example, when 36 is divided by 4, the remainder is zero. In this case, we say that "4 is a divisor of 36." On the other hand, when 25 is divided by 4, the remainder is **not** zero. In this case, we say that "4 is **not** a divisor of 25."

Greatest Common Divisor. Let a and b be natural numbers. The *common divisors* of a and b are those natural numbers that divide both a and b. The *greatest common divisor* is the largest of these common divisors.

You Try It!

Find the greatest common divisor of 12 and 18

EXAMPLE 1. Find the greatest common divisor of 18 and 24.

Solution. First list the divisors of each number, the numbers that divide each number with zero remainder.

$$\text{Divisors of 18}: \ 1,\ 2,\ 3,\ 6,\ 9,\ \text{and } 18$$
$$\text{Divisors of 24}: \ 1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ \text{and } 24$$

The common divisors are:

$$\text{Common Divisors}: \ 1,\ 2,\ 3,\ \text{and } 6$$

The greatest common divisor is the largest of the common divisors. That is,

$$\text{Greatest Common Divisor} = 6.$$

That is, the largest number that divides both 18 and 24 is the number 6. □

Answer: 6

Reducing a Fraction to Lowest Terms

First, a definition.

Lowest Terms. A fraction is said to be *reduced to lowest terms* if the greatest common divisor of both numerator and denominator is 1.

Thus, for example, 2/3 is reduced to lowest terms because the greatest common divisior of 2 and 3 is 1. On the other hand, 4/6 is **not** reduced to lowest terms because the greatest common divisor of 4 and 6 is 2.

You Try It!

Reduce the fraction 12/18 to lowest terms.

EXAMPLE 2. Reduce the fraction 18/24 to lowest terms.

Solution. One technique that works well is dividing both numerator and denominator by the greatest common divisor of the numerator and denominator. In Example 1, we saw that the greatest common divisor of 18 and 24 is 6. We divide both numerator and denominator by 6 to get

$$\begin{aligned} \frac{18}{24} &= \frac{18 \div 6}{24 \div 6} && \text{Divide numerator and denominator by 6.} \\ &= \frac{3}{4} && \text{Simplify numerator and denominator.} \end{aligned}$$

Note that the greatest common divisor of 3 and 4 is now 1. Thus, 3/4 is reduced to lowest terms.

There is a second way we can show division of numerator and denominator by 6. First, factor both numerator and denominator as follows:

$$\frac{18}{24} = \frac{3 \cdot 6}{4 \cdot 6} \qquad \text{Factor out a 6.}$$

You can then show "division" of both numerator and denominator by 6 by "crossing out" or "canceling" a 6 in the numerator for a 6 in the denominator, like this:

$$\begin{aligned} &= \frac{3 \cdot \not{6}}{4 \cdot \not{6}} && \text{Cancel common factor.} \\ &= \frac{3}{4} \end{aligned}$$

Note that we get the same equivalent fraction, reduced to lowest terms, namely 3/4.

Answer: 2/3

□

Important Point. In Example 2 we saw that 6 was both a *divisor* and a *factor* of 18. The words *divisor* and *factor* are equivalent.

We used the following technique in our second solution in Example 2.

Cancellation Rule. If you express numerator and denominator as a **product**, then you may cancel common factors from the numerator and denominator. The result will be an equivalent fraction.

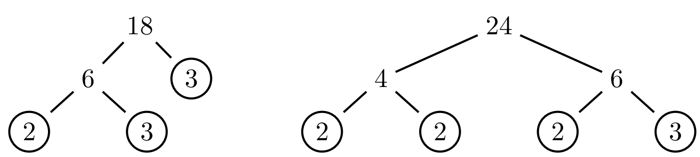
Because of the "Cancellation Rule," one of the most effective ways to reduce a fraction to lowest terms is to first find prime factorizations for both numerator and denominator, then cancel all common factors.

You Try It!

Reduce the fraction 28/35 to lowest terms.

EXAMPLE 3. Reduce the fraction 18/24 to lowest terms.

Solution. Use factor trees to prime factor numerator and denominator.



Once we've factored the numerator and denominator, we cancel common factors.

$$\begin{aligned}
\frac{18}{24} &= \frac{2\cdot 3\cdot 3}{2\cdot 2\cdot 2\cdot 3} && \text{Prime factor numerator and denominator.}\\
&= \frac{\cancel{2}\cdot \cancel{3}\cdot 3}{\cancel{2}\cdot 2\cdot 2\cdot \cancel{3}} && \text{Cancel common factors.}\\
&= \frac{3}{2\cdot 2} && \text{Remaining factors.}\\
&= \frac{3}{4} && \text{Simplify denominator.}
\end{aligned}$$

Thus, $18/24 = 3/4$.

Answer: 4/5

□

You Try It!

Reduce the fraction 36/60 to lowest terms.

EXAMPLE 4. Reduce the fraction 28/42 to lowest terms.

Solution. Use factor trees to prime factor numerator and denominator.

Now we can cancel common factors.

$$\begin{aligned}
\frac{28}{42} &= \frac{2 \cdot 2 \cdot 7}{2 \cdot 3 \cdot 7} && \text{Prime factor numerator and denominator.} \\
&= \frac{\cancel{2} \cdot 2 \cdot \cancel{7}}{\cancel{2} \cdot 3 \cdot \cancel{7}} && \text{Cancel common factors.} \\
&= \frac{2}{3}
\end{aligned}$$

Thus, $28/42 = 2/3$.

Answer: $3/5$

Reducing Fractions with Variables

We use exactly the same technique to reduce fractions whose numerators and denominators contain variables.

You Try It!

EXAMPLE 5. Reduce

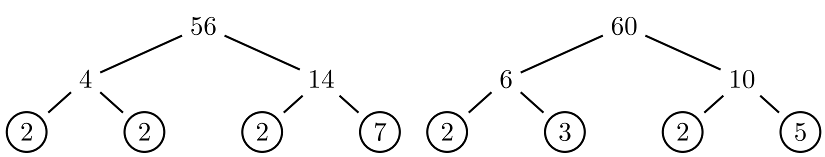
$$\frac{56x^2y}{60xy^2}$$

to lowest terms.

Reduce:

$$\frac{25a^3b}{40a^2b^3}$$

Solution. Use factor trees to factor the coefficients of numerator and denominator.



Now cancel common factors.

$$\begin{aligned}
\frac{56x^2y}{60xy^2} &= \frac{2 \cdot 2 \cdot 2 \cdot 7 \cdot x \cdot x \cdot y}{2 \cdot 2 \cdot 3 \cdot 5 \cdot x \cdot y \cdot y} && \text{Prime factor numerator and denominator.} \\
&= \frac{\cancel{2} \cdot \cancel{2} \cdot 2 \cdot 7 \cdot \cancel{x} \cdot x \cdot \cancel{y}}{\cancel{2} \cdot \cancel{2} \cdot 3 \cdot 5 \cdot \cancel{x} \cdot y \cdot \cancel{y}} && \text{Cancel common factors.} \\
&= \frac{2 \cdot 7 \cdot x}{3 \cdot 5 \cdot y} && \text{Remaining factors.} \\
&= \frac{14x}{15y} && \text{Simplify numerator and denominator.}
\end{aligned}$$

Thus, $56x^2y/(60xy^2) = 14x/(15y)$.

Answer: $\dfrac{5a}{8b^2}$

A Word on Mathematical Notation.

There are two types of mathematical notation: (1) inline mathematical notation, and (2) displayed mathematical notation.

Inline Mathematical Notation. The notation $14x/(15y)$ is called *inline mathematical notation*. When the same expression is centered on its own line, as in

$$\frac{14x}{15y},$$

this type of notation is called *displayed mathematical notation.*

When you work a problem by hand, using pencil and paper calculations, the preferred format is displayed notation, like the displayed notation used to simplify the given expression in Example 5. However, computers and calculators require that you enter your expressions using inline mathematical notation. Therefore, it is extremely important that you are equally competent with either mathematical notation: displayed or inline.

By the way, order of operations, when applied to the inline expression $14x/(15y)$, requires that we perform the multiplication inside the parentheses first. Then we must perform multiplications and divisions as they occur, as we move from left to right through the expression. This is why the inline notation $14x/(15y)$ is equivalent to the displayed notation

$$\frac{14x}{15y}.$$

However, the expression $14x/15y$ is a different beast. There are no parentheses, so we perform multiplication and division as they occur, moving left to right through the expression. Thus, we must first take the product of 14 and x, divide the result by 15, then multiply by y. In displayed notation, this result is equivalent to

$$\frac{14x}{15} \cdot y,$$

which is a different result.

Some readers might wonder why we did not use the notation $(14x)/(15y)$ to describe the solution in Example 5. After all, this inline notation is also equivalent to the displayed notation

$$\frac{14x}{15y}.$$

However, the point is that we don't need to, as order of operations already requires that we take the product of 14 and x before dividing by $15y$. If this is hurting your head, know that it's quite acceptable to use the equivalent notation $(14x)/(15y)$ instead of $14x/(15y)$. Both are correct.

Equivalent Fractions in Higher Terms

Sometimes the need arises to find an equivalent fraction with a different, larger denominator.

You Try It!

EXAMPLE 6. Express 3/5 as an equivalent fraction having denominator 20.

Solution. The key here is to remember that multiplying numerator and denominator by the same number produces an equivalent fraction. To get an equivalent fraction with a denominator of 20, we'll have to multiply numerator and denominator of 3/5 by 4.

$$\begin{aligned} \frac{3}{5} &= \frac{3 \cdot 4}{5 \cdot 4} && \text{Multiply numerator and denominator by 4.} \\ &= \frac{12}{20} && \text{Simplify numerator and denominator.} \end{aligned}$$

Therefore, 3/5 equals 12/20.

Express 2/3 as an equivalent fraction having denominator 21.

Answer: 14/21

□

You Try It!

EXAMPLE 7. Express 8 as an equivalent fraction having denominator 5.

Solution. The key here is to note that

$$8 = \frac{8}{1} \qquad \text{Understood denominator is 1.}$$

To get an equivalent fraction with a denominator of 5, we'll have to multiply numerator and denominator of 8/1 by 5.

$$\begin{aligned} &= \frac{8 \cdot 5}{1 \cdot 5} && \text{Multiply numerator and denominator by 5.} \\ &= \frac{40}{5} && \text{Simplify numerator and denominator.} \end{aligned}$$

Therefore, 8 equals 40/5.

Express 5 as an equivalent fraction having denominator 7.

Answer: 35/7

□

You Try It!

Express 3/8 as an equivalent fraction having denominator $24a$.

EXAMPLE 8. Express 2/9 as an equivalent fraction having denominator $18a$.

Solution. To get an equivalent fraction with a denominator of $18a$, we'll have to multiply numerator and denominator of 2/9 by $2a$.

$$\begin{aligned} \frac{2}{9} &= \frac{2 \cdot 2a}{9 \cdot 2a} && \text{Multiply numerator and denominator by } 2a. \\ &= \frac{4a}{18a} && \text{Simplify numerator and denominator.} \end{aligned}$$

Therefore, 2/9 equals $4a/(18a)$, or equivalently, $(4a)/(18a)$.

Answer: $\dfrac{9a}{24a}$

□

Negative Fractions

We have to also deal with fractions that are negative. First, let's discuss placement of the negative sign.

- Positive divided by negative is negative, so
$$\frac{3}{-5} = -\frac{3}{5}.$$
- But it is also true that negative divided by positive is negative. Thus,
$$\frac{-3}{5} = -\frac{3}{5}.$$

These two observations imply that all three of the following fractions are equivalent (the same number):

$$\frac{3}{-5} = -\frac{3}{5} = \frac{-3}{5}.$$

Note that there are three possible placements for the negative sign: (1) the denominator, (2) the fraction bar, or (3) the numerator. Any one of these placements produces an equivalent fraction.

Fractions and Negative Signs. Let a and b be any integers. All three of the following fractions are equivalent (same number):

$$\frac{a}{-b} = -\frac{a}{b} = \frac{-a}{b}.$$

Mathematicians prefer to place the negative sign either in the numerator or on the fraction bar. The use of a negative sign in the denominator is discouraged.

You Try It!

EXAMPLE 9. Reduce:

$$\frac{50x^3}{-75x^5}$$

to lowest terms.

Reduce:

$$\frac{14y^5}{-35y^3}$$

Solution. Prime factor numerator and denominator and cancel.

$$\begin{aligned}\frac{50x^3}{-75x^5} &= \frac{2 \cdot 5 \cdot 5 \cdot x \cdot x \cdot x}{-3 \cdot 5 \cdot 5 \cdot x \cdot x \cdot x \cdot x \cdot x} \\ &= \frac{2 \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{-3 \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x} \\ &= \frac{2}{-3 \cdot x \cdot x} \\ &= \frac{2}{-3x^2}\end{aligned}$$

However, it is preferred that there be no negative signs in the denominator, so let's place the negative sign on the fraction bar (the numerator would suit as well). Thus,

$$\frac{50x^3}{-75x^5} = -\frac{2}{3x^2}$$

Answer: $-\dfrac{2y^2}{5}$

□

We also have the following result.

Fractions and Negative Signs. Let a and b be any integers. Then,

$$\frac{-a}{-b} = \frac{a}{b}.$$

You Try It!

EXAMPLE 10. Reduce:

$$\frac{-12xy^2}{-18x^2y}$$

Reduce:

$$\frac{-21a^2b^3}{-56a^3b}$$

Solution. Unlike Example 9, some like to take care of the sign of the answer first.

$$\frac{-12xy^2}{-18x^2y} = \frac{12xy^2}{18x^2y}$$

Now we can factor numerator and denominator and cancel common factors.

$$\begin{aligned} &= \frac{2 \cdot 2 \cdot 3 \cdot x \cdot y \cdot y}{2 \cdot 3 \cdot 3 \cdot x \cdot x \cdot y} \\ &= \frac{\cancel{2} \cdot 2 \cdot \cancel{3} \cdot \cancel{x} \cdot y \cdot \cancel{y}}{\cancel{2} \cdot \cancel{3} \cdot 3 \cdot \cancel{x} \cdot x \cdot \cancel{y}} \\ &= \frac{2y}{3x} \end{aligned}$$

Thus,

$$\frac{-12xy^2}{-18x^2y} = \frac{2y}{3x}.$$

Answer: $\dfrac{3b^2}{8a}$

□

Exercises

In Exercises 1-12, find the GCD of the given numbers.

1. 72, 8

2. 76, 52

3. 52, 20

4. 56, 96

5. 36, 63

6. 63, 21

7. 72, 44

8. 10, 40

9. 16, 56

10. 54, 66

11. 84, 24

12. 75, 45

In Exercises 13-28, reduce the given fraction to lowest terms.

13. $\frac{22}{98}$

14. $\frac{28}{56}$

15. $\frac{93}{15}$

16. $\frac{90}{39}$

17. $\frac{69}{21}$

18. $\frac{74}{62}$

19. $\frac{74}{12}$

20. $\frac{66}{10}$

21. $\frac{66}{57}$

22. $\frac{34}{30}$

23. $\frac{33}{99}$

24. $\frac{20}{58}$

25. $\frac{69}{24}$

26. $\frac{18}{96}$

27. $\frac{46}{44}$

28. $\frac{92}{24}$

---

29. Express 3 as an equivalent fraction having denominator 24.

30. Express 3 as an equivalent fraction having denominator 8.

31. Express $\frac{25}{19}$ as an equivalent fraction having denominator 57.

32. Express $\frac{29}{22}$ as an equivalent fraction having denominator 44.

33. Express 2 as an equivalent fraction having denominator 2.

34. Express 2 as an equivalent fraction having denominator 8.

35. Express $\frac{18}{19}$ as an equivalent fraction having denominator 95.

36. Express $\frac{17}{22}$ as an equivalent fraction having denominator 44.

37. Express $\frac{1}{3}$ as an equivalent fraction having denominator 24.

38. Express $\frac{15}{19}$ as an equivalent fraction having denominator 95.

39. Express 16 as an equivalent fraction having denominator 4.

40. Express 5 as an equivalent fraction having denominator 2.

In Exercises 41-56, reduce the given fraction to lowest terms.

41. $\frac{34}{-86}$

42. $\frac{-48}{14}$

43. $\frac{-72}{-92}$

44. $\frac{27}{-75}$

45. $\frac{-92}{82}$

46. $\frac{-44}{-62}$

47. $\frac{-21}{33}$

48. $\frac{57}{-99}$

49. $\frac{22}{-98}$

50. $\frac{-33}{69}$

51. $\frac{42}{-88}$

52. $\frac{-100}{48}$

53. $\frac{94}{-6}$

54. $\frac{-36}{-38}$

55. $\frac{10}{-86}$

56. $\frac{-100}{-46}$

---

57. Express $\frac{3}{2}$ as an equivalent fraction having denominator $62n$.

58. Express $\frac{6}{25}$ as an equivalent fraction having denominator $50a$.

59. Express $\frac{13}{10}$ as an equivalent fraction having denominator $60m$.

60. Express $\frac{1}{16}$ as an equivalent fraction having denominator $80p$.

61. Express $\frac{3}{2}$ as an equivalent fraction having denominator $50n$.

62. Express $\frac{43}{38}$ as an equivalent fraction having denominator $76a$.

63. Express 11 as an equivalent fraction having denominator $4m$.

64. Express 13 as an equivalent fraction having denominator $6n$.

65. Express 3 as an equivalent fraction having denominator $10m$.

66. Express 10 as an equivalent fraction having denominator $8b$.

67. Express 6 as an equivalent fraction having denominator $5n$.

68. Express 16 as an equivalent fraction having denominator $2y$.

In Exercises 69-84, reduce the given fraction to lowest terms.

69. $\dfrac{82y^5}{-48y}$

70. $\dfrac{-40y^5}{-55y}$

71. $\dfrac{-77x^5}{44x^4}$

72. $\dfrac{-34x^6}{-80x}$

73. $\dfrac{-14y^5}{54y^2}$

74. $\dfrac{96y^4}{-40y^2}$

75. $\dfrac{42x}{81x^3}$

76. $\dfrac{26x^2}{32x^6}$

77. $\dfrac{-12x^5}{14x^6}$

78. $\dfrac{-28y^4}{72y^6}$

79. $\dfrac{-74x}{22x^2}$

80. $\dfrac{56x^2}{26x^3}$

81. $\dfrac{-12y^5}{98y^6}$

82. $\dfrac{96x^2}{14x^4}$

83. $\dfrac{18x^6}{-54x^2}$

84. $\dfrac{32x^6}{62x^2}$

In Exercises 85-100, reduce the given fraction to lowest terms.

85. $\dfrac{26y^2x^4}{-62y^6x^2}$

86. $\dfrac{6x^2y^3}{40x^3y^2}$

87. $\dfrac{-2y^6x^4}{-94y^2x^5}$

88. $\dfrac{90y^6x^3}{39y^3x^5}$

89. $\dfrac{30y^5x^5}{-26yx^4}$

90. $\dfrac{74x^6y^4}{-52xy^3}$

91. $\dfrac{36x^3y^2}{-98x^4y^5}$

92. $\dfrac{84x^3y}{16x^4y^2}$

93. $\dfrac{-8x^6y^3}{54x^3y^5}$

94. $\dfrac{70y^5x^2}{16y^4x^5}$

95. $\dfrac{34yx^6}{-58y^5x^4}$

96. $\dfrac{99y^2x^3}{88y^6x}$

97. $\dfrac{-36y^3x^5}{51y^2x}$

98. $\dfrac{44y^5x^5}{-88y^4x}$

99. $\dfrac{91y^3x^2}{-28y^5x^5}$

100. $\dfrac{-76y^2x}{-57y^5x^6}$

---

101. **Hurricanes.** According to the National Atmospheric and Oceanic Administration, in 2008 there were 16 named storms, of which 8 grew into hurricanes and 5 were major.

i) What fraction of named storms grew into hurricanes? Reduce your answer to lowest terms.

ii) What fraction of named storms were major hurricanes? Reduce your answer to lowest terms.

iii) What fraction of hurricanes were major? Reduce your answer to lowest terms.

102. **Tigers.** Tigers are in critical decline because of human encroachment, the loss of more than nine-tenths of their habitat, and the growing trade in tiger skins and body parts. *Associated Press-Times-Standard 01/24/10 Pressure mounts to save the tiger.*

i) Write the loss of habitat as a fraction.

ii) Describe in words what the numerator and denominator of this fraction represent.

iii) If the fraction represents the loss of the whole original habitat, how much of the original habitat remains?

Answers

1. 8

3. 4

5. 9

7. 4

9. 8

11. 12

13. $\dfrac{11}{49}$

15. $\dfrac{31}{5}$

17. $\dfrac{23}{7}$

19. $\dfrac{37}{6}$

21. $\dfrac{22}{19}$

23. $\frac{1}{3}$

25. $\frac{23}{8}$

27. $\frac{23}{22}$

29. $\frac{72}{24}$

31. $\frac{75}{57}$

33. $\frac{4}{2}$

35. $\frac{90}{95}$

37. $\frac{8}{24}$

39. $\frac{64}{4}$

41. $-\frac{17}{43}$

43. $\frac{18}{23}$

45. $-\frac{46}{41}$

47. $-\frac{7}{11}$

49. $-\frac{11}{49}$

51. $-\frac{21}{44}$

53. $-\frac{47}{3}$

55. $-\frac{5}{43}$

57. $\frac{93n}{62n}$

59. $\frac{78m}{60m}$

61. $\frac{75n}{50n}$

63. $\frac{44m}{4m}$

65. $\frac{30m}{10m}$

67. $\frac{30n}{5n}$

69. $-\frac{41y^4}{24}$

71. $-\frac{7x}{4}$

73. $-\frac{7y^3}{27}$

75. $\frac{14}{27x^2}$

77. $-\frac{6}{7x}$

79. $-\frac{37}{11x}$

81. $-\frac{6}{49y}$

83. $-\frac{x^4}{3}$

85. $-\frac{13x^2}{31y^4}$

87. $\frac{y^4}{47x}$

89. $-\frac{15y^4x}{13}$

91. $-\dfrac{18}{49xy^3}$

93. $-\dfrac{4x^3}{27y^2}$

95. $-\dfrac{17x^2}{29y^4}$

97. $-\dfrac{12yx^4}{17}$

99. $-\dfrac{13}{4y^2x^3}$

101. i) $\dfrac{1}{2}$

ii) $\dfrac{5}{16}$

iii) $\dfrac{5}{8}$

4.2 Multiplying Fractions

Consider the image in Figure 4.5, where the vertical lines divide the rectangular region into three equal pieces. If we shade one of the three equal pieces, the shaded area represents 1/3 of the whole rectangular region.

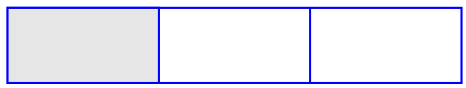


Figure 4.5: The shaded region is 1/3 of the whole region.

We'd like to visualize taking 1/2 of 1/3. To do that, we draw an additional horizontal line which divides the shaded region in half horizontally. This is shown in Figure 4.6. The shaded region that represented 1/3 is now divided into two smaller rectangular regions, one of which is shaded with a different color. This region represents 1/2 of 1/3.

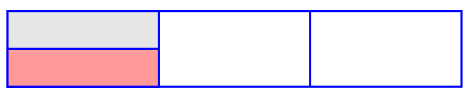


Figure 4.6: Shading 1/2 of 1/3.

Next, extend the horizontal line the full width of the rectangular region, as shown in Figure 4.7.

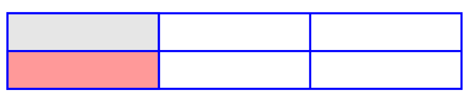


Figure 4.7: Shading 1/2 of 1/3.

Note that drawing the horizontal line, coupled with the three original vertical lines, has succeeded in dividing the full rectangular region into six smaller but equal pieces, only one of which (the one representing 1/2 of 1/3) is shaded in a new color. Hence, this newly shaded piece represents 1/6 of the whole region. The conclusion of our visual argument is the fact that 1/2 of 1/3 equals 1/6. In symbols,

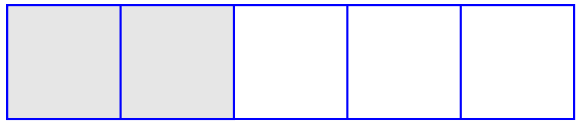
$$\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}.$$

You Try It!

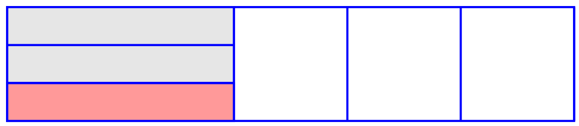
EXAMPLE 1. Create a visual argument showing that 1/3 of 2/5 is 2/15.

Create a visual argument showing that 1/2 of 1/4 is 1/8.

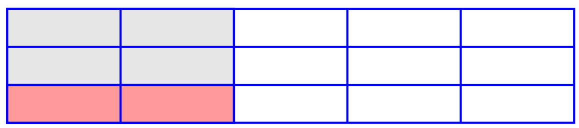
Solution. First, divide a rectangular region into five equal pieces and shade two of them. This represents the fraction 2/5.



Next, draw two horizontal lines that divide the shaded region into three equal pieces and shade 1 of the three equal pieces. This represents taking 1/3 of 2/5.



Next, extend the horizontal lines the full width of the region and return the original vertical line from the first image.



Note that the three horizontal lines, coupled with the five original vertical lines, have succeeded in dividing the whole region into 15 smaller but equal pieces, only two of which (the ones representing 1/3 of 2/5) are shaded in the new color. Hence, this newly shaded piece represents 2/15 of the whole region. The conclusion of this visual argument is the fact that 1/3 of 2/5 equals 2/15. In symbols,

$$\frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15}.$$

Answer: 

□

Multiplication Rule

In Figure 4.7, we saw that 1/2 of 1/3 equals 1/6. Note what happens when we multiply the numerators and multiply the denominators of the fractions 1/2 and 1/3.

$$\begin{aligned} \frac{1}{2} \cdot \frac{1}{3} &= \frac{1 \cdot 1}{2 \cdot 3} && \text{Multiply numerators; multiply denominators.} \\ &= \frac{1}{6} && \text{Simplify numerators and denominators.} \end{aligned}$$

We get 1/6!

Could this be coincidence or luck? Let's try that again with the fractions from Example 1, where we saw that 1/3 of 2/5 equals 2/15. Again, multiply the numerators and denominators of 1/3 and 2/5.

$$\begin{aligned}\frac{1}{3}\cdot\frac{2}{5} &= \frac{1\cdot 2}{3\cdot 5} && \text{Multiply numerators; multiply denominators.}\\ &= \frac{2}{15} && \text{Simplify numerators and denominators.}\end{aligned}$$

Again, we get 2/15!

These two examples motivate the following definition.

Multiplication Rule. To find the product of the fractions a/b and c/d, multiply their numerators and denominators. In symbols,

$$\frac{a}{b}\cdot\frac{c}{d} = \frac{a\cdot c}{b\cdot d}$$

You Try It!

Multiply:

$$\frac{1}{3}\cdot\frac{2}{5}$$

EXAMPLE 2. Multiply 1/5 and 7/9.

Solution. Multiply numerators and multiply denominators.

$$\begin{aligned}\frac{1}{5}\cdot\frac{7}{9} &= \frac{1\cdot 7}{5\cdot 9} && \text{Multiply numerators; multiply denominators.}\\ &= \frac{7}{45} && \text{Simplify numerators and denominators.}\end{aligned}$$

Multiply: $\frac{6}{15}$

□

You Try It!

Multiply:

$$-\frac{3}{5}\cdot\frac{2}{7}$$

EXAMPLE 3. Find the product of $-2/3$ and $7/9$.

Solution. The usual rules of signs apply to products. Unlike signs yield a negative result.

$$\begin{aligned}-\frac{2}{3}\cdot\frac{7}{9} &= -\frac{2\cdot 7}{3\cdot 9} && \text{Multiply numerators; multiply denominators.}\\ &= -\frac{14}{27} && \text{Simplify numerators and denominators.}\end{aligned}$$

It is not required that you physically show the middle step. If you want to do that mentally, then you can simply write

$$-\frac{2}{3}\cdot\frac{7}{9} = -\frac{14}{27}.$$

Answer: $-\frac{6}{35}$

□

Multiply and Reduce

After multiplying two fractions, make sure your answer is reduced to lowest terms (see Section 4.1).

You Try It!

Multiply:

$$\frac{3}{7} \cdot \frac{14}{9}$$

EXAMPLE 4. Multiply 3/4 times 8/9.

Solution. After multiplying, divide numerator and denominator by the greatest common divisor of the numerator and denominator.

$$\begin{aligned} \frac{3}{4} \cdot \frac{8}{9} &= \frac{3 \cdot 8}{4 \cdot 9} && \text{Multiply numerators and denominators.} \\ &= \frac{24}{36} && \text{Simplify numerator and denominator.} \\ &= \frac{24 \div 12}{36 \div 12} && \text{Divide numerator and denominator by GCD.} \\ &= \frac{2}{3} && \text{Simplify numerator and denominator.} \end{aligned}$$

Alternatively, after multiplying, you can prime factor both numerator and denominator, then cancel common factors.

$$\begin{aligned} \frac{3}{4} \cdot \frac{8}{9} &= \frac{24}{36} && \text{Multiply numerators and denominators.} \\ &= \frac{2 \cdot 2 \cdot 2 \cdot 3}{2 \cdot 2 \cdot 3 \cdot 3} && \text{Prime factor numerator and denominator.} \\ &= \frac{\not{2} \cdot \not{2} \cdot 2 \cdot \not{3}}{\not{2} \cdot \not{2} \cdot 3 \cdot \not{3}} && \text{Cancel common factors.} \\ &= \frac{2}{3} \end{aligned}$$

Answer: $\dfrac{2}{3}$

□

You Try It!

Multiply:

$$-\frac{3x}{2} \cdot \frac{6}{21x^3}$$

EXAMPLE 5. Multiply $-7x/2$ and $5/(14x^2)$.

Solution. After multiplying, prime factor both numerator and denominator, then cancel common factors. Note that unlike signs yields a negative product.

$$\begin{aligned} -\frac{7x}{2} \cdot \frac{5}{14x^2} &= -\frac{35x}{28x^2} && \text{Multiply numerators and denominators.} \\ &= -\frac{5 \cdot 7 \cdot x}{2 \cdot 2 \cdot 7 \cdot x \cdot x} && \text{Prime factor numerator and denominator.} \\ &= -\frac{5 \cdot \not{7} \cdot \not{x}}{2 \cdot 2 \cdot \not{7} \cdot \not{x} \cdot x} && \text{Cancel common factors.} \\ &= -\frac{5}{4x} \end{aligned}$$

Answer: $-\dfrac{3}{7x^2}$

□

Multiply and Cancel or Cancel and Multiply

When you are working with larger numbers, it becomes a bit harder to multiply, factor, and cancel. Consider the following argument.

$$\begin{aligned}
\frac{18}{30} \cdot \frac{35}{6} &= \frac{630}{180} && \text{Multiply numerators; multiply denominators.} \\
&= \frac{2 \cdot 3 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 3 \cdot 3 \cdot 5} && \text{Prime factor numerators and denominators.} \\
&= \frac{\cancel{2} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{5} \cdot 7}{2 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{5}} && \text{Cancel common factors.} \\
&= \frac{7}{2} && \text{Remaining factors.}
\end{aligned}$$

There are a number of difficulties with this approach. First, you have to multiply large numbers, and secondly, you have to prime factor the even larger results.

One possible workaround is to not bother multiplying numerators and denominators, leaving them in factored form.

$$\frac{18}{30} \cdot \frac{35}{6} = \frac{18 \cdot 35}{30 \cdot 6} \qquad \text{Multiply numerators; multiply denominators.}$$

Finding the prime factorization of these smaller factors is easier.

$$= \frac{(2 \cdot 3 \cdot 3) \cdot (5 \cdot 7)}{(2 \cdot 3 \cdot 5) \cdot (2 \cdot 3)} \qquad \text{Prime factor.}$$

Now we can cancel common factors. Parentheses are no longer needed in the numerator and denominator because both contain a product of prime factors, so order and grouping do not matter.

$$\begin{aligned}
&= \frac{\cancel{2} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{5} \cdot 7}{\cancel{2} \cdot \cancel{3} \cdot \cancel{5} \cdot 2 \cdot \cancel{3}} && \text{Cancel common factors.} \\
&= \frac{7}{2} && \text{Remaining factors.}
\end{aligned}$$

Another approach is to factor numerators and denominators in place, cancel common factors, then multiply.

$$\begin{aligned}
\frac{18}{30} \cdot \frac{35}{6} &= \frac{2 \cdot 3 \cdot 3}{2 \cdot 3 \cdot 5} \cdot \frac{5 \cdot 7}{2 \cdot 3} && \text{Factor numerators and denominators.} \\
&= \frac{\cancel{2} \cdot \cancel{3} \cdot \cancel{3}}{\cancel{2} \cdot \cancel{3} \cdot \cancel{5}} \cdot \frac{\cancel{5} \cdot 7}{2 \cdot \cancel{3}} && \text{Cancel common factors.} \\
&= \frac{7}{2} && \text{Remaining factors.}
\end{aligned}$$

Note that this yields exactly the same result, 7/2.

Cancellation Rule. When multiplying fractions, cancel common factors according to the following rule: "Cancel a factor in a numerator for an identical factor in a denominator."

You Try It!

Multiply:

$$\frac{6}{35} \cdot \frac{70}{36}$$

Answer: $\dfrac{1}{3}$

EXAMPLE 6. Find the product of 14/15 and 30/140.

Solution. Multiply numerators and multiply denominators. Prime factor, cancel common factors, then multiply.

$$\begin{aligned}
\frac{14}{15} \cdot \frac{30}{140} &= \frac{14 \cdot 30}{15 \cdot 140} && \text{Multiply numerators; multiply denominators.} \\
&= \frac{(2 \cdot 7) \cdot (2 \cdot 3 \cdot 5)}{(3 \cdot 5) \cdot (2 \cdot 2 \cdot 5 \cdot 7)} && \text{Prime factor numerators and denominators.} \\
&= \frac{\cancel{2} \cdot \cancel{7} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{5}}{\cancel{3} \cdot 5 \cdot \cancel{2} \cdot \cancel{2} \cdot \cancel{5} \cdot \cancel{7}} && \text{Cancel common factors.} \\
&= \frac{1}{5} && \text{Multiply.}
\end{aligned}$$

Note: Everything in the numerator cancels because you've divided the numerator by itself. Hence, the answer has a 1 in its numerator.

□

When Everything Cancels. When all the factors in the numerator cancel, this means that you are dividing the numerator by itself. Hence, you are left with a 1 in the numerator. The same rule applies to the denominator. If everything in the denominator cancels, you're left with a 1 in the denominator.

You Try It!

Simplify:

$$\frac{6a}{15b} \cdot \left(-\frac{35b^2}{10a^2}\right)$$

EXAMPLE 7. Simplify the product: $-\dfrac{6x}{55y} \cdot \left(-\dfrac{110y^2}{105x^2}\right)$.

Solution. The product of two negatives is positive.

$$-\frac{6x}{55y} \cdot \left(-\frac{110y^2}{105x^2}\right) = \frac{6x}{55y} \cdot \frac{110y^2}{105x^2} \qquad \text{Like signs gives a positive.}$$

Prime factor numerators and denominators, then cancel common factors.

$$= \frac{2 \cdot 3 \cdot x}{5 \cdot 11 \cdot y} \cdot \frac{2 \cdot 5 \cdot 11 \cdot y \cdot y}{3 \cdot 5 \cdot 7 \cdot x \cdot x}$$ Prime factor numerators & denominators.

$$= \frac{2 \cdot \cancel{3} \cdot \cancel{x}}{5 \cdot \cancel{11} \cdot \cancel{y}} \cdot \frac{2 \cdot \cancel{5} \cdot \cancel{11} \cdot \cancel{y} \cdot y}{\cancel{3} \cdot \cancel{5} \cdot 7 \cdot \cancel{x} \cdot x}$$ Cancel common factors.

$$= \frac{2 \cdot 2 \cdot y}{5 \cdot 7 \cdot x}$$ Remaining factors

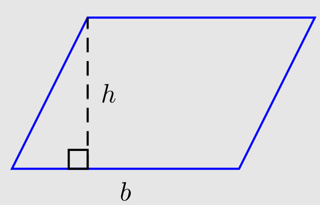
$$= \frac{4y}{35x}$$ Multiply numerators; multiply denominators.

Answer: $-\frac{21b}{5a}$

Parallelograms

In this section, we are going to learn how to find the area of a *parallelogram*. Let's begin with the definition of a parallelogram. Recall that a quadrilateral is a polygon having four sides. A parallelogram is a very special type of quadrilateral.

Parallelogram. A parallelogram is a quadrilateral whose opposite sides are parallel.



The side on which the parallelogram rests is called its *base* (labeled b in the figure) and the distance from its base to the opposite side is called its *height* (labeled h in the figure). Note that the altitude is *perpendicular* to the base (meets the base at a 90° angle).

Figure 4.8 shows a *rectangle* having length b and width h. Therefore, the area of the rectangle in Figure 4.8 is $A = bh$, which is found by taking the product of the length and width. Take a pair of scissors and cut a triangle from the right end of the rectangle as shown in Figure 4.9(a), then paste the cut triangle to the left end as shown in Figure 4.9(b). The result, seen in Figure 4.9(b) is a parallelogram having base b and height h.

Because we've *thrown no material away* in creating the parallelogram from the rectangle, the parallelogram has the same area as the original rectangle. That is, the area of the parallelogram is $A = bh$.

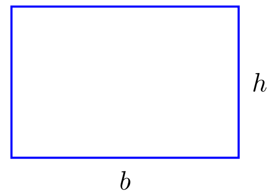
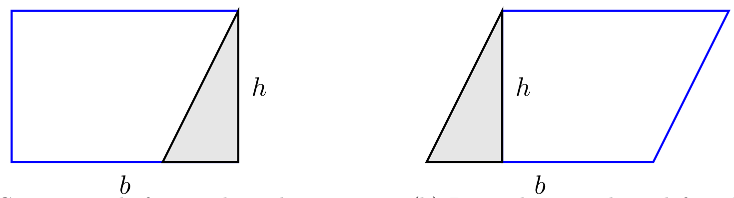


Figure 4.8: The area of the rectangle is $A = bh$.



(a) Cut a triangle from right end. (b) Paste the triangle on left end.

Figure 4.9: Creating a parallelogram from a rectangle.

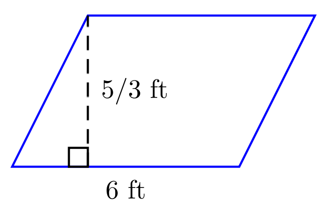
Area of a Parallelogram. A parallelogram having base b and height h has area $A = bh$. That is, to find the area of a parallelogram, take the product of its base and height.

You Try It!

The base of a parallelogram measures 14 inches. The height is 8/7 of an inch. What is the area of the parallelogram?

Answer: 16 square inches

EXAMPLE 8. Find the area of the parallelogram pictured below.



Solution. The area of the parallelogram is equal to the product of its base and height. That is,

$$\begin{aligned} A &= bh && \text{Area formula for parallelogram.} \\ &= (6\,\text{ft})\left(\frac{5}{3}\,\text{ft}\right) && \text{Substitute: } 6\,\text{ft for } b,\ 5/3\,\text{ft for } h. \\ &= \frac{30}{3}\,\text{ft}^2. && \text{Multiply numerators and denominators.} \\ &= 10\,\text{ft}^2. && \text{Divide.} \end{aligned}$$

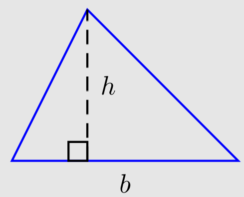
Thus, the area of the parallelogram is 10 square feet.

□

Triangles

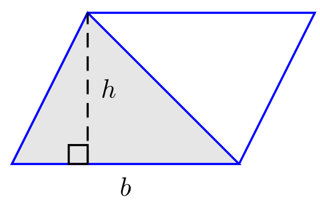
Let's turn our attention to learning how to find the area of a *triangle*.

Triangle. A triangle is a three-sided polygon. It is formed by plotting three points and connecting them with three line segments. Each of the three points is called a *vertex* of the triangle and each of the three line segments is called a *side* of the triangle.



The side on which the triangle rests is called its *base*, and the distance between its base and opposite vertex is called its *height* of *altitude*. The altitude is always *perpendicular* to the base; that is, it forms a 90° angle with the base.

It's easily seen that a triangle has half the area of a parallelogram.

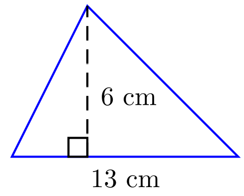


The parallelogram has area $A = bh$. Therefore, the triangle has one-half that area. That is, the area of the triangle is $A = (1/2)bh$.

Area of a Triangle. A triangle having base b and height h has area $A = (1/2)bh$. That is, to find the area of a triangle, take one-half the product of the base and height.

You Try It!

EXAMPLE 9. Find the area of the triangle pictured below.



The base of a triangle measures 15 meters. The height is 12 meters. What is the area of the triangle?

Solution. To find the area of the triangle, take one-half the product of the base and height.

$$
\begin{aligned}
A &= \frac{1}{2}bh && \text{Area of a triangle formula.}\\
&= \frac{1}{2}(13\,\text{cm})(6\,\text{cm}) && \text{Substitute: } 13\,\text{cm for } b,\ 6\,\text{cm for } h.\\
&= \frac{78\,\text{cm}^2}{2} && \text{Multiply numerators; multiply denominators.}\\
&= 39\,\text{cm}^2. && \text{Simplify.}
\end{aligned}
$$

Answer: 90 square meters

Therefore, the area of the triangle is 39 square centimeters.

□

Identifying the Base and Altitude. Sometimes it can be a bit difficult to determine the base and altitude (height) of a triangle. For example, consider the triangle in Figure 4.10(a). Let's say we choose the bottom edge of the triangle as the base and denote its length with the variable b, as shown in Figure 4.10(a).

(a) The base is easy.

(b) Adding the altitude (height).

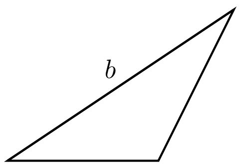
Figure 4.10: Identifying the base and altitude (height) of a triangle.

The altitude (height) of the triangle is defined as the distance between the base of the triangle and its opposite vertex. To identify this altitude, we must first extend the base, as seen in the dashed extension in Figure 4.10(b), then drop a perpendicular dashed line from the opposite vertex to the extended base, also shown in Figure 4.10(b). This perpendicular is the altitude (height) of the triangle and we denote its length by h.

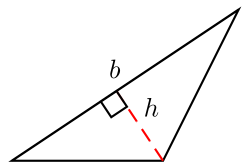
But we can go further. Any of the three sides of a triangle may be designated as the base of the triangle. Suppose, as shown in Figure 4.11(a), we identify a different side as the base, with length denoted by the variable b.

The altitude to this new base will be a segment from the opposite vertex, perpendicular to the base. Its length in Figure 4.11(b) is denoted by h.

In like manner, there is a third side of the triangle that could also be used as the base. The altitude to this third side is found by dropping a perpendicular from the vertex of the triangle directly opposite from this base. This would also require extending the base. We leave this to our readers to explore.



(a) A different base.



(b) Adding the altitude (height).

Figure 4.11: Identifying the base and altitude (height) of a triangle.

Key Point. Any of the three sides of a triangle may be used as the base. The altitude is drawn by dropping a perpendicular from the opposite vertex to the chosen base. This sometimes requires that we extend the base. Regardless of which side we use for the base, the formula $A = bh/2$ will produce the same area result.

Exercises

1. Create a diagram, such as that shown in Figure 4.7, to show that 1/3 of 1/3 is 1/9.

2. Create a diagram, such as that shown in Figure 4.7, to show that 1/2 of 1/4 is 1/8.

3. Create a diagram, such as that shown in Figure 4.7, to show that 1/3 of 1/4 is 1/12.

4. Create a diagram, such as that shown in Figure 4.7, to show that 2/3 of 1/3 is 2/9.

In Exercises 1-28, multiply the fractions, and simplify your result.

5. $\frac{-21}{4} \cdot \frac{22}{19}$

6. $\frac{-4}{19} \cdot \frac{21}{8}$

7. $\frac{20}{11} \cdot \frac{-17}{22}$

8. $\frac{-9}{2} \cdot \frac{6}{7}$

9. $\frac{21}{8} \cdot \frac{-14}{15}$

10. $\frac{-17}{18} \cdot \frac{-3}{4}$

11. $\frac{-5}{11} \cdot \frac{7}{20}$

12. $\frac{-5}{2} \cdot \frac{-20}{19}$

13. $\frac{8}{13} \cdot \frac{-1}{6}$

14. $\frac{-12}{7} \cdot \frac{5}{9}$

15. $\frac{2}{15} \cdot \frac{-9}{8}$

16. $\frac{2}{11} \cdot \frac{-21}{8}$

17. $\frac{17}{12} \cdot \frac{3}{4}$

18. $\frac{7}{13} \cdot \frac{10}{21}$

19. $\frac{-6}{23} \cdot \frac{9}{10}$

20. $\frac{12}{11} \cdot \frac{-5}{2}$

21. $\frac{-23}{24} \cdot \frac{-6}{17}$

22. $\frac{4}{9} \cdot \frac{-21}{19}$

23. $\frac{24}{7} \cdot \frac{5}{2}$

24. $\frac{-20}{23} \cdot \frac{-1}{2}$

25. $\frac{1}{2} \cdot \frac{-8}{11}$

26. $\frac{-11}{18} \cdot \frac{-20}{3}$

27. $\frac{-24}{13} \cdot \frac{-7}{18}$

28. $\frac{21}{20} \cdot \frac{-4}{5}$

In Exercises 29-40, multiply the fractions, and simplify your result.

29. $\dfrac{-12y^3}{13} \cdot \dfrac{2}{9y^6}$

30. $\dfrac{-8x^3}{3} \cdot \dfrac{-6}{5x^5}$

31. $\dfrac{11y^3}{24} \cdot \dfrac{6}{5y^5}$

32. $\dfrac{11y}{18} \cdot \dfrac{21}{17y^6}$

33. $\dfrac{-8x^2}{21} \cdot \dfrac{-18}{19x}$

34. $\dfrac{2y^4}{11} \cdot \dfrac{-7}{18y}$

35. $\dfrac{13x^6}{15} \cdot \dfrac{9}{16x^2}$

36. $\dfrac{-22x^6}{15} \cdot \dfrac{17}{16x^3}$

37. $\dfrac{-6y^3}{5} \cdot \dfrac{-20}{7y^6}$

38. $\dfrac{-21y}{5} \cdot \dfrac{-8}{3y^2}$

39. $\dfrac{-3y^3}{4} \cdot \dfrac{23}{12y}$

40. $\dfrac{-16y^6}{15} \cdot \dfrac{-21}{13y^4}$

In Exercises 41-56, multiply the fractions, and simplify your result.

41. $\dfrac{13y^6}{20x^4} \cdot \dfrac{2x}{7y^2}$

42. $\dfrac{-8y^3}{13x^6} \cdot \dfrac{7x^2}{10y^2}$

43. $\dfrac{23y^4}{21x} \cdot \dfrac{-7x^6}{4y^2}$

44. $\dfrac{-2x^6}{9y^4} \cdot \dfrac{y^5}{20x}$

45. $\dfrac{11y^6}{12x^6} \cdot \dfrac{-2x^4}{7y^2}$

46. $\dfrac{16x^3}{13y^4} \cdot \dfrac{11y^2}{18x}$

47. $\dfrac{x^6}{21y^3} \cdot \dfrac{-7y^4}{9x^5}$

48. $\dfrac{-3y^3}{5x} \cdot \dfrac{14x^5}{15y^2}$

49. $\dfrac{19y^2}{18x} \cdot \dfrac{10x^3}{7y^3}$

50. $\dfrac{-20x}{9y^3} \cdot \dfrac{-y^6}{4x^3}$

51. $\dfrac{-4y^3}{5x^5} \cdot \dfrac{-10x}{21y^4}$

52. $\dfrac{11y^2}{14x^4} \cdot \dfrac{-22x}{21y^3}$

53. $\dfrac{-16x}{21y^2} \cdot \dfrac{-7y^3}{5x^2}$

54. $\dfrac{-4y}{5x} \cdot \dfrac{10x^3}{7y^6}$

55. $\dfrac{17x^3}{3y^6} \cdot \dfrac{-12y^2}{7x^4}$

56. $\dfrac{-6x^4}{11y^3} \cdot \dfrac{13y^2}{8x^5}$

In Exercises 57-62, find the area of the parallelogram having the given base and altitude.

57. base = 8 cm, altitude = 7 cm

58. base = 2 cm, altitude = 11 cm

59. base = 6 cm, altitude = 13 cm

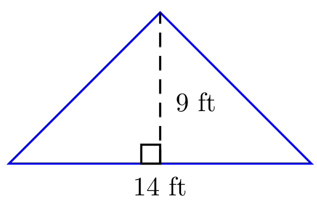
60. base = 2 cm, altitude = 6 cm

61. base = 18 cm, altitude = 14 cm

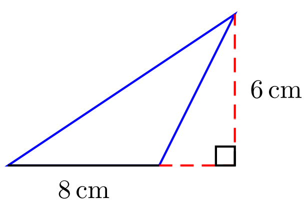
62. base = 20 cm, altitude = 2 cm

In Exercises 63-68, find the area of the triangle shown in the figure. (Note: Figures are not drawn to scale.)

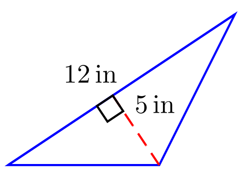
63.



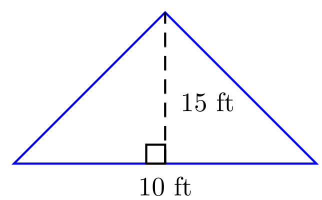
64.



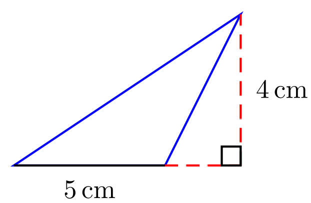
65.



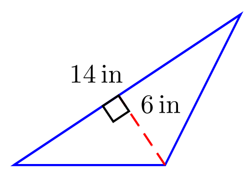
66.



67.



68.



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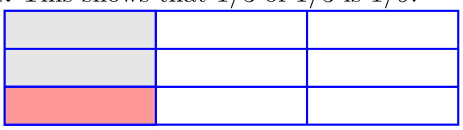
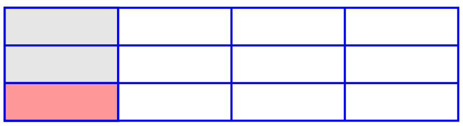
69. **Weight on the Moon.** On the moon, you would only weigh 1/6 of what you weigh on earth. If you weigh 138 pounds on earth, what would your weight on the moon be?

❧ ❧ ❧ Answers ☙ ☙ ☙

1. This shows that 1/3 of 1/3 is 1/9.



3. This shows that 1/3 of 1/4 is 1/12.



5. $-\dfrac{231}{38}$

7. $-\dfrac{170}{121}$

9. $-\dfrac{49}{20}$

11. $-\dfrac{7}{44}$

13. $-\dfrac{4}{39}$

15. $-\dfrac{3}{20}$

17. $\dfrac{17}{16}$

19. $-\dfrac{27}{115}$

21. $\dfrac{23}{68}$

23. $\dfrac{60}{7}$

25. $-\dfrac{4}{11}$

27. $\dfrac{28}{39}$

29. $-\dfrac{8}{39y^3}$

31. $\dfrac{11}{20y^2}$

33. $\dfrac{48x}{133}$

35. $\dfrac{39x^4}{80}$

37. $\dfrac{24}{7y^3}$

39. $-\dfrac{23y^2}{16}$

41. $\dfrac{13y^4}{70x^3}$

43. $-\dfrac{23y^2x^5}{12}$

45. $-\dfrac{11y^4}{42x^2}$

47. $-\dfrac{xy}{27}$

49. $\dfrac{95x^2}{63y}$

51. $\dfrac{8}{21yx^4}$

53. $\dfrac{16y}{15x}$

55. $-\dfrac{68}{7xy^4}$

57. $56\,\text{cm}^2$

59. $78\,\text{cm}^2$

61. $252\,\text{cm}^2$

63. $63\,\text{ft}^2$

65. $30\,\text{in}^2$

67. $10\,\text{cm}^2$

69. 23 pounds

4.3 Dividing Fractions

Suppose that you have four pizzas and each of the pizzas has been sliced into eight equal slices. Therefore, each slice of pizza represents 1/8 of a whole pizza.

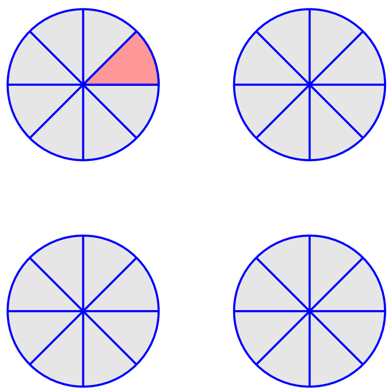


Figure 4.12: One slice of pizza is 1/8 of one whole pizza.

Now for the question: How many one-eighths are there in four? This is a division statement. To find how many one-eighths there are in 4, divide 4 by 1/8. That is,

$$\text{Number of one-eighths in four} = 4 \div \frac{1}{8}.$$

On the other hand, to find the number of one-eights in four, Figure 4.12 clearly demonstrates that this is equivalent to asking how many slices of pizza are there in four pizzas. Since there are 8 slices per pizza and four pizzas,

$$\text{Number of pizza slices} = 4 \cdot 8.$$

The conclusion is the fact that $4 \div (1/8)$ is equivalent to $4 \cdot 8$. That is,

$$\begin{aligned} 4 \div \frac{1}{8} &= 4 \cdot 8 \\ &= 32. \end{aligned}$$

Therefore, we conclude that there are 32 one-eighths in 4.

Reciprocals

The number 1 is still the *multiplicative identity* for fractions.

Multiplicative Identity Property. Let a/b be any fraction. Then,

$$\frac{a}{b} \cdot 1 = \frac{a}{b} \qquad \text{and} \qquad 1 \cdot \frac{a}{b} = \frac{a}{b}.$$

The number 1 is called the *multiplicative identity* because the identical number is returned when you multiply by 1.

Next, if we invert 3/4, that is, if we turn 3/4 upside down, we get 4/3. Note what happens when we multiply 3/4 by 4/3.

$$\begin{aligned} \frac{3}{4} \cdot \frac{4}{3} &= \frac{3 \cdot 4}{4 \cdot 3} && \text{Multiply numerators; multiply denominators.} \\ &= \frac{12}{12} && \text{Simplify numerators and denominators.} \\ &= 1 && \text{Divide.} \end{aligned}$$

The number 4/3 is called the *multiplicative inverse* or *reciprocal* of 3/4. The product of reciprocals is always 1.

Multiplicative Inverse Property. Let a/b be any fraction. The number b/a is called the *multiplicative inverse* or *reciprocal* of a/b. The product of reciprocals is 1.

$$\frac{a}{b} \cdot \frac{b}{a} = 1$$

Note: To find the multiplicative inverse (reciprocal) of a number, simply invert the number (turn it upside down).

For example, the number 1/8 is the multiplicative inverse (reciprocal) of 8 because

$$8 \cdot \frac{1}{8} = 1.$$

Note that 8 can be thought of as 8/1. Invert this number (turn it upside down) to find its multiplicative inverse (reciprocal) 1/8.

You Try It!

Find the reciprocals of: (a) $-3/7$ and (b) 15

EXAMPLE 1. Find the multiplicative inverses (reciprocals) of: (a) 2/3, (b) $-3/5$, and (c) -12.

Solution.

a) Because

$$\frac{2}{3} \cdot \frac{3}{2} = 1,$$

the multiplicative inverse (reciprocal) of 2/3 is 3/2.

b) Because

$$-\frac{3}{5} \cdot \left(-\frac{5}{3}\right) = 1,$$

the multiplicative inverse (reciprocal) of $-3/5$ is $-5/3$. Again, note that we simply inverted the number $-3/5$ to get its reciprocal $-5/3$.

c) Because

$$-12 \cdot \left(-\frac{1}{12}\right) = 1,$$

the multiplicative inverse (reciprocal) of -12 is $-1/12$. Again, note that we simply inverted the number -12 (understood to equal $-12/1$) to get its reciprocal $-1/12$.

Answer: (a) $-7/3$, (b) $1/15$

Division

Recall that we computed the number of one-eighths in four by doing this calculation:

$$\begin{aligned} 4 \div \frac{1}{8} &= 4 \cdot 8 \\ &= 32. \end{aligned}$$

Note how we inverted the divisor (second number), then changed the division to multiplication. This motivates the following definition of division.

Division Definition. If a/b and c/d are any fractions, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.$$

That is, we invert the divisor (second number) and change the division to multiplication. *Note: We like to use the phrase "invert and multiply" as a memory aid for this definition.*

You Try It!

Divide:

$$\frac{2}{3} \div \frac{10}{3}$$

EXAMPLE 2. Divide $1/2$ by $3/5$.

Solution. To divide $1/2$ by $3/5$, invert the divisor (second number), then multiply.

$$\begin{aligned} \frac{1}{2} \div \frac{3}{5} &= \frac{1}{2} \cdot \frac{5}{3} && \text{Invert the divisor (second number).} \\ &= \frac{5}{6} && \text{Multiply.} \end{aligned}$$

Answer: $1/5$

You Try It!

Divide:

$$\frac{15}{7} \div 5$$

Answer: $\dfrac{3}{7}$

EXAMPLE 3. Simplify the following expressions: (a) $3 \div \dfrac{2}{3}$ and (b) $\dfrac{4}{5} \div 5$.

Solution. In each case, invert the divisor (second number), then multiply.

a) Note that 3 is understood to be 3/1.

$$\begin{aligned} 3 \div \frac{2}{3} &= \frac{3}{1} \cdot \frac{3}{2} && \text{Invert the divisor (second number).} \\ &= \frac{9}{2} && \text{Multiply numerators; multiply denominators.} \end{aligned}$$

b) Note that 5 is understood to be 5/1.

$$\begin{aligned} \frac{4}{5} \div 5 &= \frac{4}{5} \cdot \frac{1}{5} && \text{Invert the divisor (second number).} \\ &= \frac{4}{25} && \text{Multiply numerators; multiply denominators.} \end{aligned}$$

□

After inverting, you may need to factor and cancel, as we learned to do in Section 4.2.

You Try It!

Divide:

$$\frac{6}{15} \div \left(-\frac{42}{35}\right)$$

Answer: $-1/3$

EXAMPLE 4. Divide $-6/35$ by $33/55$.

Solution. Invert, multiply, factor, and cancel common factors.

$$\begin{aligned} -\frac{6}{35} \div \frac{33}{55} &= -\frac{6}{35} \cdot \frac{55}{33} && \text{Invert the divisor (second number).} \\ &= -\frac{6 \cdot 55}{35 \cdot 33} && \text{Multiply numerators; multiply denominators.} \\ &= -\frac{(2 \cdot 3) \cdot (5 \cdot 11)}{(5 \cdot 7) \cdot (3 \cdot 11)} && \text{Factor numerators and denominators.} \\ &= -\frac{2 \cdot \cancel{3} \cdot \cancel{5} \cdot \cancel{11}}{\cancel{5} \cdot 7 \cdot \cancel{3} \cdot \cancel{11}} && \text{Cancel common factors.} \\ &= -\frac{2}{7} && \text{Remaining factors.} \end{aligned}$$

Note that unlike signs produce a negative answer.

□

Of course, you can also choose to factor numerators and denominators in place, then cancel common factors.

You Try It!

EXAMPLE 5. Divide $-6/x$ by $-12/x^2$.

Solution. Invert, factor numerators and denominators, cancel common factors, then multiply.

$$\begin{aligned} -\frac{6}{x} \div \left(-\frac{12}{x^2}\right) &= -\frac{6}{x} \cdot \left(-\frac{x^2}{12}\right) && \text{Invert second number.} \\ &= -\frac{2 \cdot 3}{x} \cdot -\frac{x \cdot x}{2 \cdot 2 \cdot 3} && \text{Factor numerators and denominators.} \\ &= -\frac{\cancel{2} \cdot \cancel{3}}{\cancel{x}} \cdot -\frac{\cancel{x} \cdot x}{2 \cdot \cancel{2} \cdot \cancel{3}} && \text{Cancel common factors.} \\ &= \frac{x}{2} && \text{Multiply.} \end{aligned}$$

Note that like signs produce a positive answer.

Divide:

$$-\frac{12}{a} \div \left(-\frac{15}{a^3}\right)$$

Answer: $-\dfrac{4a^2}{5}$

□

Exercises

In Exercises 1-16, find the reciprocal of the given number.

1. $-16/5$

2. $-3/20$

3. -17

4. -16

5. $15/16$

6. $7/9$

7. 30

8. 28

9. -46

10. -50

11. $-9/19$

12. $-4/7$

13. $3/17$

14. $3/5$

15. 11

16. 48

In Exercises 17-32, determine which property of multiplication is depicted by the given identity.

17. $\frac{2}{9} \cdot \frac{9}{2} = 1$

18. $\frac{12}{19} \cdot \frac{19}{12} = 1$

19. $\frac{-19}{12} \cdot 1 = \frac{-19}{12}$

20. $\frac{-19}{8} \cdot 1 = \frac{-19}{8}$

21. $-6 \cdot \left(-\frac{1}{6}\right) = 1$

22. $-19 \cdot \left(-\frac{1}{19}\right) = 1$

23. $\frac{-16}{11} \cdot 1 = \frac{-16}{11}$

24. $\frac{-7}{6} \cdot 1 = \frac{-7}{6}$

25. $-\frac{4}{1} \cdot \left(-\frac{1}{4}\right) = 1$

26. $-\frac{9}{10} \cdot \left(-\frac{10}{9}\right) = 1$

27. $\frac{8}{1} \cdot 1 = \frac{8}{1}$

28. $\frac{13}{15} \cdot 1 = \frac{13}{15}$

29. $14 \cdot \frac{1}{14} = 1$

30. $4 \cdot \frac{1}{4} = 1$

31. $\frac{13}{8} \cdot 1 = \frac{13}{8}$

32. $\frac{1}{13} \cdot 1 = \frac{1}{13}$

In Exercises 33-56, divide the fractions, and simplify your result.

33. $\frac{8}{23} \div \frac{-6}{11}$

34. $\frac{-10}{21} \div \frac{-6}{5}$

35. $\frac{18}{19} \div \frac{-16}{23}$

36. $\frac{13}{10} \div \frac{17}{18}$

37. $\frac{4}{21} \div \frac{-6}{5}$

38. $\frac{2}{9} \div \frac{-12}{19}$

39. $\frac{-1}{9} \div \frac{8}{3}$

40. $\frac{1}{2} \div \frac{-15}{8}$

41. $\frac{-21}{11} \div \frac{3}{10}$

42. $\frac{7}{24} \div \frac{-23}{2}$

43. $\frac{-12}{7} \div \frac{2}{3}$

44. $\frac{-9}{16} \div \frac{6}{7}$

45. $\frac{2}{19} \div \frac{24}{23}$

46. $\frac{7}{3} \div \frac{-10}{21}$

47. $\frac{-9}{5} \div \frac{-24}{19}$

48. $\frac{14}{17} \div \frac{-22}{21}$

49. $\frac{18}{11} \div \frac{14}{9}$

50. $\frac{5}{6} \div \frac{20}{19}$

51. $\frac{13}{18} \div \frac{4}{9}$

52. $\frac{-3}{2} \div \frac{-7}{12}$

53. $\frac{11}{2} \div \frac{-21}{10}$

54. $\frac{-9}{2} \div \frac{-13}{22}$

55. $\frac{3}{10} \div \frac{12}{5}$

56. $\frac{-22}{7} \div \frac{-18}{17}$

In Exercises 57-68, divide the fractions, and simplify your result.

57. $\frac{20}{17} \div 5$

58. $\frac{21}{8} \div 7$

59. $-7 \div \frac{21}{20}$

60. $-3 \div \frac{12}{17}$

61. $\frac{8}{21} \div 2$

62. $\frac{-3}{4} \div (-6)$

63. $8 \div \frac{-10}{17}$

64. $-6 \div \frac{20}{21}$

65. $-8 \div \frac{18}{5}$

66. $6 \div \frac{-21}{8}$

67. $\frac{3}{4} \div (-9)$

68. $\frac{2}{9} \div (-8)$

In Exercises 69-80, divide the fractions, and simplify your result.

69. $\dfrac{11x^2}{12} \div \dfrac{8x^4}{3}$

70. $\dfrac{-4x^2}{3} \div \dfrac{11x^6}{6}$

71. $\dfrac{17y}{9} \div \dfrac{10y^6}{3}$

72. $\dfrac{-5y}{12} \div \dfrac{-3y^5}{2}$

73. $\dfrac{-22x^4}{13} \div \dfrac{12x}{11}$

74. $\dfrac{-9y^6}{4} \div \dfrac{24y^5}{13}$

75. $\dfrac{-3x^4}{10} \div \dfrac{-4x}{5}$

76. $\dfrac{18y^4}{11} \div \dfrac{4y^2}{7}$

77. $\dfrac{-15y^2}{14} \div \dfrac{-10y^5}{13}$

78. $\dfrac{3x}{20} \div \dfrac{2x^3}{5}$

79. $\dfrac{-15x^5}{13} \div \dfrac{20x^2}{19}$

80. $\dfrac{18y^6}{7} \div \dfrac{14y^4}{9}$

In Exercises 81-96, divide the fractions, and simplify your result.

81. $\dfrac{11y^4}{14x^2} \div \dfrac{-9y^2}{7x^3}$

82. $\dfrac{-5x^2}{12y^3} \div \dfrac{-22x}{21y^5}$

83. $\dfrac{10x^4}{3y^4} \div \dfrac{7x^5}{24y^2}$

84. $\dfrac{20x^3}{11y^5} \div \dfrac{5x^5}{6y^3}$

85. $\dfrac{22y^4}{21x^5} \div \dfrac{-5y^2}{6x^4}$

86. $\dfrac{-7y^5}{8x^6} \div \dfrac{21y}{5x^5}$

87. $\dfrac{-22x^4}{21y^3} \div \dfrac{-17x^3}{3y^4}$

88. $\dfrac{-7y^4}{4x} \div \dfrac{-15y}{22x^4}$

89. $\dfrac{-16y^2}{3x^3} \div \dfrac{2y^6}{11x^5}$

90. $\dfrac{-20x}{21y^2} \div \dfrac{-22x^5}{y^6}$

91. $\dfrac{-x}{12y^4} \div \dfrac{-23x^3}{16y^3}$

92. $\dfrac{20x^2}{17y^3} \div \dfrac{8x^3}{15y}$

93. $\dfrac{y^2}{4x} \div \dfrac{-9y^5}{8x^3}$

94. $\dfrac{-10y^4}{13x^2} \div \dfrac{-5y^6}{6x^3}$

95. $\dfrac{-18x^6}{13y^4} \div \dfrac{3x}{y^2}$

96. $\dfrac{20x^4}{9y^6} \div \dfrac{14x^2}{17y^4}$

❧ ❧ ❧ Answers ☙ ☙ ☙

1. $-\dfrac{5}{16}$

3. $-\dfrac{1}{17}$

5. $\dfrac{16}{15}$

7. $\dfrac{1}{30}$

9. $-\dfrac{1}{46}$

11. $-\dfrac{19}{9}$

13. $\dfrac{17}{3}$

15. $\dfrac{1}{11}$

17. multiplicative inverse property

19. multiplicative identity property

21. multiplicative inverse property

23. multiplicative identity property

25. multiplicative inverse property

27. multiplicative identity property

29. multiplicative inverse property

31. multiplicative identity property

33. $-\dfrac{44}{69}$

35. $-\dfrac{207}{152}$

37. $-\dfrac{10}{63}$

39. $-\dfrac{1}{24}$

41. $-\dfrac{70}{11}$

43. $-\dfrac{18}{7}$

45. $\dfrac{23}{228}$

47. $\dfrac{57}{40}$

49. $\dfrac{81}{77}$

51. $\dfrac{13}{8}$

53. $-\dfrac{55}{21}$

55. $\dfrac{1}{8}$

57. $\dfrac{4}{17}$

59. $-\dfrac{20}{3}$

61. $\dfrac{4}{21}$

63. $-\dfrac{68}{5}$

65. $-\dfrac{20}{9}$

67. $-\dfrac{1}{12}$

69. $\frac{11}{32x^2}$

71. $\frac{17}{30y^5}$

73. $-\frac{121x^3}{78}$

75. $\frac{3x^3}{8}$

77. $\frac{39}{28y^3}$

79. $-\frac{57x^3}{52}$

81. $-\frac{11y^2x}{18}$

83. $\frac{80}{7xy^2}$

85. $-\frac{44y^2}{35x}$

87. $\frac{22xy}{119}$

89. $-\frac{88x^2}{3y^4}$

91. $\frac{4}{69x^2y}$

93. $-\frac{2x^2}{9y^3}$

95. $-\frac{6x^5}{13y^2}$

4.4 Adding and Subtracting Fractions

Paul and Tony order a pizza which has been cut into eight equal slices. Thus, each slice is 1/8 of the whole pizza. Paul eats two slices (shaded in light gray in Figure 4.13), or 2/8 of the whole pizza. Tony eats three slices (shaded in light red (or a darker shade of gray in black-and-white printing) in Figure 4.13), or 3/8 of the whole pizza.

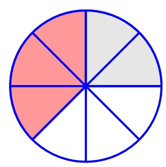


Figure 4.13: Paul eats two slices (2/8) and Tony eats three slices (3/8).

It should be clear that together Paul and Tony eat five slices, or 5/8 of the whole pizza. This reflects the fact that

$$\frac{2}{8}+\frac{3}{8}=\frac{5}{8}.$$

This demonstrates how to add two fractions with a common (same) denominator. Keep the common denominator and add the numerators. That is,

$$\begin{aligned}\frac{2}{8}+\frac{3}{8}&=\frac{2+3}{8} && \text{Keep denominator; add numerators.}\\ &=\frac{5}{8} && \text{Simplify numerator.}\end{aligned}$$

Adding Fractions with Common Denominators. Let a/c and b/c be two fractions with a common (same) denominator. Their sum is defined as

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}.$$

That is, to add two fractions having common denominators, keep the common denominator and add their numerators.

A similar rule holds for subtraction.

Subtracting Fractions with Common Denominators. Let a/c and b/c be two fractions with a common (same) denominator. Their difference is defined as

$$\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}.$$

That is, to subtract two fractions having common denominators, keep the common denominator and subtract their numerators.

You Try It!

Add:

$$\frac{1}{8}+\frac{2}{8}$$

Answer: 3/8

EXAMPLE 1. Find the sum of 4/9 and 3/9.

Solution. Keep the common denominator and add the numerators.

$$\begin{aligned}\frac{4}{9}+\frac{3}{9} &= \frac{4+3}{9} && \text{Keep denominator; add numerators.}\\ &= \frac{7}{9} && \text{Simplify numerator.}\end{aligned}$$

□

You Try It!

Subtract:

$$\frac{11}{12}-\frac{7}{12}$$

Answer: 1/3

EXAMPLE 2. Subtract 5/16 from 13/16.

Solution. Keep the common denominator and subtract the numerators.

$$\begin{aligned}\frac{13}{16}-\frac{5}{16} &= \frac{13-5}{16} && \text{Keep denominator; subtract numerators.}\\ &= \frac{8}{16} && \text{Simplify numerator.}\end{aligned}$$

Of course, as we learned in Section 4.1, we should always reduce our final answer to lowest terms. One way to accomplish that in this case is to divide numerator and denominator by 8, the greatest common divisor of 8 and 16.

$$\begin{aligned} &= \frac{8\div 8}{16\div 8} && \text{Divide numerator and denominator by 8.}\\ &= \frac{1}{2} && \text{Simplify numerator and denominator.}\end{aligned}$$

□

You Try It!

Subtract:

$$-\frac{5}{y}-\left(-\frac{2}{y}\right)$$

EXAMPLE 3. Simplify: $\dfrac{3}{x}-\left(-\dfrac{7}{x}\right)$.

Solution. Both fractions share a common denominator.

$$\begin{aligned} \frac{3}{x} - \left(-\frac{7}{x}\right) &= \frac{3}{x} + \frac{7}{x} && \text{Add the opposite.} \\ &= \frac{3+7}{x} && \text{Keep denominator, add numerators.} \\ &= \frac{10}{x} && \text{Simplify.} \end{aligned}$$

Answer: $-3/y$

□

Adding Fractions with Different Denominators

Consider the sum

$$\frac{4}{9} + \frac{1}{6}.$$

We cannot add these fractions because they do not have a common denominator. So, what to do?

Goal. In order to add two fractions with different denominators, we need to:

1. Find a common denominator for the given fractions.
2. Make fractions with the common denominator that are equivalent to the original fractions.

If we accomplish the two items in the "Goal," we will be able to find the sum of the given fractions.

So, how to start? We need to find a common denominator, but not just any common denominator. Let's agree that we want to keep the numbers as small as possible and find a *least common denominator*.

Least Common Denominator. The *least common denominator* (LCD) for a set of fractions is the smallest number divisible by each of the denominators of the given fractions.

Consider again the sum we wish to find:

$$\frac{4}{9} + \frac{1}{6}.$$

The denominators are 9 and 6. We wish to find a least common denominator, the smallest number that is divisible by both 9 and 6. A number of candidates come to mind: 36, 54, and 72 are all divisible by 9 and 6, to name a few. But the smallest number that is divisible by both 9 and 6 is 18. This is the least common denominator for 9 and 6.

We now proceed to the second item in "Goal." We need to make fractions having 18 as a denominator that are equivalent to 4/9 and 1/6. In the case of 4/9, if we multiply both numerator and denominator by 2, we get

$$\begin{aligned} \frac{4}{9} &= \frac{4 \cdot 2}{9 \cdot 2} && \text{Multiply numerator and denominator by 2.} \\ &= \frac{8}{18}. && \text{Simplify numerator and denominator.} \end{aligned}$$

In the case of 1/6, if we multiply both numerator and denominator by 3, we get

$$\begin{aligned} \frac{1}{6} &= \frac{1 \cdot 3}{6 \cdot 3} && \text{Multiply numerator and denominator by 3.} \\ &= \frac{3}{18}. && \text{Simplify numerator and denominator.} \end{aligned}$$

Typically, we'll arrange our work as follows.

$$\begin{aligned} \frac{4}{9} + \frac{1}{6} &= \frac{4 \cdot 2}{9 \cdot 2} + \frac{1 \cdot 3}{6 \cdot 3} && \text{Equivalent fractions with LCD} = 18. \\ &= \frac{8}{18} + \frac{3}{18} && \text{Simplify numerators and denominators.} \\ &= \frac{8+3}{18} && \text{Keep common denominator; add numerators.} \\ &= \frac{11}{18} && \text{Simplify numerator.} \end{aligned}$$

Let's summarize the procedure.

Adding or Subtracting Fractions with Different Denominators.

1. Find the LCD, the smallest number divisible by all the denominators of the given fractions.
2. Create fractions using the LCD as the denominator that are equivalent to the original fractions.
3. Add or subtract the resulting equivalent fractions. Simplify, including reducing the final answer to lowest terms.

You Try It!

EXAMPLE 4. Simplify: $\dfrac{3}{5} - \dfrac{2}{3}$.

Subtract:

$$\frac{3}{4} - \frac{7}{5}$$

Solution. The smallest number divisible by both 5 and 3 is 15.

$$\begin{aligned}
\frac{3}{5} - \frac{2}{3} &= \frac{3 \cdot 3}{5 \cdot 3} - \frac{2 \cdot 5}{3 \cdot 5} && \text{Equivalent fractions with LCD = 15.} \\
&= \frac{9}{15} - \frac{10}{15} && \text{Simplify numerators and denominators.} \\
&= \frac{9 - 10}{15} && \text{Keep LCD; subtract numerators.} \\
&= \frac{-1}{15} && \text{Simplify numerator.}
\end{aligned}$$

Although this answer is perfectly acceptable, negative divided by positive gives us a negative answer, so we could also write

$$= -\frac{1}{15}.$$

Answer: $-13/20$

□

You Try It!

EXAMPLE 5. Simplify: $-\dfrac{1}{4} - \dfrac{5}{6}$.

Subtract:

$$-\frac{3}{8} - \frac{1}{12}$$

Solution. The smallest number divisible by both 4 and 6 is 12.

$$\begin{aligned}
-\frac{1}{4} - \frac{5}{6} &= -\frac{1 \cdot 3}{4 \cdot 3} - \frac{5 \cdot 2}{6 \cdot 2} && \text{Equivalent fractions with LCD = 12.} \\
&= -\frac{3}{12} - \frac{10}{12} && \text{Simplify numerators and denominators.} \\
&= \frac{-3 - 10}{12} && \text{Keep LCD; subtract numerators.} \\
&= \frac{-13}{12} && \text{Simplify numerator.}
\end{aligned}$$

Answer: $-11/24$

□

You Try It!

Add:

$$\frac{5}{z}+\frac{2}{3}$$

EXAMPLE 6. Simplify: $\dfrac{5}{x}+\dfrac{3}{4}$.

Solution. The smallest number divisible by both 4 and x is $4x$.

$$\begin{aligned} \frac{5}{x}+\frac{3}{4} &= \frac{5\cdot 4}{x\cdot 4}+\frac{3\cdot x}{4\cdot x} && \text{Equivalent fractions with LCD} = 4x. \\ &= \frac{20}{4x}+\frac{3x}{4x} && \text{Simplify numerators and denominators.} \\ &= \frac{20+3x}{4x} && \text{Keep LCD; add numerators.} \end{aligned}$$

Answer: $\dfrac{15+2z}{3z}$

□

You Try It!

Simplify:

$$\frac{3}{7}-\frac{y}{4}$$

EXAMPLE 7. Simplify: $\dfrac{2}{3}-\dfrac{x}{5}$.

Solution. The smallest number divisible by both 3 and 5 is 15.

$$\begin{aligned} \frac{2}{3}-\frac{x}{5} &= \frac{2\cdot 5}{3\cdot 5}-\frac{x\cdot 3}{5\cdot 3} && \text{Equivalent fractions with LCD} = 15. \\ &= \frac{10}{15}-\frac{3x}{15} && \text{Simplify numerators and denominators.} \\ &= \frac{10-3x}{15} && \text{Keep LCD; subtract numerators.} \end{aligned}$$

Answer: $\dfrac{12-7y}{28}$

□

Least Common Multiple

First we define the *multiple* of a number.

Multiples. The multiples of a number d are $1d$, $2d$, $3d$, $4d$, etc. That is, the multiples of d are the numbers nd, where n is a natural number.

For example, the multiples of 8 are $1\cdot 8$, $2\cdot 8$, $3\cdot 8$, $4\cdot 8$, etc., or equivalently, 8, 16, 24, 32, etc.

Least Common Multiple. The *least common multiple* (LCM) of a set of numbers is the smallest number that is a multiple of each number of the given set. The procedure for finding an LCM follows:

1. List all of the multiples of each number in the given set of numbers.
2. List the multiples that are in common.
3. Pick the least of the multiples that are in common.

You Try It!

Find the least common denominator of 6 and 9.

EXAMPLE 8. Find the least common multiple (LCM) of 12 and 16.

Solution. List the multiples of 12 and 16.

$$\begin{aligned}\text{Multiples of } 12: &\quad 12,\ 24,\ 36,\ 48,\ 60,\ 72,\ 84,\ 96, \ldots \\ \text{Multiples of } 16: &\quad 16,\ 32,\ 48,\ 64,\ 80,\ 96,\ 112, \ldots\end{aligned}$$

Pick the common multiples.

$$\text{Common Multiples}: \quad 48,\ 96, \ldots$$

The LCM is the least of the common multiples.

$$\text{LCM}(12{,}16) = 48$$

Answer: 18

□

Important Observation. The *least common denominator* is the *least common multiple* of the denominators.

For example, suppose your problem is $5/12 + 5/16$. The LCD is the smallest number divisible by both 12 and 16. That number is 48, which is also the LCM of 12 and 16. Therefore, the procedure for finding the LCM can also be used to find the LCD.

Least Common Multiple Using Prime Factorization

You can also find the LCM using prime factorization.

LCM by Prime Factorization. To find an LCM for a set of numbers, follow this procedure:

1. Write down the prime factorization for each number in compact form using exponents.
2. The LCM is found by writing down every factor that appears in step 1 to the highest power of that factor that appears.

You Try It!

Use prime factorization to find the least common denominator of 18 and 24.

EXAMPLE 9. Use prime factorization to find the least common multiple (LCM) of 12 and 16.

Solution. Prime factor 12 and 16.

$$12 = 2 \cdot 2 \cdot 3$$
$$16 = 2 \cdot 2 \cdot 2 \cdot 2$$

Write the prime factorizations in compact form using exponents.

$$12 = 2^2 \cdot 3^1$$
$$16 = 2^4$$

To find the LCM, write down each factor that appears to the highest power of that factor that appears. The factors that appear are 2 and 3. The highest power of 2 that appears is 2^4. The highest power of 3 that appears is 3^1.

$$\text{LCM} = 2^4 \cdot 3^1 \quad \text{Keep highest power of each factor.}$$

Now we expand this last expression to get our LCM.

$$= 16 \cdot 3 \quad \text{Expand: } 2^4 = 16 \text{ and } 3^1 = 3.$$
$$= 48. \quad \text{Multiply.}$$

Note that this answer is identical to the LCM found in Example 8 that was found by listing multiples and choosing the smallest multiple in common.

Answer: 72

□

You Try It!

Simplify:

$$\frac{5}{24} + \frac{5}{36}$$

EXAMPLE 10. Simplify: $\dfrac{5}{28} + \dfrac{11}{42}$.

Solution. Prime factor the denominators in compact form using exponents.

$$\begin{aligned} 28 &= 2 \cdot 2 \cdot 7 = 2^2 \cdot 7 \\ 42 &= 2 \cdot 3 \cdot 7 = 2^1 \cdot 3^1 \cdot 7^1 \end{aligned}$$

To find the LCD, write down each factor that appears to the highest power of that factor that appears. The factors that appear are 2, 3, and 7. The highest power of 2 that appears is 2^2. The highest power of 3 that appears is 3^1. The highest power of 7 that appears is 7^1.

$$\begin{aligned} \text{LCM} &= 2^2 \cdot 3^1 \cdot 7^1 && \text{Keep highest power of each factor.} \\ &= 4 \cdot 3 \cdot 7 && \text{Expand: } 2^2 = 4,\ 3^1 = 3,\ 7^1 = 7. \\ &= 84 && \text{Multiply.} \end{aligned}$$

Create equivalent fractions with the new LCD, then add.

$$\begin{aligned} \frac{5}{28} + \frac{11}{42} &= \frac{5 \cdot 3}{28 \cdot 3} + \frac{11 \cdot 2}{42 \cdot 2} && \text{Equivalent fractions with LCD=84.} \\ &= \frac{15}{84} + \frac{22}{84} && \text{Simplify numerators and denominators.} \\ &= \frac{37}{84} && \text{Keep LCD; add numerators.} \end{aligned}$$

Answer: 25/72

□

You Try It!

EXAMPLE 11. Simplify: $-\frac{11}{24} - \frac{1}{18}$.

Simplify:

$$-\frac{5}{24} - \frac{11}{36}$$

Solution. Prime factor the denominators in compact form using exponents.

$$\begin{aligned} 24 &= 2 \cdot 2 \cdot 2 \cdot 3 = 2^3 \cdot 3^1 \\ 18 &= 2 \cdot 3 \cdot 3 = 2^1 \cdot 3^2 \end{aligned}$$

To find the LCD, write down each factor that appears to the highest power of that factor that appears. The factors that appear are 2 and 3. The highest power of 2 that appears is 2^3. The highest power of 3 that appears is 3^2.

$$\begin{aligned} \text{LCM} &= 2^3 \cdot 3^2 && \text{Keep highest power of each factor.} \\ &= 8 \cdot 9 && \text{Expand: } 2^3 = 8 \text{ and } 3^2 = 9. \\ &= 72. && \text{Multiply.} \end{aligned}$$

Create equivalent fractions with the new LCD, then subtract.

$$\begin{aligned}
-\frac{11}{24} - \frac{1}{18} &= -\frac{11 \cdot 3}{24 \cdot 3} - \frac{1 \cdot 4}{18 \cdot 4} && \text{Equivalent fractions with LCD=72.} \\
&= -\frac{33}{72} - \frac{4}{72} && \text{Simplify numerators and denominators.} \\
&= \frac{-33 - 4}{72} && \text{Keep LCD; subtract numerators.} \\
&= \frac{-37}{72} && \text{Simplify numerator.}
\end{aligned}$$

Of course, negative divided by positive yields a negative answer, so we can also write our answer in the form

$$-\frac{11}{24} - \frac{1}{18} = -\frac{37}{72}.$$

Answer: $-37/72$

Comparing Fractions

The simplest way to compare fractions is to create equivalent fractions.

You Try It!

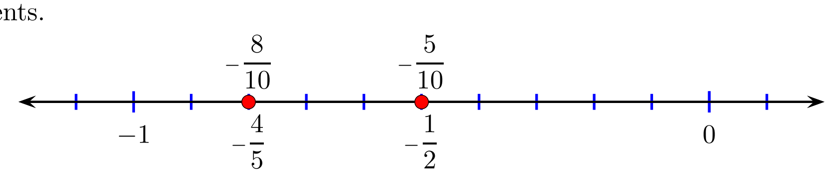
Compare $-3/8$ and $-1/2$.

EXAMPLE 12. Arrange the fractions $-1/2$ and $-4/5$ on a number line, then compare them by using the appropriate inequality symbol.

Solution. The least common denominator for 2 and 5 is the number 10. First, make equivalent fractions with a LCD equal to 10.

$$\begin{aligned}
-\frac{1}{2} &= -\frac{1 \cdot 5}{2 \cdot 5} = -\frac{5}{10} \\
-\frac{4}{5} &= -\frac{4 \cdot 2}{5 \cdot 2} = -\frac{8}{10}
\end{aligned}$$

To plot tenths, subdivide the interval between -1 and 0 into ten equal increments.



Because $-4/5$ lies to the left of $-1/2$, we have that $-4/5$ is *less than* $-1/2$, so we write

$$-\frac{4}{5} < -\frac{1}{2}.$$

Answer: $-\frac{1}{2} < -\frac{3}{8}$

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-10, list the multiples the given numbers, then list the common multiples. Select the LCM from the list of common multiples.

1. 9 and 15

2. 15 and 20

3. 20 and 8

4. 15 and 6

5. 16 and 20

6. 6 and 10

7. 20 and 12

8. 12 and 8

9. 10 and 6

10. 10 and 12

In Exercises 11-20, for the given numbers, calculate the LCM using prime factorization.

11. 54 and 12

12. 108 and 24

13. 18 and 24

14. 36 and 54

15. 72 and 108

16. 108 and 72

17. 36 and 24

18. 18 and 12

19. 12 and 18

20. 12 and 54

In Exercises 21-32, add or subtract the fractions, as indicated, and simplify your result.

21. $\frac{7}{12} - \frac{1}{12}$

22. $\frac{3}{7} - \frac{5}{7}$

23. $\frac{1}{9} + \frac{1}{9}$

24. $\frac{1}{7} + \frac{3}{7}$

25. $\frac{1}{5} - \frac{4}{5}$

26. $\frac{3}{5} - \frac{2}{5}$

27. $\frac{3}{7} - \frac{4}{7}$

28. $\frac{6}{7} - \frac{2}{7}$

29. $\frac{4}{11} + \frac{9}{11}$

30. $\frac{10}{11} + \frac{4}{11}$

31. $\frac{3}{11} + \frac{4}{11}$

32. $\frac{3}{7} + \frac{2}{7}$

In Exercises 33-56, add or subtract the fractions, as indicated, and simplify your result.

33. $\frac{1}{6} - \frac{1}{8}$

34. $\frac{7}{9} - \frac{2}{3}$

35. $\frac{1}{5} + \frac{2}{3}$

36. $\frac{7}{9} + \frac{2}{3}$

37. $\frac{2}{3} + \frac{5}{8}$

38. $\frac{3}{7} + \frac{5}{9}$

39. $\frac{4}{7} - \frac{5}{9}$

40. $\frac{3}{5} - \frac{7}{8}$

41. $\frac{2}{3} - \frac{3}{8}$

42. $\frac{2}{5} - \frac{1}{8}$

43. $\frac{6}{7} - \frac{1}{6}$

44. $\frac{1}{2} - \frac{1}{4}$

45. $\frac{1}{6} + \frac{2}{3}$

46. $\frac{4}{9} + \frac{7}{8}$

47. $\frac{7}{9} + \frac{1}{8}$

48. $\frac{1}{6} + \frac{1}{7}$

49. $\frac{1}{3} + \frac{1}{7}$

50. $\frac{5}{6} + \frac{1}{4}$

51. $\frac{1}{2} - \frac{2}{7}$

52. $\frac{1}{3} - \frac{1}{8}$

53. $\frac{5}{6} - \frac{4}{5}$

54. $\frac{1}{2} - \frac{1}{9}$

55. $\frac{1}{3} + \frac{1}{8}$

56. $\frac{1}{6} + \frac{7}{9}$

In Exercises 57-68, add or subtract the fractions, as indicated, by first using prime factorization to find the least common denominator.

57. $\frac{7}{36} + \frac{11}{54}$

58. $\frac{7}{54} + \frac{7}{24}$

59. $\frac{7}{18} - \frac{5}{12}$

60. $\frac{5}{54} - \frac{7}{12}$

61. $\frac{7}{36} + \frac{7}{54}$

62. $\frac{5}{72} + \frac{5}{108}$

63. $\frac{7}{24} - \frac{5}{36}$

64. $\frac{11}{54} + \frac{7}{72}$

65. $\frac{11}{12} + \frac{5}{18}$

66. $\frac{11}{24} + \frac{11}{108}$

67. $\frac{11}{54} - \frac{5}{24}$

68. $\frac{7}{54} - \frac{5}{24}$

In Exercises 69-80, add or subtract the fractions, as indicated, and simplify your result.

69. $-\frac{3}{7}+\left(-\frac{3}{7}\right)$

70. $-\frac{5}{9}+\left(-\frac{1}{9}\right)$

71. $\frac{7}{9}-\left(-\frac{1}{9}\right)$

72. $\frac{8}{9}-\left(-\frac{4}{9}\right)$

73. $\frac{7}{9}+\left(-\frac{2}{9}\right)$

74. $\frac{2}{3}+\left(-\frac{1}{3}\right)$

75. $-\frac{3}{5}-\frac{4}{5}$

76. $-\frac{7}{9}-\frac{1}{9}$

77. $-\frac{7}{8}+\frac{1}{8}$

78. $-\frac{2}{3}+\frac{1}{3}$

79. $-\frac{1}{3}-\left(-\frac{2}{3}\right)$

80. $-\frac{7}{8}-\left(-\frac{5}{8}\right)$

In Exercises 81-104, add or subtract the fractions, as indicated, and simplify your result.

81. $-\frac{2}{7}+\frac{4}{5}$

82. $-\frac{1}{4}+\frac{2}{7}$

83. $-\frac{1}{4}-\left(-\frac{4}{9}\right)$

84. $-\frac{3}{4}-\left(-\frac{1}{8}\right)$

85. $-\frac{2}{7}+\frac{3}{4}$

86. $-\frac{1}{3}+\frac{5}{8}$

87. $-\frac{4}{9}-\frac{1}{3}$

88. $-\frac{5}{6}-\frac{1}{3}$

89. $-\frac{5}{7}-\left(-\frac{1}{5}\right)$

90. $-\frac{6}{7}-\left(-\frac{1}{8}\right)$

91. $\frac{1}{9}+\left(-\frac{1}{3}\right)$

92. $\frac{1}{8}+\left(-\frac{1}{2}\right)$

93. $\frac{2}{3}+\left(-\frac{1}{9}\right)$

94. $\frac{3}{4}+\left(-\frac{2}{3}\right)$

95. $-\frac{1}{2}+\left(-\frac{6}{7}\right)$

96. $-\frac{4}{5}+\left(-\frac{1}{2}\right)$

97. $-\frac{1}{2}+\left(-\frac{3}{4}\right)$

98. $-\frac{3}{5}+\left(-\frac{1}{2}\right)$

99. $-\frac{1}{4}-\frac{1}{2}$

100. $-\frac{8}{9}-\frac{2}{3}$

101. $\frac{5}{8} - \left(-\frac{3}{4}\right)$

102. $\frac{3}{4} - \left(-\frac{3}{8}\right)$

103. $\frac{1}{8} - \left(-\frac{1}{3}\right)$

104. $\frac{1}{2} - \left(-\frac{4}{9}\right)$

In Exercises 105-120, add or subtract the fractions, as indicated, and write your answer is lowest terms.

105. $\frac{1}{2} + \frac{3q}{5}$

106. $\frac{4}{7} - \frac{b}{3}$

107. $\frac{4}{9} - \frac{3a}{4}$

108. $\frac{4}{9} - \frac{b}{2}$

109. $\frac{2}{s} + \frac{1}{3}$

110. $\frac{2}{s} + \frac{3}{7}$

111. $\frac{1}{3} - \frac{7}{b}$

112. $\frac{1}{2} - \frac{9}{s}$

113. $\frac{4b}{7} + \frac{2}{3}$

114. $\frac{2a}{5} + \frac{5}{8}$

115. $\frac{2}{3} - \frac{9}{t}$

116. $\frac{4}{7} - \frac{1}{y}$

117. $\frac{9}{s} + \frac{7}{8}$

118. $\frac{6}{t} - \frac{1}{9}$

119. $\frac{7b}{8} - \frac{5}{9}$

120. $\frac{3p}{4} - \frac{1}{8}$

In Exercises 121-132, determine which of the two given statements is true.

121. $-\frac{2}{3} < -\frac{8}{7}$ or $-\frac{2}{3} > -\frac{8}{7}$

122. $-\frac{1}{7} < -\frac{8}{9}$ or $-\frac{1}{7} > -\frac{8}{9}$

123. $\frac{6}{7} < \frac{7}{3}$ or $\frac{6}{7} > \frac{7}{3}$

124. $\frac{1}{2} < \frac{2}{7}$ or $\frac{1}{2} > \frac{2}{7}$

125. $-\frac{9}{4} < -\frac{2}{3}$ or $-\frac{9}{4} > -\frac{2}{3}$

126. $-\frac{3}{7} < -\frac{9}{2}$ or $-\frac{3}{7} > -\frac{9}{2}$

127. $\frac{5}{7} < \frac{5}{9}$ or $\frac{5}{7} > \frac{5}{9}$

128. $\frac{1}{2} < \frac{1}{3}$ or $\frac{1}{2} > \frac{1}{3}$

129. $-\frac{7}{2} < -\frac{1}{5}$ or $-\frac{7}{2} > -\frac{1}{5}$

130. $-\frac{3}{4} < -\frac{5}{9}$ or $-\frac{3}{4} > -\frac{5}{9}$

131. $\frac{5}{9} < \frac{6}{5}$ or $\frac{5}{9} > \frac{6}{5}$

132. $\frac{3}{2} < \frac{7}{9}$ or $\frac{3}{2} > \frac{7}{9}$

❧ ❧ ❧ Answers ☙ ☙ ☙

1. 45

3. 40

5. 80

7. 60

9. 30

11. 108

13. 72

15. 216

17. 72

19. 36

21. $\frac{1}{2}$

23. $\frac{2}{9}$

25. $-\frac{3}{5}$

27. $-\frac{1}{7}$

29. $\frac{13}{11}$

31. $\frac{7}{11}$

33. $\frac{1}{24}$

35. $\frac{13}{15}$

37. $\frac{31}{24}$

39. $\frac{1}{63}$

41. $\frac{7}{24}$

43. $\frac{29}{42}$

45. $\frac{5}{6}$

47. $\frac{65}{72}$

49. $\frac{10}{21}$

51. $\frac{3}{14}$

53. $\frac{1}{30}$

55. $\frac{11}{24}$

57. $\frac{43}{108}$

59. $-\frac{1}{36}$

61. $\frac{35}{108}$

63. $\frac{11}{72}$

65. $\frac{43}{36}$

67. $-\frac{1}{216}$

69. $-\frac{6}{7}$

71. $\frac{8}{9}$

73. $\frac{5}{9}$

75. $-\frac{7}{5}$

77. $-\frac{3}{4}$

79. $\frac{1}{3}$

81. $\frac{18}{35}$

83. $\frac{7}{36}$

85. $\frac{13}{28}$

87. $-\frac{7}{9}$

89. $-\frac{18}{35}$

91. $-\frac{2}{9}$

93. $\frac{5}{9}$

95. $-\frac{19}{14}$

97. $-\frac{5}{4}$

99. $-\frac{3}{4}$

101. $\frac{11}{8}$

103. $\frac{11}{24}$

105. $\frac{5+6q}{10}$

107. $\frac{16-27a}{36}$

109. $\frac{6+s}{3s}$

111. $\frac{b-21}{3b}$

113. $\frac{12b+14}{21}$

115. $\frac{2t-27}{3t}$

117. $\frac{72+7s}{8s}$

119. $\frac{63b-40}{72}$

121. $-\frac{2}{3} > -\frac{8}{7}$

123. $\frac{6}{7} < \frac{7}{3}$

125. $-\frac{9}{4} < -\frac{2}{3}$

127. $\frac{5}{7} > \frac{5}{9}$

129. $-\frac{7}{2} < -\frac{1}{5}$

131. $\frac{5}{9} < \frac{6}{5}$

4.5 Multiplying and Dividing Mixed Fractions

We begin with definitions of *proper* and *improper* fractions.

Proper and Improper Fractions. A *proper fraction* is a fraction whose numerator is smaller than its denominator. An *improper fraction* is a fraction whose numerator is larger than its denominator.

For example,

$$\frac{2}{3}, \ -\frac{23}{39}, \quad \text{and} \quad \frac{119}{127}$$

are all examples of proper fractions. On the other hand,

$$\frac{4}{3}, \ -\frac{317}{123}, \quad \text{and} \quad -\frac{233}{101}$$

are all examples of improper fractions.

A *mixed fraction*[1] is part whole number, part fraction.

Mixed Fractions. The number

$$5\frac{3}{4}$$

is called a *mixed fraction.* It is defined to mean

$$5\frac{3}{4} = 5 + \frac{3}{4}.$$

In the mixed fraction $5\frac{3}{4}$, the 5 is the *whole number part* and the 3/4 is the *fractional part.*

Changing Mixed Fractions to Improper Fractions

We have all the tools required to change a mixed fraction into an improper fraction. We begin with an example.

You Try It!

EXAMPLE 1. Change the mixed fraction $4\frac{7}{8}$ into an improper fraction.

Change $5\frac{3}{4}$ to an improper fraction.

[1] A mixed fractions is sometimes called a *mixed number.*

Solution. We employ the definition of a mixed fraction, make an equivalent fraction for the whole number part, then add.

$$\begin{aligned} 4\frac{7}{8} &= 4 + \frac{7}{8} && \text{By definition.} \\ &= \frac{4 \cdot 8}{8} + \frac{7}{8} && \text{Equivalent fraction with LCD} = 8. \\ &= \frac{4 \cdot 8 + 7}{8} && \text{Add numerators over common denominator.} \\ &= \frac{39}{8} && \text{Simplify the numerator.} \end{aligned}$$

Answer: 23/4

Thus, $4\frac{7}{8}$ is equal to 39/8.

□

There is a quick technique you can use to change a mixed fraction into an improper fraction.

Quick Way to Change a Mixed Fraction to an Improper Fraction. To change a mixed fraction to an improper fraction, multiply the whole number part by the denominator, add the numerator, then place the result over the denominator.

Thus, to quickly change $4\frac{7}{8}$ to an improper fraction, multiply the whole number 4 by the denominator 8, add the numerator 7, then place the result over the denominator. In symbols, this would look like this:

$$4\frac{7}{8} = \frac{4 \cdot 8 + 7}{8}.$$

This is precisely what the third step in Example 1 looks like; we're just eliminating a lot of the work.

You Try It!

Change $7\frac{3}{8}$ to an improper fraction.

EXAMPLE 2. Change $4\frac{2}{3}$ to an improper fraction.

Solution. Take $4\frac{2}{3}$, multiply the whole number part by the denominator, add the numerator, then put the result over the denominator.

$$4\frac{2}{3} = \frac{4 \cdot 3 + 2}{3}$$

Thus, the result is

$$4\frac{2}{3} = \frac{14}{3}.$$

Answer: 59/8

□

It is very easy to do the intermediate step in Example 2 mentally, allowing you to skip the intermediate step and go directly from the mixed fraction to the improper fraction without writing down a single bit of work.

You Try It!

EXAMPLE 3. Without writing down any work, use mental arithmetic to change $-2\frac{3}{5}$ to an improper fraction.

Change $-3\frac{5}{12}$ to an improper fraction.

Solution. To change $-2\frac{3}{5}$ to an improper fraction, ignore the minus sign, proceed as before, then prefix the minus sign to the resulting improper fraction. So, multiply 5 times 2 and add 3. Put the result 13 over the denominator 5, then prefix the resulting improper fraction with a minus sign. That is,

$$-2\frac{3}{5} = -\frac{13}{5}.$$

Answer: $-41/12$

□

Changing Improper Fractions to Mixed Fractions

The first step in changing the improper fraction 27/5 to a mixed fraction is to write the improper fraction as a sum.

$$\frac{27}{5} = \frac{25}{5} + \frac{2}{5} \tag{4.1}$$

Simplifying equation 4.1, we get

$$\begin{aligned}\frac{27}{5} &= 5 + \frac{2}{5} \\ &= 5\frac{2}{5}.\end{aligned}$$

Comment. You can't just choose any sum. The sum used in equation 4.1 is constructed so that the first fraction will equal a whole number and the second fraction is proper. Any other sum will fail to produce the correct mixed fraction. For example, the sum

$$\frac{27}{5} = \frac{23}{5} + \frac{4}{5}$$

is useless, because 23/5 is not a whole number. Likewise, the sum

$$\frac{27}{5} = \frac{20}{5} + \frac{7}{5}$$

is no good. Even though $20/5 = 4$ is a whole number, the second fraction 7/5 is still improper.

You Try It!

Change 25/7 to a mixed fraction.

EXAMPLE 4. Change 25/9 to a mixed fraction.

Solution. Break 25/9 into the appropriate sum.

$$\begin{aligned}\frac{25}{9} &= \frac{18}{9} + \frac{7}{9} \\ &= 2 + \frac{7}{9} \\ &= 2\frac{7}{9}\end{aligned}$$

Answer: $3\frac{4}{7}$

□

Comment. A pattern is emerging.

- In the case of 27/5, note that 27 divided by 5 is equal to 5 with a remainder of 2. Compare this with the mixed fraction result: $27/5 = 5\frac{2}{5}$.
- In the case of Example 4, note that 25 divided by 9 is 2 with a remainder of 7. Compare this with the mixed fraction result: $25/9 = 2\frac{7}{9}$.

These observations motivate the following technique.

Quick Way to Change an Improper Fraction to a Mixed Fraction. To change an improper fraction to a mixed fraction, divide the numerator by the denominator. The quotient will be the whole number part of the mixed fraction. If you place the remainder over the denominator, this will be the fractional part of the mixed fraction.

You Try It!

Change 38/9 to a mixed fraction.

EXAMPLE 5. Change 37/8 to a mixed fraction.

Solution. 37 divided by 8 is 4, with a remainder of 5. That is:

$$\begin{array}{r} 4 \\ 8\overline{)37} \\ \underline{32} \\ 5 \end{array}$$

The quotient becomes the whole number part and we put the remainder over the divisor. Thus,

$$\frac{37}{8} = 4\frac{5}{8}.$$

Note: You can check your result with the "Quick Way to Change a Mixed Fraction to an Improper Fraction." 8 times 4 plus 5 is 37. Put this over 8 to get 37/8.

Answer: $4\frac{2}{9}$

□

You Try It!

EXAMPLE 6. Change $-43/5$ to a mixed fraction.

Change $-27/8$ to a mixed fraction.

Solution. Ignore the minus sign and proceed in the same manner as in Example 5. 43 divided by 5 is 8, with a remainder of 3.

$$\begin{array}{r} 8 \\ 5\overline{)43} \\ \underline{40} \\ 3 \end{array}$$

The quotient is the whole number part, then we put the remainder over the divisor. Finally, prefix the minus sign.

$$-\frac{43}{5} = -8\frac{3}{5}.$$

Answer: $-3\frac{3}{8}$

□

Multiplying and Dividing Mixed Fractions

You have all the tools needed to multiply and divide mixed fractions. First, change the mixed fractions to improper fractions, then multiply or divide as you did in previous sections.

You Try It!

EXAMPLE 7. Simplify: $-2\frac{1}{12} \cdot 2\frac{4}{5}$.

Simplify:

$$-3\frac{3}{4} \cdot 2\frac{2}{5}$$

Solution. Change to improper fractions, factor, cancel, and simplify.

$$\begin{aligned} -2\frac{1}{12} \cdot 2\frac{4}{5} &= -\frac{25}{12} \cdot \frac{14}{5} && \text{Change to improper fractions.} \\ &= -\frac{25 \cdot 14}{12 \cdot 5} && \text{Multiply numerators; multiply denominators.} \\ & && \text{Unlike signs; product is negative.} \\ &= -\frac{(5 \cdot 5) \cdot (2 \cdot 7)}{(2 \cdot 2 \cdot 3) \cdot (5)} && \text{Prime factor.} \\ &= -\frac{\cancel{5} \cdot 5 \cdot \cancel{2} \cdot 7}{\cancel{2} \cdot 2 \cdot 3 \cdot \cancel{5}} && \text{Cancel common factors.} \\ &= -\frac{35}{6} && \text{Multiply numerators and denominators.} \end{aligned}$$

This is a perfectly good answer, but if you want a mixed fraction answer, 35 divided by 6 is 5, with a remainder of 5. Hence,

$$-2\frac{1}{12} \cdot 2\frac{4}{5} = -5\frac{5}{6}.$$

Answer: -9

You Try It!

Simplify:

$$-2\frac{4}{9} \cdot 3\frac{2}{3}$$

Answer: $-2/3$

EXAMPLE 8. Simplify: $-4\frac{4}{5} \div 5\frac{3}{5}$.

Solution. Change to improper fractions, invert and multiply, factor, cancel, and simplify.

$$\begin{aligned}
-4\frac{4}{5} \div 5\frac{3}{5} &= -\frac{24}{5} \div \frac{28}{5} && \text{Change to improper fractions.} \\
&= -\frac{24}{5} \cdot \frac{5}{28} && \text{Invert and multiply.} \\
&= -\frac{2 \cdot 2 \cdot 2 \cdot 3}{5} \cdot \frac{5}{2 \cdot 2 \cdot 7} && \text{Prime factor.} \\
&= -\frac{\cancel{2} \cdot \cancel{2} \cdot 2 \cdot 3}{\cancel{5}} \cdot \frac{\cancel{5}}{\cancel{2} \cdot \cancel{2} \cdot 7} && \text{Cancel common factors.} \\
&= -\frac{6}{7} && \text{Multiply numerators and denominators.}
\end{aligned}$$

Exercises

In Exercises 1-12, convert the mixed fraction to an improper fraction.

1. $2\frac{1}{3}$

2. $1\frac{8}{11}$

3. $1\frac{1}{19}$

4. $-1\frac{1}{5}$

5. $-1\frac{3}{7}$

6. $1\frac{3}{17}$

7. $1\frac{1}{9}$

8. $1\frac{5}{11}$

9. $-1\frac{1}{2}$

10. $-1\frac{5}{8}$

11. $1\frac{1}{3}$

12. $-1\frac{5}{7}$

In Exercises 13-24, convert the improper fraction to a mixed fraction.

13. $\frac{13}{7}$

14. $-\frac{17}{9}$

15. $-\frac{13}{5}$

16. $-\frac{10}{3}$

17. $-\frac{16}{5}$

18. $\frac{16}{13}$

19. $\frac{9}{8}$

20. $\frac{16}{5}$

21. $-\frac{6}{5}$

22. $-\frac{17}{10}$

23. $-\frac{3}{2}$

24. $-\frac{7}{4}$

In Exercises 25-48, multiply the numbers and express your answer as a mixed fraction.

25. $1\frac{1}{7} \cdot 2\frac{1}{2}$

26. $1\frac{1}{8} \cdot 1\frac{1}{6}$

27. $4 \cdot 1\frac{1}{6}$

28. $1\frac{7}{10} \cdot 4$

29. $\left(-1\frac{1}{12}\right)\left(3\frac{3}{4}\right)$

30. $\left(-3\frac{1}{2}\right)\left(3\frac{1}{3}\right)$

31. $7\frac{1}{2} \cdot 1\frac{1}{13}$

32. $2\frac{1}{4} \cdot 1\frac{5}{11}$

33. $\left(1\frac{2}{13}\right)\left(-4\frac{2}{3}\right)$

34. $\left(1\frac{1}{14}\right)\left(-2\frac{2}{5}\right)$

35. $\left(1\frac{3}{7}\right)\left(-3\frac{3}{4}\right)$

36. $\left(1\frac{4}{5}\right)\left(-3\frac{3}{4}\right)$

37. $9 \cdot \left(-1\frac{2}{15}\right)$

38. $4 \cdot \left(-2\frac{5}{6}\right)$

39. $\left(-2\frac{1}{8}\right)(-6)$

40. $(-9)\left(-3\frac{1}{6}\right)$

41. $\left(-4\frac{1}{2}\right)\left(-2\frac{2}{5}\right)$

42. $\left(-1\frac{3}{7}\right)\left(-3\frac{3}{4}\right)$

43. $\left(-2\frac{1}{6}\right) \cdot 4$

44. $(-6) \cdot \left(1\frac{1}{9}\right)$

45. $\left(-1\frac{4}{15}\right)\left(2\frac{1}{2}\right)$

46. $\left(-1\frac{1}{5}\right)\left(1\frac{5}{9}\right)$

47. $\left(-2\frac{1}{2}\right)\left(-1\frac{7}{11}\right)$

48. $\left(-1\frac{7}{11}\right)\left(-1\frac{7}{12}\right)$

In Exercises 49-72, divide the mixed fractions and express your answer as a mixed fraction.

49. $8 \div 2\frac{2}{9}$

50. $4\frac{2}{3} \div 4$

51. $\left(-3\frac{1}{2}\right) \div \left(1\frac{1}{16}\right)$

52. $\left(-1\frac{2}{5}\right) \div \left(1\frac{1}{15}\right)$

53. $6\frac{1}{2} \div 1\frac{7}{12}$

54. $5\frac{1}{2} \div 1\frac{9}{10}$

55. $(-4) \div \left(1\frac{5}{9}\right)$

56. $\left(-4\frac{2}{3}\right) \div 4$

57. $\left(-5\frac{2}{3}\right) \div \left(-2\frac{1}{6}\right)$

58. $\left(-2\frac{1}{2}\right) \div \left(-2\frac{2}{9}\right)$

59. $\left(-6\frac{1}{2}\right) \div \left(4\frac{1}{4}\right)$

60. $\left(-1\frac{1}{6}\right) \div \left(1\frac{1}{8}\right)$

61. $(-6) \div \left(-1\frac{3}{11}\right)$

62. $\left(-6\frac{2}{3}\right) \div (-6)$

63. $\left(4\frac{2}{3}\right) \div (-4)$

64. $\left(6\frac{2}{3}\right) \div (-6)$

65. $\left(1\frac{3}{4}\right) \div \left(-1\frac{1}{12}\right)$

66. $\left(2\frac{4}{7}\right) \div \left(-1\frac{1}{5}\right)$

67. $5\frac{2}{3} \div 1\frac{1}{9}$

68. $1\frac{2}{3} \div 1\frac{2}{9}$

69. $\left(-7\frac{1}{2}\right) \div \left(-2\frac{2}{5}\right)$

70. $\left(-5\frac{1}{3}\right) \div \left(-2\frac{5}{6}\right)$

71. $\left(3\frac{2}{3}\right) \div \left(-1\frac{1}{9}\right)$

72. $\left(8\frac{1}{2}\right) \div \left(-1\frac{3}{4}\right)$

---

73. **Small Lots.** How many quarter-acre lots can be made from $6\frac{1}{2}$ acres of land?

74. **Big Field.** A field was formed from $17\frac{1}{2}$ half-acre lots. How many acres was the resulting field ?

75. **Jewelry.** To make some jewelry, a bar of silver $4\frac{1}{2}$ inches long was cut into pieces $\frac{1}{12}$ inch long. How many pieces were made?

76. **Muffins.** This recipe will make 6 muffins: 1 cup milk, $1\frac{2}{3}$ cups flour, 2 eggs, 1/2 teaspoon salt, $1\frac{1}{2}$ teaspoons baking powder. Write the recipe for six dozen muffins.

❧ ❧ ❧ Answers ☙ ☙ ☙

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1. $\frac{7}{3}$

3. $\frac{20}{19}$

5. $-\frac{10}{7}$

7. $\frac{10}{9}$

9. $-\frac{3}{2}$

11. $\frac{4}{3}$

13. $1\frac{6}{7}$

15. $-2\frac{3}{5}$

17. $-3\frac{1}{5}$

19. $1\frac{1}{8}$

21. $-1\frac{1}{5}$

23. $-1\frac{1}{2}$

25. $2\frac{6}{7}$

27. $4\frac{2}{3}$

29. $-4\frac{1}{16}$

31. $8\frac{1}{13}$

33. $-5\frac{5}{13}$

35. $-5\frac{5}{14}$

37. $-10\frac{1}{5}$

39. $12\frac{3}{4}$

41. $10\frac{4}{5}$

43. $-8\frac{2}{3}$

45. $-3\frac{1}{6}$

47. $4\frac{1}{11}$

49. $3\frac{3}{5}$

51. $-3\frac{5}{17}$

53. $4\frac{2}{19}$

55. $-2\frac{4}{7}$

57. $2\frac{8}{13}$

59. $-1\frac{9}{17}$

61. $4\frac{5}{7}$

63. $-1\frac{1}{6}$

65. $-1\frac{8}{13}$

67. $5\frac{1}{10}$

69. $3\frac{1}{8}$

71. $-3\frac{3}{10}$

73. 26 quarter-acre lots

75. 54 pieces

4.6 Adding and Subtracting Mixed Fractions

In this section, we will learn how to add and subtract mixed fractions.

Adding Mixed Fractions

We can use tools we've already developed to add two or more mixed fractions.

You Try It!

EXAMPLE 1. Simplify: $2\frac{7}{8}+1\frac{3}{4}$.

Solution. Change the mixed fractions to improper fractions, make equivalent fractions with a common denominator, then add.

$$\begin{aligned} 2\,\frac{7}{8}+1\,\frac{3}{4} &= \frac{23}{8}+\frac{7}{4} && \text{Change to improper fractions.}\\ &= \frac{23}{8}+\frac{7\cdot 2}{4\cdot 2} && \text{Equivalent fractions with LCD} = 8.\\ &= \frac{23}{8}+\frac{14}{8} && \text{Simplify numerators and denominators.}\\ &= \frac{37}{8} && \text{Add numerators over common denominator.} \end{aligned}$$

Although this answer is perfectly acceptable, let's change the answer to a mixed fraction: 37 divided by 8 is 4, with a remainder of 5. Thus,

$$2\,\frac{7}{8}+1\,\frac{3}{4}=4\,\frac{5}{8}.$$

Simplify:

$$3\frac{2}{3}+4\frac{1}{8}$$

Answer: $7\frac{19}{24}$

□

You Try It!

EXAMPLE 2. Simplify: $3\frac{1}{4}+2\frac{1}{3}$.

Solution. Change the mixed fractions to improper fractions, make equivalent fractions with a common denominator, then add.

$$\begin{aligned} 3\,\frac{1}{4}+2\,\frac{1}{3} &= \frac{13}{4}+\frac{7}{3} && \text{Change to improper fractions.}\\ &= \frac{13\cdot 3}{4\cdot 3}+\frac{7\cdot 4}{3\cdot 4} && \text{Equivalent fractions with LCD} = 12.\\ &= \frac{39}{12}+\frac{28}{12} && \text{Simplify numerators and denominators.}\\ &= \frac{67}{12} && \text{Add numerators over common denominator.} \end{aligned}$$

Simplify:

$$8\frac{1}{2}+2\frac{2}{3}$$

Although this answer is perfectly acceptable, let's change the answer to a mixed fraction: 67 divided by 12 is 5, with a remainder of 7. Thus,

$$3\,\frac{1}{4} + 2\,\frac{1}{3} = 5\,\frac{7}{12}.$$

Answer: $11\frac{1}{6}$

Mixed Fraction Approach. There is another possible approach, based on the fact that a mixed fraction is a sum. Let's revisit Example 2.

You Try It!

Simplify:

$$7\frac{2}{5} + 3\frac{1}{8}$$

EXAMPLE 3. Simplify: $3\frac{1}{4} + 2\frac{1}{3}$.

Solution. Use the commutative and associative properties to change the order of addition, make equivalent fractions with a common denominator, then add.

$$\begin{aligned}
3\,\frac{1}{4} + 2\,\frac{1}{3} &= \left(3 + \frac{1}{4}\right) + \left(2 + \frac{1}{3}\right) && \text{Mixed fractions as sums.}\\
&= (3+2) + \left(\frac{1}{4} + \frac{1}{3}\right) && \text{Reorder and regroup.}\\
&= 5 + \left(\frac{1\cdot 3}{4\cdot 3} + \frac{1\cdot 4}{3\cdot 4}\right) && \text{Add whole numbers: } 3+2=5.\\
& && \text{Equivalent fractions; LCD} = 12.\\
&= 5 + \left(\frac{3}{12} + \frac{4}{12}\right) && \text{Simplify numerators and denominators.}\\
&= 5 + \frac{7}{12} && \text{Add numerators over common denominator.}
\end{aligned}$$

This result can be written in mixed fraction form. Thus,

$$3\,\frac{1}{4} + 2\,\frac{1}{3} = 5\,\frac{7}{12}.$$

Answer: $10\frac{21}{40}$

Note that this solution is identical to the result found in Example 2.

Example 3 leads us to the following result.

Adding Mixed Fractions. To add two mixed fractions, add the whole number parts, then add the fractional parts.

Working in Vertical Format

When adding mixed fractions, many prefer to work in a vertical format. For example, here is how we would arrange the solution from Example 2 and Example 3 in vertical format. We create equivalent fractions, then add the whole number parts and fractional parts.

$$\begin{array}{ccccc} 3\,\dfrac{1}{4} & = & 3\,\dfrac{1\cdot 3}{4\cdot 3} & = & 3\,\dfrac{3}{12} \\ +2\,\dfrac{1}{3} & = & +2\,\dfrac{1\cdot 4}{3\cdot 4} & = & +2\,\dfrac{4}{12} \\ \hline & & & & 5\,\dfrac{7}{12} \end{array}$$

Note that the answer is identical to the answer found in Example 2 and Example 3. That is,

$$3\,\frac{1}{4} + 2\,\frac{1}{3} = 5\,\frac{7}{12}.$$

You Try It!

EXAMPLE 4. Sarah is making window curtains for two rooms in her house. The kitchen will require $5\frac{2}{3}$ yards of material and the dining room will require $6\frac{5}{8}$ yards of material. How much total material is required?

Solution. To find the total material required for the two rooms, we must add $5\frac{2}{3}$ and $6\frac{5}{8}$. Create equivalent fractions with a common denominator, then add whole number parts and fractional parts.

$$\begin{array}{ccccc} 5\,\dfrac{2}{3} & = & 5\,\dfrac{2\cdot 8}{3\cdot 8} & = & 5\,\dfrac{16}{24} \\ +6\,\dfrac{5}{8} & = & +6\,\dfrac{5\cdot 3}{8\cdot 3} & = & +6\,\dfrac{15}{24} \\ \hline & & & & 11\,\dfrac{31}{24} \end{array}$$

An answer that is part mixed fraction, part improper fraction, is **not** allowed. To finish, we need to change the improper fractional part to a mixed fraction, then add. 31 divided by 24 is 1, with a remainder of 7. That is, $31/24 = 1\,\frac{7}{24}$. Now we can add whole number parts and fractional parts.

$$\begin{aligned} 11\,\frac{31}{24} &= 11 + 1\,\frac{7}{24} \\ &= 12\,\frac{7}{24}. \end{aligned}$$

Thus, the total material required is $12\frac{7}{24}$ yards.

Jim is working on a project that requires two boards, the first cut to a length of $6\frac{1}{2}$ feet, the second cut to a length of $5\frac{7}{8}$ feet. How many total feet of board is required?

Answer: $12\frac{3}{8}$ feet

□

Subtracting Mixed Fractions

Let's look at some examples that subtract two mixed fractions.

You Try It!

Simplify:

$$5\frac{2}{3} - 3\frac{1}{5}$$

EXAMPLE 5. Simplify: $4\frac{5}{8} - 2\frac{1}{16}$.

Solution. Change the mixed fractions to improper fractions, make equivalent fractions with a common denominator, then subtract.

$$\begin{aligned} 4\,\frac{5}{8} - 2\,\frac{1}{16} &= \frac{37}{8} - \frac{33}{16} && \text{Change to improper fractions.} \\ &= \frac{37 \cdot 2}{8 \cdot 2} - \frac{33}{16} && \text{Equivalent fractions with LCD} = 16. \\ &= \frac{74}{16} - \frac{33}{16} && \text{Simplify numerators and denominators.} \\ &= \frac{41}{16} && \text{Add numerators over common denominator.} \end{aligned}$$

Although this answer is perfectly acceptable, let's change the answer to a mixed fraction: 41 divided by 16 is 2, with a remainder of 9. Thus,

$$4\,\frac{5}{8} - 2\,\frac{1}{16} = 2\,\frac{9}{16}.$$

Answer: $2\frac{7}{15}$

□

You Try It!

Simplify:

$$4\frac{7}{9} - 2\frac{3}{18}$$

EXAMPLE 6. Simplify: $5\frac{3}{4} - 2\frac{1}{3}$.

Solution. Change the mixed fractions to improper fractions, make equivalent fractions with a common denominator, then subtract.

$$\begin{aligned} 5\,\frac{3}{4} - 2\,\frac{1}{3} &= \frac{23}{4} - \frac{7}{3} && \text{Change to improper fractions.} \\ &= \frac{23 \cdot 3}{4 \cdot 3} - \frac{7 \cdot 4}{3 \cdot 4} && \text{Equivalent fractions with LCD} = 12. \\ &= \frac{69}{12} - \frac{28}{12} && \text{Simplify numerators and denominators.} \\ &= \frac{41}{12} && \text{Add numerators over common denominator.} \end{aligned}$$

Although this answer is perfectly acceptable, let's change the answer to a mixed fraction: 41 divided by 12 is 3, with a remainder of 5. Thus,

$$5\,\frac{3}{4} - 2\,\frac{1}{3} = 3\,\frac{5}{12}.$$

Answer: $2\frac{11}{18}$

□

Mixed Fraction Approach. There is another possible approach, based on the fact that a mixed fraction is a sum. Let's revisit Example 6.

You Try It!

EXAMPLE 7. Simplify: $5\frac{3}{4} - 2\frac{1}{3}$.

Simplify:

$$8\frac{5}{6} - 4\frac{3}{8}$$

Solution. A mixed fraction is a sum.

$$5\,\frac{3}{4} - 2\,\frac{1}{3} = \left(5 + \frac{3}{4}\right) - \left(2 + \frac{1}{3}\right)$$

Distribute the negative sign.

$$= 5 + \frac{3}{4} - 2 - \frac{1}{3}$$

We could change the subtraction to adding the opposite, change the order of addition, then change the adding of opposites back to subtraction. However, it is much easier if we look at this last line as a request to add four numbers, two of which are positive and two of which are negative. Changing the order does not affect the answer.

$$= (5 - 2) + \left(\frac{3}{4} - \frac{1}{3}\right)$$

Note that we did not change the signs of any of the four numbers. We just changed the order. Subtract the whole number parts. Make equivalent fractions with a common denominator, then subtract the fractional parts.

$$\begin{aligned}
&= 3 + \left(\frac{3 \cdot 3}{4 \cdot 3} - \frac{1 \cdot 4}{3 \cdot 4}\right) && \text{Create equivalent fractions.}\\
&= 3 + \left(\frac{9}{12} - \frac{4}{12}\right) && \text{Simplify numerators and denominators.}\\
&= 3 + \frac{5}{12} && \text{Subtract fractional parts.}
\end{aligned}$$

Thus,

$$5\,\frac{3}{4} - 2\,\frac{1}{3} = 3\,\frac{5}{12}.$$

Note that this is exactly the same answer as that found in Example 6.

Answer: $4\frac{11}{24}$

□

In Example 6, we see that we handle subtraction of mixed fractions in exactly the same manner that we handle addition of mixed fractions.

Subtracting Mixed Fractions. To subtract two mixed fractions, subtract their whole number parts, then subtract their fractional parts.

Working in Vertical Format

When subtracting mixed fractions, many prefer to work in a vertical format. For example, here is how we would arrange the solution from Example 6 and Example 7 in vertical format. We create equivalent fractions, then subtract the whole number parts and fractional parts.

$$\begin{array}{rcrcr} 5\,\dfrac{3}{4} & = & 5\,\dfrac{3\cdot 3}{4\cdot 3} & = & 5\,\dfrac{9}{12} \\ -2\,\dfrac{1}{3} & = & -2\,\dfrac{1\cdot 4}{3\cdot 4} & = & -2\,\dfrac{4}{12} \\ \hline & & & & 3\,\dfrac{5}{12} \end{array}$$

Note that the answer is identical to the answer found in Example 6 and Example 7. That is,

$$5\,\frac{3}{4} - 2\,\frac{1}{3} = 3\,\frac{5}{12}.$$

Borrowing in Vertical Format

Consider the following example.

You Try It!

Simplify:

$$7\frac{1}{14} - 2\frac{5}{21}$$

EXAMPLE 8. Simplify: $8\frac{1}{4} - 5\frac{5}{6}$.

Solution. Create equivalent fractions with a common denominator.

$$\begin{array}{rcrcr} 8\,\dfrac{1}{4} & = & 8\,\dfrac{1\cdot 3}{4\cdot 3} & = & 8\,\dfrac{3}{12} \\ -5\,\dfrac{5}{6} & = & -5\,\dfrac{5\cdot 2}{6\cdot 2} & = & -5\,\dfrac{10}{12} \\ \hline \end{array}$$

You can see the difficulty. On the far right, we cannot subtract 10/12 from 3/12. The fix is to borrow 1 from 8 in the form of 12/12 and add it to the 3/12.

$$\begin{array}{rcrcr} 8\,\dfrac{3}{12} & = & 7+\dfrac{12}{12}+\dfrac{3}{12} & = & 7\,\dfrac{15}{12} \\ -5\,\dfrac{10}{12} & = & -5\,\dfrac{10}{12} & = & -5\,\dfrac{10}{12} \\ \hline & & & & 2\,\dfrac{5}{12} \end{array}$$

Now we can subtract. Hence, $8\frac{1}{4} - 5\frac{5}{6} = 2\frac{5}{12}$.

Answer: $4\frac{5}{6}$

You Try It!

EXAMPLE 9. Jim has a metal rod of length 10 inches. He cuts a length from the metal rod measuring $2\frac{7}{8}$ inches. What is the length of the remaining piece?

Sarah has a length of curtain material that measures 12 feet. She cuts a length of $6\frac{2}{3}$ feet from her curtain material. What is the length of the remaining piece?

Solution. To find the length of the remaining piece, we must subtract $2\frac{7}{8}$ from 10. There is no fractional part on the first number. To remedy this absence, we borrow 1 from 10 in the form of 8/8. Then we can subtract.

$$\begin{array}{rcrcr} 10 & = & 9+\dfrac{8}{8} & = & 9\,\dfrac{8}{8} \\ -2\,\dfrac{7}{8} & = & -2\,\dfrac{7}{8} & = & -2\,\dfrac{7}{8} \\ \hline & & & & 7\,\dfrac{1}{8} \end{array}$$

Hence, the length of the remaining piece of the metal rod is $7\frac{1}{8}$ inches.

Answer: $5\frac{1}{3}$ feet

Exercises

In Exercises 1-24, add or subtract the mixed fractions, as indicated, by first converting each mixed fraction to an improper fraction. Express your answer as a mixed fraction.

1. $9\frac{1}{4}+9\frac{1}{2}$

2. $2\frac{1}{3}+9\frac{1}{2}$

3. $6\frac{1}{2}-1\frac{1}{3}$

4. $5\frac{1}{3}-1\frac{3}{4}$

5. $9\frac{1}{2}+7\frac{1}{4}$

6. $1\frac{1}{3}+9\frac{3}{4}$

7. $5\frac{2}{3}+4\frac{1}{2}$

8. $1\frac{9}{16}+2\frac{3}{4}$

9. $3\frac{1}{3}-1\frac{1}{4}$

10. $2\frac{1}{2}-1\frac{1}{4}$

11. $8\frac{1}{2}-1\frac{1}{3}$

12. $5\frac{1}{2}-1\frac{2}{3}$

13. $4\frac{1}{2}-1\frac{1}{8}$

14. $2\frac{1}{2}-1\frac{1}{3}$

15. $4\frac{7}{8}+1\frac{3}{4}$

16. $1\frac{1}{8}+5\frac{1}{2}$

17. $2\frac{1}{3}-1\frac{1}{4}$

18. $5\frac{1}{3}-1\frac{1}{4}$

19. $9\frac{1}{2}-1\frac{3}{4}$

20. $5\frac{1}{2}-1\frac{3}{16}$

21. $4\frac{2}{3}+1\frac{1}{4}$

22. $1\frac{1}{4}+1\frac{1}{3}$

23. $9\frac{1}{2}+3\frac{1}{8}$

24. $1\frac{1}{4}+1\frac{2}{3}$

In Exercises 25-48, add or subtract the mixed fractions, as indicated, by using vertical format. Express your answer as a mixed fraction.

25. $3\frac{1}{2}+3\frac{3}{4}$

26. $1\frac{1}{2}+2\frac{2}{3}$

27. $1\frac{3}{8}+1\frac{1}{4}$

28. $2\frac{1}{4}+1\frac{2}{3}$

29. $1\frac{7}{8}+1\frac{1}{2}$

30. $1\frac{3}{4}+4\frac{1}{2}$

31. $8\,\dfrac{1}{2} - 5\,\dfrac{2}{3}$

32. $8\,\dfrac{1}{2} - 1\,\dfrac{2}{3}$

33. $7\,\dfrac{1}{2} - 1\,\dfrac{3}{16}$

34. $5\,\dfrac{1}{2} - 1\,\dfrac{1}{3}$

35. $9\,\dfrac{1}{2} - 1\,\dfrac{1}{3}$

36. $2\,\dfrac{1}{2} - 1\,\dfrac{3}{16}$

37. $5\,\dfrac{1}{3} - 2\,\dfrac{1}{2}$

38. $4\,\dfrac{1}{4} - 1\,\dfrac{1}{2}$

39. $9\,\dfrac{1}{2} - 2\,\dfrac{2}{3}$

40. $7\,\dfrac{1}{2} - 4\,\dfrac{2}{3}$

41. $1\,\dfrac{1}{16} + 1\,\dfrac{3}{4}$

42. $1\,\dfrac{1}{4} + 1\,\dfrac{1}{3}$

43. $8\,\dfrac{1}{2} + 3\,\dfrac{2}{3}$

44. $1\,\dfrac{2}{3} + 2\,\dfrac{1}{2}$

45. $6\,\dfrac{1}{2} - 1\,\dfrac{3}{16}$

46. $4\,\dfrac{1}{2} - 1\,\dfrac{1}{3}$

47. $2\,\dfrac{2}{3} + 1\,\dfrac{1}{4}$

48. $1\,\dfrac{1}{2} + 1\,\dfrac{1}{16}$

❧ ❧ ❧ Answers ☙ ☙ ☙

1. $18\,\dfrac{3}{4}$

3. $5\,\dfrac{1}{6}$

5. $16\,\dfrac{3}{4}$

7. $10\,\dfrac{1}{6}$

9. $2\,\dfrac{1}{12}$

11. $7\,\dfrac{1}{6}$

13. $3\,\dfrac{3}{8}$

15. $6\,\dfrac{5}{8}$

17. $1\,\dfrac{1}{12}$

19. $7\,\dfrac{3}{4}$

21. $5\,\dfrac{11}{12}$

23. $12\,\dfrac{5}{8}$

25. $7\dfrac{1}{4}$

27. $2\dfrac{5}{8}$

29. $3\dfrac{3}{8}$

31. $2\dfrac{5}{6}$

33. $6\frac{5}{16}$

35. $8\frac{1}{6}$

37. $2\frac{5}{6}$

39. $6\frac{5}{6}$

41. $2\frac{13}{16}$

43. $12\frac{1}{6}$

45. $5\frac{5}{16}$

47. $3\frac{11}{12}$

4.7 Order of Operations with Fractions

Let's begin by taking powers of fractions. Recall that

$$a^m = \underbrace{a \cdot a \cdot \ldots \cdot a}_{m \text{ times}}$$

You Try It!

EXAMPLE 1. Simplify: $(-3/4)^2$.

Solution. By definition,

$$\begin{aligned} \left(-\frac{3}{4}\right)^2 &= \left(-\frac{3}{4}\right)\left(-\frac{3}{4}\right) && \text{Fact: } a^2 = a \cdot a. \\ &= \frac{3 \cdot 3}{4 \cdot 4} && \text{Multiply numerators and denominators. Product of even number of negative factors is positive.} \\ &= \frac{9}{16} && \text{Simplify.} \end{aligned}$$

Simplify:

$$\left(-\frac{2}{5}\right)^2$$

Answer: $4/25$

□

You Try It!

EXAMPLE 2. Simplify: $(-2/3)^3$.

Solution. By definition,

$$\begin{aligned} \left(-\frac{2}{3}\right)^3 &= \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) && \text{Fact: } a^3 = a \cdot a \cdot a. \\ &= -\frac{2 \cdot 2 \cdot 2}{3 \cdot 3 \cdot 3} && \text{Multiply numerators and denominators. Product of odd number of negative factors is negative.} \\ &= -\frac{8}{27} && \text{Simplify.} \end{aligned}$$

Simplify:

$$\left(-\frac{1}{6}\right)^3$$

Answer: $-1/216$

□

The last two examples reiterate a principle learned earlier.

Odd and Even.

- The product of an **even** number of negative factors is positive.
- The product of an **odd** number of negative factors is negative.

Order of Operations

For convenience, we repeat here the rules guiding order of operations.

Rules Guiding Order of Operations. When evaluating expressions, proceed in the following order.

1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
2. Evaluate all exponents that appear in the expression.
3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.
4. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

You Try It!

Simplify:

$$-\frac{2}{3}+\frac{3}{4}\left(-\frac{1}{2}\right)$$

Answer: $-25/24$

EXAMPLE 3. Simplify: $-\frac{1}{2}+\frac{1}{4}\left(-\frac{1}{3}\right)$.

Solution. Multiply first, then add.

$$\begin{aligned}
-\frac{1}{2}+\frac{1}{4}\left(-\frac{1}{3}\right) &= -\frac{1}{2}+\left(-\frac{1}{12}\right) && \text{Multiply: } \frac{1}{4}\left(-\frac{1}{3}\right)=-\frac{1}{12}. \\
&= -\frac{1\cdot 6}{2\cdot 6}+\left(-\frac{1}{12}\right) && \text{Equivalent fractions, LCD} = 12. \\
&= -\frac{6}{12}+\left(-\frac{1}{12}\right) && \text{Simplify numerator and denominator.} \\
&= -\frac{7}{12} && \text{Add over common denominator.}
\end{aligned}$$

□

You Try It!

EXAMPLE 4. Simplify: $2\left(-\frac{1}{2}\right)^2 + 4\left(-\frac{1}{2}\right)$.

Simplify:

$$3\left(-\frac{1}{3}\right)^2 - 2\left(-\frac{1}{3}\right)$$

Solution. Exponents first, then multiply, then add.

$$\begin{aligned}
2\left(-\frac{1}{2}\right)^2 + 4\left(-\frac{1}{2}\right) &= 2\left(\frac{1}{4}\right) + 4\left(-\frac{1}{2}\right) && \text{Exponent first: } \left(-\frac{1}{2}\right)^2 = \frac{1}{4}. \\
&= \frac{1}{2} + \left(-\frac{2}{1}\right) && \text{Multiply: } 2\left(\frac{1}{4}\right) = \frac{1}{2} \\
& && \text{and } 4\left(-\frac{1}{2}\right) = -\frac{2}{1}. \\
&= \frac{1}{2} + \left(-\frac{2 \cdot 2}{1 \cdot 2}\right) && \text{Equivalent fractions, LCD} = 2. \\
&= \frac{1}{2} + \left(-\frac{4}{2}\right) && \text{Simplify numerator and denominator.} \\
&= -\frac{3}{2} && \text{Add over common denominator.}
\end{aligned}$$

Answer: 1

□

You Try It!

EXAMPLE 5. Given $a = -3/4$, $b = 1/2$, $c = 1/3$, and $d = -1/4$, evaluate the expression $ab - cd$.

Given $a = -1/2$, $b = 1/3$, and $c = -1/5$, evaluate $a + bc$.

Solution. Recall that it is good practice to prepare parentheses before substituting.

$$ad - bc = (\quad)(\quad) - (\quad)(\quad)$$

Substitute the given values into the algebraic expression, then simplify using order of operations.

$$\begin{aligned} ab - cd &= \left(-\frac{3}{4}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{3}\right)\left(-\frac{1}{4}\right) && \text{Substitute: } -3/4 \text{ for } a,\ 1/2 \text{ for } b, \\ & && 1/3 \text{ for } c, \text{ and } -1/4 \text{ for } d. \\ &= -\frac{3}{8} - \left(-\frac{1}{12}\right) && \text{Multiply first: } \left(-\frac{3}{4}\right)\left(\frac{1}{2}\right) = -\frac{3}{8} \\ & && \text{and } \left(\frac{1}{3}\right)\left(-\frac{1}{4}\right) = -\frac{1}{12}. \\ &= -\frac{3}{8} + \frac{1}{12} && \text{Subtract by adding opposite.} \\ &= -\frac{3 \cdot 3}{8 \cdot 3} + \frac{1 \cdot 2}{12 \cdot 2} && \text{Equivalent fractions; LCD} = 24. \\ &= -\frac{9}{24} + \frac{2}{24} && \text{Simplify numerators and denominators.} \\ &= -\frac{7}{24} && \text{Add over common denominator.} \end{aligned}$$

Answer: $-17/30$

□

You Try It!

Give $a = -1/2$ and $b = -1/3$, evaluate $ab \div (a + b)$.

EXAMPLE 6. Given $a = -1/4$ and $b = 1/2$, evaluate $(a^2 - b^2) \div (a + b)$.

Solution. Recall that it is good practice to prepare parentheses before substituting.

$$(a^2 - b^2) \div (a + b) = \Big((\quad)^2 - (\quad)^2\Big) \div \Big((\quad) + (\quad)\Big)$$

Substitute the given values into the algebraic expression, then evaluate exponents first.

$$\begin{aligned} (a^2 - b^2) \div (a + b) &= \left(\left(-\frac{1}{4}\right)^2 - \left(\frac{1}{2}\right)^2\right) \div \left(\left(-\frac{1}{4}\right) + \left(\frac{1}{2}\right)\right) \\ &= \left(\frac{1}{16} - \frac{1}{4}\right) \div \left(-\frac{1}{4} + \frac{1}{2}\right) \end{aligned}$$

We must evaluate parentheses first. Inside each set of parentheses, create equivalent fractions and perform subtractions and additions next.

$$\begin{aligned} &= \left(\frac{1}{16} - \frac{1 \cdot 4}{4 \cdot 4}\right) \div \left(-\frac{1}{4} + \frac{1 \cdot 2}{2 \cdot 2}\right) \\ &= \left(\frac{1}{16} - \frac{4}{16}\right) \div \left(-\frac{1}{4} + \frac{2}{4}\right) \\ &= -\frac{3}{16} \div \frac{1}{4} \end{aligned}$$

Invert and multiply.

$$= -\frac{3}{16} \cdot \frac{4}{1}$$
$$= -\frac{12}{16}$$

Reduce.

$$= -\frac{12 \div 4}{16 \div 4}$$
$$= -\frac{3}{4}$$

Note: In the last step, you could also reduce by prime factoring numerator and denominator and canceling common factors.

Answer: $-1/5$

□

Complex Fractions

Complex Fractions. When the numerator and denominator of a fraction contain fractions themselves, such an expression is called a *complex fraction.*

You can use the standard order of operations to simplify a complex fraction. Recall the advice when a fraction is present.

Fractional Expressions. If a fractional expression is present, simplify the numerator and denominator separately, then divide.

You Try It!

EXAMPLE 7. Simplify:

$$\frac{-\frac{1}{2} + \frac{1}{3}}{\frac{3}{4} - \frac{3}{2}}$$

Simplify:

$$\frac{\frac{1}{4} - \frac{1}{3}}{\frac{1}{4} + \frac{1}{3}}$$

Solution. We have addition in the numerator, subtraction in the denominator. In each case, we need equivalent fractions with a common denominator.

$$\frac{-\frac{1}{2}+\frac{1}{3}}{\frac{3}{4}-\frac{3}{2}} = \frac{-\frac{1\cdot 3}{2\cdot 3}+\frac{1\cdot 2}{3\cdot 2}}{\frac{3}{4}-\frac{3\cdot 2}{2\cdot 2}} \quad \text{Create equivalent fractions.}$$

$$= \frac{-\frac{3}{6}+\frac{2}{6}}{\frac{3}{4}-\frac{6}{4}} \quad \text{Simplify numerator and denominator.}$$

$$= \frac{-\frac{1}{6}}{-\frac{3}{4}} \quad \text{Numerator: } -\frac{3}{6}+\frac{2}{6}=-\frac{1}{6}. \quad \text{Denominator: } \frac{3}{4}-\frac{6}{4}=-\frac{3}{4}.$$

The last expression asks us to divide. Invert and multiply.

$$= -\frac{1}{6} \div \left(-\frac{3}{4}\right) \quad \text{A complex fraction means divide.}$$

$$= -\frac{1}{6} \cdot \left(-\frac{4}{3}\right) \quad \text{Invert and multiply.}$$

Like signs (two negatives) give a positive product. Multiply numerators and denominators, then reduce.

$$= \frac{4}{18} \quad \text{Like signs yields positive answer. Multiply numerators and denominators.}$$

$$= \frac{4 \div 2}{18 \div 2} \quad \text{Divide both numerator and denominator by 2.}$$

$$= \frac{2}{9} \quad \text{Simplify.}$$

Alternatively, one could prime factor and cancel to reduce to lowest terms; that is,

$$\frac{4}{18} = \frac{2\cdot 2}{2\cdot 3\cdot 3} \quad \text{Prime factor.}$$

$$= \frac{\cancel{2}\cdot 2}{\cancel{2}\cdot 3\cdot 3} \quad \text{Cancel common factors.}$$

$$= \frac{2}{9} \quad \text{Simplify.}$$

Answer: $-1/7$

□

Clearing Fractions. An alternate technique for simplifying complex fractions is available.

Clearing Fractions from Complex Fractions. You can clear fractions from a complex fraction using the following algorithm:

1. Determine an LCD_1 for the numerator.
2. Determine an LCD_2 for the denominator.
3. Determine an LCD for both LCD_1 and LCD_2.
4. Multiply both numerator and denominator by this "combined" LCD.

Let's apply this technique to the complex fraction of Example 7.

You Try It!

EXAMPLE 8. Simplify:

$$\frac{-\frac{1}{2}+\frac{1}{3}}{\frac{3}{4}-\frac{3}{2}}$$

Simplify:

$$\frac{-\frac{2}{3}+\frac{1}{5}}{\frac{4}{5}-\frac{1}{2}}$$

Solution. As we saw in the solution in Example 7, common denominators of 6 and 4 were used for the numerator and denominator, respectively. Thus, a common denominator for both numerator and denominator would be 12. We begin the alternate solution technique by multiplying both numerator and denominator by 12.

$$\frac{-\frac{1}{2}+\frac{1}{3}}{\frac{3}{4}-\frac{3}{2}} = \frac{12\left(-\frac{1}{2}+\frac{1}{3}\right)}{12\left(\frac{3}{4}-\frac{3}{2}\right)}$$

Multiply numerator and denominator by 12.

$$= \frac{12\left(-\frac{1}{2}\right)+12\left(\frac{1}{3}\right)}{12\left(\frac{3}{4}\right)-12\left(\frac{3}{2}\right)}$$

Distribute the 12.

$$= \frac{-6+4}{9-18}$$

Multiply: $12(-1/2) = -6$, $12(1/2) = 4$. $12(3/4) = 9$, and $12(3/2) = 18$.

$$= \frac{-2}{-9}$$

Simplify.

$$= \frac{2}{9}$$

Like signs yields positive.

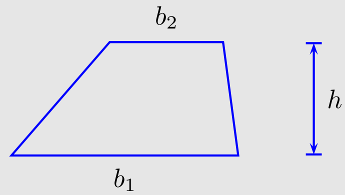
Answer: $-14/9$

□

Application — Trapezoid

A trapezoid is a special type of quadrilateral (four-sided polygon).

Trapezoid. A quadrilateral with one pair of parallel opposite sides is called a *trapezoid.*



The pair of parallel sides are called the *bases* of the trapezoid. Their lengths are marked by the variables b_1 and b_2 in the figure above. The distance between the parallel bases is called the *height* or *altitude* of the trapezoid. The height is marked by the variable h in the figure above.

Mathematicians use *subscripts* to create new variables. Thus, b_1 (“b sub 1”) and b_2 (“b sub 2”) are two distinct variables, used in this case to represent the length of the bases of the trapezoid.

By drawing in a diagonal, we can divide the trapezoid into two triangles (see Figure 4.14).

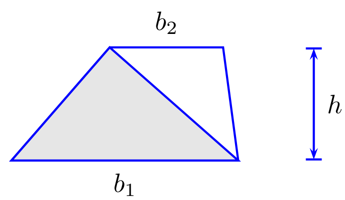


Figure 4.14: Dividing the trapezoid into two triangles.

We can find the area of the trapezoid by summing the areas of the two triangles.

- The shaded triangle in Figure 4.14 has base b_1 and height h. Hence, the area of the shaded triangle is $(1/2)b_1h$.

- The unshaded triangle in Figure 4.14 has base b_2 and height h. Hence, the area of the unshaded triangle is $(1/2)b_2h$.

Summing the areas, the area of the trapezoid is

$$\text{Area of Trapezoid} = \frac{1}{2}b_1h + \frac{1}{2}b_2h.$$

We can use the distributive property to factor out a $(1/2)h$.

Area of a Trapezoid. A trapezoid with bases b_1 and b_2 and height h has area

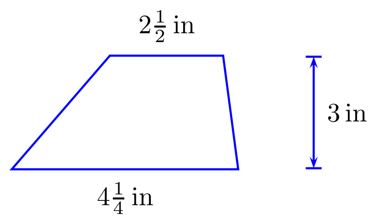
$$A = \frac{1}{2} h \left(b_1 + b_2\right).$$

That is, to find the area, sum the bases, multiply by the height, and take one-half of the result.

You Try It!

A trapezoid has bases measuring 6 and 15 feet, respectively. The height of the trapezoid is 5 feet. Find the area of the trapezoid.

EXAMPLE 9. Find the area of the trapezoid pictured below.



Solution. The formula for the area of a trapezoid is

$$A = \frac{1}{2} h \left(b_1 + b_2\right)$$

Substituting the given bases and height, we get

$$A = \frac{1}{2}(3)\left(4\frac{1}{4} + 2\frac{1}{2}\right).$$

Simplify the expression inside the parentheses first. Change mixed fractions to improper fractions, make equivalent fractions with a common denominator, then add.

$$\begin{aligned} A &= \frac{1}{2}(3)\left(\frac{17}{4} + \frac{5}{2}\right) \\ &= \frac{1}{2}(3)\left(\frac{17}{4} + \frac{5 \cdot 2}{2 \cdot 2}\right) \\ &= \frac{1}{2}(3)\left(\frac{17}{4} + \frac{10}{4}\right) \\ &= \frac{1}{2}\left(\frac{3}{1}\right)\left(\frac{27}{4}\right) \end{aligned}$$

Multiply numerators and denominators.

$$= \frac{81}{8}$$

This improper fraction is a perfectly good answer, but let's change this result to a mixed fraction (81 divided by 8 is 10 with a remainder of 1). Thus, the area of the trapezoid is

$$A = 10\,\frac{1}{8}\,\text{square inches.}$$

Answer: $52\frac{1}{2}$ square feet

□

Exercises

In Exercises 1-8, simplify the expression.

1. $\left(-\frac{7}{3}\right)^3$

2. $\left(\frac{1}{2}\right)^3$

3. $\left(\frac{5}{3}\right)^4$

4. $\left(-\frac{3}{5}\right)^4$

5. $\left(\frac{1}{2}\right)^5$

6. $\left(\frac{3}{4}\right)^5$

7. $\left(\frac{4}{3}\right)^2$

8. $\left(-\frac{8}{5}\right)^2$

---

9. If $a = 7/6$, evaluate a^3.

10. If $e = 1/6$, evaluate e^3.

11. If $e = -2/3$, evaluate $-e^2$.

12. If $c = -1/5$, evaluate $-c^2$.

13. If $b = -5/9$, evaluate b^2.

14. If $c = 5/7$, evaluate c^2.

15. If $b = -1/2$, evaluate $-b^3$.

16. If $a = -2/9$, evaluate $-a^3$.

In Exercises 17-36, simplify the expression.

17. $\left(-\frac{1}{2}\right)\left(\frac{1}{6}\right) - \left(\frac{7}{8}\right)\left(-\frac{7}{9}\right)$

18. $\left(-\frac{3}{4}\right)\left(\frac{1}{2}\right) - \left(\frac{3}{5}\right)\left(\frac{1}{4}\right)$

19. $\left(-\frac{9}{8}\right)^2 - \left(-\frac{3}{2}\right)\left(\frac{7}{3}\right)$

20. $\left(\frac{3}{2}\right)^2 - \left(\frac{7}{8}\right)\left(-\frac{1}{2}\right)$

21. $\left(-\frac{1}{2}\right)\left(-\frac{7}{4}\right) - \left(-\frac{1}{2}\right)^2$

22. $\left(\frac{1}{5}\right)\left(-\frac{9}{4}\right) - \left(\frac{7}{4}\right)^2$

23. $-\frac{7}{6} - \frac{1}{7} \cdot \frac{7}{9}$

24. $-\frac{4}{9} - \frac{8}{5} \cdot \frac{8}{9}$

25. $\frac{3}{4} + \frac{9}{7}\left(-\frac{7}{6}\right)$

26. $\frac{3}{2} + \frac{1}{4}\left(-\frac{9}{8}\right)$

27. $\left(-\frac{1}{3}\right)^2 + \left(\frac{7}{8}\right)\left(-\frac{1}{3}\right)$

28. $\left(-\frac{2}{9}\right)^2 + \left(\frac{2}{3}\right)\left(\frac{1}{2}\right)$

29. $\frac{5}{9} + \frac{5}{9} \cdot \frac{7}{9}$

30. $-\frac{1}{2} + \frac{9}{8} \cdot \frac{1}{3}$

31. $\left(-\frac{5}{6}\right)\left(\frac{3}{8}\right)+\left(-\frac{7}{9}\right)\left(-\frac{3}{4}\right)$

32. $\left(\frac{7}{4}\right)\left(\frac{6}{5}\right)+\left(-\frac{2}{5}\right)\left(\frac{8}{3}\right)$

33. $\frac{4}{3}-\frac{2}{9}\left(-\frac{3}{4}\right)$

34. $-\frac{1}{3}-\frac{1}{5}\left(-\frac{4}{3}\right)$

35. $\left(-\frac{5}{9}\right)\left(\frac{1}{2}\right)+\left(-\frac{1}{6}\right)^2$

36. $\left(\frac{1}{4}\right)\left(\frac{1}{6}\right)+\left(-\frac{5}{6}\right)^2$

---

37. Given $a=-5/4$, $b=1/2$, and $c=3/8$, evaluate $a+bc$.

38. Given $a=-3/5$, $b=1/5$, and $c=1/3$, evaluate $a+bc$.

39. Given $x=-1/8$, $y=5/2$, and $z=-1/2$, evaluate the expression $x+yz$.

40. Given $x=-5/9$, $y=1/4$, and $z=-2/3$, evaluate the expression $x+yz$.

41. Given $a=3/4$, $b=5/7$, and $c=1/2$, evaluate the expression $a-bc$.

42. Given $a=5/9$, $b=2/3$, and $c=2/9$, evaluate the expression $a-bc$.

43. Given $x=-3/2$, $y=1/4$, and $z=-5/7$, evaluate x^2-yz.

44. Given $x=-3/2$, $y=-1/2$, and $z=5/3$, evaluate x^2-yz.

45. Given $a=6/7$, $b=2/3$, $c=-8/9$, and $d=-6/7$, evaluate $ab+cd$.

46. Given $a=4/9$, $b=-3/2$, $c=7/3$, and $d=-8/9$, evaluate $ab+cd$.

47. Given $w=-1/8$, $x=-2/7$, $y=-1/2$, and $z=8/7$, evaluate $wx-yz$.

48. Given $w=2/7$, $x=-9/4$, $y=-3/4$, and $z=-9/2$, evaluate $wx-yz$.

49. Given $x=3/8$, $y=3/5$, and $z=-3/2$, evaluate $xy+z^2$.

50. Given $x=-1/2$, $y=7/5$, and $z=-3/2$, evaluate $xy+z^2$.

51. Given $u=9/7$, $v=2/3$, and $w=-3/7$, evaluate $uv-w^2$.

52. Given $u=8/7$, $v=-4/3$, and $w=2/3$, evaluate $uv-w^2$.

53. Given $a=7/8$, $b=-1/4$, and $c=-3/2$, evaluate a^2+bc.

54. Given $a=-5/8$, $b=3/2$, and $c=-3/2$, evaluate a^2+bc.

55. Given $u=1/3$, $v=5/2$, and $w=-2/9$, evaluate the expression $u-vw$.

56. Given $u=-1/2$, $v=1/4$, and $w=-1/4$, evaluate the expression $u-vw$.

In Exercises 57-68, simplify the complex rational expression.

57. $$\frac{\frac{8}{3}+\frac{7}{6}}{-\frac{9}{2}-\frac{1}{4}}$$

58. $$\frac{\frac{7}{8}+\frac{1}{9}}{\frac{8}{9}-\frac{1}{6}}$$

59. $$\frac{\frac{3}{4}+\frac{4}{3}}{\frac{1}{9}+\frac{5}{3}}$$

60. $$\frac{-\frac{9}{8}-\frac{6}{5}}{\frac{7}{4}+\frac{1}{2}}$$

61. $$\frac{\frac{7}{5}+\frac{5}{2}}{-\frac{1}{4}+\frac{1}{2}}$$

62. $$\frac{\frac{5}{6}+\frac{2}{3}}{\frac{3}{5}+\frac{2}{3}}$$

63. $$\frac{-\frac{3}{2}-\frac{2}{3}}{-\frac{7}{4}-\frac{2}{3}}$$

64. $$\frac{\frac{8}{9}+\frac{3}{4}}{-\frac{2}{3}-\frac{1}{6}}$$

65. $$\frac{-\frac{1}{2}-\frac{4}{7}}{-\frac{5}{7}+\frac{1}{6}}$$

66. $$\frac{-\frac{3}{2}-\frac{5}{8}}{\frac{3}{4}-\frac{1}{2}}$$

67. $$\frac{-\frac{3}{7}-\frac{1}{3}}{\frac{1}{3}-\frac{6}{7}}$$

68. $$\frac{-\frac{5}{8}-\frac{6}{5}}{-\frac{5}{4}-\frac{3}{8}}$$

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69. A trapezoid has bases measuring $3\frac{3}{8}$ and $5\frac{1}{2}$ feet, respectively. The height of the trapezoid is 7 feet. Find the area of the trapezoid.

70. A trapezoid has bases measuring $2\frac{1}{2}$ and $6\frac{7}{8}$ feet, respectively. The height of the trapezoid is 3 feet. Find the area of the trapezoid.

71. A trapezoid has bases measuring $2\frac{1}{4}$ and $7\frac{3}{8}$ feet, respectively. The height of the trapezoid is 7 feet. Find the area of the trapezoid.

72. A trapezoid has bases measuring $3\frac{1}{8}$ and $6\frac{1}{2}$ feet, respectively. The height of the trapezoid is 3 feet. Find the area of the trapezoid.

73. A trapezoid has bases measuring $2\frac{3}{4}$ and $6\frac{5}{8}$ feet, respectively. The height of the trapezoid is 3 feet. Find the area of the trapezoid.

74. A trapezoid has bases measuring $2\frac{1}{4}$ and $7\frac{1}{8}$ feet, respectively. The height of the trapezoid is 5 feet. Find the area of the trapezoid.

Answers

1. $-\frac{343}{27}$

3. $\frac{625}{81}$

5. $\frac{1}{32}$

7. $\frac{16}{9}$

9. $\frac{343}{216}$

11. $-\frac{4}{9}$

13. $\frac{25}{81}$

15. $\frac{1}{8}$

17. $\frac{43}{72}$

19. $\frac{305}{64}$

21. $\frac{5}{8}$

23. $-\frac{23}{18}$

25. $-\frac{3}{4}$

27. $-\frac{13}{72}$

29. $\frac{80}{81}$

31. $\frac{13}{48}$

33. $\frac{3}{2}$

35. $-\frac{1}{4}$

37. $-\frac{17}{16}$

39. $-\frac{11}{8}$

41. $\frac{11}{28}$

43. $\frac{17}{7}$

45. $\frac{4}{3}$

47. $\frac{17}{28}$

49. $\frac{99}{40}$

51. $\frac{33}{49}$

53. $\frac{73}{64}$

55. $\frac{8}{9}$

57. $-\dfrac{46}{57}$

59. $\dfrac{75}{64}$

61. $\dfrac{78}{5}$

63. $\dfrac{26}{29}$

65. $\dfrac{45}{23}$

67. $\dfrac{16}{11}$

69. $31\dfrac{1}{16}$

71. $33\dfrac{11}{16}$

73. $14\dfrac{1}{16}$

4.8 Solving Equations with Fractions

Undoing Subtraction

We can still add the same amount to both sides of an equation without changing the solution.

You Try It!

Solver for x:

$$x - \frac{2}{3} = \frac{1}{5}$$

EXAMPLE 1. Solve for x: $x - \frac{5}{6} = \frac{1}{3}$.

Solution. To "undo" subtracting 5/6, add 5/6 to both sides of the equation and simplify.

$$\begin{aligned} x - \frac{5}{6} &= \frac{1}{3} && \text{Original equation.} \\ x - \frac{5}{6} + \frac{5}{6} &= \frac{1}{3} + \frac{5}{6} && \text{Add } \frac{5}{6} \text{ to both sides.} \\ x &= \frac{1 \cdot 2}{3 \cdot 2} + \frac{5}{6} && \text{Equivalent fractions, LCD} = 6. \\ x &= \frac{2}{6} + \frac{5}{6} && \text{Simplify.} \\ x &= \frac{7}{6} && \text{Add.} \end{aligned}$$

It is perfectly acceptable to leave your answer as an improper fraction. If you desire, or if you are instructed to do so, you can change your answer to a mixed fraction (7 divided by 6 is 1 with a remainder of 1). That is $x = 1\frac{1}{6}$.

Checking the Solution. Substitute 7/6 for x in the original equation and simplify.

$$\begin{aligned} x - \frac{5}{6} &= \frac{1}{3} && \text{Original equation.} \\ \frac{7}{6} - \frac{5}{6} &= \frac{1}{3} && \text{Substitute } 7/6 \text{ for } x. \\ \frac{2}{6} &= \frac{1}{3} && \text{Subtract.} \\ \frac{1}{3} &= \frac{1}{3} && \text{Reduce.} \end{aligned}$$

Because the last statement is true, we conclude that 7/6 is a solution of the equation $x - 5/6 = 1/3$.

Answer: 13/15

□

Undoing Addition

You can still subtract the same amount from both sides of an equation without changing the solution.

You Try It!

EXAMPLE 2. Solve for x: $x + \frac{2}{3} = -\frac{3}{5}$.

Solve for x:

$$x + \frac{3}{4} = -\frac{1}{2}$$

Solution. To "undo" adding 2/3, subtract 2/3 from both sides of the equation and simplify.

$$\begin{aligned} x + \frac{2}{3} &= -\frac{3}{5} && \text{Original equation.} \\ x + \frac{2}{3} - \frac{2}{3} &= -\frac{3}{5} - \frac{2}{3} && \text{Subtract } \frac{2}{3} \text{ from both sides.} \\ x &= -\frac{3 \cdot 3}{5 \cdot 3} - \frac{2 \cdot 5}{3 \cdot 5} && \text{Equivalent fractions, LCD} = 15. \\ x &= -\frac{9}{15} - \frac{10}{15} && \text{Simplify.} \\ x &= -\frac{19}{15} && \text{Subtract.} \end{aligned}$$

Readers are encouraged to check this solution in the original equation.

Answer: $-5/4$

□

Undoing Multiplication

We "undo" multiplication by dividing. For example, to solve the equation $2x = 6$, we would divide both sides of the equation by 2. In similar fashion, we could divide both sides of the equation

$$\frac{3}{5}x = \frac{4}{10}$$

by 3/5. However, it is more efficient to take advantage of reciprocals. For convenience, we remind readers of the *Multiplicative Inverse Property.*

Multiplicative Inverse Property. Let a/b be any fraction. The number b/a is called the *multiplicative inverse* or *reciprocal* of a/b. The product of reciprocals is 1.

$$\frac{a}{b} \cdot \frac{b}{a} = 1.$$

Let's put our knowledge of reciprocals to work.

You Try It!

EXAMPLE 3. Solve for x: $\frac{3}{5}x = \frac{4}{10}$.

Solve for y:

$$\frac{2}{3}y = \frac{4}{5}$$

Solution. To "undo" multiplying by 3/5, multiply both sides by the reciprocal 5/3 and simplify.

$$\begin{aligned}
\frac{3}{5}x &= \frac{4}{10} && \text{Original equation.}\\
\frac{5}{3}\left(\frac{3}{5}x\right) &= \frac{5}{3}\left(\frac{4}{10}\right) && \text{Multiply both sides by } 5/3.\\
\left(\frac{5}{3}\cdot\frac{3}{5}\right)x &= \frac{20}{30} && \text{On the left, use the associative property to regroup. On the right, multiply.}\\
1x &= \frac{2}{3} && \text{On the left, } \frac{5}{3}\cdot\frac{3}{5} = 1. \text{ On the right, reduce: } \frac{20}{30} = \frac{2}{3}.\\
x &= \frac{2}{3} && \text{On the left, } 1x = x.
\end{aligned}$$

Checking the Solution. Substitute 2/3 for x in the original equation and simplify.

$$\begin{aligned}
\frac{3}{5}x &= \frac{4}{10} && \text{Original equation.}\\
\frac{3}{5}\left(\frac{2}{3}\right) &= \frac{4}{10} && \text{Substitute } 2/3 \text{ for } x.\\
\frac{6}{15} &= \frac{4}{10} && \text{Multiply numerators; multiply denominators.}\\
\frac{2}{5} &= \frac{2}{5} && \text{Reduce both sides to lowest terms.}
\end{aligned}$$

Because this last statement is true, we conclude that 2/3 is a solution of the equation $(3/5)x = 4/10$.

Answer: 6/5

□

You Try It!

Solve for z:

$$-\frac{2}{7}z = \frac{4}{21}$$

EXAMPLE 4. Solve for x: $-\dfrac{8}{9}x = \dfrac{5}{18}$.

Solution. To "undo" multiplying by −8/9, multiply both sides by the reciprocal −9/8 and simplify.

$$-\frac{8}{9}x = \frac{5}{18}$$ Original equation.

$$-\frac{9}{8}\left(-\frac{8}{9}x\right) = -\frac{9}{8}\left(\frac{5}{18}\right)$$ Multiply both sides by $-9/8$.

$$\left[-\frac{9}{8}\cdot\left(-\frac{8}{9}\right)\right]x = -\frac{3\cdot 3}{2\cdot 2\cdot 2}\cdot\frac{5}{2\cdot 3\cdot 3}$$ On the left, use the associative property to regroup. On the right, prime factor.

$$1x = -\frac{\cancel{3}\cdot\cancel{3}}{2\cdot 2\cdot 2}\cdot\frac{5}{2\cdot\cancel{3}\cdot\cancel{3}}$$ On the left, $-\frac{9}{8}\cdot\left(-\frac{8}{9}\right) = 1$. On the right, cancel common factors.

$$x = -\frac{5}{16}$$ On the left, $1x = x$. Multiply on right.

Readers are encouraged to check this solution in the original equation.

Answer: $-2/3$

Clearing Fractions from the Equation

Although the technique demonstrated in the previous examples is a solid mathematical technique, working with fractions in an equation is not always the most efficient use of your time.

Clearing Fractions from the Equation. To clear all fractions from an equation, multiply both sides of the equation by the least common denominator of the fractions that appear in the equation.

Let's put this idea to work.

You Try It!

EXAMPLE 5. In Example 1, we were asked to solve the following equation for x:

$$x - \frac{5}{6} = \frac{1}{3}.$$

Take a moment to review the solution technique in Example 1. We will now solve this equation by first clearing all fractions from the equation.

Solve for t:

$$t - \frac{2}{7} = -\frac{1}{4}$$

Solution. Multiply both sides of the equation by the least common denominator for the fractions appearing in the equation.

$x - \frac{5}{6} = \frac{1}{3}$	Original equation.
$6\left(x - \frac{5}{6}\right) = 6\left(\frac{1}{3}\right)$	Multiply both sides by 6.
$6x - 6\left(\frac{5}{6}\right) = 6\left(\frac{1}{3}\right)$	Distribute the 6.
$6x - 5 = 2$	On each side, multiply first. $6\left(\frac{5}{6}\right) = 5$ and $6\left(\frac{1}{3}\right) = 2$.

Note that the equation is now entirely clear of fractions, making it a much simpler equation to solve.

$6x - 5 + 5 = 2 + 5$	Add 5 to both sides.
$6x = 7$	Simplify both sides.
$\frac{6x}{6} = \frac{7}{6}$	Divide both sides by 6.
$x = \frac{7}{6}$	Simplify.

Note that this is the same solution found in Example 1.

Answer: 1/28

□

You Try It!

Solve for u:

$$-\frac{7}{9}u = \frac{14}{27}$$

EXAMPLE 6. In Example 4, we were asked to solve the following equation for x.

$$-\frac{8}{9}x = \frac{5}{18}$$

Take a moment to review the solution in Example 4. We will now solve this equation by first clearing all fractions from the equation.

Solution. Multiply both sides of the equation by the least common denominator for the fractions that appear in the equation.

$-\frac{8}{9}x = \frac{5}{18}$	Original equation.
$18\left(-\frac{8}{9}x\right) = 18\left(\frac{5}{18}\right)$	Multiply both sides by 18.
$-16x = 5$	On each side, cancel and multiply. $18\left(-\frac{8}{9}\right) = -16$ and $18\left(\frac{5}{18}\right) = 5$.

Note that the equation is now entirely free of fractions. Continuing,

$$\frac{-16x}{-16} = \frac{5}{-16} \qquad \text{Divide both sides by } -16.$$

$$x = -\frac{5}{16} \qquad \text{Simplify.}$$

Note that this is the same as the solution found in Example 4.

Answer: $-2/3$

□

You Try It!

EXAMPLE 7. Solve for x: $\frac{2}{3}x + \frac{3}{4} = \frac{1}{2}$.

Solve for r:

$$\frac{3}{4}r + \frac{2}{3} = \frac{1}{2}$$

Solution. Multiply both sides of the equation by the least common denominator for the fractions appearing in the equation.

$$\frac{2}{3}x + \frac{3}{4} = \frac{1}{2} \qquad \text{Original equation.}$$

$$12\left(\frac{2}{3}x + \frac{3}{4}\right) = 12\left(\frac{1}{2}\right) \qquad \text{Multiply both sides by 12.}$$

$$12\left(\frac{2}{3}x\right) + 12\left(\frac{3}{4}\right) = 12\left(\frac{1}{2}\right) \qquad \text{On the left, distribute 12.}$$

$$8x + 9 = 6 \qquad \text{Multiply: } 12\left(\frac{2}{3}x\right) = 8x,\ 12\left(\frac{3}{4}\right) = 9, \text{ and } 12\left(\frac{1}{2}\right) = 6.$$

Note that the equation is now entirely free of fractions. We need to isolate the terms containing x on one side of the equation.

$$8x + 9 - 9 = 6 - 9 \qquad \text{Subtract 9 from both sides.}$$

$$8x = -3 \qquad \text{Simplify both sides.}$$

$$\frac{8x}{8} = \frac{-3}{8} \qquad \text{Divide both sides by 8.}$$

$$x = -\frac{3}{8} \qquad \text{Simplify both sides.}$$

Readers are encouraged to check this solution in the original equation.

Answer: $-2/9$

□

You Try It!

EXAMPLE 8. Solve for x: $\frac{2}{3} - \frac{3x}{4} = \frac{x}{2} - \frac{1}{8}$.

Solve for s:

$$\frac{3}{2} - \frac{2s}{5} = \frac{s}{3} - \frac{1}{5}$$

Solution. Multiply both sides of the equation by the least common denominator for the fractions in the equation.

$$\frac{2}{3} - \frac{3x}{4} = \frac{x}{2} - \frac{1}{8} \qquad \text{Original equation.}$$

$$24\left(\frac{2}{3} - \frac{3x}{4}\right) = 24\left(\frac{x}{2} - \frac{1}{8}\right) \qquad \text{Multiply both sides by 24.}$$

$$24\left(\frac{2}{3}\right) - 24\left(\frac{3x}{4}\right) = 24\left(\frac{x}{2}\right) - 24\left(\frac{1}{8}\right) \qquad \text{On both sides, distribute 24.}$$

$$16 - 18x = 12x - 3 \qquad \text{Left: } 24\left(\frac{2}{3}\right) = 16,\ 24\left(\frac{3x}{4}\right) = 18x.$$

$$\text{Right: } 24\left(\frac{x}{2}\right) = 12x,\ 24\left(\frac{1}{8}\right) = 3.$$

Note that the equation is now entirely free of fractions. We need to isolate the terms containing x on one side of the equation.

$$16 - 18x - 12x = 12x - 3 - 12x \qquad \text{Subtract } 12x \text{ from both sides.}$$

$$16 - 30x = -3 \qquad \text{Left: } -18x - 12x = -30x. \quad \text{Right: } 12x - 12x = 0.$$

$$16 - 30x - 16 = -3 - 16 \qquad \text{Subtract 16 from both sides.}$$

$$-30x = -19 \qquad \text{Left: } 16 - 16 = 0. \quad \text{Right: } -3 - 16 = -19.$$

$$\frac{-30x}{-30} = \frac{-19}{-30} \qquad \text{Divide both sides by } -30.$$

$$x = \frac{19}{30} \qquad \text{Simplify both sides.}$$

Answer: 51/22

Readers are encouraged to check this solution in the original equation.

□

Applications

Let's look at some applications that involve equations containing fractions. For convenience, we repeat the *Requirements for Word Problem Solutions.*

Requirements for Word Problem Solutions.

1. **Set up a Variable Dictionary.** You must let your readers know what each variable in your problem represents. This can be accomplished in a number of ways:
   - Statements such as "Let P represent the perimeter of the rectangle."

- Labeling unknown values with variables in a table.
- Labeling unknown quantities in a sketch or diagram.

2. Set up an Equation. Every solution to a word problem must include a carefully crafted equation that accurately describes the constraints in the problem statement.

3. Solve the Equation. You must always solve the equation set up in the previous step.

4. Answer the Question. This step is easily overlooked. For example, the problem might ask for Jane's age, but your equation's solution gives the age of Jane's sister Liz. Make sure you answer the original question asked in the problem. Your solution should be written in a sentence with appropriate units.

5. Look Back. It is important to note that this step does not imply that you should simply check your solution in your equation. After all, it's possible that your equation incorrectly models the problem's situation, so you could have a valid solution to an incorrect equation. The important question is: "Does your answer make sense based on the words in the original problem statement."

You Try It!

EXAMPLE 9. In the third quarter of a basketball game, announcers informed the crowd that attendance for the game was 12,250. If this is two-thirds of the capacity, find the full seating capacity for the basketball arena.

Attendance for the Celtics game was 9,510. If this is 3/4 of capacity, what is the capacity of the Celtics' arena?

Solution. We follow the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* Let F represent the full seating capacity. *Note: It is much better to use a variable that "sounds like" the quantity that it represents. In this case, letting F represent the full seating capacity is much more descriptive than using x to represent the full seating capacity.*

2. *Set up an Equation.* Two-thirds of the full seating capacity is 12,250.

Two-thirds	of	Full Seating Capacity	is	12,250
$\frac{2}{3}$	$\cdot$	F	$=$	12,250

Hence, the equation is

$$\frac{2}{3}F = 12250.$$

3. *Solve the Equation.* Multiply both sides by 3 to clear fractions, then solve.

$$\begin{aligned}
\frac{2}{3}F &= 12250 && \text{Original equation.}\\
3\left(\frac{2}{3}F\right) &= 3(12250) && \text{Multiply both sides by 3.}\\
2F &= 36750 && \text{Simplify both sides.}\\
\frac{2F}{2} &= \frac{36750}{2} && \text{Divide both sides by 2.}\\
F &= 18375 && \text{Simplify both sides.}
\end{aligned}$$

4. *Answer the Question.* The full seating capacity is 18,375.

5. *Look Back.* The words of the problem state that 2/3 of the seating capacity is 12,250. Let's take two-thirds of our answer and see what we get.

$$\begin{aligned}
\frac{2}{3} \cdot 18375 &= \frac{2}{3} \cdot \frac{18375}{1}\\
&= \frac{2}{3} \cdot \frac{3 \cdot 6125}{1}\\
&= \frac{2}{\cancel{3}} \cdot \frac{\cancel{3} \cdot 6125}{1}\\
&= 12250
\end{aligned}$$

This is the correct attendance, so our solution is correct.

Answer: 12,680

□

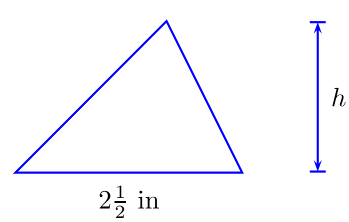
You Try It!

The area of a triangle is 161 square feet. If the base of the triangle measures $40\frac{1}{4}$ feet, find the height of the triangle.

EXAMPLE 10. The area of a triangle is 20 square inches. If the length of the base is $2\frac{1}{2}$ inches, find the height (altitude) of the triangle.

Solution. We follow the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* Our variable dictionary will take the form of a well labeled diagram.



2. *Set up an Equation.* The area A of a triangle with base b and height h is
$$A = \frac{1}{2}bh.$$
Substitute $A = 20$ and $b = 2\frac{1}{2}$.
$$20 = \frac{1}{2}\left(2\frac{1}{2}\right)h.$$

3. *Solve the Equation.* Change the mixed fraction to an improper fraction, then simplify.
$$\begin{aligned} 20 &= \frac{1}{2}\left(2\frac{1}{2}\right)h && \text{Original equation.} \\ 20 &= \frac{1}{2}\left(\frac{5}{2}\right)h && \text{Mixed to improper: } 2\frac{1}{2} = \frac{5}{2}. \\ 20 &= \left(\frac{1}{2}\cdot\frac{5}{2}\right)h && \text{Associative property.} \\ 20 &= \frac{5}{4}h && \text{Multiply: } \frac{1}{2}\cdot\frac{5}{2} = \frac{5}{4}. \end{aligned}$$
Now, multiply both sides by 4/5 and solve.
$$\begin{aligned} \frac{4}{5}(20) &= \frac{4}{5}\left(\frac{5}{4}h\right) && \text{Multiply both sides by 4/5.} \\ 16 &= h && \text{Simplify: } \frac{4}{5}(20) = 16 \\ & && \text{and } \frac{4}{5}\cdot\frac{5}{4} = 1. \end{aligned}$$

4. *Answer the Question.* The height of the triangle is 16 inches.

5. *Look Back.* If the height is 16 inches and the base is $2\frac{1}{2}$ inches, then the area is
$$\begin{aligned} A &= \frac{1}{2}\left(2\frac{1}{2}\right)(16) \\ &= \frac{1}{2}\cdot\frac{5}{2}\cdot\frac{16}{1} \\ &= \frac{5\cdot 16}{2\cdot 2} \\ &= \frac{(5)\cdot(2\cdot 2\cdot 2\cdot 2)}{(2)\cdot(2)} \\ &= \frac{5\cdot \cancel{2}\cdot \cancel{2}\cdot 2\cdot 2}{\cancel{2}\cdot\cancel{2}} \\ &= 20 \end{aligned}$$
This is the correct area (20 square inches), so our solution is correct. Answer: 8 feet

□

Exercises

1. Is 1/4 a solution of the equation

$$x + \frac{5}{8} = \frac{5}{8}\ ?$$

2. Is 1/4 a solution of the equation

$$x + \frac{1}{3} = \frac{5}{12}\ ?$$

3. Is −8/15 a solution of the equation

$$\frac{1}{4}x = -\frac{1}{15}\ ?$$

4. Is −18/7 a solution of the equation

$$-\frac{3}{8}x = \frac{25}{28}\ ?$$

5. Is 1/2 a solution of the equation

$$x + \frac{4}{9} = \frac{17}{18}\ ?$$

6. Is 1/3 a solution of the equation

$$x + \frac{3}{4} = \frac{13}{12}\ ?$$

7. Is 3/8 a solution of the equation

$$x - \frac{5}{9} = -\frac{13}{72}\ ?$$

8. Is 1/2 a solution of the equation

$$x - \frac{3}{5} = -\frac{1}{10}\ ?$$

9. Is 2/7 a solution of the equation

$$x - \frac{4}{9} = -\frac{8}{63}\ ?$$

10. Is 1/9 a solution of the equation

$$x - \frac{4}{7} = -\frac{31}{63}\ ?$$

11. Is 8/5 a solution of the equation

$$\frac{11}{14}x = \frac{44}{35}\ ?$$

12. Is 16/9 a solution of the equation

$$\frac{13}{18}x = \frac{104}{81}\ ?$$

In Exercises 13-24, solve the equation and simplify your answer.

13. $2x - 3 = 6x + 7$

14. $9x - 8 = -9x - 3$

15. $-7x + 4 = 3x$

16. $6x + 9 = -6x$

17. $-2x = 9x - 4$

18. $-6x = -9x + 8$

19. $-8x = 7x - 7$

20. $-6x = 5x + 4$

21. $-7x + 8 = 2x$

22. $-x - 7 = 3x$

23. $-9x + 4 = 4x - 6$

24. $-2x + 4 = x - 7$

In Exercises 25-48, solve the equation and simplify your answer.

25. $x+\frac{3}{2}=\frac{1}{2}$

26. $x-\frac{3}{4}=\frac{1}{4}$

27. $-\frac{9}{5}x=\frac{1}{2}$

28. $\frac{7}{3}x=-\frac{7}{2}$

29. $\frac{3}{8}x=\frac{8}{7}$

30. $-\frac{1}{9}x=-\frac{3}{5}$

31. $\frac{2}{5}x=-\frac{1}{6}$

32. $\frac{1}{6}x=\frac{2}{9}$

33. $-\frac{3}{2}x=\frac{8}{7}$

34. $-\frac{3}{2}x=-\frac{7}{5}$

35. $x+\frac{3}{4}=\frac{5}{9}$

36. $x-\frac{1}{9}=-\frac{3}{2}$

37. $x-\frac{4}{7}=\frac{7}{8}$

38. $x+\frac{4}{9}=-\frac{3}{4}$

39. $x+\frac{8}{9}=\frac{2}{3}$

40. $x-\frac{5}{6}=\frac{1}{4}$

41. $x+\frac{5}{2}=-\frac{9}{8}$

42. $x+\frac{1}{2}=\frac{5}{3}$

43. $-\frac{8}{5}x=\frac{7}{9}$

44. $-\frac{3}{2}x=-\frac{5}{9}$

45. $x-\frac{1}{4}=-\frac{1}{8}$

46. $x-\frac{9}{2}=-\frac{7}{2}$

47. $-\frac{1}{4}x=\frac{1}{2}$

48. $-\frac{8}{9}x=-\frac{8}{3}$

In Exercises 49-72, solve the equation and simplify your answer.

49. $-\frac{7}{3}x-\frac{2}{3}=\frac{3}{4}x+\frac{2}{3}$

50. $\frac{1}{2}x-\frac{1}{2}=\frac{3}{2}x+\frac{3}{4}$

51. $-\frac{7}{2}x-\frac{5}{4}=\frac{4}{5}$

52. $-\frac{7}{6}x+\frac{5}{6}=-\frac{8}{9}$

53. $-\frac{9}{7}x+\frac{9}{2}=-\frac{5}{2}$

54. $\frac{5}{9}x-\frac{7}{2}=\frac{1}{4}$

55. $\frac{1}{4}x-\frac{4}{3}=-\frac{2}{3}$

56. $\frac{8}{7}x+\frac{3}{7}=\frac{5}{3}$

57. $\frac{5}{3}x+\frac{3}{2}=-\frac{1}{4}$

58. $\frac{1}{2}x-\frac{8}{3}=-\frac{2}{5}$

59. $-\frac{1}{3}x+\frac{4}{5}=-\frac{9}{5}x-\frac{5}{6}$

60. $-\frac{2}{9}x-\frac{3}{5}=\frac{4}{5}x-\frac{3}{2}$

61. $-\frac{4}{9}x - \frac{8}{9} = \frac{1}{2}x - \frac{1}{2}$

62. $-\frac{5}{4}x - \frac{5}{3} = \frac{8}{7}x + \frac{7}{3}$

63. $\frac{1}{2}x - \frac{1}{8} = -\frac{1}{8}x + \frac{5}{7}$

64. $-\frac{3}{2}x + \frac{8}{3} = \frac{7}{9}x - \frac{1}{2}$

65. $-\frac{3}{7}x - \frac{1}{3} = -\frac{1}{9}$

66. $\frac{2}{3}x + \frac{2}{9} = -\frac{9}{5}$

67. $-\frac{3}{4}x + \frac{2}{7} = \frac{8}{7}x - \frac{1}{3}$

68. $\frac{1}{2}x + \frac{1}{3} = -\frac{5}{2}x - \frac{1}{4}$

69. $-\frac{3}{4}x - \frac{2}{3} = -\frac{2}{3}x - \frac{1}{2}$

70. $\frac{1}{3}x - \frac{5}{7} = \frac{3}{2}x + \frac{4}{3}$

71. $-\frac{5}{2}x + \frac{9}{5} = \frac{5}{8}$

72. $\frac{9}{4}x + \frac{4}{3} = -\frac{1}{6}$

---

73. At a local soccer game, announcers informed the crowd that attendance for the game was 4,302. If this is 2/9 of the capacity, find the full seating capacity for the soccer stadium.

74. At a local basketball game, announcers informed the crowd that attendance for the game was 5,394. If this is 2/7 of the capacity, find the full seating capacity for the basketball stadium.

75. The area of a triangle is 51 square inches. If the length of the base is $8\,\frac{1}{2}$ inches, find the height (altitude) of the triangle.

76. The area of a triangle is 20 square inches. If the length of the base is $2\,\frac{1}{2}$ inches, find the height (altitude) of the triangle.

77. The area of a triangle is 18 square inches. If the length of the base is $4\,\frac{1}{2}$ inches, find the height (altitude) of the triangle.

78. The area of a triangle is 44 square inches. If the length of the base is $5\,\frac{1}{2}$ inches, find the height (altitude) of the triangle.

79. At a local hockey game, announcers informed the crowd that attendance for the game was 4,536. If this is 2/11 of the capacity, find the full seating capacity for the hockey stadium.

80. At a local soccer game, announcers informed the crowd that attendance for the game was 6,970. If this is 2/7 of the capacity, find the full seating capacity for the soccer stadium.

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81. **Pirates.** About one-third of the world's pirate attacks in 2008 occurred off the Somali coast. If there were 111 pirate attacks off the Somali coast, estimate the number of pirate attacks worldwide in 2008.

82. **Nuclear arsenal.** The U.S. and Russia agreed to cut nuclear arsenals of long-range nuclear weapons by about a third, down to $1,550$. How many long-range nuclear weapons are there now? *Associated Press-Times-Standard 04/04/10 Nuclear heartland anxious about missile cuts.*

83. **Seed vault.** The Svalbard Global Seed Vault has amassed half a million seed samples, and now houses at least one-third of the world's crop seeds. Estimate the total number of world's crop seeds. *Associated Press-Times-Standard 03/15/10 Norway doomsday seed vault hits half-million mark.*

84. **Freight train.** The three and one-half mile long Union Pacific train is about $2\frac{1}{2}$ times the length of a typical freight train. How long is a typical freight train? *Associated Press-Times-Standard 01/13/10 Unususally long train raises safety concerns.*

Answers

1. No

3. No

5. Yes

7. Yes

9. No

11. Yes

13. $-\frac{5}{2}$

15. $\frac{2}{5}$

17. $\frac{4}{11}$

19. $\frac{7}{15}$

21. $\frac{8}{9}$

23. $\frac{10}{13}$

25. -1

27. $-\frac{5}{18}$

29. $\frac{64}{21}$

31. $-\frac{5}{12}$

33. $-\frac{16}{21}$

35. $-\frac{7}{36}$

37. $\frac{81}{56}$

39. $-\frac{2}{9}$

41. $-\frac{29}{8}$

43. $-\frac{35}{72}$

45. $\frac{1}{8}$

47. -2

49. $-\frac{16}{37}$

51. $-\frac{41}{70}$

53. $\frac{49}{9}$

55. $\frac{8}{3}$

57. $-\frac{21}{20}$

59. $-\frac{49}{44}$

61. $-\frac{7}{17}$

63. $\frac{47}{35}$

65. $-\frac{14}{27}$

67. $\frac{52}{159}$

69. -2

71. $\frac{47}{100}$

73. 19,359

75. 12

77. 8

79. 24,948

81. There were about 333 pirate attacks worldwide.

83. $1,500,000$

Chapter 5

Decimals

On January 29, 2001, the New York Stock exchange ended its 200-year tradition of quoting stock prices in fractions and switched to decimals.

It was said that pricing stocks the same way other consumer items were priced would make it easier for investors to understand and compare stock prices. Foreign exchanges had been trading in decimals for decades. Supporters of the change claimed that trading volume, the number of shares of stock traded, would increase and improve efficiency.

But switching to decimals would have another effect of *narrowing the spread*. The *spread* is the difference between the best price offered by buyers, called the *bid*, and the price requested by sellers called the *ask*. Stock brokers make commissions as a percentage of the spread which, using fractions, could be anywhere upwards from 12 cents per share.

When the New York Stock Exchange began back in 1792, the dollar was based on the Spanish *real*, (pronounced ray-al), also called *pieces of eight* as these silver coins were often cut into quarters or eighths to make change. This is what led to stock prices first denominated in eighths. Thus, the smallest spread that could occur would be 1/8 of a dollar, or 12.5 cents. That may seem like small change, but buying 1000 shares for $1 per share with a $0.125 spread is a $125.00 commision. Not bad for a quick trade!

Decimalization of stock pricing allowed for spreads as small as 1 cent. Since the number of shares traded on stock market exchanges have skyrocketed, with trillions of shares traded daily, stock broker commissions have not suffered. And the ease with which investors can quickly grasp the price of stock shares has contributed to the opening of markets for all classes of people.

In this chapter, we'll learn about how to compute and solve problems with decimals, and see how they relate to fractions.

5.1 Introduction to Decimals

Recall that whole numbers are constructed by using *digits*.

The Digits. The set

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

is called the set of *digits*.

As an example, the whole number 55,555 ("fifty-five thousand five hundred fifty-five") is constructed by using a single digit. However, the position of the digit 5 determines its value in the number 55,555. The first occurrence of the

5	5	5	5	5
ten thousands	thousands	hundreds	tens	ones
10,000	1,000	100	10	1

Table 5.1: Place value.

digit 5 happens in the ten thousands place, so its value is 5 ten thousands, or 50,000. The next occurrence of the digit 5 is in the thousands place, so its value is 5 thousands, or 5,000. Indeed, the whole number 55,555 in expanded form is

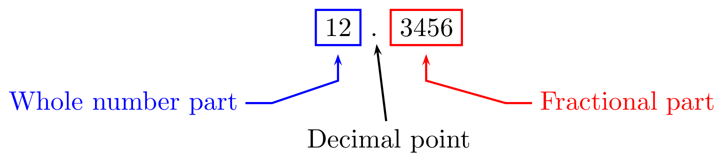
$$50000 + 5000 + 500 + 50 + 5,$$

which reflects the value of the digit 5 in each place.

Decimal Notation

In Table 5.1, each time you move one column to the left, the place value is 10 times larger than the place value of the preceding column. Vice-versa, each time you move one column to the right, the place value is 1/10 of the place value of the preceding column.

Now, consider the *decimal number* 12.3456, which consists of three parts: the whole number part, the decimal point, and the fractional part.



The whole number part of the decimal number is the part that lies strictly to the left of the decimal point, and the place value of each digit in the whole number part is given by the columns shown in Table 5.1.

The fractional part of the decimal number is the part that lies strictly to the right of the decimal point. As we saw in Table 5.1, each column has a value equal to 1/10 of the value of the column that lies to its immediate left. Thus, it should come as no surprise that:

- The first column to the right of the decimal point has place value 1/10 (tenths).
- The second column to the right of the decimal point has place value 1/100 (hundredths).
- The third column to the right of the decimal point has place value 1/1000 (thousandths).
- The fourth column to the right of the decimal point has place value 1/10000 (ten-thousandths).

These results are summarized for the decimal number 12.3456 in Table 5.2.

1	2	.	3	4	5	6
tens	ones	decimal point	tenths	hundredths	thousandths	ten-thousandths
10	1	.	1/10	1/100	1/1000	1/10000

Table 5.2: Place value.

Pronouncing Decimal Numbers

The decimal number 12.3456 is made up of 1 ten, 2 ones, 3 tenths, 4 hundredths, 5 thousandths, and 6 ten-thousandths (see Table 5.2), and can be written in *expanded form* as

$$12.3456 = 10 + 2 + \frac{3}{10} + \frac{4}{100} + \frac{5}{1000} + \frac{6}{10000}.$$

Note that the whole numbers can be combined and the fractions can be written with a common denominator and summed.

$$\begin{aligned} 12.3456 &= 12 + \frac{3 \cdot 1000}{10 \cdot 1000} + \frac{4 \cdot 100}{100 \cdot 100} + \frac{5 \cdot 10}{1000 \cdot 10} + \frac{6}{10000} \\ &= 12 + \frac{3000}{10000} + \frac{400}{10000} + \frac{50}{10000} + \frac{6}{10000} \\ &= 12 + \frac{3456}{10000} \end{aligned}$$

The result tells us how to pronounce the number 12.3456. It is pronounced "twelve and three thousand, four hundred fifty-six ten-thousandths."

You Try It!

Place the decimal number 3,502.23 in expanded form, then combine the whole number part and sum the fractional part over a common denominator

EXAMPLE 1. Place the decimal number 1,234.56 in expanded form, then combine the whole number part and sum the fractional part over a common denominator. Use the result to help pronounce the decimal number.

Solution. In expanded form,

$$1,234.56 = 1,000 + 200 + 30 + 4 + \frac{5}{10} + \frac{6}{100}$$

Sum the whole number parts. Express the fractional parts as equivalent fractions and combine over one common denominator.

$$\begin{aligned} &= 1,234 + \frac{5 \cdot 10}{10 \cdot 10} + \frac{6}{100} \\ &= 1,234 + \frac{50}{100} + \frac{6}{100} \\ &= 1,234 + \frac{56}{100} \end{aligned}$$

Hence, 1,234.56 is pronounced "one thousand, two hundred thirty-four and fifty-six hundredths."

Answer: $3,502 + \frac{23}{100}$

□

You Try It!

EXAMPLE 2. Place the decimal number 56.128 in expanded form, then combine the whole number part and sum the fractional part over a common denominator. Use the result to help pronounce the decimal number.

Place the decimal number 235.568 in expanded form, then combine the whole number part and sum the fractional part over a common denominator

Solution. In expanded form,

$$56.128 = 50 + 6 + \frac{1}{10} + \frac{2}{100} + \frac{8}{1000}$$

Sum the whole number parts. Express the fractional parts as equivalent fractions and combine over one common denominator.

$$\begin{aligned} &= 56 + \frac{1 \cdot 100}{10 \cdot 100} + \frac{2 \cdot 10}{100 \cdot 10} + \frac{8}{1000} \\ &= 56 + \frac{100}{1000} + \frac{20}{1000} + \frac{8}{1000} \\ &= 56 + \frac{128}{1000} \end{aligned}$$

Thus, 56.128 is pronounced "fifty-six and one hundred twenty-eight thousandths."

Answer: $235 + \frac{568}{1000}$

□

The discussion and example leads to the following result.

How to Read a Decimal Number

1. Pronounce the whole number part to the left of the decimal as you would any whole number.
2. Say the word "and" for the decimal point.
3. State the fractional part to the right of the decimal as you would any whole number, followed by the place value of the digit in the rightmost column.

You Try It!

EXAMPLE 3. Pronounce the decimal number 34.12.

Pronounce 28.73

Solution. The rightmost digit in the fractional part of 34.12 is in the hundredths column. Thus, 34.12 is pronounced "thirty-four and twelve hundredths."

Answer: "Twenty-eight and seventy-three hundredths"

□

Important Point. In pronouncing decimal numbers, the decimal point is read as "and." No other instance of the word "and" should appear in the pronunciation.

You Try It!

Pronounce 286.9.

Answer: "Four hundred thirty-four and two tenths"

EXAMPLE 4. Explain why "four hundred and thirty-four and two tenths" is an *incorrect* pronunciation of the decimal number 434.2.

Solution. The decimal point is read as "and." No other occurrence of the word "and" is allowed in the pronunciation. The correct pronunciation should be "four hundred thirty-four and two tenths."

□

You Try It!

Pronounce $7,002.207$.

Answer: "Seven thousand two and two hundred seven thousandths."

EXAMPLE 5. Pronounce the decimal number 5,678.123.

Solution. The rightmost digit in the fractional part of 5,678.123 is in the thousandths column. Hence, 5,678.123 is pronounced "5 thousand six hundred seventy-eight and one hundred twenty-three thousandths."

□

You Try It!

Pronounce 500.1205.

Asnwer: "Five hundred and one thousand two hundred five ten-thousandths."

EXAMPLE 6. Pronounce the decimal number 995.4325.

Solution. The rightmost digit in the fractional part of 995.4325 is in the ten-thousandths column. Hence, 995.4325 is pronounced "nine hundred ninety-five and four thousand three hundred twenty-five ten-thoudandths."

□

Decimals to Fractions

Because we now have the ability to pronounce decimal numbers, it is a simple exercise to change a decimal to a fraction.[1] For example, 134.12 is pronounced

[1]Changing fractions to decimals will be covered in Section 5.5.

"one hundred thirty-four and twelve hundredths," so it can easily be written as a mixed fraction.

$$134.12 = 134\,\frac{12}{100}$$

But this mixed fraction can be changed to an improper fraction.

$$\begin{aligned} 134\,\frac{12}{100} &= \frac{100 \cdot 134 + 12}{100} \\ &= \frac{13400 + 12}{100} \\ &= \frac{13412}{100} \end{aligned}$$

Note that the numerator is our original number without the decimal point. There are two decimal places in the original number and the denominator of the final improper fraction contains two zeros.

This discussion leads to the following result.

Changing Decimals to Improper Fractions. To change a decimal number to an improper fraction, proceed as follows:

1. Create a fraction.
2. Place the decimal number in the numerator **without the decimal point**.
3. Count the number of decimal places. Place an equal number of zeros in the denominator.

You Try It!

Change 17.205 to an improper fraction.

EXAMPLE 7. Change the following decimal numbers to improper fractions: (a) 1.2345, and (b) 27.198.

Solution. In each case, place the number in the numerator without the decimal point. In the denominator, add a number of zeros equal to the number of decimal places.

a) The decimal number 1.2345 has four decimal places. Hence,

$$1.2345 = \frac{12345}{10000}$$

b) The decimal number 27.198 has three decimal places. Hence,

$$27.198 = \frac{27198}{1000}$$

Answer: $\dfrac{17205}{100}$

□

You Try It!

Change 0.375 to a fraction, reduced to lowest terms.

EXAMPLE 8. Change each of the following decimals to fractions reduced to lowest terms: (a) 0.35, and (b) 0.125.

Solution. Place each number in the numerator without the decimal point. Place a number of zeros in the denominator equal to the number of decimal places. Reduce to lowest terms.

a) First, place 35 over 100.

$$0.35 = \frac{35}{100}$$

We can divide both numerator and denominator by the greatest common divisor.

$$\begin{aligned} &= \frac{35 \div 5}{100 \div 5} && \text{Divide numerator and denominator by 5.} \\ &= \frac{7}{20} && \text{Simplify numerator and denominator.} \end{aligned}$$

b) First, place 125 over 1000.

$$0.125 = \frac{125}{1000}$$

Prime factor and cancel common factors.

$$\begin{aligned} &= \frac{5 \cdot 5 \cdot 5}{2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5} && \text{Prime factor numerator and denominator.} \\ &= \frac{\cancel{5} \cdot \cancel{5} \cdot \cancel{5}}{2 \cdot 2 \cdot 2 \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{5}} && \text{Cancel common factors.} \\ &= \frac{1}{8} && \text{Simplify.} \end{aligned}$$

Answer: 3/8

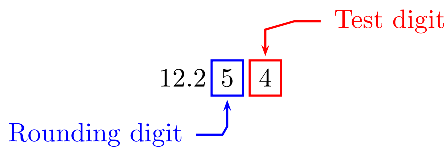
□

Rounding

The rules for rounding decimal numbers are almost identical to the rules for rounding whole numbers. First, a bit of terminology.

Rounding Digit and Test Digit. The digit in the place to which we wish to round is called the *rounding digit* and the digit that follows on its immediate right is called the *test* digit.

If we want to round the decimal number 12.254 to the nearest hundredth, then the rounding digit is 5 and the test digit is 4.



If we used the rules for rounding whole numbers, because the test digit 4 is less than 5, we would replace all digits to the right of the rounding digit with zeros to obtain the following approximation.

$$12.254 \approx 12.250$$

However, because

$$12.250 = 12\,\frac{250}{1000} = 12\,\frac{25}{100},$$

the trailing zero at the end of the fractional part is irrelevant. Hence, we *truncate* every digit after the rounding digit and use the following approximation.

$$12.254 \approx 12.25$$

Important Observation. Deleting trailing zeros from the end of the fractional part of a decimal number does not change its value.

The above discussion motivates the following algorithm for rounding decimal numbers.

Rounding Decimal Numbers. Locate the *rounding digit* and the *test digit.*

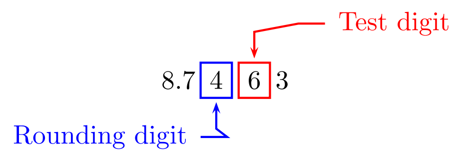
- If the test digit is greater than or equal to 5, add 1 to the rounding digit and truncate all digits to the right of the rounding digit.
- If the test digit is less than 5, simply truncate all digits to the right of the rounding digit.

You Try It!

Round 9.2768 to the nearest hundredth.

EXAMPLE 9. Round 8.7463 to the nearest hundredth.

Solution. Locate the rounding digit in the hundredths place and the test digit to its immediate right.



Because the test digit is greater than 5, add 1 to the rounding digit and truncate all digits to the right of the rounding digit. Hence, to the nearest hundredth:

$$8.7463 \approx 8.75$$

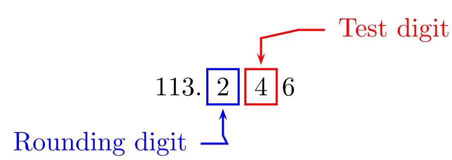
Answer: 9.28

You Try It!

Round 58.748 to the nearest tenth.

EXAMPLE 10. Round 113.246 to the nearest tenth.

Solution. Locate the rounding digit in the tenths place and the test digit to its immediate right.



Because the test digit is less than 5, truncate all digits to the right of the rounding digit. Hence, to the nearest tenth:

$$113.246 \approx 113.2$$

Answer: 58.7

Comparing Decimals

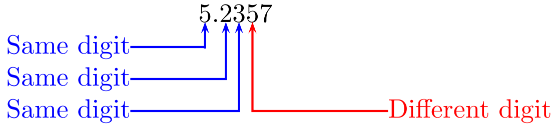
We can compare two positive decimals by comparing digits in each place as we move from left to right, place by place. For example, suppose we wish to compare the decimal numbers 5.234 and 5.2357. First, add enough trailing zeros to the decimal number with the fewer decimal places so that the numbers have the same number of decimal places. In this case, note that

$$5.234 = 5\,\frac{234}{1000} = 5\,\frac{2340}{10000} = 5.2340.$$

Important Observation. Adding trailing zeros to the end of the fractional part of a decimal number does not change its value.

Next, align the numbers as follows.

5.2340
5.2357



As you scan the columns, moving left to right, the first place that has different digits occurs in the thousandths place, where the digit 5 is the second number is greater than the digit 4 in the first number in the same place. Because 5 is greater than 4, the second number is larger than the first. That is:

$$5.234 < 5.2357$$

This discussion suggests the following algorithm.

Comparing Positive Decimal Numbers.

1. Add enough trailing zeros so that both numbers have the same number of decimal places.
2. Compare the digits in each place, moving left to right.
3. As soon as you find two digits in the same place that are different, the decimal number with the greatest digit in this place is the **larger** number.

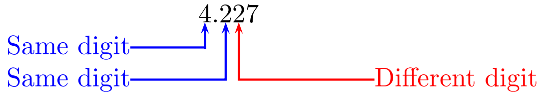
You Try It!

Compare 8.34 and 8.348.

EXAMPLE 11. Compare 4.25 and 4.227.

Solution. Add a trailing zero to the first decimal number and align the numbers as follows.

4.250
4.227



The first difference is in the hundredths place, where the digit 5 in the first number is greater than the digit 2 in the same place of the second number. Hence, the first number is **larger** than the second; that is:

$$4.25 > 4.227$$

Answer: $8.34 < 8.348$

When comparing negative numbers, the number with the larger magnitude is the smaller number. Hence, we have to adjust our algorithm for comparing negative decimal numbers.

Comparing Negative Decimal Numbers.

1. Add enough trailing zeros so that both numbers have the same number of decimal places.
2. Compare the digits in each place, moving left to right.
3. As soon as you find two digits in the same place that are different, the decimal number with the greatest digit in this place is the **smaller** number.

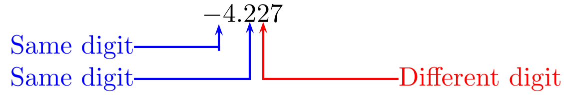
You Try It!

Compare −7.86 and −7.85.

EXAMPLE 12. Compare −4.25 and −4.227.

Solution. Add a trailing zero to the first decimal number and align the numbers as follows.

$$\begin{array}{r} -4.250 \\ -4.227 \end{array}$$



The first difference is in the hundredths place, where the digit 5 in the first number is greater than the digit 2 in the same place of the second number. Hence, the first number is **smaller** than the second; that is:

$$-4.25 < -4.227$$

Answer: $-7.86 < -7.85$

Exercises

1. Which digit is in the tenths column of the number 4,552.0908?

2. Which digit is in the thousandths column of the number 7,881.6127?

3. Which digit is in the tenths column of the number 4,408.2148?

4. Which digit is in the tenths column of the number 9,279.0075?

5. Which digit is in the ten-thousandths column of the number 2,709.5097?

6. Which digit is in the hundredths column of the number 1,743.1634?

7. Which digit is in the hundredths column of the number 3,501.4456?

8. Which digit is in the ten-thousandths column of the number 9,214.3625?

9. Which digit is in the hundredths column of the number 5,705.2193?

10. Which digit is in the hundredths column of the number 7,135.2755?

11. Which digit is in the tenths column of the number 8,129.3075?

12. Which digit is in the thousandths column of the number 6,971.4289?

In Exercises 13-20, write the given decimal number in expanded form.

13. 46.139

14. 68.392

15. 643.19

16. 815.64

17. 14.829

18. 45.913

19. 658.71

20. 619.38

In Exercises 21-28, follow the procedure shown in Examples 1 and 2 to write the decimal number in expanded form, then combine the whole number part and sum the fractional part over a common denominator.

21. 32.187

22. 35.491

23. 36.754

24. 89.357

25. 596.71

26. 754.23

27. 527.49

28. 496.15

In Exercises 29-40, pronounce the given decimal number. Write your answer out in words.

29. 0.9837

30. 0.6879

31. 0.2653

32. 0.8934

33. 925.47

34. 974.35

35. 83.427

36. 32.759

37. 63.729

38. 85.327

39. 826.57

40. 384.72

In Exercises 41-52, convert the given decimal to a mixed fraction. Do *not* simplify your answer.

41. 98.1

42. 625.591

43. 781.7

44. 219.999

45. 915.239

46. 676.037

47. 560.453

48. 710.9

49. 414.939

50. 120.58

51. 446.73

52. 653.877

In Exercises 53-60, convert the given decimal to an improper fraction. Do *not* simplify your answer.

53. 8.7

54. 3.1

55. 5.47

56. 5.27

57. 2.133

58. 2.893

59. 3.9

60. 1.271

In Exercises 61-68, convert the given decimal to a fraction. Reduce your answer to lowest terms.

61. 0.35

62. 0.38

63. 0.06

64. 0.84

65. 0.98

66. 0.88

67. 0.72

68. 0.78

---

69. Round 79.369 to the nearest hundredth.

70. Round 54.797 to the nearest hundredth.

71. Round 71.2427 to the nearest thousandth.

72. Round 59.2125 to the nearest thousandth.

73. Round 29.379 to the nearest tenth.

74. Round 42.841 to the nearest tenth.

75. Round 89.3033 to the nearest thousandth.

76. Round 9.0052 to the nearest thousandth.

77. Round 20.655 to the nearest tenth.

78. Round 53.967 to the nearest tenth.

79. Round 19.854 to the nearest hundredth.

80. Round 49.397 to the nearest hundredth.

In Exercises 81-92, determine which of the two given statements is true.

81.

$$0.30387617 < 0.3036562$$
or
$$0.30387617 > 0.3036562$$

82.

$$8.5934 < 8.554$$
or
$$8.5934 > 8.554$$

83.

$$-0.034 < -0.040493$$
or
$$-0.034 > -0.040493$$

84.

$$-0.081284 < -0.08118$$
or
$$-0.081284 > -0.08118$$

85.

$$-8.3527 < -8.36553$$
or
$$-8.3527 > -8.36553$$

86.

$$-0.00786 < -0.0051385$$
or
$$-0.00786 > -0.0051385$$

87.

$$18.62192 < 18.6293549$$
or
$$18.62192 > 18.6293549$$

88.

$$514.873553 < 514.86374$$
or
$$514.873553 > 514.86374$$

89.

$$36.8298 < 36.8266595$$
or
$$36.8298 > 36.8266595$$

90.

$$0.000681 < 0.00043174$$
or
$$0.000681 > 0.00043174$$

91.

$$-15.188392 < -15.187157$$
or
$$-15.188392 > -15.187157$$

92.

$$-0.049785 < -0.012916$$
or
$$-0.049785 > -0.012916$$

93. Write the decimal number in words.

i) A recently discovered 7.03-carat blue diamond auctioned at Sotheby's.

ii) The newly launched European Planck telescope will stay in orbit 1.75 years measuring radiation from the Big Bang.

iii) The sun composes 0.9985 of the mass in our solar system.

iv) Clay particles are small - only 0.0001 inch.

94. **Light speed.** The *index of refraction* for a given material is a value representing the number of times slower a light wave travels in that particular material than it travels in the vacuum of space.

i) Reorder the materials by their index of refraction from lowest to highest.

ii) How many times slower is a lightwave in a diamond compared with a vacuum?

Material	Index of Refraction
Diamond	2.417
Vacuum	1.0000
Plexiglas	1.51
Air	1.0003
Water	1.333
Zircon	1.923
Crown Glass	1.52
Ice	1.31

95. **Shorter day?** Scientists at NASA's Jet Propulsion Laboratory calculated that the earthquake in Chile may have shortened the length of a day on Earth by 1.26 millionths of a second.

i) Write that number completely as a decimal.

ii) Actual observations of the length of the day are accurate to five millionths of a second. Write this fraction as a decimal.

iii) Comparing the two decimals above and determine which is smaller. Do you think scientists can observe and measure the calculated slowing of the earth?

Answers

1. 0

3. 2

5. 7

7. 4

9. 1

11. 3

13. $40 + 6 + \frac{1}{10} + \frac{3}{100} + \frac{9}{1000}$

15. $600 + 40 + 3 + \frac{1}{10} + \frac{9}{100}$

17. $10 + 4 + \frac{8}{10} + \frac{2}{100} + \frac{9}{1000}$

19. $600 + 50 + 8 + \frac{7}{10} + \frac{1}{100}$

21. $32 + \frac{187}{1000}$

23. $36 + \frac{754}{1000}$

25. $596 + \frac{71}{100}$

27. $527 + \frac{49}{100}$

29. nine thousand eight hundred thirty-seven ten-thousandths

31. two thousand six hundred fifty-three ten-thousandths

33. nine hundred twenty-five and forty-seven hundredths

35. eighty-three and four hundred twenty-seven thousandths

37. sixty-three and seven hundred twenty-nine thousandths

39. eight hundred twenty-six and fifty-seven hundredths

41. $98\frac{1}{10}$

43. $781\frac{7}{10}$

45. $915\frac{239}{1000}$

47. $560\frac{453}{1000}$

49. $414\frac{939}{1000}$

51. $446\frac{73}{100}$

53. $\frac{87}{10}$

55. $\frac{547}{100}$

57. $\frac{2133}{1000}$

59. $\frac{39}{10}$

61. $\frac{7}{20}$

63. $\frac{3}{50}$

65. $\frac{49}{50}$

67. $\frac{18}{25}$

69. 79.37

71. 71.243

73. 29.4

75. 89.303

77. 20.7

79. 19.85

81. $0.30387617 > 0.3036562$

83. $-0.034 > -0.040493$

85. $-8.3527 > -8.36553$

87. $18.62192 < 18.6293549$

89. $36.8298 > 36.8266595$

91. $-15.188392 < -15.187157$

93. i) seven and three hundredths

ii) one and seventy-five hundredths

iii) nine thousand nine hundred eighty-five ten-thousandths

iv) one ten-thousandth of an inch

95. i) 0.00000126

ii) 0.000005

iii) $0.00000126 < 0.000005$; scientists would be unable to measure the calculated change in the length of a day.

5.2 Adding and Subtracting Decimals

Adding Decimals

Addition of decimal numbers is quite similar to addition of whole numbers. For example, suppose that we are asked to add 2.34 and 5.25. We could change these decimal numbers to mixed fractions and add.

$$\begin{aligned} 2.34 + 5.25 &= 2\frac{34}{100} + 5\frac{25}{100} \\ &= 7\frac{59}{100} \end{aligned}$$

However, we can also line the decimal numbers on their decimal points and add vertically, as follows.

$$\begin{array}{r} 2.34 \\ +\,5.25 \\ \hline 7.59 \end{array}$$

Note that this alignment procedure produces the same result, "seven and fifty nine hundredths." This motivates the following procedure for adding decimal numbers.

Adding Decimals. To add decimal numbers, proceed as follows:

1. Place the numbers to be added in vertical format, aligning the decimal points.
2. Add the numbers as if they were whole numbers.
3. Place the decimal point in the answer in the *same column* as the decimal points above it.

You Try It!

EXAMPLE 1. Add 3.125 and 4.814.

Add: $2.864 + 3.029$

Solution. Place the numbers in vertical format, aligning on their decimal points. Add, then place the decimal point in the answer in the same column as the decimal points that appear above the answer.

$$\begin{array}{r} 3.125 \\ +\,4.814 \\ \hline 7.939 \end{array}$$

Thus, $3.125 + 4.814 = 7.939$.

Answer: 5.893

□

You Try It!

Alice has \$8.63 in her purse and Joanna has \$2.29. If they combine sum their money, what is the total?

EXAMPLE 2. Jane has \$4.35 in her purse. Jim has \$5.62 in his wallet. If they sum their money, what is the total?

Solution. Arrange the numbers in vertical format, aligning decimal points, then add.

$$\begin{array}{r} \$4.35 \\ +\,\$5.62 \\ \hline \$9.97 \end{array}$$

Together they have \$9.97, nine dollars and ninety seven cents.

Answer: \$10.91

□

Before looking at another example, let's recall an important observation.

Important Observation. Adding zeros to the end of the fractional part of a decimal number does not change its value. Similarly, deleting trailing zeros from the end of a decimal number does not change its value.

For example, we could add two zeros to the end of the fractional part of 7.25 to obtain 7.2500. The numbers 7.25 and 7.2500 are identical as the following argument shows:

$$\begin{aligned} 7.2500 &= 7\frac{2500}{10000} \\ &= 7\frac{25}{100} \\ &= 7.25 \end{aligned}$$

You Try It!

Add: $9.7 + 15.86$

EXAMPLE 3. Add 7.5 and 12.23.

Solution. Arrange the numbers in vertical format, aligning their decimal points in a column. Note that we add a trailing zero to improve columnar alignment.

$$\begin{array}{r} 7.50 \\ +\,12.23 \\ \hline 19.73 \end{array}$$

Hence, $7.5 + 12.23 = 19.73$.

Answer: 25.56

□

You Try It!

EXAMPLE 4. Find the sum: $12.2 + 8.352 + 22.44$.

Add: $12.9 + 4.286 + 33.97$

Solution. Arrange the numbers in vertical format, aligning their decimal points in a column. Note that we add trailing zeros to improve the columnar alignment.

$$\begin{array}{r} 12.200 \\ 8.352 \\ +\,22.440 \\ \hline 42.992 \end{array}$$

Hence, $12.2 + 8.352 + 22.44 = 42.992$.

Answer: 51.156

Subtracting Decimals

Subtraction of decimal numbers proceeds in much the same way as addition of decimal numbers.

Subtracting Decimals. To subtract decimal numbers, proceed as follows:

1. Place the numbers to be subtracted in vertical format, aligning the decimal points.
2. Subtract the numbers as if they were whole numbers.
3. Place the decimal point in the answer in the *same column* as the decimal points above it.

You Try It!

EXAMPLE 5. Subtract 12.23 from 33.57.

Subtract: $58.76 - 38.95$

Solution. Arrange the numbers in vertical format, aligning their decimal points in a column, then subtract. Note that we subtract 12.23 **from** 33.57.

$$\begin{array}{r} 33.57 \\ -\,12.23 \\ \hline 21.34 \end{array}$$

Hence, $33.57 - 12.23 = 21.34$.

Asnwer: 19.81

As with addition, we add trailing zeros to the fractional part of the decimal numbers to help columnar alignment.

You Try It!

Subtract: $15.2 - 8.756$

EXAMPLE 6. Find the difference: $13.3 - 8.572$.

Solution. Arrange the numbers in vertical format, aligning their decimal points in a column. Note that we add trailing zeros to the fractional part of 13.3 to improve columnar alignment.

$$\begin{array}{r} 13.300 \\ -\,8.572 \\ \hline 4.728 \end{array}$$

Answer: 6.444

Hence, $13.3 - 8.572 = 4.728$.

□

Adding and Subtracting Signed Decimal Numbers

We use the same rules for addition of signed decimal numbers as we did for the addition of integers.

Adding Two Decimals with Like Signs. To add two decimals with like signs, proceed as follows:

1. Add the magnitudes of the decimal numbers.
2. Prefix the common sign.

You Try It!

Simplify: $-5.7 + (-83.85)$

EXAMPLE 7. Simplify: $-3.2 + (-18.95)$.

Solution. To add like signs, first add the magnitudes.

$$\begin{array}{r} 3.20 \\ +\,18.95 \\ \hline 22.15 \end{array}$$

Answer: -89.55

Prefix the common sign. Hence, $-3.2 + (-18.95) = -22.15$

□

We use the same rule as we did for integers when adding decimals with unlike signs.

Adding Two Decimals with Unlike Signs. To add two decimals with unlike signs, proceed as follows:

1. Subtract the smaller magnitude from the larger magnitude.
2. Prefix the sign of the decimal number with the larger magnitude.

You Try It!

EXAMPLE 8. Simplify: $-3 + 2.24$.

Simplify: $-8 + 5.74$

Solution. To add unlike signs, first subtract the smaller magnitude from the larger magnitude.

$$\begin{array}{r} 3.00 \\ -\ 2.24 \\ \hline 0.76 \end{array}$$

Prefix the sign of the decimal number with the larger magnitude. Hence, $-3 + 2.24 = -0.76$.

Answer: -2.26

Subtraction still means *add the opposite*.

You Try It!

EXAMPLE 9. Simplify: $-8.567 - (-12.3)$.

Simplify: $-2.384 - (-15.2)$

Solution. Subtraction must first be changed to addition by adding the opposite.

$$-8.567 - (-12.3) = -8.567 + 12.3$$

We have unlike signs. First, subtract the smaller magnitude from the larger magnitude.

$$\begin{array}{r} 12.300 \\ -\ 8.567 \\ \hline 3.733 \end{array}$$

Prefix the sign of the decimal number with the larger magnitude. Hence:

$$\begin{aligned} -8.567 - (-12.3) &= -8.567 + 12.3 \\ &= 3.733 \end{aligned}$$

Answer: 12.816

Order of operations demands that we simplify expressions contained in parentheses first.

You Try It!

EXAMPLE 10. Simplify: $-11.2 - (-8.45 + 2.7)$.

Simplify: $-12.8 - (-7.44 + 3.7)$

Solution. We need to add inside the parentheses first. Because we have unlike signs, subtract the smaller magnitude from the larger magnitude.

$$\begin{array}{r} 8.45 \\ -\ 2.70 \\ \hline 5.75 \end{array}$$

Prefix the sign of the number with the larger magnitude. Therefore,

$$-11.2 - (-8.45 + 2.7) = -11.2 - (-5.75)$$

Subtraction means add the opposite.

$$-11.2 - (-5.75) = -11.2 + 5.75$$

Again, we have unlike signs. Subtract the smaller magnitude from the larger magnitude.

$$\begin{array}{r} 11.20 \\ -\ 5.75 \\ \hline 5.45 \end{array}$$

Prefix the sign of the number with the large magnitude.

$$-11.2 + 5.75 = -5.45$$

Answer: -9.06

□

Writing Mathematics. The solution to the previous example should be written as follows:

$$\begin{aligned} -11.2 - (-8.45 + 2.7) &= -11.2 - (-5.75) \\ &= -11.2 + 5.75 \\ &= -5.45 \end{aligned}$$

Any scratch work, such as the computations in vertical format in the previous example, should be done in the margin or on a scratch pad.

You Try It!

EXAMPLE 11. Simplify: $-12.3 - |-4.6 - (-2.84)|$.

Simplify:

$$-8.6 - |-5.5 - (-8.32)|$$

Solution. We simplify the expression inside the absolute value bars first, take the absolute value of the result, then subtract.

$$\begin{aligned}
&-12.3 - |-4.6 - (-2.84)| \\
&\quad = -12.3 - |-4.6 + 2.84| && \text{Add the opposite.} \\
&\quad = -12.3 - |-1.76| && \text{Add: } -4.6 + 2.84 = -1.76. \\
&\quad = -12.3 - 1.76 && |-1.76| = 1.76. \\
&\quad = -12.3 + (-1.76) && \text{Add the opposite.} \\
&\quad = -14.06 && \text{Add: } -12.3 + (-1.76) = -14.06.
\end{aligned}$$

Answer: -11.42

□

Exercises

In Exercises 1-12, add the decimals.

1. $31.9 + 84.7$

2. $9.39 + 7.7$

3. $4 + 97.18$

4. $2.645 + 2.444$

5. $4 + 87.502$

6. $23.69 + 97.8$

7. $95.57 + 7.88$

8. $18.7 + 7$

9. $52.671 + 5.97$

10. $9.696 + 28.2$

11. $4.76 + 2.1$

12. $1.5 + 46.4$

In Exercises 13-24, subtract the decimals.

13. $9 - 2.261$

14. $98.14 - 7.27$

15. $80.9 - 6$

16. $9.126 - 6$

17. $55.672 - 3.3$

18. $4.717 - 1.637$

19. $60.575 - 6$

20. $8.91 - 2.68$

21. $39.8 - 4.5$

22. $8.210 - 3.7$

23. $8.1 - 2.12$

24. $7.675 - 1.1$

In Exercises 25-64, add or subtract the decimals, as indicated.

25. $-19.13 - 7$

26. $-8 - 79.8$

27. $6.08 - 76.8$

28. $5.76 - 36.8$

29. $-34.7 + (-56.214)$

30. $-7.5 + (-7.11)$

31. $8.4 + (-6.757)$

32. $-1.94 + 72.85$

33. $-50.4 + 7.6$

34. $1.4 + (-86.9)$

35. $-43.3 + 2.2$

36. $0.08 + (-2.33)$

37. $0.19 - 0.7$

38. $9 - 18.01$

39. $-7 - 1.504$

40. $-4.28 - 2.6$

41. $-4.47 + (-2)$

42. $-9 + (-43.67)$

43. $71.72 - (-6)$

44. $6 - (-8.4)$

45. $-9.829-(-17.33)$

46. $-95.23-(-71.7)$

47. $2.001-4.202$

48. $4-11.421$

49. $2.6-2.99$

50. $3.57-84.21$

51. $-4.560-2.335$

52. $-4.95-96.89$

53. $-54.3-3.97$

54. $-2-29.285$

55. $-6.32+(-48.663)$

56. $-8.8+(-34.27)$

57. $-8-(-3.686)$

58. $-2.263-(-72.3)$

59. $9.365+(-5)$

60. $-0.12+6.973$

61. $2.762-(-7.3)$

62. $65.079-(-52.6)$

63. $-96.1+(-9.65)$

64. $-1.067+(-4.4)$

In Exercises 65-80, simplify the given expression.

65. $-12.05-|17.83-(-17.16)|$

66. $15.88-|-5.22-(-19.94)|$

67. $-6.4+|9.38-(-9.39)|$

68. $-16.74+|16.64-2.6|$

69. $-19.1-(1.51-(-17.35))$

70. $17.98-(10.07-(-10.1))$

71. $11.55+(6.3-(-1.9))$

72. $-8.14+(16.6-(-15.41))$

73. $-1.7-(1.9-(-16.25))$

74. $-4.06-(4.4-(-10.04))$

75. $1.2+|8.74-16.5|$

76. $18.4+|16.5-7.6|$

77. $-12.4-|3.81-16.4|$

78. $13.65-|11.55-(-4.44)|$

79. $-11.15+(11.6-(-16.68))$

80. $8.5+(3.9-6.98)$

---

81. **Big Banks.** Market capitalization of nation's four largest banks (*as of April 23, 2009*)

JPMorgan Chase & Co	\$124.8 billion
Wells Fargo & Co.	\$85.3 billion
Goldman Sachs Group Inc.	\$61.8 billion
Bank of America	\$56.4 billion

What is the total value of the nation's four largest banks? *Associated Press Times-Standard 4/22/09*

82. **Telescope Mirror.** The newly launched Herschel Telescope has a mirror 11.5 feet in diameter while Hubble's mirror is 7.9 feet in diameter. How much larger is Herschel's mirror in diameter than Hubble's?

83. **Average Temperature.** The average temperatures in Sacramento, California in July are a high daytime temperature of 93.8 degrees Fahrenheit and a low nighttime temperature of 60.9 degrees Fahrenheit. What is the change in temperature from day to night? *Hint: See Section 2.3 for the formula for comparing temperatures.*

84. **Average Temperature.** The average temperatures in Redding, California in July are a high daytime temperature of 98.2 degrees Fahrenheit and a low nighttime temperature of 64.9 degrees Fahrenheit. What is the change in temperature from day to night? *Hint: See Section 2.3 for the formula for comparing temperatures.*

85. **Net Worth.** Net worth is defined as *assets* minus *liabilities*. *Assets* are everything of value that can be converted to cash while *liabilities* are the total of debts. General Growth Properties, the owners of the Bayshore Mall, have $29.6 billion in assets and $27 billion in liabilities, and have gone bankrupt. What was General Growth Properties net worth before bankruptcy? *Times-Standard 4/17/2009*

86. **Grape crush.** The California Department of Food and Agriculture's preliminary grape crush report shows that the state produced 3.69 million tons of wine grapes in 2009. That's just shy of the record 2005 crush of 3.76 million tons. By how many tons short of the record was the crush of 2009? *Associated Press-Times-Standard Calif. winegrapes harvest jumped 23% in '09.*

87. **Turnover.** The Labor Department's Job Openings and Labor Turnover Survey claims that employers hired about 4.08 million people in January 2010 while 4.12 million people were fired or otherwise left their jobs. How many more people lost jobs than were hired? Convert your answer to a whole number. *Associated Press-Times-Standard 03/10/10 Job openings up sharply in January to* 2.7*M.*

Answers

1. 116.6

3. 101.18

5. 91.502

7. 103.45

9. 58.641

11. 6.86

13. 6.739

15. 74.9

17. 52.372

19. 54.575

21. 35.3

23. 5.98

25. −26.13

27. −70.72

29. −90.914

31. 1.643

33. −42.8

35. -41.1

37. -0.51

39. -8.504

41. -6.47

43. 77.72

45. 7.501

47. -2.201

49. -0.39

51. -6.895

53. -58.27

55. -54.983

57. -4.314

59. 4.365

61. 10.062

63. -105.75

65. -47.04

67. 12.37

69. -37.96

71. 19.75

73. -19.85

75. 8.96

77. -24.99

79. 17.13

81. \$328.3 billion

83. -32.9 degrees Fahrenheit

85. \$2.6 billion

87. $40,000$

5.3 Multiplying Decimals

Multiplying decimal numbers involves two steps: (1) multiplying the numbers as whole numbers, ignoring the decimal point, and (2) placing the decimal point in the correct position in the product or answer.

For example, consider (0.7)(0.08), which asks us to find the product of "seven tenths" and "eight hundredths." We could change these decimal numbers to fractions, then multiply.

$$\begin{aligned}(0.7)(0.08) &= \frac{7}{10} \cdot \frac{8}{100} \\ &= \frac{56}{1000} \\ &= 0.056\end{aligned}$$

The product is 56/1000, or "fifty six thousandths," which as a decimal is written 0.056.

Important Observations. There are two very important observations to be made about the example (0.7)(0.08).

1. In fractional form
$$\frac{7}{10} \cdot \frac{8}{100} = \frac{56}{1000},$$
note that the numerator of the product is found by taking the product of the whole numbers 7 and 8. That is, you ignore the decimal points in 0.7 and 0.08 and multiply 7 and 8 as if they were whole numbers.

2. The first factor, 0.7, has one digit to the right of the decimal point. Its fractional equivalent, 7/10, has one zero in its denominator. The second factor, 0.08, has two digits to the right of the decimal point. Its fractional equivalent, 8/100, has two zeros in its denominator. Therefore, the product 56/1000 is forced to have three zeros in its denominator and its decimal equivalent, 0.056, must therefore have three digits to the right of the decimal point.

Let's look at another example.

You Try It!

Multiply: (1.86)(9.5)

EXAMPLE 1. Simplify: (2.34)(1.2).

Solution. Change the decimal numbers "two and thirty four hundredths" and "one and two tenths" to fractions, then multiply.

$$\begin{aligned}(2.34)(1.2) &= 2\frac{34}{100}\cdot 1\frac{2}{10} && \text{Change decimals to fractions.}\\ &= \frac{234}{100}\cdot\frac{12}{10} && \text{Change mixed to improper fractions.}\\ &= \frac{2808}{1000} && \text{Multiply numerators and denominators.}\\ &= 2\frac{808}{1000} && \text{Change to mixed fraction.}\\ &= 2.808 && \text{Change back to decimal form.}\end{aligned}$$

Answer: 17.67

Important Observations. We make the same two observations as in the previous example.

1. If we treat the decimal numbers as whole numbers without decimal points, then $(234)(12) = 2808$, which is the numerator of the fraction $2808/1000$ in the solution shown in Example 1. These are also the same digits shown in the answer 2.808.

2. There are two digits to the right of the decimal point in the first factor 2.34 and one digit to the right of the decimal point in the second factor 1.2. This is a total of three digits to the right of the decimal points in the factors, which is precisely the same number of digits that appear to the right of the decimal point in the answer 2.808.

The observations made at the end of the previous two examples lead us to the following method.

Multiplying Decimal Numbers. To multiply two decimal numbers, perform the following steps:

1. Ignore the decimal points in the factors and multiply the two factors as if they were whole numbers.

2. Count the number of digits to the right of the decimal point in each factor. Sum these two numbers.

3. Place the decimal point in the product so that the number of digits to the right of the decimal point equals the sum found in step 2.

You Try It!

Multiply: (5.98)(3.7)

EXAMPLE 2. Use the steps outlined in *Multiplying Decimal Numbers* to find the product in Example 1.

Solution. We follow the steps outlined in *Multiplying Decimal Numbers*.

1. The first step is to multiply the factors 2.34 and 1.2 as whole numbers, ignoring the decimal points.

$$\begin{array}{r} 234 \\ \times\ 12 \\ \hline 468 \\ 234 \\ \hline 2808 \end{array}$$

2. The second step is to find the sum of the number of digits to the right of the decimal points in each factor. Note that 2.34 has two digits to the right of the decimal point, while 1.2 has one digit to the right of the decimal point. Thus, we have a total of three digits to the right of the decimal points in the factors.

3. The third and final step is to place the decimal point in the product or answer so that there are a total of three digits to the right of the decimal point. Thus,

$$(2.34)(1.2) = 2.808.$$

   Note that this is precisely the same solution found in Example 1.

What follows is a convenient way to arrange your work in vertical format.

$$\begin{array}{r} 2.34 \\ \times\ 1.2 \\ \hline 468 \\ 2\ 34 \\ \hline 2.808 \end{array}$$

Answer: 22.126

□

You Try It!

Multiply: (9.582)(8.6)

EXAMPLE 3. Simplify: (8.235)(2.3).

Solution. We use the convenient vertical format introduced at the end of Example 2.

$$\begin{array}{r} 8.235 \\ \times\, 2.3 \\ \hline 2\,4705 \\ 16\,470 \\ \hline 18.9405 \end{array}$$

The factor 8.235 has three digits to the right of the decimal point; the factor 2.3 has one digit to the right of the decimal point. Therefore, there must be a total of four digits to the right of the decimal point in the product or answer.

Answer: 82.4052

Multiplying Signed Decimal Numbers

The rules governing multiplication of signed decimal numbers are identical to the rules governing multiplication of integers.

Like Signs. The product of two decimal numbers with like signs is positive. That is:

$$(+)(+) = + \qquad \text{and} \qquad (-)(-) = +$$

Unike Signs. The product of two decimal numbers with unlike signs is negative. That is:

$$(+)(-) = - \qquad \text{and} \qquad (-)(+) = -$$

You Try It!

EXAMPLE 4. Simplify: $(-2.22)(-1.23)$.

Multiply: $(-3.86)(-5.77)$

Solution. Ignore the signs to do the multiplication on the left, then consider the signs in the final answer on the right.

As each factor has two digits to the right of the decimal point, there should be a total of 4 decimals to the right of the decimal point in the product.

$$\begin{array}{r} 2.22 \\ \times\, 1.23 \\ \hline 666 \\ 444 \\ 1\,11 \\ \hline 1.6206 \end{array}$$

Like signs give a positive product. Hence:

$$(-2.22)(-1.23) = 1.6206$$

Answer: 22.2722

You Try It!

Multiply: $(9.23)(-0.018)$

EXAMPLE 5. Simplify: $(5.68)(-0.012)$.

Solution. Ignore the signs to do the multiplication on the left, then consider the signs in the final answer on the right.

The first factor has two digits to the right of the decimal point, the second factor has three. Therefore, there must be a total of five digits to the right of the decimal point in the product or answer. This necessitates prepending an extra zero in front of our product.

$$\begin{array}{r} 5.68 \\ \times\, 0.012 \\ \hline 1136 \\ 568 \\ \hline 0.06816 \end{array}$$

Unlike signs give a negative product. Hence:

$$(5.68)(-0.012) = -0.06816$$

Answer: -0.16614

□

Order of Operations

The same *Rules Guiding Order of Operations* also apply to decimal numbers.

Rules Guiding Order of Operations. When evaluating expressions, proceed in the following order.

1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
2. Evaluate all exponents that appear in the expression.
3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.
4. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

You Try It!

If $a = 3.8$ and $b = -4.6$, evaluate the expression:

$$2.5a^2 - b^2$$

EXAMPLE 6. If $a = 3.1$ and $b = -2.4$, evaluate $a^2 - 3.2b^2$.

Solution. Prepare the expression for substitution using parentheses.

$$a^2 - 3.2b^2 = (\quad)^2 - 3.2(\quad)^2$$

Substitute 3.1 for a and -2.4 for b and simplify.

$a^2 - 3.2b^2 = (3.1)^2 - 3.2(-2.4)^2$	Substitute: 3.1 for a, -2.4 for b.
$= 9.61 - 3.2(5.76)$	Exponents first: $(3.1)^2 = 9.61$, $(-2.4)^2 = 5.76$
$= 9.61 - 18.432$	Multiply: $3.2(5.76) = 18.432$
$= -8.822$	Subtract: $9.61 - 18.432 = -8.822$

Answer: 14.94

□

Powers of Ten

Consider:

$$10^1 = 10$$
$$10^2 = 10 \cdot 10 = 100$$
$$10^3 = 10 \cdot 10 \cdot 10 = 1,000$$
$$10^4 = 10 \cdot 10 \cdot 10 \cdot 10 = 10,000$$

Note the answer for 10^4, a one followed by four zeros! Do you see the pattern?

Powers of Ten. In the expression 10^n, the exponent matches the number of zeros in the answer. Hence, 10^n will be a 1 followed by n zeros.

You Try It!

EXAMPLE 7. Simplify: 10^9.

Simplify: 10^6

Solution. 10^9 should be a 1 followed by 9 zeros. That is,

$$10^9 = 1,000,000,000,$$

or "one billion."

Answer: 1,000,000

□

Multiplying Decimal Numbers by Powers of Ten

Let's multiply 1.234567 by 10^3, or equivalently, by 1,000. Ignore the decimal point and multiply the numbers as whole numbers.

$$\begin{array}{r} 1.234567 \\ \times 1000 \\ \hline 1234.567000 \end{array}$$

The sum total of digits to the right of the decimal point in each factor is 6. Therefore, we place the decimal point in the product so that there are six digits to the right of the decimal point.

However, the trailing zeros may be removed without changing the value of the product. That is, 1.234567 times 1000 is 1234.567. Note that the decimal point in the product is three places further to the right than in the original factor. This observation leads to the following result.

Multiplying a Decimal Number by a Power of Ten. Multiplying a decimal number by 10^n will move the decimal point n places to the right.

You Try It!

Simplify: $1.234567 \cdot 10^2$

Answer: 123.4567

EXAMPLE 8. Simplify: $1.234567 \cdot 10^4$

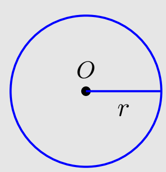
Solution. Multiplying by 10^4 (or equivalently, by 10,000) moves the decimal 4 places to the right. Thus, $1.234567 \cdot 10,000 = 12345.67$.

□

The Circle

Let's begin with a definition.

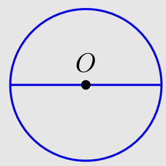
The Circle. A *circle* is the collection of all points equidistant from a given point O, called the *center* of the circle.



The segment joining any point on the circle to the center of the circle is called a *radius* of the circle. In the figure above, the variable r represents the length of the radius of the circle.

We need another term, the *diameter* of a circle.

The Diameter of a Circle. If two points on a circle are connected with a line segment, then that segment is called a *chord* of the circle. If the chord passes through the center of the circle, then the chord is called the *diameter* of the circle.



In the figure above, the variable d represents the length of the diameter of the circle. Note that the diameter is twice the length of the radius; in symbols,

$$d = 2r.$$

The Circumference of a Circle

When we work with polygons, the *perimeter* of the polygon is found by summing the lengths of its edges. The circle uses a different name for its perimeter.

The Circumference of a Circle. The length of the circle is called its *circumference*. We usually use the variable C to denote the circumference of a circle.

That is, if one were to walk along the circle, the total distance traveled in one revolution is the circumference of the circle.

The ancient mathematicians of Egypt and Greece noted a striking relation between the circumference of a circle and its diameter. They discovered that whenever you divide a circle's circumference by its diameter, you get a constant. That is, if you take a very large circle and divide its circumference by its diameter, you get exactly the same number if you take a very small circle and divide its circumference by its diameter. This common constant was named π ("pi").

Relating the Circumference and Diameter. Whenever a circle's circumference is divided by its diameter, the answer is the constant π. That is, if C is the circumference of the circle and d is the circle's diameter, then

$$\frac{C}{d} = \pi.$$

In modern times, we usually multiply both sides of this equation by d to obtain the formula for the circumference of a circle.

$$C = \pi d$$

Because the diameter of a circle is twice the length of its radius, we can substitute $d = 2r$ in the last equation to get an alternate form of the circumference equation.

$$C = \pi(2r) = 2\pi r$$

The number π has a rich and storied history. Ancient geometers from Egypt, Babylonia, India, and Greece knew that π was slightly larger than 3. The earliest known approximations date from around 1900 BC (Wikipedia); they are 25/8 (Babylonia) and 256/81 (Egypt). The Indian text *Shatapatha Brahmana* gives π as $339/108 \approx 3.139$. Archimedes (287-212 BC) was the first to estimate π rigorously, approximating the circumference of a circle with inscribed and circumscribed polygons. He was able to prove that $223/71 < \pi < 22/7$. Taking the average of these values yields $\pi \approx 3.1419$. Modern mathematicians have proved that π is an *irrational* number, an infinite decimal that never repeats any pattern. Mathematicians, with the help of computers, routinely produce approximations of π with billions of digits after the decimal point.

Digits of Pi. Here is π, correct to the first fifty digits.

$$\pi = 3.14159265358979323846264338327950288419716939937510\ldots$$

The number of digits of π used depends on the application. Working at very small scales, one might keep many digits of π, but if you are building a circular garden fence in your backyard, then fewer digits of π are needed.

You Try It!

Find the radius of a circle having radius 14 inches. Use $\pi \approx 3.14$

EXAMPLE 9. Find the circumference of a circle given its radius is 12 feet.

Solution. The circumference of the circle is given by the formula $C = \pi d$, or, because $d = 2r$,

$$C = 2\pi r.$$

Substitute 12 for r.

$$C = 2\pi r = 2\pi(12) = 24\pi$$

Therefore, the circumference of the circle is *exactly* $C = 24\pi$ feet.

We can approximate the circumference by entering an approximation for π. Let's use $\pi \approx 3.14$. *Note: The symbol ≈ is read "approximately equal to."*

$$C = 24\pi \approx 24(3.14) \approx 75.36 \text{ feet}$$

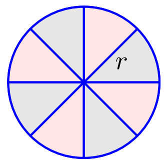
It is important to understand that the solution $C = 24\pi$ feet is the **exact** circumference, while $C \approx 75.36$ feet is only an approximation.

Answer: 87.92 inches

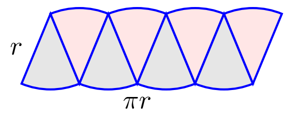
□

The Area of a Circle

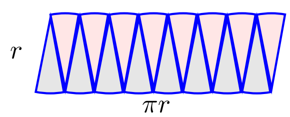
Here's a reasonable argument used to help develop a formula for the area of a circle. Start with a circle of radius r and divide it into 8 equal wedges, as shown in the figure that follows.



Rearrange the pieces as shown in the following figure.



Note that the rearranged pieces almost form a rectangle with length approximately half the circumference of the circle, πr, and width approximately r. The area would be approximately $A \approx (\text{length})(\text{width}) \approx (\pi r)(r) \approx \pi r^2$. This approximation would be even better if we doubled the number of wedges of the circle.



If we doubled the number of wedges again, the resulting figure would even more closely resemble a rectangle with length πr and width r. This leads to the following conclusion.

The Area of a Circle. The area of a circle of radius r is given by the formula

$$A = \pi r^2.$$

You Try It!

EXAMPLE 10. Find the area of a circle having a diameter of 12.5 meters. Use 3.14 for π and round the answer for the area to the nearest tenth of a square meter.

Find the area of a circle having radius 12.2 centimeters. Use $\pi \approx 3.14$

Solution. The diameter is twice the radius.

$$d = 2r$$

Substitute 12.5 for d and solve for r.

$$\begin{aligned} 12.5 &= 2r && \text{Substitute 12.5 for } d. \\ \frac{12.5}{2} &= \frac{2r}{2} && \text{Divide both sides by 2.} \\ 6.25 &= r && \text{Simiplify.} \end{aligned}$$

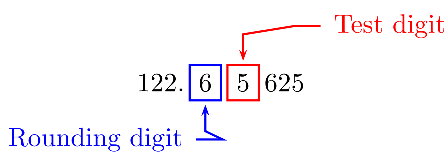
Hence, the radius is 6.25 meters. To find the area, use the formula

$$A = \pi r^2$$

and substutite: 3.14 for π and 6.25 for r.

$$\begin{aligned} A &= (3.14)(6.25)^2 && \text{Substitute: 3.14 for } \pi\text{, 6.25 for } r. \\ &= (3.14)(39.0625) && \text{Square first: } (6.25)^2 = 39.0625. \\ &= 122.65625 && \text{Multiply: } (3.14)(39.0625) = 122.65625. \end{aligned}$$

Hence, the approximate area of the circle is $A = 122.65625$ square meters. To round to the nearest tenth of a square meter, identify the rounding digit and the test digit.



Because the test digit is greater than or equal to 5, add 1 to the rounding digit and truncate. Thus, correct to the nearest tenth of a square meter, the area of the circle is approximately

$$A \approx 122.7\,\text{m}^2.$$

Answer: $467.3576\,\text{cm}^2$

□

Exercises

In Exercises 1-28, multiply the decimals.

1. $(6.7)(0.03)$

2. $(2.4)(0.2)$

3. $(28.9)(5.9)$

4. $(33.5)(2.1)$

5. $(4.1)(4.6)$

6. $(2.6)(8.2)$

7. $(75.3)(0.4)$

8. $(21.4)(0.6)$

9. $(6.98)(0.9)$

10. $(2.11)(0.04)$

11. $(57.9)(3.29)$

12. $(3.58)(16.3)$

13. $(47.3)(0.9)$

14. $(30.7)(0.4)$

15. $(9.9)(6.7)$

16. $(7.2)(6.1)$

17. $(19.5)(7.9)$

18. $(43.4)(8.9)$

19. $(6.9)(0.3)$

20. $(7.7)(0.7)$

21. $(35.3)(3.81)$

22. $(5.44)(9.58)$

23. $(2.32)(0.03)$

24. $(4.48)(0.08)$

25. $(3.02)(6.7)$

26. $(1.26)(9.4)$

27. $(4.98)(6.2)$

28. $(3.53)(2.9)$

In Exercises 29-56, multiply the decimals.

29. $(-9.41)(0.07)$

30. $(4.45)(-0.4)$

31. $(-7.4)(-0.9)$

32. $(-6.9)(0.05)$

33. $(-8.2)(3.7)$

34. $(-7.5)(-6.6)$

35. $(9.72)(-9.1)$

36. $(6.22)(-9.4)$

37. $(-6.4)(2.6)$

38. $(2.3)(-4.4)$

39. $(-39.3)(-0.8)$

40. $(57.7)(-0.04)$

41. $(63.1)(-0.02)$

42. $(-51.1)(-0.8)$

43. $(-90.8)(3.1)$

44. $(-74.7)(2.9)$

45. $(47.5)(-82.1)$

46. $(-14.8)(-12.7)$

47. $(-31.1)(-4.8)$

48. $(-28.7)(-6.8)$

49. $(-2.5)(-0.07)$

50. $(-1.3)(-0.05)$

51. $(1.02)(-0.2)$

52. $(-7.48)(-0.1)$

53. $(7.81)(-5.5)$

54. $(-1.94)(4.2)$

55. $(-2.09)(37.9)$

56. $(20.6)(-15.2)$

In Exercises 57-68, multiply the decimal by the given power of 10.

57. $24.264 \cdot 10$

58. $65.722 \cdot 100$

59. $53.867 \cdot 10^4$

60. $23.216 \cdot 10^4$

61. $5.096 \cdot 10^3$

62. $60.890 \cdot 10^3$

63. $37.968 \cdot 10^3$

64. $43.552 \cdot 10^3$

65. $61.303 \cdot 100$

66. $83.837 \cdot 1000$

67. $74.896 \cdot 1000$

68. $30.728 \cdot 100$

In Exercises 69-80, simplify the given expression.

69. $(0.36)(7.4) - (-2.8)^2$

70. $(-8.88)(-9.2) - (-2.3)^2$

71. $9.4 - (-7.7)(1.2)^2$

72. $0.7 - (-8.7)(-9.4)^2$

73. $5.94 - (-1.2)(-1.8)^2$

74. $-2.6 - (-9.8)(9.9)^2$

75. $6.3 - 4.2(9.3)^2$

76. $9.9 - (-4.1)(8.5)^2$

77. $(6.3)(1.88) - (-2.2)^2$

78. $(-4.98)(-1.7) - 3.5^2$

79. $(-8.1)(9.4) - 1.8^2$

80. $(-3.63)(5.2) - 0.8^2$

---

81. Given $a = -6.24$, $b = 0.4$, and $c = 7.2$, evaluate the expression $a - bc^2$.

82. Given $a = 4.1$, $b = -1.8$, and $c = -9.5$, evaluate the expression $a - bc^2$.

83. Given $a = -2.4$, $b = -2.1$, and $c = -4.6$, evaluate the expression $ab - c^2$.

84. Given $a = 3.3$, $b = 7.3$, and $c = 3.4$, evaluate the expression $ab - c^2$.

85. Given $a = -3.21$, $b = 3.5$, and $c = 8.3$, evaluate the expression $a - bc^2$.

86. Given $a = 7.45$, $b = -6.1$, and $c = -3.5$, evaluate the expression $a - bc^2$.

87. Given $a = -4.5$, $b = -6.9$, and $c = 4.6$, evaluate the expression $ab - c^2$.

88. Given $a = -8.3$, $b = 8.2$, and $c = 5.4$, evaluate the expression $ab - c^2$.

---

89. A circle has a diameter of 8.56 inches. Using $\pi \approx 3.14$, find the circumference of the circle, correct to the nearest tenth of an inch.

90. A circle has a diameter of 14.23 inches. Using $\pi \approx 3.14$, find the circumference of the circle, correct to the nearest tenth of an inch.

91. A circle has a diameter of 12.04 inches. Using $\pi \approx 3.14$, find the circumference of the circle, correct to the nearest tenth of an inch.

92. A circle has a diameter of 14.11 inches. Using $\pi \approx 3.14$, find the circumference of the circle, correct to the nearest tenth of an inch.

93. A circle has a diameter of 10.75 inches. Using $\pi \approx 3.14$, find the area of the circle, correct to the nearest hundredth of a square inch.

94. A circle has a diameter of 15.49 inches. Using $\pi \approx 3.14$, find the area of the circle, correct to the nearest hundredth of a square inch.

95. A circle has a diameter of 13.96 inches. Using $\pi \approx 3.14$, find the area of the circle, correct to the nearest hundredth of a square inch.

96. A circle has a diameter of 15.95 inches. Using $\pi \approx 3.14$, find the area of the circle, correct to the nearest hundredth of a square inch.

---

97. Sue has decided to build a circular fish pond near her patio. She wants it to be 15 feet in diameter and 1.5 feet deep. What is the volume of water it will hold? Use $\pi \approx 3.14$. *Hint: The volume of a cylinder is given by the formula $V = \pi r^2 h$, which is the area of the circular base times the height of the cylinder.*

98. John has a decision to make regarding his employment. He currently has a job at Taco Loco in Fortuna. After taxes, he makes about $9.20 per hour and works about 168 hours a month. He currently pays $400 per month for rent. He has an opportunity to move to Santa Rosa and take a job at Mi Ultimo Refugio which would pay $10.30 per hour after taxes for 168 hours a month, but his rent would cost $570 per month.

a) After paying for housing in Fortuna, how much does he have left over each month for other expenditures?

b) After paying for housing in Santa Rosa, how much would he have left over each month for other expenditures?

c) For which job would he have more money left after paying rent and how much would it be?

99. John decided to move to Santa Rosa and take the job at Mi Ultimo Refugio (see Exercise 98). He was able to increase his income because he could work 4 Sundays a month at time-and-a half. So now he worked 32 hours a month at time-and-a-half and 136 hours at the regular rate of $10.30 (all after taxes were removed). *Note: He previously had worked 168 hours per month at $10.30 per hour.*

a) What was his new monthly income?

b) How much did his monthly income increase?

100. **Electric Bill.** On a recent bill, PGE charged \$0.11531 per Kwh for the first 333 Kwh of electrical power used. If a household used 312 Kwh of power, what was their electrical bill?

101. **Cabernet.** In Napa Valley, one acre of good land can produce about 3.5 tons of quality grapes. At an average price of \$3,414 per ton for premium cabernet, how much money could you generate on one acre of cabernet farming? *Associated Press-Times-Standard 03/11/10 Grape moth threatens Napa Valley growing method.*

102. **Fertilizer.** Using the 2008 Ohio Farm Custom Rates, the average cost for spreading dry bulk fertilizer is about \$4.50 per acre. What is the cost to fertilize 50 acres?

103. **Agribusiness.** Huge corporate agribarns house 1000 pigs each.

a) If each pig weighs approximately 100 pounds, how many pounds of pig is in each warehouse?

b) At an average \$1.29 per pound, what is the total cash value for a corporate agribarn? *Associated Press-Times-Standard 12/29/09 Pressure rises to stop antibiotics in agriculture.*

104. **Shipwrecks.** A dozen centuries-old shipwrecks were found in the Baltic Sea by a gas company building an underwater pipeline between Russia and Germany. The 12 wrecks were found in a 30-mile-long and 1.2-mile-wide corridor at a depth of 430 feet. Model the corridor with a rectangle and find the approximate area of the region where the ships were found. *Associated Press-Times-Standard 03/10/10 Centuries-old shipwrecks found in Baltic Sea.*

105. **Radio dish.** The diameter of the "workhorse fleet" of radio telescopes, like the one in Goldstone, California, is 230 feet. What is the circumference of the radio telescope dish to the nearest tenth? *Associated Press-Times-Standard 03/09/2010 NASA will repair deep space antenna in California desert.*

❧ ❧ ❧ Answers ☙ ☙ ☙

1. 0.201

3. 170.51

5. 18.86

7. 30.12

9. 6.282

11. 190.491

13. 42.57

15. 66.33

17. 154.05

19. 2.07

21. 134.493

23. 0.0696

25. 20.234

27. 30.876

29. −0.6587

31. 6.66

33. −30.34

35. −88.452

37. −16.64

39. 31.44

41. −1.262

43. −281.48

45. −3899.75

47. 149.28

49. 0.175

51. −0.204

53. −42.955

55. −79.211

57. 242.64

59. 538670

61. 5096

63. 37968

65. 6130.3

67. 74896

69. −5.176

71. 20.488

73. 9.828

75. −356.958

77. 7.004

79. −79.38

81. −26.976

83. −16.12

85. −244.325

87. 9.89

89. 26.9 in.

91. 37.8 in.

93. 90.72 square inches

95. 152.98 square inches

97. 264.9375 cubic feet

99.
a) $1895.20
b) $164.80

101. $11,949

103.
a) 100,000 pounds
b) $129,000

105. 722.2 feet

5.4 Dividing Decimals

In this and following sections we make use of the terms *divisor*, *dividend*, *quotient*, and *remainder*.

Divisor, Dividend, Quotient, and Remainder. This schematic reminds readers of the position of these terms in the division process.

$$\begin{array}{r} \text{quotient} \\ \text{divisor}\overline{)\,\text{dividend}} \\ \ldots \\ \hline \text{remainder} \end{array}$$

Now that these terms are defined, we begin the discussion of division of decimal numbers.

Suppose that we wish to divide 637 by 100. We could do this in fraction form, change the result to a mixed fraction, then the mixed fraction to decimal form.

$$\frac{637}{100} = 6\,\frac{37}{100} = 6.37$$

We can also arrange the division much as we would the division of two whole numbers.

$$\begin{array}{r} 6.37 \\ 100\overline{)637.00} \\ \underline{600} \\ 37\,0 \\ \underline{30\,0} \\ 7\,00 \\ \underline{7\,00} \\ 0 \end{array}$$

Note that adding two zeros after the decimal point in the dividend does not change the value of 637. Further, note that we proceed as if we are dividing two whole numbers, placing the decimal point in the quotient directly above the decimal point in the dividend.

These observations lead to the following algorithm.

Dividing a Decimal by a Whole Number. To divide a decimal number by a whole number, proceed as follows:

1. Set up the long division as you would the division of two whole numbers.
2. Perform the division as if the numbers were both whole numbers, adding zeros to the right of the decimal point in the dividend as necessary to complete the division.
3. Place the decimal point in the quotient immediately above the decimal point in the dividend.

You Try It!

EXAMPLE 1. Divide 23 by 20.

Solution. Arrange as if using long division to divide whole numbers, adding enough zeros to the right of the decimal point in the dividend to complete the division.

$$\begin{array}{r} 1.15 \\ 20\overline{)23.00} \\ \underline{20} \\ 3\;0 \\ \underline{2\;0} \\ 1\;00 \\ \underline{1\;00} \\ 0 \end{array}$$

Hence, 23 divided by 20 is 1.15.

Adding Zeros to the Right of the Decimal Point. Usually, one does not immediately see how many zeros to the right of the decimal point in the dividend are needed. These zeros are usually added at each step of the division, until the division is completed or the user is willing to terminate the process and accept only an estimate of the quotient.

You Try It!

Divide: $42.55 \div 23$

EXAMPLE 2. Divide: $155.2 \div 25$.

Solution. Arrange as if using long division to divide whole numbers, and begin.

$$\begin{array}{r} 6.2 \\ 25\overline{)155.2} \\ \underline{150} \\ 5\;2 \\ \underline{5\;0} \\ 2 \end{array}$$

We still have a nonzero remainder. Adding another zero does no good.

$$\begin{array}{r} 6.20 \\ 25\overline{)155.20} \\ \underline{150} \\ 5\;2 \\ \underline{5\;0} \\ 20 \end{array}$$

However, if we add one more additional zero, the division completes with a zero remainder.

$$\begin{array}{r} 6.208 \\ 25\overline{)155.200} \\ \underline{150} \\ 5\,2 \\ \underline{5\,0} \\ 200 \\ \underline{200} \\ 0 \end{array}$$

Answer: 1.85

Thus, 155.2 divided by 25 is 6.208.

Decimal Divisors

When the divisor contains a decimal point, we have a little work to do before we begin the division process. Suppose that we wish to divide 1.25 by 2.5. In fraction form, we could start with

$$\frac{1.25}{2.5},$$

then clear the decimals from the denominator by multiplying both numerator and denominator by 10. *Note: Recall that multiplying by 10 moves the decimal point one place to the right.*

$$\begin{aligned} \frac{1.25}{2.5} &= \frac{1.25 \cdot 10}{2.5 \cdot 10} \\ &= \frac{12.5}{25} \end{aligned}$$

Thus, dividing 1.25 by 2.5 is equivalent to dividing 12.5 by 25. This we know how to do.

$$\begin{array}{r} 0.5 \\ 25\overline{)12.5} \\ \underline{12\,5} \\ 0 \end{array}$$

Thus, 1.25 divided by 2.5 is 0.5.

Writing Mathematics. Never write .5. Always add the leading zero in the ones place and write 0.5.

Instead of working in fraction form, we can take care of positioning the decimal point in the long division framework. Start with:

$$2.5\overline{)1.25}$$

Move the decimal point in the divisor to the end of the divisor, then move the decimal point in the dividend an equal number of places.

$$2.5\,\overline{)1.2\,5}$$

Thus, the division becomes

$$25\overline{)12.5}$$

and we proceed as above to find the quotient.

This discussion motivates the following algorithm.

Dividing by a Decimal Divisor. If the divisor contains a decimal, proceed as follows:

1. Move the decimal to the end of the divisor.
2. Move the decimal in the dividend an equal number of places.

We can then complete the division using the rules for dividing a decimal by a whole number.

You Try It!

Divide: $0.45\overline{)36.99}$

EXAMPLE 3. Divide: $0.36\overline{)4.392}$

Solution. Move the decimal in the divisor to the end of the divisor. Move the decimal in the dividend an equal number of places (two places) to the right.

$$0.36\,\overline{)4.39\,2}$$

Now we can follow the algorithm for dividing a decimal number by a whole number.

$$\begin{array}{r}
12.2 \\
36\overline{)439.2} \\
\underline{36} \\
79 \\
\underline{72} \\
7\,2 \\
\underline{7\,2} \\
0
\end{array}$$

Answer: 82.2

Thus, 4.392 divided by 0.36 is 12.2. □

Dividing Signed Decimal Numbers

The rules governing division of signed decimal numbers are identical to the rules governing division of integers.

Like Signs. The quotient of two decimal numbers with like signs is positive. That is:

$$\frac{(+)}{(+)} = + \qquad \text{and} \qquad \frac{(-)}{(-)} = +$$

Unlike Signs. The quotient of two decimal numbers with unlike signs is negative. That is:

$$\frac{(+)}{(-)} = - \qquad \text{and} \qquad \frac{(-)}{(+)} = -$$

You Try It!

Divide: $-0.0113 \div 0.05$

EXAMPLE 4. Divide: $-0.03 \div 0.024$.

Solution. First, divide the magnitudes. Move the decimal in the divisor to the end of the divisor. Move the decimal in the dividend an equal number of places (three places) to the right. Note that this requires an extra trailing zero in the dividend.

$$0.024\,\overline{)0.030}$$

Our problem then becomes:

$$24\overline{)30}$$

We can now follow the algorithm for dividing a decimal number by a whole number. Note that we have to add two trailing zeros in the dividend to complete the division with a zero remainder.

$$\begin{array}{r} 1.25 \\ 24\overline{)30.00} \\ \underline{24} \\ 6\,0 \\ \underline{4\,8} \\ 1\,20 \\ \underline{1\,20} \\ 0 \end{array}$$

Finally, because the quotient of unlike signs is negative, -0.03 divided by 0.024 is -1.25. That is,

$$\frac{-0.03}{0.024} = -1.25.$$

Answer: -0.226

Rounding

Sometimes an exact decimal representation of a fraction is not needed and is an approximation is more than adequate.

You Try It!

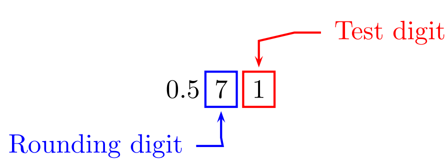
EXAMPLE 5. Convert 4/7 to a decimal. Round your answer to the nearest hundredth.

Convert 5/7 to a decimal. Round your answer to the nearest hundredth.

Solution. We need to carry the division one place beyond the hundredths place.

```
   0.571
7)4.000
  3 5
    50
    49
     10
      7
      3
```

To round to the nearest hundredth, first identify the rounding and test digits.



Because the "test digit" is less 5, leave the rounding digit alone and truncate. Therefore, correct to the nearest hundredth, $4/7 \approx 0.57$.

Answer: 0.71

Dividing by Powers of Ten

Recall:

$$\begin{aligned} 10^1 &= 10 \\ 10^2 &= 10 \cdot 10 = 100 \\ 10^3 &= 10 \cdot 10 \cdot 10 = 1000 \end{aligned}$$

Powers of Ten. In the expression 10^n, the exponent matches the number of zeros in the answer. Hence, 10^n will be a 1 followed by n zeros.

Thus, $10^4 = 10,000$, $10^5 = 100,000$, etc. The exponent tells us how many zeros will follow the 1.

Let's divide 123456.7 by 1000.

$$\begin{array}{r}
123.4567 \\
1000\overline{)123456.7000} \\
\underline{1000} \\
2345 \\
\underline{2000} \\
3456 \\
\underline{3000} \\
456\ 7 \\
\underline{400\ 0} \\
56\ 70 \\
\underline{50\ 00} \\
6\ 700 \\
\underline{6\ 000} \\
7000 \\
\underline{7000} \\
0
\end{array}$$

Note the result: 123456.7 divided by 1000 is 123.4567. Dividing by 1000 moves the decimal point 3 places to the left!

$$123456.7 \div 1000 = 123\ 456.7 = 123.4567$$

This discussion leads to the following result.

Dividing a Decimal by a Power of Ten. Dividing a decimal number by 10^n will move the decimal point n places to the left.

You Try It!

EXAMPLE 6. Simplify: $123456.7 \div 10^4$

Solution. Dividing by 10^4 (or equivalently, 10,000) moves the decimal point four places to the left. Thus, $123456.7 \div 10^4 = 12.34567$.

Simplify: $123456.7 \div 10^2$

Answer: 1234.567

Order of Operations

We remind readers of the *Rules Guiding Order of Operations*.

Rules Guiding Order of Operations. When evaluating expressions, proceed in the following order.

1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
2. Evaluate all exponents that appear in the expression.
3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.
4. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

In addition, when fractions are present:

Fractional Expressions. If a fractional expression is present, simplify the numerator and denominator separately, then divide.

You Try It!

EXAMPLE 7. Evaluate $ab/(c+d)$, given that $a = 2.1$, $b = -3.4$, $c = -1.3$, and $d = 1.1$.

Solution. Recall that it is good practice to prepare an expression for substitution by using parentheses.

$$ab/(c+d) = (\quad)(\quad)/((\quad)+(\quad))$$

If $a = -2.1$, $b = 1.7$, $c = 4$, and $d = 0.05$, evaluate:

$$\frac{a+b}{cd}$$

Substitute the given values for a, b, c, and d, then use the *Rules Guiding Order of Operations* to simplify the resulting expression.

$$\begin{aligned} ab/(c+d) &= (2.1)(-3.4)/((-1.3)+(1.1)) && \text{2.1, −3.4, −1.3, 1.1 for } a, b, c, d. \\ &= (2.1)(-3.4)/(-0.2) && \text{Parens: } (-1.3)+(1.1) = -0.2. \\ &= -7.14/(-0.2) && \text{Multiply: } (2.1)(-3.4) = -7.14. \\ &= 35.7 && \text{Divide: } -7.14/(-0.2) = 35.7. \end{aligned}$$

Answer: -2

□

You Try It!

If $a = 0.5$ and $b = -0.125$, evaluate:

$$\frac{2a-b}{a+2b}$$

EXAMPLE 8. Given $a = 0.1$ and $b = -0.3$, evaluate the expression

$$\frac{a+2b}{2a+b}.$$

Solution. Substitute the given values, then use the *Rules Guiding Order of Operations* to simplify the resulting expression.

$$\frac{a+2b}{2a+b} = \frac{(0.1)+2(-0.3)}{2(0.1)+(-0.3)} \qquad \text{0.1 for } a, -0.3 \text{ for } b.$$

Simplify the numerator, simplify the denominator, then divide.

$$\begin{aligned} &= \frac{0.1+(-0.6)}{0.2+(-0.3)} && \text{Numerator: } 2(-0.3) = -0.6. \\ & && \text{Denominator: } 2(0.1) = 0.2. \\ &= \frac{-0.5}{-0.1} && \text{Numerator: } 0.1+(-0.6) = -0.5. \\ & && \text{Denominator: } 0.2+(-0.3) = -0.1. \\ &= 5 && \text{Divide: } -0.5/(-0.1) = 5. \end{aligned}$$

Answer: 4.5

□

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-16, divide the numbers.

1. $\frac{39}{52}$

2. $\frac{16}{25}$

3. $\frac{755.3}{83}$

4. $\frac{410.4}{76}$

5. $\frac{333}{74}$

6. $\frac{117}{65}$

7. $\frac{32.12}{73}$

8. $\frac{12.32}{44}$

9. $\frac{37.63}{71}$

10. $\frac{20.46}{31}$

11. $\frac{138}{92}$

12. $\frac{110}{25}$

13. $\frac{17}{25}$

14. $\frac{18}{75}$

15. $\frac{229.5}{51}$

16. $\frac{525.6}{72}$

In Exercises 17-40, divide the decimals.

17. $\frac{0.3478}{0.47}$

18. $\frac{0.4559}{0.97}$

19. $\frac{1.694}{2.2}$

20. $\frac{1.008}{1.8}$

21. $\frac{43.61}{4.9}$

22. $\frac{22.78}{3.4}$

23. $\frac{1.107}{0.41}$

24. $\frac{2.465}{0.29}$

25. $\frac{2.958}{0.51}$

26. $\frac{5.141}{0.53}$

27. $\frac{71.76}{7.8}$

28. $\frac{14.08}{8.8}$

29. $\frac{0.8649}{0.93}$

30. $\frac{0.3901}{0.83}$

31. $\frac{0.6958}{0.71}$

32. $\frac{0.1829}{0.31}$

33. $\frac{1.248}{0.52}$

34. $\frac{6.375}{0.85}$

35. $\frac{62.56}{9.2}$

36. $\frac{28.08}{7.8}$

37. $\frac{6.278}{8.6}$

38. $\frac{3.185}{4.9}$

39. $\frac{2.698}{7.1}$

40. $\frac{4.959}{8.7}$

In Exercises 41-64, divide the decimals.

41. $\frac{-11.04}{1.6}$

42. $\frac{-31.27}{5.3}$

43. $\frac{-3.024}{5.6}$

44. $\frac{-3.498}{5.3}$

45. $\frac{-0.1056}{0.22}$

46. $\frac{-0.2952}{-0.72}$

47. $\frac{0.3204}{-0.89}$

48. $\frac{0.3306}{-0.38}$

49. $\frac{-1.419}{0.43}$

50. $\frac{-1.625}{-0.25}$

51. $\frac{-16.72}{-2.2}$

52. $\frac{-66.24}{9.2}$

53. $\frac{-2.088}{-0.87}$

54. $\frac{-2.025}{-0.75}$

55. $\frac{-1.634}{-8.6}$

56. $\frac{-3.094}{3.4}$

57. $\frac{-0.119}{0.85}$

58. $\frac{0.5766}{-0.62}$

59. $\frac{-3.591}{-6.3}$

60. $\frac{-3.016}{5.8}$

61. $\frac{36.96}{-4.4}$

62. $\frac{-78.26}{-8.6}$

63. $\frac{-2.156}{-0.98}$

64. $\frac{-6.072}{0.66}$

In Exercises 65-76, divide the decimal by the given power of 10.

65. $\frac{524.35}{100}$

66. $\frac{849.39}{100}$

67. $\dfrac{563.94}{10^3}$

68. $\dfrac{884.15}{10^3}$

69. $\dfrac{116.81}{10^2}$

70. $\dfrac{578.01}{10^3}$

71. $\dfrac{694.55}{10}$

72. $\dfrac{578.68}{100}$

73. $\dfrac{341.16}{10^3}$

74. $\dfrac{46.63}{10^4}$

75. $\dfrac{113.02}{1000}$

76. $\dfrac{520.77}{1000}$

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77. Compute the quotient 52/83, and round your answer to the nearest tenth.

78. Compute the quotient 43/82, and round your answer to the nearest tenth.

79. Compute the quotient 51/59, and round your answer to the nearest tenth.

80. Compute the quotient 17/69, and round your answer to the nearest tenth.

81. Compute the quotient 5/74, and round your answer to the nearest hundredth.

82. Compute the quotient 3/41, and round your answer to the nearest hundredth.

83. Compute the quotient 5/94, and round your answer to the nearest hundredth.

84. Compute the quotient 3/75, and round your answer to the nearest hundredth.

85. Compute the quotient 7/72, and round your answer to the nearest hundredth.

86. Compute the quotient 4/57, and round your answer to the nearest hundredth.

87. Compute the quotient 16/86, and round your answer to the nearest tenth.

88. Compute the quotient 21/38, and round your answer to the nearest tenth.

In Exercises 89-100, simplify the given expression.

89. $\dfrac{7.5 \cdot 7.1 - 19.5}{0.54}$

90. $\dfrac{1.5(-8.8) - (-18.6)}{1.8}$

91. $\dfrac{17.76 - (-11.7)}{0.5^2}$

92. $\dfrac{-14.8 - 2.1}{2.6^2}$

93. $\dfrac{-18.22 - 6.7}{14.75 - 7.75}$

94. $\dfrac{1.4 - 13.25}{-6.84 - (-2.1)}$

95. $\dfrac{-12.9 - (-10.98)}{0.5^2}$

96. $\dfrac{5.1 - (-16.5)}{(-1.5)^2}$

97. $\dfrac{-9.5 \cdot 1.6 - 3.7}{-3.6}$

98. $\dfrac{6.5(-1.6) - 3.35}{-2.75}$

99. $\dfrac{-14.98 - 9.6}{17.99 - 19.99}$

100. $\dfrac{-5.6 - 7.5}{-5.05 - 1.5}$

101. Given $a = -2.21$, $c = 3.3$, and $d = 0.5$, evaluate and simplify the following expression.

$$\frac{a-c}{d^2}$$

102. Given $a = 2.8$, $c = -14.68$, and $d = 0.5$, evaluate and simplify the following expression.

$$\frac{a-c}{d^2}$$

103. Given $a = -5.8$, $b = 10.37$, $c = 4.8$, and $d = 5.64$, evaluate and simplify the following expression:

$$\frac{a-b}{c-d}$$

104. Given $a = -10.79$, $b = 3.94$, $c = -3.2$, and $d = -8.11$, evaluate and simplify the following expression:

$$\frac{a-b}{c-d}$$

105. Given $a = -1.5$, $b = 4.7$, $c = 18.8$, and $d = -11.75$, evaluate and simplify the following expression.

$$\frac{ab-c}{d}$$

106. Given $a = 9.3$, $b = 6.6$, $c = 14.27$, and $d = 0.2$, evaluate and simplify the following expression.

$$\frac{ab-c}{d}$$

107. **Biodiesel plants.** There are about 180 biodiesel plants operating in about 40 states. Of the states that have them, what is the average number of biodiesel plants per state? *Associated Press-Times-Standard 01/02/10 Fledgling biofuel industry ends year on a dour note.*

108. **Bat fungus.** A fungus called "white-nose syndrome" has killed an estimated 500,000 bats throughout the country. This means about 2,400,000 pounds of bugs aren't eaten over the year, says Forest Service biologist Becky Ewing. How many pounds of insects does an average bat eat annually? *Associated Press-Times-Standard 5/2/09*

109. **Patent backlog.** In the U.S. Patent and Trademark Office, 6000 examiners have a backlog of 770,000 new, unexamined applications for patents. How many applications is that for each examiner to catch up on? Round your answer to the nearest tenth. *Associated Press-Times-Standard 5/5/09*

110. **Doing well.** The large health insurer Wellpoint, Inc., owner of Anthem Blue Cross, earned \$536 million in the last three months of 2009. What was the average earnings per month for the insurer over that period? Round to the nearest million. *Associated Press-Times-Standard 02/09/10 HHS secretary asks Anthem Blue Cross to justify rate hike.*

111. **Cyber attacks.** The Pentagon has spent \$100 million over a six-month period responding to and repairing damage from cyber-attacks and other computer network problems. What's the average amount of money spent per month over that time? Round your answer to the nearest hundredth of a million. *Associated Press-Times-Standard 4/19/09*

112. **Daily milk.** The average California cow can produce 2,305 gallons of milk annually. How much milk can a cow produce each day? Round your answer to the nearest hundredth of a gallon. *http://www.moomilk.com/faq.htm*

113. **Media mail.** To promote her business, Theresa mails several packages via Media Mail. One package weighing 2 lbs. costs \$2.77, another package weighing 3 lbs. costs \$3.16, and the third package weighing 5 lbs. costs \$3.94 to mail. What was the average cost per pound to mail the packages? Round your result to the nearest penny. *http://www.usps.com/prices/media-mail-prices.htm*

❧ ❧ ❧ Answers ☙ ☙ ☙

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1. 0.75

3. 9.1

5. 4.5

7. 0.44

9. 0.53

11. 1.5

13. 0.68

15. 4.5

17. 0.74

19. 0.77

21. 8.9

23. 2.7

25. 5.8

27. 9.2

29. 0.93

31. 0.98

33. 2.4

35. 6.8

37. 0.73

39. 0.38

41. −6.9

43. −0.54

45. −0.48

47. −0.36

49. −3.3

51. 7.6

53. 2.4

55. 0.19

57. −0.14

59. 0.57

61. −8.4

63. 2.2

65. 5.2435

67. 0.56394

69. 1.1681

71. 69.455

73. 0.34116

75. 0.11302

77. 0.6

79. 0.9

81. 0.07

83. 0.05

85. 0.10

87. 0.2

89. 62.5

91. 117.84

93. −3.56

95. −7.68

97. 5.25

99. 12.29

101. −22.04

103. 19.25

105. 2.2

107. 4.5 biodiesel plants

109. 128.3

111. $16.67 million

113. $0.99 per pound

5.5 Fractions and Decimals

When converting a fraction to a decimal, only one of two things can happen. Either the process will terminate or the decimal representation will begin to repeat a pattern of digits. In each case, the procedure for changing a fraction to a decimal is the same.

Changing a Fraction to a Decimal. To change a fraction to a decimal, divide the numerator by the denominator. *Hint: If you first reduce the fraction to lowest terms, the numbers will be smaller and the division will be a bit easier as a result.*

Terminating Decimals

Terminating Decimals. First reduce the fraction to lowest terms. If the denominator of the resulting fraction has a prime factorization consisting of strictly twos and/or fives, then the decimal representation will "terminate."

You Try It!

EXAMPLE 1. Change 15/48 to a decimal.

Change 10/16 to a decimal.

Solution. First, reduce the fraction to lowest terms.

$$\begin{aligned} \frac{15}{48} &= \frac{3 \cdot 5}{3 \cdot 16} \\ &= \frac{5}{16} \end{aligned}$$

Next, note that the denominator of 5/16 has prime factorization $16 = 2 \cdot 2 \cdot 2 \cdot 2$. It consists only of twos. Hence, the decimal representation of 5/16 should terminate.

$$\begin{array}{r} 0.3125 \\ 16\overline{)5.0000} \\ \underline{4\,8} \\ 20 \\ \underline{16} \\ 40 \\ \underline{32} \\ 80 \\ \underline{80} \\ 0 \end{array}$$

The zero remainder terminates the process. Hence, $5/16 = 0.3125$.

Answer: 0.625

You Try It!

Change $7\frac{11}{20}$ to a decimal.

EXAMPLE 2. Change $3\,\frac{7}{20}$ to a decimal.

Solution. Note that 7/20 is reduced to lowest terms and its denominator has prime factorization $20 = 2 \cdot 2 \cdot 5$. It consists only of twos and fives. Hence, the decimal representation of 7/20 should terminate.

$$\begin{array}{r} 0.35 \\ 20\overline{)7.00} \\ \underline{6\ 0} \\ 1\ 00 \\ \underline{1\ 00} \\ 0 \end{array}$$

The zero remainder terminates the process. Hence, $7/20 = 0.35$. Therefore, $3\,\frac{7}{20} = 3.35$.

Answer: 7.55

□

Repeating Decimals

Repeating Decimals. First reduce the fraction to lowest terms. If the prime factorization of the resulting denominator does not consist strictly of twos and fives, then the division process will never have a remainder of zero. However, repeated patterns of digits must eventually reveal themselves.

You Try It!

Change 5/12 to a decimal.

EXAMPLE 3. Change 1/12 to a decimal.

Solution. Note that 1/12 is reduced to lowest terms and the denominator has a prime factorization $12 = 2 \cdot 2 \cdot 3$ that does **not** consist strictly of twos and fives. Hence, the decimal representation of 1/12 will not "terminate." We need to carry out the division until a remainder reappears for a second time. This will indicate repetition is beginning.

$$\begin{array}{r} .083 \\ 12\overline{)1.000} \\ \underline{96} \\ 40 \\ \underline{36} \\ 4 \end{array}$$

Note the second appearance of 4 as a remainder in the division above. This is an indication that repetition is beginning. However, to be sure, let's carry the division out for a couple more places.

$$\begin{array}{r}
.08333 \\
12\overline{)1.00000} \\
\underline{96} \\
40 \\
\underline{36} \\
40 \\
\underline{36} \\
40 \\
\underline{36} \\
4
\end{array}$$

Note how the remainder 4 repeats over and over. In the quotient, note how the digit 3 repeats over and over. It is pretty evident that if we were to carry out the division a few more places, we would get

$$\frac{1}{12} = 0.833333\ldots$$

The ellipsis is a symbolic way of saying that the threes will repeat forever. It is the mathematical equivalent of the word "etcetera."

Answer: $0.41666\ldots$

There is an alternative notation to the ellipsis, namely

$$\frac{1}{12} = 0.08\overline{3}.$$

The bar over the 3 (called a "repeating bar") indicates that the 3 will repeat indefinitely. That is,

$$0.08\overline{3} = 0.083333\ldots.$$

Using the Repeating Bar. To use the repeating bar notation, take whatever block of digits are under the repeating bar and duplicate that block of digits infinitely to the right.

Thus, for example:

- $5.3\overline{45} = 5.3454545\ldots.$
- $0.\overline{142857} = 0.142857142857142857\ldots.$

Important Observation. Although $0.8\overline{33}$ will also produce $0.8333333\ldots$, as a rule we should use as few digits as possible under the repeating bar. Thus, $0.8\overline{3}$ is preferred over $0.8\overline{33}$.

You Try It!

Change 5/33 to a decimal.

EXAMPLE 4. Change 23/111 to a decimal.

Solution. The denominator of 23/111 has prime factorization $111 = 3 \cdot 37$ and does **not** consist strictly of twos and fives. Hence, the decimal representation will not "terminate." We need to perform the division until we spot a repeated remainder.

$$\begin{array}{r} 0.207 \\ 111\overline{)23.000} \\ \underline{22\;2} \\ 800 \\ \underline{777} \\ 23 \end{array}$$

Note the return of 23 as a remainder. Thus, the digit pattern in the quotient should start anew, but let's add a few places more to our division to be sure.

$$\begin{array}{r} 0.207207 \\ 111\overline{)23.000000} \\ \underline{22\;2} \\ 800 \\ \underline{777} \\ 230 \\ \underline{222} \\ 800 \\ \underline{777} \\ 23 \end{array}$$

Aha! Again a remainder of 23. Repetition! At this point, we are confident that

$$\frac{23}{111} = 0.207207\ldots.$$

Using a "repeating bar," this result can be written

$$\frac{23}{111} = 0.\overline{207}.$$

Answer: $0.151515\ldots$

□

Expressions Containing Both Decimals and Fractions

At this point we can convert fractions to decimals, and vice-versa, we can convert decimals to fractions. Therefore, we should be able to evaluate expressions that contain a mix of fraction and decimal numbers.

You Try It!

Simplify: $-\dfrac{7}{8} - 6.5$

EXAMPLE 5. Simplify: $-\dfrac{3}{8} - 1.25$.

Solution. Let's change 1.25 to an improper fraction.

$$\begin{aligned} 1.25 &= \frac{125}{100} && \text{Two decimal places} \implies \text{two zeros.} \\ &= \frac{5}{4} && \text{Reduce to lowest terms.} \end{aligned}$$

In the original problem, replace 1.25 with 5/4, make equivalent fractions with a common denominator, then subtract.

$$\begin{aligned} -\frac{3}{8} - 1.25 &= -\frac{3}{8} - \frac{5}{4} && \text{Replace 1.25 with 5/4.} \\ &= -\frac{3}{8} - \frac{5 \cdot 2}{4 \cdot 2} && \text{Equivalent fractions, LCD = 8.} \\ &= -\frac{3}{8} - \frac{10}{8} && \text{Simplify numerator and denominator.} \\ &= -\frac{3}{8} + \left(-\frac{10}{8}\right) && \text{Add the opposite.} \\ &= -\frac{13}{8} && \text{Add.} \end{aligned}$$

Thus, $-3/8 - 1.25 = -13/8$.

Alternate Solution. Because $-3/8$ is reduced to lowest terms and $8 = 2 \cdot 2 \cdot 2$ consists only of twos, the decimal representation of $-3/8$ will terminate.

$$\begin{array}{r} 0.375 \\ 8\overline{)3.000} \\ \underline{2\,4} \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \end{array}$$

Hence, $-3/8 = -0.375$. Now, replace $-3/8$ in the original problem with -0.375, then simplify.

$$\begin{aligned} -\frac{3}{8} - 1.25 &= -0.375 - 1.25 && \text{Replace } -3/8 \text{ with } -0.375. \\ &= -0.375 + (-1.25) && \text{Add the opposite.} \\ &= -1.625 && \text{Add.} \end{aligned}$$

Thus, $-3/8 - 1.25 = -1.625$.

Are They the Same? The first method produced $-13/8$ as an answer; the second method produced -1.625. Are these the same results? One way to find out is to change -1.625 to an improper fraction.

$$\begin{aligned} -1.625 &= -\frac{1625}{1000} && \text{Three places} \implies \text{three zeros.} \\ &= -\frac{5 \cdot 5 \cdot 5 \cdot 5 \cdot 13}{2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5} && \text{Prime factor.} \\ &= -\frac{13}{2 \cdot 2 \cdot 2} && \text{Cancel common factors.} \\ &= -\frac{13}{8} && \text{Simplify.} \end{aligned}$$

Answer: $-7\frac{3}{8}$ or -7.375

Thus, the two answers are the same.

□

Let's look at another example.

You Try It!

Simplify: $-\dfrac{4}{9} + 0.25$

EXAMPLE 6. Simplify: $-\dfrac{2}{3} + 0.35$.

Solution. Let's attack this expression by first changing 0.35 to a fraction.

$$\begin{aligned} -\frac{2}{3} + 0.35 &= -\frac{2}{3} + \frac{35}{100} && \text{Change 0.35 to a fraction.} \\ &= -\frac{2}{3} + \frac{7}{20} && \text{Reduce 35/100 to lowest terms.} \end{aligned}$$

Find an LCD, make equivalent fractions, then add.

$$\begin{aligned} &= -\frac{2 \cdot 20}{3 \cdot 20} + \frac{7 \cdot 3}{20 \cdot 3} && \text{Equivalent fractions with LCD} = 60. \\ &= -\frac{40}{60} + \frac{21}{60} && \text{Simplify numerators and denominators.} \\ &= -\frac{19}{60} && \text{Add.} \end{aligned}$$

Answer: $-7/36$

Thus, $-\dfrac{2}{3} + 0.35 = -\dfrac{19}{60}$.

□

In Example 6, we run into trouble if we try to change $-2/3$ to a decimal. The decimal representation for $-2/3$ is a repeating decimal (the denominator is not made up of only twos and fives). Indeed, $-2/3 = -0.\overline{6}$. To add $-0.\overline{6}$ and 0.35, we have to align the decimal points, then begin adding at the right end. But $-0.\overline{6}$ has no right end! This observation leads to the following piece of advice.

Important Observation. When presented with a problem containing both decimals and fractions, if the decimal representation of any fraction repeats, its best to first change all numbers to fractions, then simplify.

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-20, convert the given fraction to a terminating decimal.

1. $\frac{59}{16}$

2. $\frac{19}{5}$

3. $\frac{35}{4}$

4. $\frac{21}{4}$

5. $\frac{1}{16}$

6. $\frac{14}{5}$

7. $\frac{6}{8}$

8. $\frac{7}{175}$

9. $\frac{3}{2}$

10. $\frac{15}{16}$

11. $\frac{119}{175}$

12. $\frac{4}{8}$

13. $\frac{9}{8}$

14. $\frac{5}{2}$

15. $\frac{78}{240}$

16. $\frac{150}{96}$

17. $\frac{25}{10}$

18. $\frac{2}{4}$

19. $\frac{9}{24}$

20. $\frac{216}{150}$

In Exercises 21-44, convert the given fraction to a repeating decimal. Use the "repeating bar" notation.

21. $\frac{256}{180}$

22. $\frac{268}{180}$

23. $\frac{364}{12}$

24. $\frac{292}{36}$

25. $\frac{81}{110}$

26. $\frac{82}{99}$

27. $\frac{76}{15}$

28. $\frac{23}{9}$

29. $\frac{50}{99}$

30. $\frac{53}{99}$

31. $\frac{61}{15}$

32. $\frac{37}{18}$

33. $\dfrac{98}{66}$

34. $\dfrac{305}{330}$

35. $\dfrac{190}{495}$

36. $\dfrac{102}{396}$

37. $\dfrac{13}{15}$

38. $\dfrac{65}{36}$

39. $\dfrac{532}{21}$

40. $\dfrac{44}{60}$

41. $\dfrac{26}{198}$

42. $\dfrac{686}{231}$

43. $\dfrac{47}{66}$

44. $\dfrac{41}{198}$

In Exercises 45-52, simplify the given expression by first converting the fraction into a terminating decimal.

45. $\dfrac{7}{4} - 7.4$

46. $\dfrac{3}{2} - 2.73$

47. $\dfrac{7}{5} + 5.31$

48. $-\dfrac{7}{4} + 3.3$

49. $\dfrac{9}{10} - 8.61$

50. $\dfrac{3}{4} + 3.7$

51. $\dfrac{6}{5} - 7.65$

52. $-\dfrac{3}{10} + 8.1$

In Exercises 53-60, simplify the given expression by first converting the decimal into a fraction.

53. $\dfrac{7}{6} - 2.9$

54. $-\dfrac{11}{6} + 1.12$

55. $-\dfrac{4}{3} - 0.32$

56. $\dfrac{11}{6} - 0.375$

57. $-\dfrac{2}{3} + 0.9$

58. $\dfrac{2}{3} - 0.1$

59. $\dfrac{4}{3} - 2.6$

60. $-\dfrac{5}{6} + 2.3$

In Exercises 61-64, simplify the given expression.

61. $\frac{5}{6} + 2.375$

62. $\frac{5}{3} + 0.55$

63. $\frac{11}{8} + 8.2$

64. $\frac{13}{8} + 8.4$

65. $-\frac{7}{10} + 1.2$

66. $-\frac{7}{5} - 3.34$

67. $-\frac{11}{6} + 0.375$

68. $\frac{5}{3} - 1.1$

Answers

---

1. 3.6875

3. 8.75

5. 0.0625

7. 0.75

9. 1.5

11. 0.68

13. 1.125

15. 0.325

17. 2.5

19. 0.375

21. $1.4\overline{2}$

23. $30.\overline{3}$

25. $0.7\overline{36}$

27. $5.0\overline{6}$

29. $0.\overline{50}$

31. $4.0\overline{6}$

33. $1.\overline{48}$

35. $0.\overline{38}$

37. $0.8\overline{6}$

39. $25.\overline{3}$

41. $0.\overline{13}$

43. $0.7\overline{12}$

45. -5.65

47. 6.71

49. -7.71

51. -6.45

53. $-\frac{26}{15}$

55. $-\frac{124}{75}$

57. $\frac{7}{30}$

59. $-\frac{19}{15}$

61. $\frac{77}{24}$

63. 9.575

65. 0.5

67. $-\frac{35}{24}$

5.6 Equations With Decimals

We can add or subtract the same decimal number from both sides of an equation without affecting the solution.

You Try It!

EXAMPLE 1. Solve for x: $x - 1.35 = -2.6$.

Solution. To undo subtracting 1.35, add 1.35 to both sides of the equation.

$$\begin{aligned} x - 1.35 &= -2.6 && \text{Original equation.} \\ x - 1.35 + 1.35 &= -2.6 + 1.35 && \text{Add 1.35 to both sides.} \\ x &= -1.25 && \text{Simplify: } -2.6 + 1.35 = -1.25. \end{aligned}$$

Solve for x:

$$x + 1.25 = 0.6$$

Answer: -0.65

□

We can still multiply both sides of an equation by the same decimal number without affecting the solution.

You Try It!

EXAMPLE 2. Solve for x: $\dfrac{x}{-0.35} = 4.2$.

Solution. To undo dividing by -0.35, multiply both sides of the equation by -0.35.

$$\begin{aligned} \frac{x}{-0.35} &= 4.2 && \text{Original equation.} \\ -0.35\left(\frac{x}{-0.35}\right) &= -0.35\,(4.2) && \text{Multiply both sides by } -0.35. \\ x &= -1.470 && \text{Simplify: } -0.35(4.2) = -1.470. \end{aligned}$$

Solve for y:

$$\frac{y}{0.37} = -1.52$$

Answer: -0.5624

□

We can still divide both sides of an equation by the same decimal number without affecting the solution.

You Try It!

EXAMPLE 3. Solve for x: $-1.2x = -4.08$.

Solution. To undo multiplying by -1.2, divide both sides of the equation by -1.2.

$$\begin{aligned} -1.2x &= -4.08 && \text{Original equation.} \\ \frac{-1.2x}{-1.2} &= \frac{-4.08}{-1.2} && \text{Divide both sides by } -1.2. \\ x &= 3.4 && \text{Simplify: } -4.08/(-1.2) = 3.4. \end{aligned}$$

Solve for z:

$$-2.5z = 1.4$$

Answer: -0.56

□

Combining Operations

We sometimes need to combine operations.

You Try It!

Solve for u:

$$-0.02u - 3.2 = -1.75$$

EXAMPLE 4. Solve for x: $-3.8x - 1.7 = -17.28$.

Solution. To undo subtracting 1.7, add 1.7 to both sides of the equation.

$$-3.8x - 1.7 = -17.28 \quad \text{Original equation.}$$
$$-3.8x - 1.7 + 1.7 = -17.28 + 1.7 \quad \text{Add 1.7 to both sides}$$
$$-3.8x = -15.58 \quad \text{Simplify: } -17.28 + 1.7 = -15.58.$$

Next, to undo multiplying by -3.8, divide both sides of the equation by -3.8.

$$\frac{-3.8x}{-3.8} = \frac{-15.58}{-3.8} \quad \text{Divide both sides by } -3.8.$$
$$x = 4.1 \quad \text{Simplify: } -15.58/(-3.8) = 4.1.$$

Answer: -72.5

□

Combining Like Terms

Combining like terms with decimal coefficients is done in the same manner as combining like terms with integer coefficients.

You Try It!

Simplify:

$$-1.185t + 3.2t$$

EXAMPLE 5. Simplify the expression: $-3.2x + 1.16x$.

Solution. To combine these like terms we must add the coefficients.

To add coefficients with unlike signs, first subtract the coefficient with the smaller magnitude from the coefficient with the larger magnitude.

$$\begin{array}{r} 3.20 \\ -\ 1.16 \\ \hline 2.04 \end{array}$$

Prefix the sign of the decimal number having the larger magnitude. Hence:

$$-3.2 + 1.16 = -2.04.$$

We can now combine like terms as follows:

$$-3.2x + 1.16x = -2.04x$$

Answer: $2.015t$

□

When solving equations, we sometimes need to combine like terms.

You Try It!

EXAMPLE 6. Solve the equation for x: $4.2 - 3.1x + 2x = -7.02$.

Solution. Combine like terms on the left-hand side of the equation.

$$\begin{aligned}
4.2 - 3.1x + 2x &= -7.02 && \text{Original equation.}\\
4.2 - 1.1x &= -7.02 && \text{Combine like terms: } -3.1x + 2x = -1.1x.\\
4.2 - 1.1x - 4.2 &= -7.02 - 4.2 && \text{Subtract 4.2 from both sides.}\\
-1.1x &= -11.02 && \text{Subtract: } -7.02 - 4.2 = -11.22.\\
\frac{-1.1x}{-1.1} &= \frac{-11.22}{-1.1} && \text{Divide both sides by } -1.1.\\
x &= 10.2 && \text{Divide: } -11.22/(-1.1) = 10.2.
\end{aligned}$$

Thus, the solution of the equation is 10.2.

Check. Like all equations, we can check our solution by substituting our answer in the original equation.

$$\begin{aligned}
4.2 - 3.1x + 2x &= -7.02 && \text{Original equation.}\\
4.2 - 3.1(10.2) + 2(10.2) &= -7.02 && \text{Substitute 10.2 for } x.\\
4.2 - 31.62 + 20.4 &= -7.02 && \text{Multiply: } 3.1(10.2) = 31.62,\ 2(10.2) = 20.4.\\
-27.42 + 20.4 &= -7.02 && \text{Order of Ops: Add, left to right.}\\
& && 4.2 - 31.62 = -27.42.\\
-7.02 &= -7.02 && \text{Add: } -27.42 + 20.4 = -7.02.
\end{aligned}$$

Because the last line is a true statement, the solution $x = 10.2$ checks.

Solve for r:

$$-4.2 + 3.6r - 4.1r = 1.86$$

Answer: -12.12

□

Using the Distributive Property

Sometimes we will need to employ the distributive property when solving equations.

Distributive Property. Let a, b, and c be any numbers. Then,

$$a(b + c) = ab + ac.$$

You Try It!

Solve for x:

$$-2.5x - 0.1(x - 2.3) = 8.03$$

EXAMPLE 7. Solve the equation for x: $-6.3x - 0.4(x - 1.2) = -0.86$.

Solution. We first distribute the -0.4 times each term in the parentheses, then combine like terms.

$$\begin{aligned}
-6.3x - 0.4(x - 1.2) &= -0.86 && \text{Original equation.}\\
-6.3x - 0.4x + 0.48 &= -0.86 && \text{Distribute. Note that } -0.4(-1.2) = 0.48.\\
-6.7x + 0.48 &= -0.86 && \text{Combine like terms.}
\end{aligned}$$

Next, subtract 0.48 from both sides, then divide both sides of the resulting equation by -6.7.

$$\begin{aligned}
-6.7x + 0.48 - 0.48 &= -0.86 - 0.48 && \text{Subtract 0.48 from both sides.}\\
-6.7x &= -1.34 && \text{Simplify: } -0.86 - 0.48 = -1.34.\\
\frac{-6.7x}{-6.7} &= \frac{-1.34}{-6.7} && \text{Divide both sides by } -6.7.\\
x &= 0.2 && \text{Simplify: } -1.34/(-6.7) = 0.2.
\end{aligned}$$

Answer: -3

□

Rounding Solutions

Sometimes an approximate solution is adequate.

You Try It!

Solve for x:

$$4.2x - 1.25 = 3.4 + 0.71x$$

EXAMPLE 8. Solve the equation $3.1x + 4.6 = 2.5 - 2.2x$ for x. Round the answer to the nearest tenth.

Solution. We need to isolate the terms containing x on one side of the equation. We begin by adding $2.2x$ to both sides of the equation.

$$\begin{aligned}
3.1x + 4.6 &= 2.5 - 2.2x && \text{Original equation.}\\
3.1x + 4.6 + 2.2x &= 2.5 - 2.2x + 2.2x && \text{Add } 2.2x \text{ to both sides.}\\
5.3x + 4.6 &= 2.5 && \text{Combine terms: } 3.1x + 2.2x = 5.3x.
\end{aligned}$$

To undo adding 4.6, subtract 4.6 from both sides of the equation.

$$\begin{aligned}
5.3x + 4.6 - 4.6 &= 2.5 - 4.6 && \text{Subtract 4.6 from both sides.}\\
5.3x &= -2.1 && \text{Simplify: } 2.5 - 4.6 = -2.1.
\end{aligned}$$

To undo the effect of multiplying by 5.3, divide both sides of the equation by 5.3.

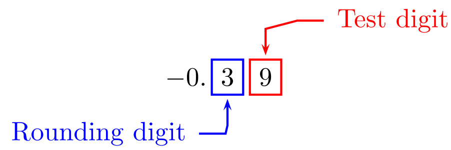
$$\frac{5.3x}{5.3} = \frac{-2.1}{5.3} \qquad \text{Divide both sides by 5.3.}$$
$$x \approx -0.4 \qquad \text{Round solution to nearest tenth.}$$

To round the answer to the nearest tenth, we must carry the division out one additional place.

$$\begin{array}{r} 0.39 \\ 53\overline{)21.00} \\ \underline{15\ 9} \\ 5\ 10 \\ \underline{4\ 77} \\ 33 \end{array}$$

Thus, $-2.1/5.3 \approx -0.39$.

Because the "test digit" is greater than or equal to 5, add 1 to the rounding digit and truncate.



Thus, $-0.39 \approx -0.4$.

Answer: 1.33

Applications

Let's look at some applications that involve equations containing decimals. For convenience, we repeat the *Requirements for Word Problem Solutions*.

Requirements for Word Problem Solutions.

1. **Set up a Variable Dictionary.** You must let your readers know what each variable in your problem represents. This can be accomplished in a number of ways:
   - Statements such as "Let P represent the perimeter of the rectangle."
   - Labeling unknown values with variables in a table.
   - Labeling unknown quantities in a sketch or diagram.
2. **Set up an Equation.** Every solution to a word problem must include a carefully crafted equation that accurately describes the constraints in the problem statement.
3. **Solve the Equation.** You must always solve the equation set up in the previous step.

4. Answer the Question. This step is easily overlooked. For example, the problem might ask for Jane's age, but your equation's solution gives the age of Jane's sister Liz. Make sure you answer the original question asked in the problem. Your solution should be written in a sentence with appropriate units.

5. Look Back. It is important to note that this step does not imply that you should simply check your solution in your equation. After all, it's possible that your equation incorrectly models the problem's situation, so you could have a valid solution to an incorrect equation. The important question is: "Does your answer make sense based on the words in the original problem statement."

Let's start with a rectangular garden problem.

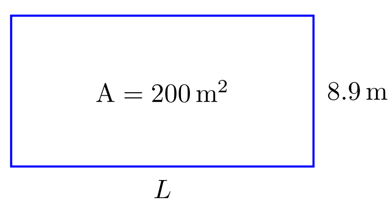
You Try It!

Eta's dog run is in the shape of a rectangle with area 500 square feet. If the length of the run is 28 feet, find the width of the run, correct to the nearest tenth of a foot.

EXAMPLE 9. Molly needs to create a rectangular garden plot covering 200 square meters (200 m^2). If the width of the plot is 8.9 meters, find the length of the plot correct to the nearest tenth of a meter.

Solution. We will follow the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* We will use a sketch to define our variables.



Note that L represents the length of the rectangle.

2. *Set Up an Equation.* The area A of a rectangle is given by the formula

$$A = LW,$$

where L and W represent the length and width of the rectangle, respectively. Substitute 200 for A and 8.9 for W in the formula to obtain

$$200 = L(8.9),$$

or equivalently,

$$200 = 8.9L.$$

3. *Solve the Equation.* Divide both sides of the last equation by 8.9, then round your answer to the nearest tenth.

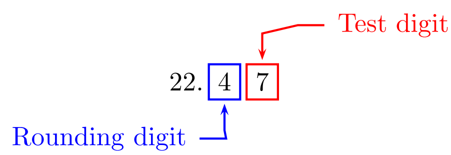
$$\frac{200}{8.9} = \frac{8.9L}{8.9} \qquad \text{Divide both sides by 8.9.}$$

$$22.5 \approx L \qquad \text{Round to nearest tenth.}$$

To round the answer to the nearest tenth, we must carry the division out one additional place.

$$\begin{array}{r}
22.47 \\
89\overline{)2000.00} \\
\underline{178} \\
220 \\
\underline{178} \\
42\;0 \\
\underline{35\;6} \\
6\;40 \\
\underline{6\;23} \\
0
\end{array}$$

Because the "test digit" is greater than or equal to 5, add 1 to the rounding digit and truncate.



Thus, $200/8.9 \approx 22.5$.

4. *Answer the Question.* To the nearest tenth of a meter, the length of the rectangular plot is $L \approx 22.5$ meters.

5. *Look Back.* We have $L \approx 22.5$ meters and $W = 8.9$ meters. Multiply length and width to find the area.

$$\text{Area} \approx (22.5\,\text{m})(8.9\,\text{m}) \approx 200.25\,\text{m}^2.$$

Note that this is very nearly the exact area of 200 square meters. The discrepancy is due to the fact that we found the length rounded to the nearest tenth of a meter.

Answer: 17.9 feet

□

You Try It!

Adult tickets to the circus cost \$12.25 apiece. If the club has \$1,200 set aside for adult ticket purchase, how many adult tickets can they purchase?

EXAMPLE 10. Children's tickets to the circus go on sale for \$6.75. The Boys and Girls club of Eureka has \$1,000 set aside to purchase these tickets. Approximately how many tickets can the Girls and Boys club purchase?

Solution. We will follow the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* Let N represent the number of tickets purchased by the Boys and Girls club of Eureka.

2. *Set Up an Equation.* Note that

Price per Ticket	times	Number of Tickets	is	Full Purchase Price
6.75	$\cdot$	N	$=$	1,000

Hence, our equation is

$$6.75N = 1000.$$

3. *Solve the Equation.* Divide both sides of the equation by 6.75.

$$\begin{aligned} \frac{6.75N}{6.75} &= \frac{1000}{6.75} && \text{Divide both sides by 6.75.} \\ N &\approx 148 && \text{Truncate to nearest unit.} \end{aligned}$$

Push the decimal point to the right-end of the divisor and the decimal point in the dividend an equal number of places.

$$6.75\overline{)1000.00}$$

We'll stop the division at the units position.

$$\begin{array}{r} 148 \\ 675\overline{)100000} \\ \underline{675} \\ 3250 \\ \underline{2700} \\ 5500 \\ \underline{5400} \\ 100 \end{array}$$

4. *Answer the Question.* The Boys and Girls club can purchase 148 tickets.

5. *Look Back.* Let's calculate the cost of 148 tickets at \$6.75 apiece.

$$\begin{array}{r} 1\,48 \\ \times\ 6.75 \\ \hline 7\,40 \\ 103\,6 \\ 888 \\ \hline 999.00 \end{array}$$

Thus, at \$6.75 apiece, 148 tickets will cost \$999. Because the Boys and Girls club of Eureka has \$1,000 to work with, note that the club doesn't have enough money left for another ticket.

Answer: 97

□

You Try It!

Dylan has a circular dog pen with circumference 100 feet. Find the radius of the pen, correct to the nearest tenth of a foot. Use $\pi \approx 3.14$

EXAMPLE 11. Marta has 20 feet of decorative fencing which she will use for the border of a small circular garden. Find the diameter of the circular garden, correct to the nearest hundredth of a foot. Use $\pi \approx 3.14$.

Solution. The formula governing the relation between the circumference and diameter of a circle is

$$C = \pi d.$$

The 20 feet of decorative fencing will be the circumference of the circular garden. Substitute 20 for C and 3.14 for π.

$$20 = 3.14d$$

Divide both sides of the equation by 3.14.

$$\begin{aligned} \frac{20}{3.14} &= \frac{3.14d}{3.14} \\ \frac{20}{3.14} &= d \end{aligned}$$

Move the decimal point to the end of the divisor, then move the decimal point in the dividend an equal number of places (two places) to the right. Note that we must add two trailing zeros in the dividend.

$$3.14\,\overline{)20.00}$$

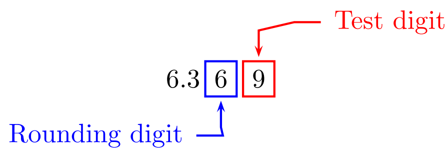
Thus, the problem becomes:

$$314\overline{)2000}$$

We need to round to the nearest hundredth. This requires that we carry the division one additional place to the right of the hundredths place (i.e., to the thousandths place).

$$\begin{array}{r} 6.369 \\ 314\overline{)2000.000} \\ \underline{1884} \\ 116\;0 \\ \underline{94\;2} \\ 21\;80 \\ \underline{18\;84} \\ 2\;960 \\ \underline{2\;826} \\ 134 \end{array}$$

For the final step, we must round 6.369 to the nearest hundredth. In the schematic that follows, we've boxed the hundredths digit (the "rounding digit") and the "test digit" that follows the "rounding digit."



Because the "test digit" is greater than or equal to 5, we add 1 to the "rounding digit," then truncate. Therefore, to the nearest hundredth of a foot, the diameter of the circle is approximately

$$d \approx 6.37\,\text{ft}.$$

Answer: 15.9 feet

Exercises

In Exercises 1-16, solve the equation.

1. $5.57x - 2.45x = 5.46$

2. $-0.3x - 6.5x = 3.4$

3. $-5.8x + 0.32 + 0.2x = -6.96$

4. $-2.2x - 0.8 - 7.8x = -3.3$

5. $-4.9x + 88.2 = 24.5$

6. $-0.2x - 32.71 = 57.61$

7. $0.35x - 63.58 = 55.14$

8. $-0.2x - 67.3 = 93.5$

9. $-10.3x + 82.4 = 0$

10. $-1.33x - 45.22 = 0$

11. $-12.5x + 13.5 = 0$

12. $44.15x - 8.83 = 0$

13. $7.3x - 8.9 - 8.34x = 2.8$

14. $0.9x + 4.5 - 0.5x = 3.5$

15. $-0.2x + 2.2x = 6.8$

16. $-7.9x + 2.9x = 8.6$

In Exercises 17-34, solve the equation.

17. $6.24x - 5.2 = 5.2x$

18. $-0.6x + 6.3 = 1.5x$

19. $-0.7x - 2.4 = -3.7x - 8.91$

20. $3.4x - 4.89 = 2.9x + 3.6$

21. $-4.9x = -5.4x + 8.4$

22. $2.5x = 4.5x + 5.8$

23. $-2.8x = -2.3x - 6.5$

24. $1.2x = 0.35x - 1.36$

25. $-2.97x - 2.6 = -3.47x + 7.47$

26. $-8.6x - 2.62 = -7.1x + 8.54$

27. $-1.7x = -0.2x - 0.6$

28. $3.89x = -5.11x + 5.4$

29. $-1.02x + 7.08 = -2.79x$

30. $1.5x - 2.4 = 0.3x$

31. $-4.75x - 6.77 = -7.45x + 3.49$

32. $-1.2x - 2.8 = -0.7x - 5.6$

33. $-4.06x - 7.38 = 4.94x$

34. $-4.22x + 7.8 = -6.3x$

In Exercises 35-52, solve the equation.

35. $2.3 + 0.1(x + 2.9) = 6.9$

36. $-6.37 + 6.3(x + 4.9) = -1.33$

37. $0.5(1.5x - 6.58) = 6.88$

38. $0.5(-2.5x - 4.7) = 16.9$

39. $-6.3x - 0.4(x - 1.8) = -16.03$

40. $-2.8x + 5.08(x - 4.84) = 19.85$

41. $2.4(0.3x + 3.2) = -11.4$

42. $-0.7(0.2x + 5.48) = 16.45$

43. $-0.8(0.3x + 0.4) = -11.3$

44. $7.5(4.4x + 7.88) = 17.19$

45. $-7.57 - 2.42(x + 5.54) = 6.95$

46. $5.9 - 0.5(x + 5.8) = 12.15$

47. $-1.7 - 5.56(x + 6.1) = 12.2$

48. $-7.93 + 0.01(x + 7.9) = 14.2$

49. $4.3x - 0.7(x + 2.1) = 8.61$

50. $1.5x - 4.5(x + 4.92) = 15.6$

51. $-4.8x + 3.3(x - 0.4) = -7.05$

52. $-1.1x + 1.3(x + 1.3) = 19.88$

In Exercises 53-58, solve the equation.

53. $0.9(6.2x - 5.9) = 3.4(3.7x + 4.3) - 1.8$

54. $0.4(-4.6x+4.7) = -1.6(-2.2x+6.9)-4.5$

55. $-1.8(-1.6x + 1.7) = -1.8(-3.6x - 4.1)$

56. $-3.3(-6.3x + 4.2) - 5.3 = 1.7(6.2x + 3.2)$

57. $0.9(0.4x + 2.5) - 2.5 = -1.9(0.8x + 3.1)$

58. $5.5(6.7x + 7.3) = -5.5(-4.2x + 2.2)$

---

59. Stacy runs a business out of her home making bird houses. Each month she has fixed costs of \$200. In addition, for each bird house she makes, she incurs an additional cost of \$3.00. If her total costs for the month were \$296.00, how many bird houses did she make?

60. Stella runs a business out of her home making curtains. Each month she has fixed costs of \$175. In addition, for each curtain she makes, she incurs an additional cost of \$2.75. If her total costs for the month were \$274.00, how many curtains did she make?

61. A stationary store has staplers on sale for \$1.50 apiece. A business purchases an unknown number of these and the total cost of their purchase is \$36.00. How many were purchased?

62. A stationary store has CD packs on sale for \$2.50 apiece. A business purchases an unknown number of these and the total cost of their purchase is \$40.00. How many were purchased?

63. Julie runs a business out of her home making table cloths. Each month she has fixed costs of \$100. In addition, for each table cloth she makes, she incurs an additional cost of \$2.75. If her total costs for the month were \$221.00, how many table cloths did she make?

64. Stella runs a business out of her home making quilts. Each month she has fixed costs of \$200. In addition, for each quilt she makes, she incurs an additional cost of \$1.75. If her total costs for the month were \$280.50, how many quilts did she make?

65. Marta has 60 feet of decorative fencing which she will use for the border of a small circular garden. Find the diameter of the circular garden, correct to the nearest hundredth of a foot. Use $\pi \approx 3.14$.

66. Trinity has 44 feet of decorative fencing which she will use for the border of a small circular garden. Find the diameter of the circular garden, correct to the nearest hundredth of a foot. Use $\pi \approx 3.14$.

67. Children's tickets to the ice capades go on sale for \$4.25. The YMCA of Sacramento has \$1,000 set aside to purchase these tickets. Approximately how many tickets can the YMCA of Sacramento purchase?

68. Children's tickets to the ice capades go on sale for $5. The Knights of Columbus has $1,200 set aside to purchase these tickets. Approximately how many tickets can the Knights of Columbus purchase?

69. A stationary store has mechanical pencils on sale for $2.25 apiece. A business purchases an unknown number of these and the total cost of their purchase is $65.25. How many were purchased?

70. A stationary store has engineering templates on sale for $2.50 apiece. A business purchases an unknown number of these and the total cost of their purchase is $60.00. How many were purchased?

71. Marta has 61 feet of decorative fencing which she will use for the border of a small circular garden. Find the diameter of the circular garden, correct to the nearest hundredth of a foot. Use $\pi \approx 3.14$.

72. Kathy has 86 feet of decorative fencing which she will use for the border of a small circular garden. Find the diameter of the circular garden, correct to the nearest hundredth of a foot. Use $\pi \approx 3.14$.

73. Kathy needs to create a rectangular garden plot covering 100 square meters (100 m^2). If the width of the plot is 7.5 meters, find the length of the plot correct to the nearest tenth of a meter.

74. Marianne needs to create a rectangular garden plot covering 223 square meters (223 m^2). If the width of the plot is 8.3 meters, find the length of the plot correct to the nearest tenth of a meter.

75. Children's tickets to the stock car races go on sale for $4.5. The Boys and Girls club of Eureka has $1,300 set aside to purchase these tickets. Approximately how many tickets can the Boys and Girls club of Eureka purchase?

76. Children's tickets to the movies go on sale for $4.75. The Lions club of Alameda has $800 set aside to purchase these tickets. Approximately how many tickets can the Lions club of Alameda purchase?

77. Ashley needs to create a rectangular garden plot covering 115 square meters (115 m^2). If the width of the plot is 6.8 meters, find the length of the plot correct to the nearest tenth of a meter.

78. Molly needs to create a rectangular garden plot covering 268 square meters (268 m^2). If the width of the plot is 6.1 meters, find the length of the plot correct to the nearest tenth of a meter.

---

79. **Crude Inventory.** US commercial crude oil inventories decreased by 3.8 million barrels in the week ending June 19. If there were 353.9 million barrels the following week, what were crude oil inventories before the decline? *rttnews.com 06/24/09*

80. **Undocumented.** In 2008, California had 2.7 million undocumented residents. This is double the number in 1990. How many undocumented residents were in California in 1990? *Associated Press Times-Standard 4/15/09*

81. **Diamonds Shining.** The *index of refraction* n indicates the number of times slower that a light wave travels in a particular medium than it travels in a vacuum. A diamond has an index of refraction of 2.4. This is about one and one-quarter times greater than the index of refraction of a zircon. What is the index of refraction of a zircon? Round your result to the nearest tenth.

Answers

1. 1.75

3. 1.3

5. 13

7. 339.2

9. 8

11. 1.08

13. −11.25

15. 3.4

17. 5

19. −2.17

21. 16.8

23. 13

25. 20.14

27. 0.4

29. −4

31. 3.8

33. −0.82

35. 43.1

37. 13.56

39. 2.5

41. −26.5

43. 45.75

45. −11.54

47. −8.6

49. 2.8

51. 3.82

53. −2.59

55. −2.9

57. −3

59. 32

61. 24

63. 44

65. 19.11 feet

67. 235 tickets

69. 29

71. 19.43 feet

73. 13.3 meters

75. 288 tickets

77. 16.9 meters

79. 357.7 million barrels

81. 1.9

5.7 Introduction to Square Roots

Recall that

$$x^2 = x \cdot x.$$

The Square of a Number. The number x^2 is called the *square* of the number x.

Thus, for example:

- $9^2 = 9 \cdot 9 = 81$. Therefore, the number 81 is the square of the number 9.
- $(-4)^2 = (-4)(-4) = 16$. Therefore, the number 16 is the square of the number -4.

In the margin, we've placed a "List of Squares" of the whole numbers ranging from 0 through 25, inclusive.

List of Squares

x	x^2
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100
11	121
12	144
13	169
14	196
15	225
16	256
17	289
18	324
19	361
20	400
21	441
22	484
23	529
24	576
25	625

Square Roots

Once you've mastered the process of squaring a whole number, then you are ready for the inverse of the squaring process, taking the *square root* of a whole number.

- Above, we saw that $9^2 = 81$. We called the number 81 the *square* of the number 9. Conversely, we call the number 9 a *square root* of the number 81.
- Above, we saw that $(-4)^2 = 16$. We called the number 16 the *square* of the number -4. Conversely, we call the number -4 a *square root* of the number 16.

Square Root. If $a^2 = b$, then a is called a *square root* of the number b.

You Try It!

EXAMPLE 1. Find the square roots of the number 49.

Find the square roots of 256.

Solution. To find a square root of 49, we must think of a number a such that $a^2 = 49$. Two numbers come to mind.

- $(-7)^2 = 49$. Therefore, -7 is a square root of 49.
- $7^2 = 49$. Therefore, 7 is a square root of 49.

Note that 49 has two square roots, one of which is positive and the other one is negative.

Answer: $-16, 16$

□

Square Roots

x	$\sqrt{x}$
0	0
1	1
4	2
9	3
16	4
25	5
36	6
49	7
64	8
81	9
100	10
121	11
144	12
169	13
196	14
225	15
256	16
289	17
324	18
361	19
400	20
441	21
484	22
529	23
576	24
625	25

You Try It!

Find the square roots of 625.

EXAMPLE 2. Find the square roots of the number 196.

Solution. To find a square root of 196, we must think of a number a such that $a^2 = 196$. With help from the "List of Squares," two numbers come to mind.

- $(-14)^2 = 196$. Therefore, -14 is a square root of 196.
- $14^2 = 196$. Therefore, 14 is a square root of 196.

Note that 196 has two square roots, one of which is positive and the other one is negative.

Answer: -25, 25

□

You Try It!

Find the square roots of 9.

EXAMPLE 3. Find the square roots of the number 0.

Solution. To find a square root of 0, we must think of a number a such that $a^2 = 0$. There is only one such number, namely zero. Hence, 0 is the square root of 0.

Answer: -3, 3

□

You Try It!

Find the square roots of -81.

EXAMPLE 4. Find the square roots of the number -25.

Solution. To find a square root of -25, we must think of a number a such that $a^2 = -25$. This is impossible because no square of a real number (whole number, integer, fraction, or decimal) can be negative. Positive times positive is positive and negative times negative is also positive. You cannot square and get a negative answer. Therefore, -25 has no square roots[2].

Answer: There are none.

□

Radical Notation

Because $(-3)^2 = 9$ and $3^2 = 9$, both -3 and 3 are square roots of 9. Special notation, called *radical notation*, is used to request these square roots.

[2] At least not in Prealgebra. In later courses, you will be introduced to the set of complex numbers, where -25 will have two square roots

- The radical notation $\sqrt{9}$, pronounced "the nonnegative square root of 9," calls for the nonnegative[3] square root of 9. Hence,

$$\sqrt{9} = 3.$$

- The radical notation $-\sqrt{9}$, pronounced "the negative square root of 9," calls for the negative square root of 9. Hence,

$$-\sqrt{9} = -3.$$

Radical Notation. In the expression $\sqrt{9}$, the symbol $\sqrt{}$ is called a *radical* and the number within the radical, in this case the number 9, is called the *radicand.*

List of Squares

x	x^2
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100
11	121
12	144
13	169
14	196
15	225
16	256
17	289
18	324
19	361
20	400
21	441
22	484
23	529
24	576
25	625

For example,

- In the expression $\sqrt{529}$, the number 529 is the radicand.
- In the expression $\sqrt{a^2 + b^2}$, the expression $a^2 + b^2$ is the radicand.

Radical Notation and Square Root. If b is a positive number, then

1. $\sqrt{b}$ calls for the *nonnegative* square root of b.
2. $-\sqrt{b}$ calls for the *negative* square root of b.

Note: Nonnegative is equivalent to saying "not negative;" i.e., positive or zero.

You Try It!

Simplify:

a) $\sqrt{144}$

b) $-\sqrt{324}$

EXAMPLE 5. Simplify: (a) $\sqrt{121}$, (b) $-\sqrt{625}$, and (c) $\sqrt{0}$.

Solution.

(a) Referring to the list of squares, we note that $11^2 = 121$ and $(-11)^2 = 121$. Therefore, both 11 and -11 are square roots of 121. However, $\sqrt{121}$ calls for the nonnegative square root of 121. Thus,

$$\sqrt{121} = 11.$$

(b) Referring to the list of squares, we note that $25^2 = 625$ and $(-25)^2 = 625$. Therefore, both 25 and -25 are square roots of 625. However, $-\sqrt{625}$ calls for the negative square root of 625. Thus,

$$-\sqrt{625} = -25.$$

[3] Nonnegative is equivalent to saying "not negative;" i.e., positive or zero.

(c) There is only one square root of zero. Therefore,

$$\sqrt{0} = 0.$$

Answer: (a) 12 (b) −18

You Try It!

Simplify:

a) $-\sqrt{36}$

b) $\sqrt{-36}$

Answer: (a) −6 (b) undefined

EXAMPLE 6. Simplify: (a) $-\sqrt{25}$, and (b) $\sqrt{-25}$.

Solution.

(a) Because $5^2 = 25$ and $(-5)^2 = 25$, both 5 and −5 are square roots of 25. However, the notation $-\sqrt{25}$ calls for the negative square root of 25. Thus,

$$-\sqrt{25} = -5.$$

(b) It is not possible to square a real number (whole number, integer, fraction, or decimal) and get −25. Therefore, there is no real square root of −25. That is,

$$\sqrt{-25}$$

is not a real number. It is undefined[4].

Square Roots

x	$\sqrt{x}$
0	0
1	1
4	2
9	3
16	4
25	5
36	6
49	7
64	8
81	9
100	10
121	11
144	12
169	13
196	14
225	15
256	16
289	17
324	18
361	19
400	20
441	21
484	22
529	23
576	24
625	25

Order of Operations

With the addition of radical notation, the *Rules Guiding Order of Operations* change slightly.

Rules Guiding Order of Operations. When evaluating expressions, proceed in the following order.

1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
2. Evaluate all exponents and radicals that appear in the expression.
3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.
4. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

The only change in the rules is in item #2, which says: "Evaluate all exponents and radicals that appear in the expression," putting radicals on the same level as exponents.

[4] At least in Prealgebra. In later courses you will be introduced to the set of complex numbers, where $\sqrt{-25}$ will take on a new meaning.

You Try It!

Simplify:

$$2\sqrt{4}-3\sqrt{9}$$

Answer: -5

EXAMPLE 7. Simplify: $-3\sqrt{9}+12\sqrt{4}$.

Solution. According to the *Rules Guiding Order of Operations*, we must evaluate the radicals in this expression first.

$$\begin{aligned} -3\sqrt{9}+12\sqrt{4} &= -3(3)+12(2) && \text{Evaluate radicals first: } \sqrt{9}=3 \text{ and } \sqrt{4}=2. \\ &= -9+24 && \text{Multiply: } -3(3)=-9 \text{ and } 12(2)=24. \\ &= 15 && \text{Add: } -9+24=15. \end{aligned}$$

□

List of Squares

x	x^2
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100
11	121
12	144
13	169
14	196
15	225
16	256
17	289
18	324
19	361
20	400
21	441
22	484
23	529
24	576
25	625

You Try It!

Simplify:

$$5-8\sqrt{169}$$

Answer: -99

EXAMPLE 8. Simplify: $-2-3\sqrt{36}$.

Solution. According to the *Rules Guiding Order of Operations*, we must evaluate the radicals in this expression first, moving left to right.

$$\begin{aligned} -2-3\sqrt{36} &= -2-3(6) && \text{Evaluate radicals first: } \sqrt{36}=6 \\ &= -2-18 && \text{Multiply: } 3(6)=18. \\ &= -20 && \text{Subtract: } -2-18=-2+(-18)=-20. \end{aligned}$$

□

You Try It!

Simplify:

a) $\sqrt{25+144}$

b) $\sqrt{25}+\sqrt{144}$

Answer: (a) 13 (b) 17

EXAMPLE 9. Simplify: (a) $\sqrt{9+16}$ and (b) $\sqrt{9}+\sqrt{16}$.

Solution. Apply the *Rules Guiding Order of Operations.*

a) In this case, the radical acts like grouping symbols, so we must evaluate what is inside the radical first.

$$\begin{aligned} \sqrt{9+16} &= \sqrt{25} && \text{Add: } 9+16=25. \\ &= 5 && \text{Take nonnegative square root: } \sqrt{25}=5. \end{aligned}$$

b) In this example, we must evaluate the square roots first.

$$\begin{aligned} \sqrt{9}+\sqrt{16} &= 3+4 && \text{Square root: } \sqrt{9}=3 \text{ and } \sqrt{16}=4. \\ &= 7 && \text{Add: } 3+4=7. \end{aligned}$$

□

Fractions and Decimals

We can also find square roots of fractions and decimals.

You Try It!

Simplify:

a) $\sqrt{\frac{25}{49}}$

b) $\sqrt{0.36}$

EXAMPLE 10. Simplify: (a) $\sqrt{\dfrac{4}{9}}$, and (b) $-\sqrt{0.49}$.

Solution.

(a) Because $\left(\dfrac{2}{3}\right)^2 = \left(\dfrac{2}{3}\right)\left(\dfrac{2}{3}\right) = \dfrac{4}{9}$, then

$$\sqrt{\frac{4}{9}} = \frac{2}{3}.$$

(b) Because $(0.7)^2 = (0.7)(0.7) = 0.49$ and $(-0.7)^2 = (-0.7)(-0.7) = 0.49$, both 0.7 and -0.7 are square roots of 0.49. However, $-\sqrt{0.49}$ calls for the negative square root of 0.49. Hence,

$$-\sqrt{0.49} = -0.7.$$

Answer: (a) 5/7 (b) 0.6

Square Roots

x	$\sqrt{x}$
0	0
1	1
4	2
9	3
16	4
25	5
36	6
49	7
64	8
81	9
100	10
121	11
144	12
169	13
196	14
225	15
256	16
289	17
324	18
361	19
400	20
441	21
484	22
529	23
576	24
625	25

Estimating Square Roots

The squares in the "List of Squares" are called *perfect squares*. Each is the square of a whole number. Not all numbers are perfect squares. For example, in the case of $\sqrt{24}$, there is no whole number whose square is equal to 24. However, this does not prevent $\sqrt{24}$ from being a perfectly good number.

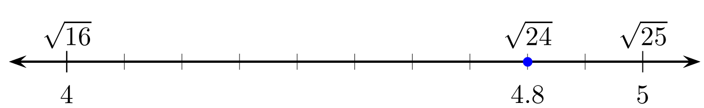
We can use the "List of Squares" to find decimal approximations when the radicand is not a perfect square.

You Try It!

Estimate: $\sqrt{83}$

EXAMPLE 11. Estimate $\sqrt{24}$ by guessing. Use a calculator to find a more accurate result and compare this result with your guess.

Solution. From the "List of Squares," note that 24 lies betwen 16 and 25, so $\sqrt{24}$ will lie between 4 and 5, with $\sqrt{24}$ much closer to 5 than it is to 4.



Let's guess

$$\sqrt{24} \approx 4.8.$$

As a check, let's square 4.8.

$$(4.8)^2 = (4.8)(4.8) = 23.04$$

Not quite 24! Clearly, $\sqrt{24}$ must be a little bit bigger than 4.8.

Let's use a scientific calculator to get a better approximation. From our calculator, using the square root button, we find

$$\sqrt{24} \approx 4.89897948557.$$

Even though this is better than our estimate of 4.8, it is still only an approximation. Our calculator was only capable of providing 11 decimal places. However, the exact decimal representation of $\sqrt{24}$ is an infinite decimal that never terminates and never establishes a pattern of repetition.

Just for fun, here is a decimal approximation of $\sqrt{24}$ that is accurate to 1000 places, courtesy of http://www.wolframalpha.com/.

4.89897948556635619639456814941178278393189496131334025686538 51
3450192075491463005307971886620928046963718920245322837824971773
0919675514683251567902474557105657825495055353142495260210541823 5
4044696262135797338170726488670509120806761761787874917113569314 9
4487226082885405404323484036766001631796156760261794014573879872 6
1674316188801600887477375098329029307878290024089452896266632587 0
2188948362702657099008893234345326285099529663624900802313209072 9
1801868717233586396733133253381826381307172753221051631235873247 2
3582205893441767091510257671059796648201117380410012830932248234 7
0679882086211598579693467906510557472083659310343660782073560076 7
2463325946466056580995478209485272014102527539509377735401281985 9
1185143465692900577618302885149260520590592647415105006845511983 0
9085256259600612934415988485060457568524106813589572009319387995 9
8711950812334271730930691249641651255377273856188261274486701772 9
6031449692674464894759090976288769586727401839482029557046575118 2
12631969215662073401907064945 3

If you were to multiply this number by itself (square the number), you would get a number that is extremely close to 24, but it would not be exactly 24. There would still be a little discrepancy. Answer: 9.1

□

Important Observation. A calculator can only produce a finite number of decimal places. If the decimal representation of your number does not terminate within this limited number of places, then the number in your calculator window is only an approximation.

- The decimal representation of 1/8 will terminate within three places, so most calculators will report the exact answer, 0.125.

- For contrast, 2/3 does not terminate. A calculator capable of reporting 11 places of accuracy produces the number 0.666666666667. However, the exact decimal representation of 2/3 is $0.\overline{6}$. Note that the calculator has rounded in the last place and only provides an approximation of 2/3. If your instructor asks for an exact answer on an exam or quiz then 0.666666666667, being an approximation, is not acceptable. You must give the exact answer 2/3.

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-16, list all square roots of the given number. If the number has no square roots, write "none".

1. 256

2. 361

3. -289

4. -400

5. 441

6. 36

7. 324

8. 0

9. 144

10. 100

11. -144

12. -100

13. 121

14. -196

15. 529

16. 400

In Exercises 17-32, compute the exact square root. If the square root is undefined, write "undefined".

17. $\sqrt{-9}$

18. $-\sqrt{-196}$

19. $\sqrt{576}$

20. $\sqrt{289}$

21. $\sqrt{-529}$

22. $\sqrt{-256}$

23. $-\sqrt{25}$

24. $\sqrt{225}$

25. $-\sqrt{484}$

26. $-\sqrt{36}$

27. $-\sqrt{196}$

28. $-\sqrt{289}$

29. $\sqrt{441}$

30. $\sqrt{324}$

31. $-\sqrt{4}$

32. $\sqrt{100}$

In Exercises 33-52, compute the exact square root.

33. $\sqrt{0.81}$

34. $\sqrt{5.29}$

35. $\sqrt{3.61}$

36. $\sqrt{0.09}$

37. $\sqrt{\dfrac{225}{16}}$

38. $\sqrt{\dfrac{100}{81}}$

39. $\sqrt{3.24}$

40. $\sqrt{5.76}$

41. $\sqrt{\dfrac{121}{49}}$

42. $\sqrt{\dfrac{625}{324}}$

43. $\sqrt{\dfrac{529}{121}}$

44. $\sqrt{\dfrac{4}{121}}$

45. $\sqrt{2.89}$

46. $\sqrt{4.41}$

47. $\sqrt{\dfrac{144}{25}}$

48. $\sqrt{\dfrac{49}{36}}$

49. $\sqrt{\dfrac{256}{361}}$

50. $\sqrt{\dfrac{529}{16}}$

51. $\sqrt{0.49}$

52. $\sqrt{4.84}$

In Exercises 53-70, compute the exact value of the given expression.

53. $6-\sqrt{576}$

54. $-2-7\sqrt{576}$

55. $\sqrt{8^2+15^2}$

56. $\sqrt{7^2+24^2}$

57. $6\sqrt{16}-9\sqrt{49}$

58. $3\sqrt{441}+6\sqrt{484}$

59. $\sqrt{5^2+12^2}$

60. $\sqrt{15^2+20^2}$

61. $\sqrt{3^2+4^2}$

62. $\sqrt{6^2+8^2}$

63. $-2\sqrt{324}-6\sqrt{361}$

64. $-6\sqrt{576}-8\sqrt{121}$

65. $-4-3\sqrt{529}$

66. $-1+\sqrt{625}$

67. $-9\sqrt{484}+7\sqrt{81}$

68. $-\sqrt{625}-5\sqrt{576}$

69. $2-\sqrt{16}$

70. $8-6\sqrt{400}$

In Exercises 71-76, complete the following tasks to estimate the given square root.

a) Determine the two integers that the square root lies between.

b) Draw a number line, and locate the approximate location of the square root between the two integers found in part (a).

c) Without using a calculator, estimate the square root to the nearest tenth.

71. $\sqrt{58}$

72. $\sqrt{27}$

73. $\sqrt{79}$

74. $\sqrt{12}$

75. $\sqrt{44}$

76. $\sqrt{88}$

In Exercises 77-82, use a calculator to approximate the square root to the nearest tenth.

77. $\sqrt{469}$

78. $\sqrt{73}$

79. $\sqrt{615}$

80. $\sqrt{162}$

81. $\sqrt{444}$

82. $\sqrt{223}$

❧ ❧ ❧ Answers ☙ ☙ ☙

1. $16, -16$

3. none

5. $21, -21$

7. $18, -18$

9. $12, -12$

11. none

13. $11, -11$

15. $23, -23$

17. undefined

19. 24

21. undefined

23. -5

25. -22

27. -14

29. 21

31. -2

33. 0.9

35. 1.9

37. $\dfrac{15}{4}$

39. 1.8

41. $\dfrac{11}{7}$

43. $\dfrac{23}{11}$

45. 1.7

47. $\dfrac{12}{5}$

49. $\dfrac{16}{19}$

51. 0.7

53. -18

55. 17

57. -39

59. 13

61. 5

63. -150

65. -73

67. -135

69. -2

71. 7.6

73. 8.9

75. 6.6

77. 21.7

79. 24.8

81. 21.1

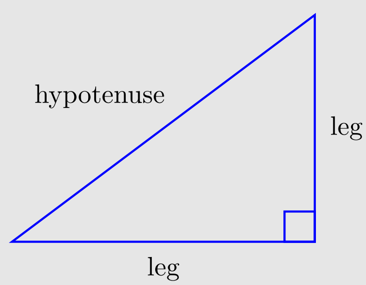
5.8 The Pythagorean Theorem

Pythagoras was a Greek mathematician and philosopher, born on the island of Samos (ca. 582 BC). He founded a number of schools, one in particular in a town in southern Italy called Crotone, whose members eventually became known as the Pythagoreans. The inner circle at the school, the *Mathematikoi*, lived at the school, rid themselves of all personal possessions, were vegetarians, and observed a strict vow of silence. They studied mathematics, philosophy, and music, and held the belief that numbers constitute the true nature of things, giving numbers a mystical or even spiritual quality.

Today, nothing is known of Pythagoras's writings, perhaps due to the secrecy and silence of the Pythagorean society. However, one of the most famous theorems in all of mathematics does bear his name, the *Pythagorean Theorem*.

Prior to revealing the contents of the Pythagorean Theorem, we pause to provide the definition of a right triangle and its constituent parts.

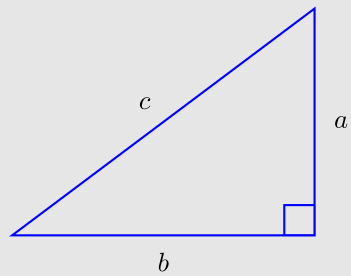
Right Triangle. A triangle with one right angle (90°) is called a *right triangle*. In the figure below, the right angle is marked with a little square.



The side of the triangle that is directly opposite the right angle is called the *hypotenuse*. The sides of the triangle that include the right angle are called the *legs* of the right triangle.

Now we can state one of the most ancient theorems of mathematics, the *Pythagorean Theorem*.

Pythagorean Theorem. Let c represent the length of the *hypotenuse* of a right triangle, and let a and b represent the lengths of its legs, as pictured in the image that follows.



The relationship involving the legs and hypotenuse of the right triangle, given by

$$a^2 + b^2 = c^2,$$

is called the *Pythagorean Theorem.*

Here are two important observations.

Observations Regarding the Hypotenuse. Two important facts regarding the hypotenuse of the right triangle are:

1. The hypotenuse is the longest side of the triangle and lies directly opposite the right angle.
2. In the Pythagorean equation $a^2 + b^2 = c^2$, the hypotenuse lies by itself on one side of the equation.

The Pythagorean Theorem can only be applied to right triangles.

Let's look at a simple application of the Pythagorean Theorem.

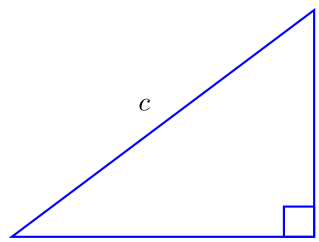
You Try It!

The legs of a right triangle measure 5 and 12 feet, respectively. Find the length of the hypotenuse.

EXAMPLE 1. The legs of a right triangle measure 3 and 4 meters, respectively. Find the length of the hypotenuse.

Solution. Let's follow the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* Let c represent the length of the hypotenuse, as pictured in the following sketch.



2. *Set up an Equation.* The Pythagorean Theorem says that

$$a^2 + b^2 = c^2.$$

In this example, the legs are known. Substitute 4 for a and 3 for b (3 for a and 4 for b works equally well) into the Pythagorean equation.

$$4^2 + 3^2 = c^2$$

3. *Solve the Equation.*

$4^2 + 3^2 = c^2$	The Pythagorean equation.
$16 + 9 = c^2$	Exponents first: $4^2 = 16$ and $3^2 = 9$.
$25 = c^2$	Add: $16 + 9 = 25$.
$5 = c$	Take the nonnegative square root.

Technically, there are two answers to $c^2 = 25$, i.e., $c = -5$ or $c = 5$. However, c represents the hypotenuse of the right triangle and must be nonnegative. Hence, we must choose $c = 5$.

4. *Answer the Question.* The hypotenuse has length 5 meters.

5. *Look Back.* Do the numbers satisfy the Pythagorean Theorem? The sum of the squares of the legs should equal the square of the hypotenuse. Let's check.

$$4^2 + 3^2 = 5^2$$
$$16 + 9 = 25$$
$$25 = 25$$

All is well!

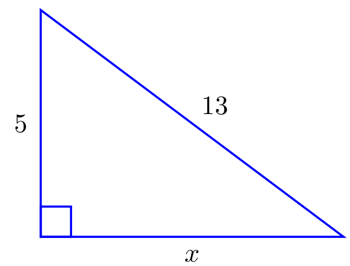
Answer: 13 feet

□

You Try It!

EXAMPLE 2. Given the following right triangle, find the length of the missing side.

The hypotenuse of a right triangle measures 25 centimeters. One leg of the right triangle measures 24 centimeters. Find the length of the remaining leg.



Solution. Note that the hypotenuse (across from the right angle) has length 13. This quantity should lie on one side of the Pythagorean equation all by itself. The sum of the squares of the legs go on the other side. Hence,

$$5^2 + x^2 = 13^2$$

Solve the equation for x.

$25 + x^2 = 169$	Exponents first: $5^2 = 25$ and $13^2 = 169$.
$25 + x^2 - 25 = 169 - 25$	Subtract 25 from both sides.
$x^2 = 144$	Simplify both sides.
$x = 12$	Take the nonnegative square root of 144.

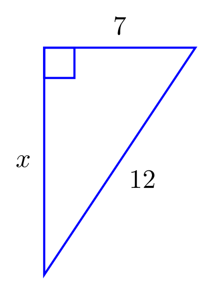
Answer: 7 centimeters

Perfect squares are nice, but not required.

You Try It!

The hypotenuse and one leg of a right triangle measure 9 and 7 inches, respectively. Find the length of the remaining leg.

EXAMPLE 3. Given the following right triangle, find the **exact** length of the missing side.



Solution. Note that the hypotenuse (across from the right angle) has length 12. This quantity should lie on one side of the Pythagorean equation all by itself. The sum of the squares of the legs go on the other side. Hence,

$$x^2 + 7^2 = 12^2$$

Solve the equation for x.

$$x^2 + 49 = 144 \qquad \text{Exponents first: } 7^2 = 49 \text{ and } 12^2 = 144.$$
$$x^2 + 49 - 49 = 144 - 49 \qquad \text{Subtract 49 from both sides.}$$
$$x^2 = 95 \qquad \text{Simplify both sides.}$$
$$x = \sqrt{95} \qquad \text{Take the nonnegative square root of 95.}$$

Hence, the **exact** length of the missing side is $\sqrt{95}$.

Answer: $\sqrt{32}$ inches

Important Observation. Any attempt to use your calculator to approximate $\sqrt{95}$ in Example 3 would be an error as the instructions asked for an **exact** answer.

Sometimes an approximate answer is desired, particularly in applications.

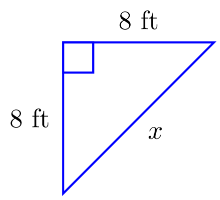
You Try It!

A 15 foot ladder leans against the wall of a building. The base of the ladder lies 5 feet from the base of the wall. How high up the wall does the top of the ladder reach? Round your answer to the nearest tenth of a foot.

EXAMPLE 4. Ginny want to create a vegetable garden in the corner of her yard in the shape of a right triangle. She cuts two boards of length 8 feet which will form the legs of her garden. Find the length of board she should cut to form the hypotenuse of her garden, correct to the nearest tenth of a foot.

Solution. We follow the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* We begin with a labeled sketch. Let x represent the length of the unknown hypotenuse.



2. *Set Up an Equation.* The hypotenuse is isolated on one side of the Pythagorean equation.

$$x^2 = 8^2 + 8^2$$

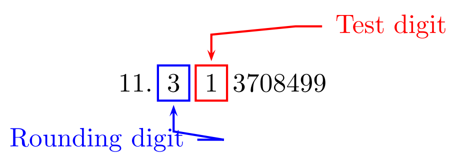
3. *Solve the Equation.*

$$x^2 = 8^2 + 8^2 \qquad \text{The Pythagorean equation.}$$
$$x^2 = 64 + 64 \qquad \text{Exponents first: } 8^2 = 64 \text{ and } 8^2 = 64.$$
$$x^2 = 128 \qquad \text{Add: } 64 + 64 = 128.$$
$$x = \sqrt{128} \qquad \text{Take the nonnegative square root.}$$

4. *Answer the Question.* The **exact** length of the hypotenuse is $\sqrt{128}$ feet, but we're asked to find the hypotenuse to the nearest tenth of a foot. Using a calculator, we find an approximation for $\sqrt{128}$.

$$\sqrt{128} \approx 11.313708499$$

To round to the nearest tenth, first identify the rounding and test digits.



The test digit is less than five. So we leave the rounding digit alone and truncate. Therefore, correct to the nearest tenth of a foot, the length of the hypotenuse is approximately 11.3 feet.

5. *Look Back.* The sum of the squares of the legs is

$$\begin{aligned} 8^2 + 8^2 &= 64 + 64 \\ &= 128. \end{aligned}$$

The square of the hypotenuse is

$$(11.3)^2 = 127.69$$

These are almost the same, the discrepancy due to the fact that we rounded to find an approximation for the hypotenuse.

Answer: 14.1 feet

□

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-16, your solutions should include a well-labeled sketch.

1. The length of one leg of a right triangle is 15 meters, and the length of the hypotenuse is 25 meters. Find the exact length of the other leg.

2. The length of one leg of a right triangle is 7 meters, and the length of the hypotenuse is 25 meters. Find the exact length of the other leg.

3. The lengths of two legs of a right triangle are 12 meters and 16 meters. Find the exact length of the hypotenuse.

4. The lengths of two legs of a right triangle are 9 meters and 12 meters. Find the exact length of the hypotenuse.

5. The length of one leg of a right triangle is 13 meters, and the length of the hypotenuse is 22 meters. Find the exact length of the other leg.

6. The length of one leg of a right triangle is 6 meters, and the length of the hypotenuse is 15 meters. Find the exact length of the other leg.

7. The lengths of two legs of a right triangle are 2 meters and 21 meters. Find the exact length of the hypotenuse.

8. The lengths of two legs of a right triangle are 7 meters and 8 meters. Find the exact length of the hypotenuse.

9. The length of one leg of a right triangle is 12 meters, and the length of the hypotenuse is 19 meters. Find the exact length of the other leg.

10. The length of one leg of a right triangle is 5 meters, and the length of the hypotenuse is 10 meters. Find the exact length of the other leg.

11. The lengths of two legs of a right triangle are 6 meters and 8 meters. Find the exact length of the hypotenuse.

12. The lengths of two legs of a right triangle are 5 meters and 12 meters. Find the exact length of the hypotenuse.

13. The length of one leg of a right triangle is 6 meters, and the length of the hypotenuse is 10 meters. Find the exact length of the other leg.

14. The length of one leg of a right triangle is 9 meters, and the length of the hypotenuse is 15 meters. Find the exact length of the other leg.

15. The lengths of two legs of a right triangle are 6 meters and 22 meters. Find the exact length of the hypotenuse.

16. The lengths of two legs of a right triangle are 9 meters and 19 meters. Find the exact length of the hypotenuse.

In Exercises 17-24, your solutions should include a well-labeled sketch.

17. The lengths of two legs of a right triangle are 3 meters and 18 meters. Find the length of the hypotenuse. Round your answer to the nearest hundredth.

18. The lengths of two legs of a right triangle are 10 feet and 16 feet. Find the length of the hypotenuse. Round your answer to the nearest tenth.

19. The length of one leg of a right triangle is 2 meters, and the length of the hypotenuse is 17 meters. Find the length of the other leg. Round your answer to the nearest tenth.

20. The length of one leg of a right triangle is 4 meters, and the length of the hypotenuse is 12 meters. Find the length of the other leg. Round your answer to the nearest hundredth.

21. The lengths of two legs of a right triangle are 15 feet and 18 feet. Find the length of the hypotenuse. Round your answer to the nearest hundredth.

22. The lengths of two legs of a right triangle are 6 feet and 13 feet. Find the length of the hypotenuse. Round your answer to the nearest tenth.

23. The length of one leg of a right triangle is 4 meters, and the length of the hypotenuse is 8 meters. Find the length of the other leg. Round your answer to the nearest hundredth.

24. The length of one leg of a right triangle is 3 meters, and the length of the hypotenuse is 15 meters. Find the length of the other leg. Round your answer to the nearest tenth.

---

25. Greta and Fritz are planting a 13-meter by 18-meter rectangular garden, and are laying it out using string. They would like to know the length of a diagonal to make sure that right angles are formed. Find the length of a diagonal. Round your answer to the nearest hundredth. Your solution should include a well-labeled sketch.

26. Markos and Angelina are planting an 11-meter by 19-meter rectangular garden, and are laying it out using string. They would like to know the length of a diagonal to make sure that right angles are formed. Find the length of a diagonal. Round your answer to the nearest tenth. Your solution should include a well-labeled sketch.

27. The base of a 24-meter long guy wire is located 10 meters from the base of the telephone pole that it is anchoring. How high up the pole does the guy wire reach? Round your answer to the nearest hundredth. Your solution should include a well-labeled sketch.

28. The base of a 30-foot long guy wire is located 9 feet from the base of the telephone pole that it is anchoring. How high up the pole does the guy wire reach? Round your answer to the nearest hundredth. Your solution should include a well-labeled sketch.

---

29. **Hiking Trail.** A hiking trail runs due south for 8 kilometers, then turns west for about 15 kilometers, and then heads northeast on a direct path to the starting point. How long is the entire trail?

30. **Animal Trail.** An animal trail runs due east from a watering hole for 12 kilometers, then goes north for 5 kilometers. Then the trail turns southwest on a direct path back to the watering hole. How long is the entire trail?

31. **Upper Window.** A 10-foot ladder leans against the wall of a house. How close to the wall must the bottom of the ladder be in order to reach a window 8 feet above the ground?

32. **How high?** A 10-foot ladder leans against the wall of a house. How high will the ladder be if the bottom of the ladder is 4 feet from the wall? Round your answer to the nearest tenth.

❧ ❧ ❧ Answers ☙ ☙ ☙

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1. 20 meters

3. 20 meters

5. $\sqrt{315}$ meters

7. $\sqrt{445}$ meters

9. $\sqrt{217}$ meters

11. 10 meters

13. 8 meters

15. $\sqrt{520}$ meters

17. 18.25 meters

19. 16.9 meters

21. 23.43 feet

23. 6.93 meters

25. 22.20 meters

27. 21.82 meters

29. 40 kilometers

31. 6 ft.

Chapter 6

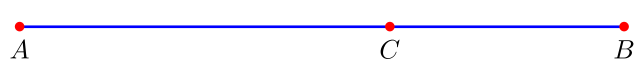
Ratio and Proportion

From the beginnings of the human race, we've long compared one quantity with another, a comparison that is called a *ratio* in mathematics. "Their tribe has twice as many cattle as ours" or "Two baskets of wheat cost 12 ducats" are examples of ratios that ring from distant times. Indeed, the concept of a *ratio* cannot be assigned to any one individual or class of individual. In his *History of Mathematics*, D. E. Smith writes:

> *It is rather profitless to speculate as to the domain in which the concept of ratio first appeared. The idea that one tribe is twice as large as another and the idea that one leather strap is only half as long as another both involve the notion of ratio; both are such as would develop early in the history of the race, and yet one has to do with ratio of numbers and the other with the ratio of geometric magnitudes. Indeed, when we come to the Greek writers we find Nicomachus including ratio in his arithmetic, Eudoxus in his geometry, and Theon of Smyrna in his chapter on music.*

Examples and applications of ratios are limitless: speed is a ratio that compares changes in distance with respect to time, acceleration is a ratio that compares changes in speed with respect to time, and percentages compare the part with the whole. We've already studied one classic ratio, the ratio of the circumference of a circle to its diameter, which gives us the definition of π.

One of the most famous ratios in history involves the division of a line segment AB into two segments AC and CB by selecting a point C on the segment AB.



The idea is to select a point C on the segment AB so that

$$\frac{AB}{AC} = \frac{AC}{CB}.$$

This ratio has a special name, the *Golden Ratio*, and has an exact value equal to $(1+\sqrt{5})/2$. The Golden Ratio has been known since the time of Euclid. Ancient and modern architects have long held that the most pleasing rectangular shape is the one whose ratio of length to width is equal to the Golden Ratio.

The comparison of two ratios, such as $AB/AC = AC/CB$, is called a *proportion*. Proportions are used in a number of practical ways. For example, if 5 cans of tomato sauce cost 2 dollars, we can find the number of cans that can be purchased with 10 dollars by comparing two ratios in a proportion:

$$\frac{5\,\text{cans of tomato sauce}}{2\,\text{dollars}} = \frac{x\,\text{cans of tomato sauce}}{10\,\text{dollars}}$$

Any discussion of ratio involves comparing two quantities, so the units of each quantity become extremely important. Two different systems of units are used when measuring length, capacity, and time: the *American* system of units and the *metric* system of units. In this chapter we will discuss both systems and explain how to convert quantities measured in one system to quantities measured in the other system.

Let's begin the journey.

6.1 Introduction to Ratios and Rates

We use *ratios* to compare two numeric quantities or quantities with the same units.

Ratio. A *ratio* is the quotient of two numerical quantities or two quantities with the same physical units.

For example, ancient Greek geometers believed that the most pleasing rectangle to the eye had length and width such that the ratio of length to width was a specific number, called the *Golden Ratio*, approximately equal to 1.6180339887.... Architects to this day use this ratio in their designs.

There are a number of equivalent ways of expressing ratios, three of which we will use in this text: fraction notation, "to" notation, and "colon" notation.

- 3/4 is a ratio, read as "the ratio of 3 to 4."
- 3 to 4 is a ratio, read as "the ratio of 3 to 4."
- 3:4 is a ratio, read as "the ratio of 3 to 4."

You Try It!

Express 0.12 : 0.3 as a fraction reduced to lowest terms.

EXAMPLE 1. Express each of the following ratios as a fraction reduced to lowest terms: (a) 36 to 24, and (b) 0.12 : 0.18.

Solution

(a) To express the ratio "36 to 24" as a fraction, place 36 over 24 and reduce.

$$\begin{aligned} \frac{36}{24} &= \frac{3 \cdot 12}{2 \cdot 12} && \text{Factor.} \\ &= \frac{3 \cdot \cancel{12}}{2 \cdot \cancel{12}} && \text{Cancel common factor.} \\ &= \frac{3}{2} \end{aligned}$$

Thus, the ratio 36 to 24 equals 3/2.

(b) To express the ratio "0.12:0.18" as a fraction, place 0.12 over 0.18 and reduce.

$$\begin{aligned}
\frac{0.12}{0.18} &= \frac{(0.12)(100)}{(0.18)(100)} && \text{Multiply numerator and denominator by 100.}\\
&= \frac{12}{18} && \text{Move each decimal 2 places right.}\\
&= \frac{2\cdot 6}{3\cdot 6} && \text{Factor.}\\
&= \frac{2\cdot \not{6}}{3\cdot \not{6}} && \text{Cancel.}\\
&= \frac{2}{3}
\end{aligned}$$

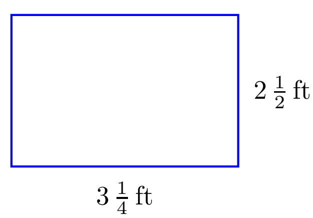
Answer: 2/5

Thus, the ratio 0.12:0.18 equals 2/3.

You Try It!

A rectangle has length $8\frac{1}{4}$ inches and width $3\frac{1}{2}$ inches. Express the ratio of length to width as a fraction reduced to lowest terms.

EXAMPLE 2. For the rectangle that follows, express the ratio of length to width as a fraction reduced to lowest terms.



Solution. The ratio length to width can be expressed as a fraction and reduced as follows.

$$\begin{aligned}
\frac{\text{length}}{\text{width}} &= \frac{3\,\frac{1}{4}\text{ ft}}{2\,\frac{1}{2}\text{ ft}} && \text{Length to width as a fraction.}\\
&= \frac{3\,\frac{1}{4}\,\not{\text{ft}}}{2\,\frac{1}{2}\,\not{\text{ft}}} && \text{Cancel common units.}\\
&= \frac{\frac{13}{4}}{\frac{5}{2}} && \text{Mixed to improper fractions.}
\end{aligned}$$

Invert and multiply, factor, and cancel common factors.

$$= \frac{13}{4} \cdot \frac{2}{5}$$ Invert and multiply.

$$= \frac{26}{20}$$ Multiply numerators and denominators.

$$= \frac{13 \cdot 2}{10 \cdot 2}$$ Factor numerator and denominator.

$$= \frac{13 \cdot \not{2}}{10 \cdot \not{2}}$$ Cancel common factors.

$$= \frac{13}{10}$$

Hence, the ratio length to width is 13/10.

Answer: 33/14

Rates

We now introduce the concept of *rate*, a special type of ratio.

Rate. A *rate* is a quotient of two measurements with different units.

The physical interpretation of a rate in terms of its units is an important skill.

You Try It!

Lanny travels 180 kilometers on 14 liters of gasoline. Express the ratio distance traveled to gas consumption as a fraction reduced to lowest terms.

EXAMPLE 3. An automobile travels 224 miles on 12 gallons of gasoline. Express the ratio distance traveled to gas consumption as a fraction reduced to lowest terms. Write a short sentence explaining the physical significance of your solution. Include units in your description.

Solution. Place miles traveled over gallons of gasoline consumed and reduce.

$$\frac{224\,\text{mi}}{12\,\text{gal}} = \frac{56 \cdot 4\,\text{mi}}{3 \cdot 4\,\text{gal}}$$ Factor

$$= \frac{56 \cdot \not{4}\,\text{mi}}{3 \cdot \not{4}\,\text{gal}}$$ Cancel common factor.

$$= \frac{56\,\text{mi}}{3\,\text{gal}}$$

Thus, the rate is 56 miles to 3 gallons of gasoline. In plain-speak, this means that the automobile travels 56 miles on 3 gallons of gasoline.

Answer: 90/7 kilometers per litre.

Unit Rates

When making comparisons, it is helpful to have a rate in a form where the denominator is 1. Such rates are given a special name.

Unit Rate. A *unit rate* is a rate whose denominator is 1.

You Try It!

Jacob drives 120 kilometers in 3 hours. Find his average rate of speed.

EXAMPLE 4. Herman drives 120 miles in 4 hours. Find his average rate of speed.

Solution. Place the distance traveled over the time it takes to drive that distance.

$$\begin{aligned}\frac{120\,\text{miles}}{4\,\text{hours}} &= \frac{30\,\text{miles}}{1\,\text{hour}} && \text{Divide: } 120/4 = 30.\\ &= 30\,\text{miles/hour}\end{aligned}$$

Hence, Herman's average rate of speed is 30 miles per hour.

Answer: 40 kilometers per hour.

□

You Try It!

Frannie works 5.5 hours and receives $120 for her efforts. What is her hourly salary rate? Round your answer to the nearest penny.

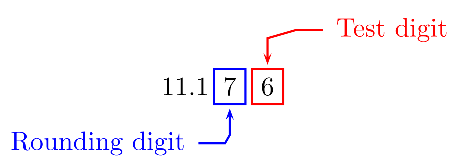
EXAMPLE 5. Aditya works 8.5 hours and receives $95 for his efforts. What is his hourly salary rate?

Solution. Let's place money earned over hours worked to get the following rate:

$$\frac{95\,\text{dollars}}{8.5\,\text{hours}}$$

We will get a much better idea of Aditya's salary rate if we express the rate with a denominator of 1. To do so, divide. Push the decimal in the divisor to the far right, then move the decimal an equal number of places in the dividend. As we are dealing with dollars and cents, we will round our answer to the nearest hundredth.

```
       11.176
   85)950.000
      85
      100
       85
       15 0
        8 5
        6 50
        5 95
          550
          510
           40
```



Because the test digit is greater than or equal to 5, we add 1 to the rounding digit and truncate; i.e., $95/8.5 \approx 11.18$. Hence,

$$\begin{aligned}\frac{95\,\text{dollars}}{8.5\,\text{hours}} &= \frac{11.18\,\text{dollars}}{1\,\text{hour}} && \text{Divide: } 95/8.5 \approx 11.18.\\ &= 11.18\,\text{dollars/hour.}\end{aligned}$$

That is, his salary rate is 11.18 dollars per hour.

Answer: $21.82 per hour

You Try It!

EXAMPLE 6. One automobile travels 422 miles on 15 gallons of gasoline. A second automobile travels 354 miles on 13 gallons of gasoline. Which automobile gets the better gas mileage?

Alicia works 8 hours and makes $100. Connie works 10 hours and makes $122. Which woman works at the larger hourly rate?

Solution. Decimal division (rounded to the nearest tenth) reveals the better gas mileage.

In the case of the first autombile, we get the following rate:

$$\frac{422\,\text{mi}}{15\,\text{gal}}$$

Divide.

```
     28.13
15)422.00
   30
   122
   120
     2 0
     1 5
       50
       45
        5
```

To the nearest tenth, 28.1.

In the case of the second autombile, we get the following rate:

$$\frac{354\,\text{mi}}{13\,\text{gal}}$$

Divide.

```
     27.23
13)354.00
   26
    94
    91
     3 0
     2 6
       40
       39
        1
```

To the nearest tenth, 27.2.

In the case of the first automobile, the mileage rate is 28.1 mi/1 gal, which can be read "28.1 miles per gallon." In the case of the second automobile, the mileage rate is 27.2 mi/1 gal, which can be read "27.2 miles per gallon." Therefore, the first automobile gets the better gas mileage.

Answer: Alicia

Exercises

In Exercises 1-24, express the given ratio as a fraction reduced to lowest terms.

1. 0.14 : 0.44

2. 0.74 : 0.2

3. 0.05 : 0.75

4. 0.78 : 0.4

5. 0.1 : 0.95

6. 0.93 : 0.39

7. $2\frac{2}{9} : 1\frac{1}{3}$

8. $3\frac{2}{3} : 2\frac{4}{9}$

9. 0.36 : 0.6

10. 0.58 : 0.42

11. 15 : 21

12. 77 : 121

13. $2\frac{8}{9} : 2\frac{2}{3}$

14. $1\frac{2}{3} : 3\frac{8}{9}$

15. $3\frac{8}{9} : 2\frac{1}{3}$

16. $1\frac{5}{9} : 1\frac{1}{3}$

17. $2\frac{5}{8} : 1\frac{3}{4}$

18. $2\frac{4}{9} : 1\frac{1}{3}$

19. 10 : 35

20. 132 : 84

21. 9 : 33

22. 35 : 10

23. 27 : 99

24. 12 : 28

---

25. One automobile travels 271.8 miles on 10.1 gallons of gasoline. A second automobile travels 257.9 miles on 11.1 gallons of gasoline. Which automobile gets the better gas mileage?

26. One automobile travels 202.9 miles on 13.9 gallons of gasoline. A second automobile travels 221.6 miles on 11.8 gallons of gasoline. Which automobile gets the better gas mileage?

27. Todd is paid 183 dollars for 8.25 hours work. What is his hourly salary rate, rounded to the nearest penny?

28. David is paid 105 dollars for 8.5 hours work. What is his hourly salary rate, rounded to the nearest penny?

29. An automobile travels 140 miles in 4 hours. Find the average rate of speed.

30. An automobile travels 120 miles in 5 hours. Find the average rate of speed.

31. Judah is paid 187 dollars for 8 hours work. What is his hourly salary rate, rounded to the nearest penny?

32. Judah is paid 181 dollars for 8.75 hours work. What is his hourly salary rate, rounded to the nearest penny?

33. One automobile travels 234.2 miles on 10.8 gallons of gasoline. A second automobile travels 270.5 miles on 10.8 gallons of gasoline. Which automobile gets the better gas mileage?

34. One automobile travels 297.6 miles on 10.7 gallons of gasoline. A second automobile travels 298.1 miles on 12.6 gallons of gasoline. Which automobile gets the better gas mileage?

35. An automobile travels 180 miles in 5 hours. Find the average rate of speed.

36. An automobile travels 220 miles in 5 hours. Find the average rate of speed.

---

37. **Antarctic trek.** Seven women on a 562-mile Antarctic ski trek reached the South Pole 38 days after they began their adventure. What was the ladies' average rate of speed per day? Round your result to the nearest tenth of a mile. *Associated Press-Times-Standard 12/31/09 After 562-mile ski trek, seven women reach the South Pole.*

Answers

1. $\frac{7}{22}$

3. $\frac{1}{15}$

5. $\frac{2}{19}$

7. $\frac{5}{3}$

9. $\frac{3}{5}$

11. $\frac{5}{7}$

13. $\frac{13}{12}$

15. $\frac{5}{3}$

17. $\frac{3}{2}$

19. $\frac{2}{7}$

21. $\frac{3}{11}$

23. $\frac{3}{11}$

25. The first automobile has the better mileage per gallon.

27. 22.18 dollars/hr

29. 35 mi/hr

31. 23.38 dollars/hr

33. The second automobile has the better mileage per gallon.

35. 36 mi/hr

37. 14.8 miles per day

6.2 Introduction to Proportion

In Section 6.1, we introduced the concepts or ratio and rate. In this section, we equate these ratios in a construct called a *proportion*.

Proportions. A *proportion* is a statement that equates two ratios or rates.

For example, each of the equations

$$\frac{1}{3}=\frac{2}{6}, \qquad \frac{15\,\text{miles}}{2\,\text{hours}}=\frac{30\,\text{miles}}{4\,\text{hours}}, \qquad \text{and} \qquad \frac{a}{b}=\frac{c}{d}$$

compare two ratios or rates and is a proportion.

The proportion

$$\frac{1}{3}=\frac{2}{6}$$

is read "one is to three as two is to six." The four numbers that make up this proportion are called the *terms* of the proportion and are ordered in a natural manner.

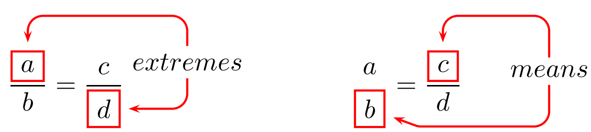
$$\begin{array}{r}\text{First term} \longrightarrow \\ \text{Second term} \longrightarrow\end{array} \frac{1}{3}=\frac{2}{6} \begin{array}{l}\longleftarrow \text{Third term} \\ \longleftarrow \text{Fourth term}\end{array}$$

Extremes and Means. The first and fourth terms are called the *extremes* of the proportion. The second and third terms are called the *means* of the proportion.

In the proportion

$$\frac{a}{b}=\frac{c}{d},$$

the terms a and d are the extremes; the terms b and c are the means.



If we multiply both sides of the proportion by the common denominator,

$$bd\left(\frac{a}{b}\right)=bd\left(\frac{c}{d}\right)$$

then cancel,

$$\cancel{b}d\left(\frac{a}{\cancel{b}}\right)=b\cancel{d}\left(\frac{c}{\cancel{d}}\right)$$

we get the following result.

$$ad=bc$$

This leads to the following observation.

Product of Extremes and Means. In the proportion

$$\frac{a}{b} = \frac{c}{d}$$

the product of the means equals the product of the extremes. That is,

$$ad = bc.$$

We can get an equivalent result using a technique called *cross multiplication.*

$$\frac{a}{b} = \frac{c}{d}$$

Product of means $= bc$

Product of extremes $= ad$

You Try It!

EXAMPLE 1. Which of the following is a valid proportion: (a) $\frac{2}{3} = \frac{7}{12}$, or (b) $\frac{4}{9} = \frac{12}{27}$.

Solution.

(a) Cross multiply

$$\frac{2}{3} = \frac{7}{12}$$

Product of means

Product of extremes

to get

$$24 = 21.$$

Hence, the product of the extremes does not equal the product of the means, so $2/3 = 7/12$ is **not** a valid proportion.

(b) Cross multiply

$$\frac{4}{9} = \frac{12}{27}$$

Product of means

Product of extremes

to get

$$108 = 108.$$

Hence, the product of the extremes equals the product of the means, so $4/9 = 12/27$ **is** a valid proportion.

Is the following a valid proportion?

$$\frac{4}{3} = \frac{16}{11}$$

Answer: No

□

Solving Proportions

We already have all the tools needed to solve proportions. Let's begin with the first example.

You Try It!

Solve the proportion for n:

$$\frac{2}{3} = \frac{n}{9}$$

EXAMPLE 2. Solve the proportion for x: $\dfrac{3}{4} = \dfrac{x}{12}$.

Solution. Cross multiply, then solve the resulting equation.

$\dfrac{3}{4} = \dfrac{x}{12}$	Original proportion.
$4 \cdot x = 3 \cdot 12$	Products of means and extremes are equal.
$4x = 36$	Simplify.
$\dfrac{4x}{4} = \dfrac{36}{4}$	Divide both sides by 4.
$x = 9$	Simplify.

Check. Substitute 9 for x into the original proportion and check.

$\dfrac{3}{4} = \dfrac{x}{12}$	Original proportion.
$\dfrac{3}{4} = \dfrac{9}{12}$	Substitute 9 for x.

Cross multiply.

Thus, the solution 9 checks.

Answer: 6

You Try It!

Solve the proportion for m:

$$\frac{9}{6} = \frac{m}{4}$$

EXAMPLE 3. Solve the proportion for n: $\dfrac{3}{2} = \dfrac{24}{n}$.

Solution. Cross multiply, then solve the resulting equation.

$\dfrac{3}{2} = \dfrac{24}{n}$	Original proportion.
$3 \cdot n = 2 \cdot 24$	Products of means and extremes are equal.
$3n = 48$	Simplify.
$\dfrac{3n}{3} = \dfrac{48}{3}$	Divide both sides by 3.
$n = 16$	Simplify.

Check. Substitute 16 for n into the original proportion and check.

$\frac{3}{2} = \frac{24}{n}$	Original proportion.
$\frac{3}{2} = \frac{24}{16}$	Substitute 16 for n.

Cross multiply.

$$\frac{3}{2} = \frac{24}{16}$$

Product of means $= 48$

Product of extremes $= 48$

Thus, the solution 16 checks.

Answer: 6

You Try It!

EXAMPLE 4. Solve the proportion for x: $\dfrac{2x+1}{15} = \dfrac{1}{3}$.

Solve the proportion for y:

$$\frac{6+2y}{18} = \frac{8}{9}$$

Solution. Cross multiply, then solve the resulting equation.

$\frac{2x+1}{15} = \frac{1}{3}$	Original proportion.
$3(2x+1) = 15(1)$	Products of means and extremes are equal.
$6x + 3 = 15$	On the left, distribute. On the right, multiply.
$6x + 3 - 3 = 15 - 3$	Subtract 3 from both sides.
$6x = 12$	Simplify.
$\frac{6x}{6} = \frac{12}{6}$	Divide both sides by 6.
$x = 2$	Simplify both sides.

Check. We'll leave it to our readers to check this solution.

Answer: 5

Applications

A number of practical applications involve solving a proportion.

You Try It!

EXAMPLE 5. If 5 oranges cost \$1.15, what will be the cost for 15 oranges (assuming an equal rate)?

If 7 apples cost \$3.15, how much will 10 apples cost (assuming an equal rate)?

Solution. Let x represent the cost for 15 oranges. Assuming the rate for 5 oranges at \$1.15 equals the rate for 15 oranges at an unknown cost x, we set up the following proportion.

$$\frac{5}{1.15} = \frac{15}{x}$$

Cross multiply

$$\frac{5}{1.15} = \frac{15}{x}$$

Product of means $= (1.15)(15)$

Product of extremes $= 5x$

to get

$$5x = 17.25.$$

Solve for x.

$$\frac{5x}{5} = \frac{17.25}{5}$$

$$x = 3.45$$

Answer: \$4.50

Thus, 15 oranges cost \$3.45.

Checking Units is Extremely Important. When setting up a proportion, check to make sure that both numerators have the same units and both denominators have the same units.

For example, in Example 5, both numerators have "oranges" as units and both denominators have "dollars" as units.

$$\begin{array}{r} \text{Oranges} \longrightarrow \\ \text{Dollars} \longrightarrow \end{array} \frac{5}{1.15} = \frac{15}{x} \begin{array}{l} \longleftarrow \text{Oranges} \\ \longleftarrow \text{Dollars} \end{array}$$

This proportion is set up correctly, because both numerators have the same units and both denominators have the same units.

On the other hand, if we had set the proportion up incorrectly as follows,

$$\begin{array}{r} \text{Oranges} \longrightarrow \\ \text{Dollars} \longrightarrow \end{array} \frac{5}{1.15} = \frac{x}{15} \begin{array}{l} \longleftarrow \text{Dollars} \\ \longleftarrow \text{Oranges} \end{array}$$

a quick check of the units reveals the error; i.e., the numerators have different units and the denominators have different units. *Checking units helps us avoid errors!*

You Try It!

Eloise and Susannah are planning a trip is Sequoia National Park. On their map, 3 inches represents 50 miles. How long is their trip if the route measures $4\frac{1}{2}$ inches on the map?

EXAMPLE 6. Dylan and David are planning a backpacking trip in Yosemite National Park. On their map, the legend indicates that 1.2 centimeters represents 2 miles. How long is their trip if the route measures 10.6 centimeters on the map? Round your answer to the nearest tenth of a mile.

Solution. Let's set up the proportion with units.

$$\frac{1.2\,\text{cm}}{2\,\text{mi}} = \frac{10.6\,\text{cm}}{x\,\text{mi}}$$

Note how including the units aids in the setup of the proportion. Now, let's drop the units and solve for x.

$\frac{1.2}{2} = \frac{10.6}{x}$	Original proportion.
$1.2x = (2)(10.6)$	Cross multiply.
$1.2x = 21.2$	Simplify right-hand side.
$\frac{1.2x}{1.2} = \frac{21.2}{1.2}$	Divide both sides by 1.2.
$x \approx 17.66$	On the right: Divide.

We carried the division on the right one decimal place past the tenths place. The rounding digit is a 6 and the following test digit is a 6. Add 1 to the rounding digit and truncate.

To the nearest tenth of a mile, the backpacking route is approximately 17.7 miles.

Answer: 75 miles

□

You Try It!

EXAMPLE 7. A recipe making 2 dozen cookies requires $1\frac{3}{4}$ cups of flour, among other ingredients. If the baker wishes to make twice that number of cookies, how much flour is required?

Dough for 3 pizzas requires $8\frac{1}{2}$ cups of flour. If the baker wishes to make 9 pizzas, how many cups of flour are required?

Solution. Twice 2 dozen is 4 dozen cookies. Let x represent the amount of flour needed for 4 dozen cookies. Assuming an equal rate for 2 dozen cookies (2 dozen requires $1\frac{3}{4}$ cups of flour), we set up the following proportion. Again, using units helps us craft the correct proportion.

$$\frac{2\,\text{dozen}}{1\frac{3}{4}\,\text{cups}} = \frac{4\,\text{dozen}}{x\,\text{cups}}$$

Note how including the units aids in the setup of the proportion. Now, let's drop the units and solve for x.

$\frac{2}{1\frac{3}{4}} = \frac{4}{x}$	Original proportion.
$2x = 1\frac{3}{4} \cdot 4$	Cross multiply.
$2x = \frac{7}{4} \cdot 4$	Change to improper fraction.
$2x = 7$	Multiply.

Divide both sides of the equation by 2 and finish.

$$\begin{aligned} \frac{2x}{2} &= \frac{7}{2} && \text{Divide both sides by 2.} \\ x &= \frac{7}{2} \end{aligned}$$

Change the improper fraction to a mixed fraction. Thus, it will take $3\frac{1}{2}$ cups of flour to make 4 dozen cookies.

Answer: $25\frac{1}{2}$ cups

□

Exercises

In Exercises 1-12, which of the following is a true proportion?

1. $\frac{9}{7}=\frac{27}{21}$, $\frac{4}{3}=\frac{9}{7}$, $\frac{7}{2}=\frac{8}{9}$, $\frac{4}{8}=\frac{9}{6}$

2. $\frac{6}{7}=\frac{18}{21}$, $\frac{2}{3}=\frac{8}{6}$, $\frac{4}{3}=\frac{3}{2}$, $\frac{8}{9}=\frac{3}{8}$

3. $\frac{7}{6}=\frac{28}{24}$, $\frac{5}{6}=\frac{5}{4}$, $\frac{9}{5}=\frac{7}{3}$, $\frac{9}{2}=\frac{8}{9}$

4. $\frac{7}{6}=\frac{2}{8}$, $\frac{4}{5}=\frac{5}{7}$, $\frac{3}{4}=\frac{15}{20}$, $\frac{8}{4}=\frac{8}{7}$

5. $\frac{6}{5}=\frac{24}{20}$, $\frac{7}{3}=\frac{2}{4}$, $\frac{2}{4}=\frac{2}{6}$, $\frac{5}{2}=\frac{2}{8}$

6. $\frac{9}{8}=\frac{4}{3}$, $\frac{5}{7}=\frac{10}{14}$, $\frac{8}{6}=\frac{5}{4}$, $\frac{8}{5}=\frac{2}{6}$

7. $\frac{3}{5}=\frac{2}{8}$, $\frac{3}{7}=\frac{6}{14}$, $\frac{5}{6}=\frac{2}{4}$, $\frac{7}{4}=\frac{5}{9}$

8. $\frac{7}{3}=\frac{7}{6}$, $\frac{4}{7}=\frac{8}{14}$, $\frac{5}{3}=\frac{7}{8}$, $\frac{5}{7}=\frac{6}{9}$

9. $\frac{5}{4}=\frac{25}{20}$, $\frac{9}{3}=\frac{9}{6}$, $\frac{7}{4}=\frac{3}{6}$, $\frac{3}{5}=\frac{9}{4}$

10. $\frac{7}{6}=\frac{6}{9}$, $\frac{7}{3}=\frac{2}{5}$, $\frac{6}{7}=\frac{30}{35}$, $\frac{4}{7}=\frac{2}{8}$

11. $\frac{9}{7}=\frac{4}{3}$, $\frac{9}{4}=\frac{7}{9}$, $\frac{3}{5}=\frac{6}{10}$, $\frac{3}{9}=\frac{9}{5}$

12. $\frac{4}{3}=\frac{8}{7}$, $\frac{2}{6}=\frac{5}{8}$, $\frac{7}{2}=\frac{3}{6}$, $\frac{9}{4}=\frac{36}{16}$

In Exercises 13-36, solve the given proportion.

13. $\frac{17}{3}=\frac{x}{18}$

14. $\frac{16}{5}=\frac{x}{20}$

15. $\frac{6x+10}{6}=\frac{11}{3}$

16. $\frac{4x+8}{12}=\frac{5}{3}$

17. $\frac{17}{9}=\frac{x}{18}$

18. $\frac{8}{9}=\frac{x}{18}$

19. $\frac{11}{2}=\frac{x}{8}$

20. $\frac{11}{4}=\frac{x}{8}$

21. $\frac{7x+15}{15}=\frac{10}{3}$

22. $\frac{7x+3}{8}=\frac{5}{4}$

23. $\frac{11}{2}=\frac{x}{10}$

24. $\frac{19}{6}=\frac{x}{18}$

25. $\frac{5x+8}{12}=\frac{2}{3}$

26. $\frac{3x+12}{6}=\frac{3}{2}$

27. $\frac{2}{15}=\frac{24}{x}$

28. $\frac{7}{8}=\frac{14}{x}$

29. $\frac{3}{16}=\frac{6}{x}$

30. $\frac{4}{21}=\frac{12}{x}$

31. $\frac{5}{22}=\frac{20}{x}$

32. $\frac{3}{22}=\frac{21}{x}$

33. $\frac{2x+10}{6}=\frac{14}{3}$

34. $\dfrac{2x+9}{9} = \dfrac{13}{3}$

35. $\dfrac{7}{2} = \dfrac{21}{x}$

36. $\dfrac{2}{15} = \dfrac{18}{x}$

---

37. If 13 dog bones cost $1.97, what will be the cost for 7 dog bones (assuming an equal rate)? Round your answer to the nearest penny.

38. If 2 watermelons cost $3.89, what will be the cost for 11 watermelons (assuming an equal rate)? Round your answer to the nearest penny.

39. If 7 bananas cost $2.55, what will be the cost for 14 bananas (assuming an equal rate)? Round your answer to the nearest penny.

40. If 2 apples cost $2.05, what will be the cost for 8 apples (assuming an equal rate)? Round your answer to the nearest penny.

41. If 13 oranges cost $3.61, what will be the cost for 11 oranges (assuming an equal rate)? Round your answer to the nearest penny.

42. If 3 watermelons cost $1.05, what will be the cost for 9 watermelons (assuming an equal rate)? Round your answer to the nearest penny.

43. If 3 dog bones cost $1.23, what will be the cost for 13 dog bones (assuming an equal rate)? Round your answer to the nearest penny.

44. If 3 watermelons cost $4.41, what will be the cost for 7 watermelons (assuming an equal rate)? Round your answer to the nearest penny.

45. If 3 apples cost $3.24, what will be the cost for 13 apples (assuming an equal rate)? Round your answer to the nearest penny.

46. If 6 apples cost $3.43, what will be the cost for 7 apples (assuming an equal rate)? Round your answer to the nearest penny.

47. If 4 dog bones cost $1.03, what will be the cost for 8 dog bones (assuming an equal rate)? Round your answer to the nearest penny.

48. If 4 oranges cost $4.28, what will be the cost for 3 oranges (assuming an equal rate)? Round your answer to the nearest penny.

---

49. **Two rolls.** In Haiti, two flat rolls cost 5 gourdes, about 12 cents. How many cents would 20 rolls cost? *Associated Press-Times-Standard 02/18/10 Haiti's earthquake camps turning into shanty towns.*

50. **Turbines.** As proposed, the Shell Wind Energy project consists of 25 ridge-top turbines that can generate up to 50 megawatts, or enough to supply electricity to about 1,000 homes. Estimate the number of ridge-top turbines that would be needed to supply electricity to 70,000 homes, the approximate number of properties in Humboldt County, CA. *John Driscoll Times-Standard 12/24/09 Wind power project goes under analysis.*

51. **Dumptrucks.** U.S. Highway 199 had a landslide where as much as 3,000 cubic yards of material fell on the road, reportedly requiring about 200 large dumptrucks to remove. Only a week earlier, 40,000 cubic yards of material fell on Highway 96. Estimate the number of dumptrucks needed for that slide rounded to the nearest whole number. *Associated Press-Times-Standard 03/09/10 Another highway closed by slide.*

52. **Timber sales.** Alaska's 26,000 square mile Tongass National Forest plan allows for timber sales of up to 267 million board-feet per year – enough for nearly 27,000 two-bedroom homes, but demand for timber is far short of that. Less than 25 million board-feet was logged in the forest in 2009. Forest Service officials have said they hope to increase logging in the Tongass to about 100 million board-feet per year. *Associated Press-Times-Standard 02/18/10 Industry loses lawsuit over logging in Alaska.*

i) Estimate the number of 2-bedroom homes that 25 million board-feet of timber would build.

ii) How many 2-bedroom homes would 100 million board-feet of timber build?

53. **Costly spill.** In Australia, penalties on ships causing oil spills are approximately $1.75 million Australian dollars, equivalent to about $1.64 million US dollars. After an oil tanker was grounded onto a coral reef, Australian officials are considering raising the fine to $10 million Australian dollars. What will the new fine be in US dollars? Round your answer to the nearest hundredth of a million dollars. *Associated Press-Times-Standard 04/13/10 Ship that leaked oil on Great Barrier Reef removed.*

Answers

1. $\dfrac{9}{7} = \dfrac{27}{21}$ is a proportion

3. $\dfrac{7}{6} = \dfrac{28}{24}$ is a proportion

5. $\dfrac{6}{5} = \dfrac{24}{20}$ is a proportion

7. $\dfrac{3}{7} = \dfrac{6}{14}$ is a proportion

9. $\dfrac{5}{4} = \dfrac{25}{20}$ is a proportion

11. $\dfrac{3}{5} = \dfrac{6}{10}$ is a proportion

13. 102

15. 2

17. 34

19. 44

21. 5

23. 55

25. 0

27. 180

29. 32

31. 88

33. 9

35. 6

37. $1.06

39. \$5.10

41. \$3.05

43. \$5.33

45. \$14.04

47. \$2.06

49. \$0.48

51. $2,667$ loads

53. \$9.37 million US dollars

6.3 Unit Conversion: American System

In this section we will develop a technique for converting units used in the American system. We begin with a discussion of common measurements of length in the United States.

Units of Length

The most common units of length are the inch, foot, yard, and mile. Our focus will be on the technique used to convert from one unit of length to another.

American Units of Length. Facts relating common units of length.

$$1\,\text{foot (ft)} = 12\,\text{inches (in)} \qquad 1\,\text{yard (yd)} = 3\,\text{feet (ft)}$$
$$1\,\text{mile (mi)} = 5280\,\text{feet (ft)}$$

Take for example, the fact that there are 3 feet in 1 yard, which can be stated as an equation, using the common abbreviations for feet (ft) and yards (yd).

$$3\,\text{ft} = 1\,\text{yd}$$

If we divide both sides of the equation by 3 ft,

$$\frac{3\,\text{ft}}{3\,\text{ft}} = \frac{1\,\text{yd}}{3\,\text{ft}},$$

or equivalently,

$$1 = \frac{1\,\text{yd}}{3\,\text{ft}}.$$

The key observation is the fact that the ratio $1\,\text{yd}/3\,\text{ft}$ equals the number 1. Consequently, multiplying by the "conversion factor" $1\,\text{yd}/3\,\text{ft}$ is equivalent to multiplying by 1. This can be used to change a measurement in feet to yards.

You Try It!

EXAMPLE 1. Change 36 feet to yards.

Change 81 feet to yards.

Solution. Multiply by the conversion factor 1 yd/3 ft.

$$\begin{aligned}
36\,\text{ft} &= 36\,\text{ft}\cdot 1 && \text{Multiplicative Identity Property.}\\
&= 36\,\text{ft}\cdot\frac{1\,\text{yd}}{3\,\text{ft}} && \text{Replace 1 with 1 yd/3 ft.}\\
&= 36\,\cancel{\text{ft}}\cdot\frac{1\,\text{yd}}{3\,\cancel{\text{ft}}} && \text{Cancel common unit.}\\
&= \frac{36\cdot 1}{3}\,\text{yd} && \text{Multiply fractions.}\\
&= \frac{36}{3}\,\text{yd} && \text{Simplify.}\\
&= 12\,\text{yd} && \text{Divide.}
\end{aligned}$$

Answer: 27 yards

Hence, 36 feet equals 12 yards.

On the other hand, we can start again with

$$3\,\text{ft} = 1\,\text{yd}$$

and divide both sides of the equation by 1 yd.

$$\frac{3\,\text{ft}}{1\,\text{yd}} = \frac{1\,\text{yd}}{1\,\text{yd}}$$

This gives the conversion factor

$$\frac{3\,\text{ft}}{1\,\text{yd}} = 1.$$

The key observation is the fact that the ratio 3 ft/1 yd equals the number 1. Consequently, multiplying by the "conversion factor" 3 ft/1 yd is equivalent to multiplying by 1. This can be used to change a measurement in yards to feet.

You Try It!

Change 15 yards to feet.

EXAMPLE 2. Change 18 yards to feet.

Solution. Multiply by the conversion factor 3 ft/1 yd.

$$\begin{aligned}
18\,\text{yd} &= 18\,\text{yd}\cdot 1 && \text{Multiplicative Identity Property.}\\
&= 18\,\text{yd}\cdot\frac{3\,\text{ft}}{1\,\text{yd}} && \text{Replace 1 with 3 ft/1 yd.}\\
&= 18\,\cancel{\text{yd}}\cdot\frac{3\,\text{ft}}{1\,\cancel{\text{yd}}} && \text{Cancel common unit.}\\
&= \frac{18\cdot 3}{1}\,\text{ft} && \text{Multiply fractions.}\\
&= 54\,\text{ft} && \text{Simplify.}
\end{aligned}$$

Hence, 18 yards equals 54 feet.

Answer: 45 feet

Another common comparison is the fact that there are 12 inches in 1 foot. This can be represented as an equation using the common abbreviation for inches (in) and feet (ft).

$$12\,\mathrm{in} = 1\,\mathrm{ft}$$

Dividing both sides by 12 in

$$\frac{12\,\mathrm{in}}{12\,\mathrm{in}} = \frac{1\,\mathrm{ft}}{12\,\mathrm{in}},$$

yields the conversion factor

$$1 = \frac{1\,\mathrm{ft}}{12\,\mathrm{in}}.$$

The key observation is the fact that the ratio 1 ft/12 in equals the number 1. Consequently, multiplying by the "conversion factor" 1 ft/12 in is equivalent to multiplying by 1. This can be used to change a measurement in inches to feet.

You Try It!

EXAMPLE 3. Change 24 inches to feet.

Change 48 inches to feet.

Solution. Multiply by the conversion factor 1 ft/12 in.

$$\begin{aligned}
24\,\mathrm{in} &= 24\,\mathrm{in}\cdot 1 && \text{Multiplicative Identity Property.}\\
&= 24\,\mathrm{in}\cdot\frac{1\,\mathrm{ft}}{12\,\mathrm{in}} && \text{Replace 1 with 1 ft/12 in.}\\
&= 24\,\cancel{\mathrm{in}}\cdot\frac{1\,\mathrm{ft}}{12\,\cancel{\mathrm{in}}} && \text{Cancel common unit.}\\
&= \frac{24\cdot 1}{12}\,\mathrm{ft} && \text{Multiply fractions.}\\
&= 2\,\mathrm{ft} && \text{Simplify.}
\end{aligned}$$

Hence, 24 inches equals 2 feet.

Answer: 4 feet

We provide a summary of conversion factors for units of length in Table 6.1.

Convert	Conversion Factor	Convert	Conversion Factor
feet to inches	12 in/1 ft	inches to feet	1 ft/12 in
yards to feet	3 ft/1 yd	feet to yards	1 yd/3 ft
miles to feet	5280 ft/1 mi	feet to miles	1 mi/5280 ft

Table 6.1: Conversion factors for units of length.

Some conversions require more than one application of a conversion factor.

You Try It!

Change 8 yards to inches.

EXAMPLE 4. Change 4 yards to inches.

Solution. We multiply by a chain of conversion factors, the first to change yards to feet, the second to change feet to inches.

$$\begin{aligned} 4\,\text{yd} &= 4\,\text{yd} \cdot \frac{3\,\text{ft}}{1\,\text{yd}} \cdot \frac{12\,\text{in}}{1\,\text{ft}} && \text{Multiply by conversion factors.} \\ &= 4\,\cancel{\text{yd}} \cdot \frac{3\,\cancel{\text{ft}}}{1\,\cancel{\text{yd}}} \cdot \frac{12\,\text{in}}{1\,\cancel{\text{ft}}} && \text{Cancel common units.} \\ &= \frac{4 \cdot 3 \cdot 12}{1 \cdot 1}\,\text{in} && \text{Multiply fractions.} \\ &= 144\,\text{in} && \text{Simplify.} \end{aligned}$$

Hence, 4 yards equals 144 inches.

Answer: 288 inches

You Try It!

Change 5 miles to yards.

EXAMPLE 5. Change 2 miles to yards.

Solution. We multiply by a chain of conversion factors, the first to change miles to feet, the second to change feet to yards.

$$\begin{aligned} 2\,\text{mi} &= 2\,\text{mi} \cdot \frac{5280\,\text{ft}}{1\,\text{mi}} \cdot \frac{1\,\text{yd}}{3\,\text{ft}} && \text{Multiply by conversion factors.} \\ &= 2\,\cancel{\text{mi}} \cdot \frac{5280\,\cancel{\text{ft}}}{1\,\cancel{\text{mi}}} \cdot \frac{1\,\text{yd}}{3\,\cancel{\text{ft}}} && \text{Cancel common units.} \\ &= \frac{2 \cdot 5280 \cdot 1}{1 \cdot 3}\,\text{yd} && \text{Multiply fractions.} \\ &= 3520\,\text{yd} && \text{Simplify.} \end{aligned}$$

Hence, 2 miles equals 3,520 yards.

Answer: 8,800 yards

Units of Weight

The most common units of weight are the ounce, pound, and ton. Our focus will remain on how to convert from one unit to another.

American Units of Weight. Facts relating common units of weight.

1 pound (lb) = 16 ounces (oz) 1 ton = 2000 pounds (lb)

The above facts lead to the conversion factors in Table 6.2.

Convert	Conversion Factor	Convert	Conversion Factor
pounds to ounces	16 oz/1 lb	ounces to pounds	1 lb/16 oz
tons to pounds	2000 lb/1 ton	pounds to tons	1 ton/2000 lb

Table 6.2: Conversion factors for units of weight.

You Try It!

EXAMPLE 6. Change $2\frac{1}{2}$ pounds to ounces.

Change $6\frac{1}{4}$ pounds to ounces.

Solution. Multiply by the appropriate conversion factor.

$$\begin{aligned} 2\frac{1}{2}\,\text{lb} &= 2\frac{1}{2}\,\text{lb} \cdot \frac{16\,\text{oz}}{1\,\text{lb}} && \text{Multiply by conversion factor.} \\ &= 2\frac{1}{2}\,\cancel{\text{lb}} \cdot \frac{16\,\text{oz}}{1\,\cancel{\text{lb}}} && \text{Cancel common units.} \\ &= \left(2\frac{1}{2} \cdot 16\right)\,\text{oz} && \text{Multiply fractions.} \\ &= \left(\frac{5}{2} \cdot 16\right)\,\text{oz} && \text{Mixed to improper fraction.} \\ &= \frac{80}{2}\,\text{oz} && \text{Multiply.} \\ &= 40\,\text{oz} && \text{Divide.} \end{aligned}$$

Hence, $2\frac{1}{2}$ pounds equals 40 ounces.

Answer: 100 ounces

□

You Try It!

EXAMPLE 7. Change 3.2 tons to ounces.

Change 4.1 tons to ounces.

Solution. This exercise requires multiplying by a chain of conversion factors.

$$\begin{aligned} 3.2\,\text{ton} &= 3.2\,\text{ton} \cdot \frac{2000\,\text{lb}}{1\,\text{ton}} \cdot \frac{16\,\text{oz}}{1\,\text{lb}} && \text{Multiply by conversion factors.} \\ &= 3.2\,\cancel{\text{ton}} \cdot \frac{2000\,\cancel{\text{lb}}}{1\,\cancel{\text{ton}}} \cdot \frac{16\,\text{oz}}{1\,\cancel{\text{lb}}} && \text{Cancel common units.} \\ &= \frac{3.2 \cdot 2000 \cdot 16}{1 \cdot 1}\,\text{oz} && \text{Multiply fractions.} \\ &= 102,400\,\text{oz} && \text{Simplify.} \end{aligned}$$

Hence, 3.2 tons equals 102,400 ounces.

Answer: 128,000 ounces

□

Units of Volume

The most common units of volume are fluid ounces, cups, pints, quarts, and gallons. We will focus on converting from one unit to another.

American Units of Volume. Facts relating common units of volume.

$$1\,\text{cup (c)} = 8\,\text{fluid ounces (fl oz)} \qquad 1\,\text{pint (pt)} = 2\,\text{cups (c)}$$
$$1\,\text{quart (qt)} = 2\,\text{pints (pt)} \qquad 1\,\text{gallon (gal)} = 4\,\text{quarts (qt)}$$

These facts lead to the conversion factors listed in Table 6.3.

Convert	Conversion Factor	Convert	Conversion Factor
cups to ounces	8 fl oz/1 c	ounces to cups	1 c/8 fl oz
pints to cups	2 c/1 pt	cups to pints	1 pt/2 c
quarts to pints	2 pt/1 qt	pints to quarts	1 qt/2 pt
gallons to quarts	4 qt/1 gal	quarts to gallons	1 gal/4 qt

Table 6.3: Conversion factors for units of volume.

You Try It!

Change 3.2 gallons to pints.

EXAMPLE 8. Change 5.6 gallons to pints.

Solution. This exercise requires multiplying by a chain of conversion factors.

$$\begin{aligned}
5.6\,\text{gal} &= 5.6\,\text{gal} \cdot \frac{4\,\text{qt}}{1\,\text{gal}} \cdot \frac{2\,\text{pt}}{1\,\text{qt}} && \text{Multiply by conversion factors.} \\
&= 5.6\,\cancel{\text{gal}} \cdot \frac{4\,\cancel{\text{qt}}}{1\,\cancel{\text{gal}}} \cdot \frac{2\,\text{pt}}{1\,\cancel{\text{qt}}} && \text{Cancel common units.} \\
&= \frac{5.6 \cdot 4 \cdot 2}{1 \cdot 1}\,\text{pt} && \text{Multiply fractions.} \\
&= 44.8\,\text{pt} && \text{Simiplify.}
\end{aligned}$$

Hence, 5.6 gallons equals 44.8 pints.

Answer: 25.6 pints

□

Units of Time

The most common units of time are seconds, minutes, hours, days, and years.

American Units of Time. Facts relating common units of time.

$$1\,\text{minute (min)} = 60\,\text{seconds (s)} \qquad 1\,\text{hour (hr)} = 60\,\text{minutes (min)}$$
$$1\,\text{day (day)} = 24\,\text{hours (hr)} \qquad 1\,\text{year (yr)} = 365\,\text{days (day)}$$

These facts lead to the conversion factors in Table 6.4.

Convert	Conversion Factor	Convert	Conversion Factor
minutes to seconds	60 s/1 min	seconds to minutes	1 min/60 s
hours to minutes	60 min/1 hr	minutes to hours	1 hr/60 min
days to hours	24 hr/1 day	hours to days	1 day/24 hr
years to days	365 day/1 yr	days to years	1 yr/365 day

Table 6.4: Conversion factors for units of time.

You Try It!

EXAMPLE 9. How many seconds in a year?

How many seconds in a day?

Solution. A chain of conversion factors is needed.

$$\begin{aligned}
1\,\text{yr} &= 1\,\text{yr} \cdot \frac{365\,\text{day}}{1\,\text{yr}} \cdot \frac{24\,\text{hr}}{1\,\text{day}} \cdot \frac{60\,\text{min}}{1\,\text{hr}} \cdot \frac{60\,\text{s}}{1\,\text{min}} && \text{Conversion factors.}\\
&= 1\,\cancel{\text{yr}} \cdot \frac{365\,\cancel{\text{day}}}{1\,\cancel{\text{yr}}} \cdot \frac{24\,\cancel{\text{hr}}}{1\,\cancel{\text{day}}} \cdot \frac{60\,\cancel{\text{min}}}{1\,\cancel{\text{hr}}} \cdot \frac{60\,\text{s}}{1\,\cancel{\text{min}}} && \text{Cancel common units.}\\
&= \frac{1 \cdot 365 \cdot 24 \cdot 60 \cdot 60}{1 \cdot 1 \cdot 1 \cdot 1}\,\text{s} && \text{Multiply fractions.}\\
&= 31{,}536{,}000\,\text{s} && \text{Simplify.}
\end{aligned}$$

Thus, 1 year equals 31,536,000 seconds.

Answer: 86,400 seconds

□

Converting Units of Speed

Ever wonder how fast a baseball is moving?

You Try It!

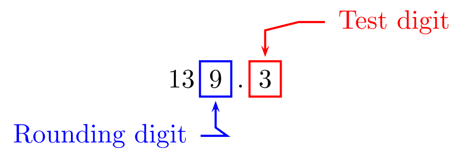
EXAMPLE 10. A professional pitcher can throw a baseball at 95 miles per hour. How fast is this in feet per second? Round your answer to the nearest foot per second.

A women's softball pitcher can throw her fastball at 60 miles per hour. How fast is this in feet per second? Round your answer to the nearest foot per second.

Solution. There are 5280 feet in a mile, 60 minutes in an hour, and 60 seconds in a minute.

$$\begin{aligned}
95\,\frac{\text{mi}}{\text{h}} &\approx 95\,\frac{\text{mi}}{\text{h}} \cdot \frac{5280\,\text{ft}}{1\,\text{mi}} \cdot \frac{1\,\text{h}}{60\,\text{min}} \cdot \frac{1\,\text{min}}{60\,\text{s}} && \text{Conversion factors.}\\
&\approx 95\,\frac{\cancel{\text{mi}}}{\cancel{\text{h}}} \cdot \frac{5280\,\text{ft}}{1\,\cancel{\text{mi}}} \cdot \frac{1\,\cancel{\text{h}}}{60\,\cancel{\text{min}}} \cdot \frac{1\,\cancel{\text{min}}}{60\,\text{s}} && \text{Cancel common units.}\\
&\approx \frac{95 \cdot 5280 \cdot 1 \cdot 1}{1 \cdot 60 \cdot 60}\,\frac{\text{ft}}{\text{s}} && \text{Multiply fractions.}\\
&\approx 139.3\,\frac{\text{ft}}{\text{s}} && \text{Multiply and divide.}
\end{aligned}$$

To round to the nearest foot per second, identify the rounding and test digits.



Because the test digit is less than 5, leave the rounding digit alone and truncate. Thus, to the nearest foot per second, the speed is approximately 139 feet per second.

Whew! Since the batter stands at home plate, which is about 60 feet from where the pitch is delivered, the batter has less than 1/2 a second to react to the pitch!

Answer: 88 feet per second

□

Exercises

1. Change 8 yards to feet.

2. Change 60 yards to feet.

3. Change 261 feet to yards.

4. Change 126 feet to yards.

5. Change 235 inches to yards. Round your answer to the nearest tenth of a yard.

6. Change 244 inches to yards. Round your answer to the nearest tenth of a yard.

7. Change 141 feet to yards.

8. Change 78 feet to yards.

9. Change 2.8 miles to feet.

10. Change 4.9 miles to feet.

11. Change 104 inches to yards. Round your answer to the nearest tenth of a yard.

12. Change 101 inches to yards. Round your answer to the nearest tenth of a yard.

13. Change 168,372 inches to miles, correct to the nearest tenth of a mile.

14. Change 198,550 inches to miles, correct to the nearest tenth of a mile.

15. Change 82 feet to inches.

16. Change 80 feet to inches.

17. Change 2.9 yards to inches. Round your answer to the nearest inch.

18. Change 4.5 yards to inches. Round your answer to the nearest inch.

19. Change 25,756 feet to miles. Round your answer to the nearest tenth of a mile.

20. Change 19,257 feet to miles. Round your answer to the nearest tenth of a mile.

21. Change 5 yards to feet.

22. Change 20 yards to feet.

23. Change 169,312 inches to miles, correct to the nearest tenth of a mile.

24. Change 162,211 inches to miles, correct to the nearest tenth of a mile.

25. Change 1.5 yards to inches. Round your answer to the nearest inch.

26. Change 2.1 yards to inches. Round your answer to the nearest inch.

27. Change 360 inches to feet.

28. Change 768 inches to feet.

29. Change 48 inches to feet.

30. Change 528 inches to feet.

31. Change 15,363 feet to miles. Round your answer to the nearest tenth of a mile.

32. Change 8,540 feet to miles. Round your answer to the nearest tenth of a mile.

33. Change 1.7 miles to inches.

34. Change 4.7 miles to inches.

35. Change 3.1 miles to inches.

36. Change 1.8 miles to inches.

37. Change 3.6 miles to feet.

38. Change 3.1 miles to feet.

39. Change 18 feet to inches.

40. Change 33 feet to inches.

41. Change $5\,\frac{1}{8}$ pounds to ounces.

42. Change $3\,\frac{1}{16}$ pounds to ounces.

43. Change 2.4 tons to ounces.

44. Change 3.4 tons to ounces.

45. Change 34 ounces to pounds. Express your answer as a fraction reduced to lowest terms.

46. Change 78 ounces to pounds. Express your answer as a fraction reduced to lowest terms.

47. Change 2.2 tons to pounds.

48. Change 4.8 tons to pounds.

49. Change 70 ounces to pounds. Express your answer as a fraction reduced to lowest terms.

50. Change 20 ounces to pounds. Express your answer as a fraction reduced to lowest terms.

51. Change 9,560 pounds to tons. Round your answer to the nearest tenth of a ton.

52. Change 9,499 pounds to tons. Round your answer to the nearest tenth of a ton.

53. Change $2\,\frac{1}{2}$ pounds to ounces.

54. Change $7\,\frac{1}{16}$ pounds to ounces.

55. Change 5.9 tons to pounds.

56. Change 2.1 tons to pounds.

57. Change 2.5 tons to ounces.

58. Change 5.3 tons to ounces.

59. Change 8,111 pounds to tons. Round your answer to the nearest tenth of a ton.

60. Change 8,273 pounds to tons. Round your answer to the nearest tenth of a ton.

---

61. Change 4.5625 pints to fluid ounces.

62. Change 2.9375 pints to fluid ounces.

63. Change 32 fluid ounces to pints.

64. Change 160 fluid ounces to pints.

65. Change 3.7 gallons to pints.

66. Change 2.4 gallons to pints.

67. Change 216 pints to gallons.

68. Change 96 pints to gallons.

69. Change 544 fluid ounces to pints.

70. Change 432 fluid ounces to pints.

71. Change 112 pints to gallons.

72. Change 200 pints to gallons.

73. Change 7.7 gallons to pints.

74. Change 5.7 gallons to pints.

75. Change 3.875 pints to fluid ounces.

76. Change 3 pints to fluid ounces.

---

77. Change 7.8 years to hours.

78. Change 4.7 years to hours.

79. Change 7.6 years to hours.

80. Change 6.6 years to hours.

81. Change 4,025,005 seconds to days. Round your answer to the nearest tenth of a day.

82. Change 4,672,133 seconds to days. Round your answer to the nearest tenth of a day.

83. Change 37,668 hours to years.

84. Change 40,296 hours to years.

85. Change 22,776 hours to years.

86. Change 29,784 hours to years.

87. Change 96 days to seconds.

88. Change 50 days to seconds.

89. Change 40 days to seconds.

90. Change 10 days to seconds.

91. Change 3,750,580 seconds to days. Round your answer to the nearest tenth of a day.

92. Change 4,493,469 seconds to days. Round your answer to the nearest tenth of a day.

---

93. Change 367 feet per second to miles per hour. Round your answer to the nearest mile per hour.

94. Change 354 feet per second to miles per hour. Round your answer to the nearest mile per hour.

95. Change 442 feet per second to miles per hour. Round your answer to the nearest mile per hour.

96. Change 388 feet per second to miles per hour. Round your answer to the nearest mile per hour.

97. Change 30 miles per hour to feet per second. Round your answer to the nearest foot per second.

98. Change 99 miles per hour to feet per second. Round your answer to the nearest foot per second.

99. Change 106 miles per hour to feet per second. Round your answer to the nearest foot per second.

100. Change 119 miles per hour to feet per second. Round your answer to the nearest foot per second.

---

101. **Strong man.** Famed strongman Joe Rollino, who was still bending quarters with his fingers at age 104, once lifted 3,200 pounds at Coney Island Amusement Park. How many tons did Joe lift that day? *Associated Press-Times-Standard 01/12/10 NYC amusement park strongman, 104, killed by van.*

102. **Earth day.** The amount of time it takes the Earth to rotate once around its axis is one day. How many seconds is that?

103. **Water break.** "The average age of Washington, DC's water pipes is 76 years, and they are not alone. Every two minutes, somewhere in the country, a pipe breaks." How many pipes break each year in the US? *New York Times 03/14/10 Saving U.S. water and sewer systems could be costly.*

❧ ❧ ❧ Answers ☙ ☙ ☙

1. 24 feet

3. 87 yards

5. 6.5 yards

7. 47 yards

9. 14,784 feet

11. 2.9 yards

13. 2.7 miles

15. 984 inches

17. 104 inches

19. 4.9 miles

21. 15 feet

23. 2.7 miles

25. 54 inches

27. 30 feet

29. 4 feet

31. 2.9 miles

33. 107,712 inches

35. 196,416 inches

37. 19,008 feet

39. 216 inches

41. 82 ounces

43. 76,800 ounces

45. $2\frac{1}{8}$ pounds

47. 4,400 pounds

49. $4\frac{3}{8}$ pounds

51. 4.8 tons

53. 40 ounces

55. 11,800 pounds

57. 80,000 ounces

59. 4.1 tons

61. 73 fluid ounces

63. 2 pints

65. 29.6 pints

67. 27 gallons

69. 34 pints

71. 14 gallons

73. 61.6 pints

75. 62 fluid ounces

77. 68,328 hours

79. 66,576 hours

81. 46.6 days

83. 4.3 years

85. 2.6 years

87. 8,294,400 seconds

89. 3,456,000 seconds

91. 43.4 days

93. 250 mi/hr

95. 301 mi/hr

97. 44 ft/s

99. 155 ft/s

101. 1.6 tons

103. $262,800$

6.4 Unit Conversion: Metric System

The metric system of units is the standard system of units preferred by scientists. It is based on the base ten number system and its decimal format is more friendly to users of this system.

There is a common set of prefixes adopted by the metric system to indicate a power of ten to apply to the base unit.

Metric System Prefixes. This is a list of standard prefixes for the metric system and their meanings.

deka = 10	deci = 1/10
hecto = 100	centi = 1/100
kilo = 1000	milli = 1/1000

Thus, for example, a decameter is 10 meters, a hectoliter is 100 liters, and a kilogram is 1000 grams. Similarly, a decimeter is 1/10 of a meter, a centiliter is 1/100 of a liter, and a milligram is 1/1000 of a gram.

Units of Length

The standard measure of length in the metric system is the meter.

> *Historically, the meter was defined by the French Academy of Sciences as the length between two marks on a platinum-iridium bar, which was designed to represent 110,000,000 of the distance from the equator to the north pole through Paris. In 1983, it was redefined by the International Bureau of Weights and Measures (BIPM) as the distance travelled by light in free space in 1299,792,458 of a second. (Wikipedia)*

We can apply the standard prefixes to get the following result.

Metric Units of Length. These units of length are used in the metric system.

Unit Length	Unit Abbreviation
1 kilometer = 1000 meters	km
1 hectometer = 100 meters	hm
1 dekameter = 10 meters	dam
1 meter	m
1 decimeter = $\frac{1}{10}$ meter	dm
1 centimeter = $\frac{1}{100}$ meter	cm
1 millimeter = $\frac{1}{1000}$ meter	mm

We can use these facts to build conversion factors as we did in Section 6.3. For example, because

$$1\,\mathrm{km} = 1000\,\mathrm{m},$$

we can divide both sides by 1000 m to produce the conversion factor

$$1 = \frac{1\,\mathrm{km}}{1000\,\mathrm{m}}.$$

This conversion factor can help change meters into kilometers.

Before using this conversion factor in an example, we repeat here the rules for multiplying and dividing by powers of ten. We will be making heavy use of these rules in this section.

Multiplying and Dividing by Powers of Ten.

- Multiplying a decimal number by 10^n will move the decimal point n places to the right. For example, $3.2567 \cdot 10^2 = 3.2567 \cdot 100 = 325.67$.
- Dividing a decimal number by 10^n will move the decimal point n places to the left. For example, $3.2567/10^2 = 3.2567/100 = 0.032567$.

And now the example.

You Try It!

Change 1,156 meters to kilometers.

EXAMPLE 1. Change 2,326 meters to kilometers.

Solution. Multiply by the conversion factor 1 km/1000 m.

$$\begin{aligned}
2326\,\mathrm{m} &= 2326\,\mathrm{m} \cdot \frac{1\,\mathrm{km}}{1000\,\mathrm{m}} && \text{Apply conversion factor.}\\
&= 2326\,\cancel{\mathrm{m}} \cdot \frac{1\,\mathrm{km}}{1000\,\cancel{\mathrm{m}}} && \text{Cancel common units.}\\
&= \frac{2326 \cdot 1}{1000}\,\mathrm{km} && \text{Multiply fractions.}\\
&= 2.326\,\mathrm{km} && \text{Simplify.}
\end{aligned}$$

In the last step, note that dividing by 1000 moves the decimal point three places to the left. Thus, 2326 meters is equal to 2.326 kilometers.

Alternate Solution. A second solution depends upon the fact that multiplying or dividing by a power of ten will move the decimal point right or left a number of places equal to the number of zeros present in the multiplier or divisor. Thus, as we saw above, dividing by 1000 moved the decimal point 3 places to the left.

Suppose that we arrange the metric units of length in order, from largest to smallest, as shown below.

km hm dam m dm cm mm

Note that we must move 3 places left to move from the meters (m) abbreviation to the kilometers (km) abbreviation. In like manner, if we write 2,326 meters as 2,326.0 meters, then we can convert to kilometers by moving the decimal 3 places to the left.

$$2,326.0\,\text{m} = 2.236\,0\,\text{km} = 2.326\,\text{km}$$

Answer: 1.156 kilometers

You Try It!

Change 276 centimeters to meters.

EXAMPLE 2. Change 537 centimeters to meters.

Solution. We know that

$$1\,\text{cm} = \frac{1}{100}\,\text{m},$$

or multiplying both sides of this result by 100,

$$100\,\text{cm} = 1\,\text{m}.$$

Dividing both sides of this last result by 100 cm, we obtain the conversion factor 1 m/100 cm.

$$\begin{aligned} 537\,\text{cm} &= 537\,\text{cm} \cdot \frac{1\,\text{m}}{100\,\text{cm}} && \text{Apply conversion factor.} \\ &= 537\,\cancel{\text{cm}} \cdot \frac{1\,\text{m}}{100\,\cancel{\text{cm}}} && \text{Cancel common units.} \\ &= \frac{537 \cdot 1}{100}\,\text{m} && \text{Multiply fractions.} \\ &= 5.37\,\text{m} \end{aligned}$$

In the last step, note that dividing by 100 moves the decimal point two places to the left.

Alternately, we can set up our ordered list of units.

km hm dam m dm cm mm

Note that we must move 2 places left to move from the centimeters (cm) abbreviation to the meters (m) abbreviation. In like manner, if we write 537 centimeters as 537.0 centimeters, then we can convert to meters by moving the decimal 2 places to the left.

$$537.0\,\text{cm} = 5.37\,0\,\text{m} = 5.37\,\text{m}$$

Answer: 2.76 meters

Sometimes more than one conversion factor is needed.

You Try It!

Change 13.5 dekameters to centimeters.

EXAMPLE 3. Change 10.2 dekameters to centimeters.

Solution. We have two facts:

- 1 dam=10 m, which yields the conversion factor 10 m/1 dam.
- 1 cm=(1/100) m or 100 cm=1 m, which yields the conversion factor 100 cm/1 m.

$$\begin{aligned} 10.2\,\text{dam} &= 10.2\,\text{dam} \cdot \frac{10\,\text{m}}{1\,\text{dam}} \cdot \frac{100\,\text{cm}}{1\,\text{m}} && \text{Apply conversion factors.} \\ &= 10.2\,\cancel{\text{dam}} \cdot \frac{10\,\cancel{\text{m}}}{1\,\cancel{\text{dam}}} \cdot \frac{100\,\text{cm}}{1\,\cancel{\text{m}}} && \text{Cancel common units.} \\ &= \frac{10.2 \cdot 10 \cdot 100}{1 \cdot 1}\,\text{cm} && \text{Multiply fractions.} \\ &= 10,200\,\text{cm} \end{aligned}$$

In the last step, note that multiplying by 10, then by 100, moves the decimal point three places to the right.

Alternately, we can set up our ordered list of units.

km hm dam m dm cm mm

Note that we must move 3 places right to move from the dekameters (dam) abbreviation to the centimeters (cm) abbreviation. In like manner, we can convert 10.2 dekameters to centimeters by moving the decimal 3 places to the right.

$$10.2\,\text{dam} = 10\,200\,\text{cm} = 10,200\,\text{cm}$$

Answer: 13,500 centimeters

Units of Mass

The fundamental unit of mass in the metric system is called a *gram.* Originally, it was defined to be equal to one cubic centimeter of water measured at the temperature of melting ice. Now it is simply defined as 1/1000 of a kilogram, which is defined by a physical prototype preserved by the International Bureau of Weights and Measures (Wikipedia). The mass of an object is not the same as an object's weight, but rather a resistance to motion when an external force is applied.

The same metric system prefixes apply.

Metric Units of Mass. These units of mass are used in the metric system.

Unit of Mass	**Unit Abbreviation**
1 kilogram = 1000 grams	kg
1 hectogram = 100 grams	hg
1 dekagram = 10 grams	dag
1 gram	g
1 decigram = $\frac{1}{10}$ gram	dg
1 centigram = $\frac{1}{100}$ gram	cg
1 milligram = $\frac{1}{1000}$ gram	mg

You Try It!

Convert 0.05 dekagrams to milligrams.

EXAMPLE 4. Convert 0.025 dekagrams to milligrams.

Solution. We'll use two conversion factors:

- 1 dag=10 g, which yields the conversion factor 10 g/1 dag.
- 1 mg=(1/1000) g, which yields the conversion factor 1000 mg/1 g.

$$\begin{aligned} 0.025\,\text{dag} &= 0.025\,\text{dag} \cdot \frac{10\,\text{g}}{1\,\text{dag}} \cdot \frac{1000\,\text{mg}}{1\,\text{g}} && \text{Apply conversion factors.} \\ &= 0.025\,\cancel{\text{dag}} \cdot \frac{10\,\cancel{\text{g}}}{1\,\cancel{\text{dag}}} \cdot \frac{1000\,\text{mg}}{1\,\cancel{\text{g}}} && \text{Cancel common units.} \\ &= \frac{0.025 \cdot 10 \cdot 1000}{1 \cdot 1}\,\text{mg} && \text{Multiply fractions.} \\ &= 250\,\text{mg} \end{aligned}$$

Alternately, we can set up our ordered list of units.

kg hg dag g dg cg mg

Note that we must move 4 places right to move from the dekagrams (dag) abbreviation to the milligrams (mg) abbreviation. In like manner, we can convert 0.025 dekagrams to milligrams by moving the decimal 4 places to the right.

$$0.0250\,\text{dag} = 0\,0250.\,\text{mg} = 250\,\text{mg}$$

Answer: 500 milligrams

Units of Volume

The fundamental unit of volume in the metric system is called a *litre*. Originally, one litre was defined as the volume of one kilogram of water measured at 4° C at 760 millimeters of mercury (Wikipedia). Currently, 1 litre is defined as 1 cubic decimeter (imagine a cube with each edge 1/10 of a meter).

The same metric system prefixes apply.

Metric Units of Volume. These units of volume are used in the metric system.

Unit of Volume	Unit Abbreviation
1 kilolitre = 1000 litres	kL
1 hectolitre = 100 litres	hL
1 dekalitre = 10 litres	daL
1 litre	L
1 decilitre = $\frac{1}{10}$ litre	dL
1 centilitre = $\frac{1}{100}$ litre	cL
1 millilitre = $\frac{1}{1000}$ litre	mL

You Try It!

Convert 5,763 millilitres to dekalitres.

EXAMPLE 5. Convert 11,725 millilitres to dekalitres.

Solution. We'll use two conversion factors:

- 1 daL=10 L, which yields the conversion factor 1 daL/10 L.
- 1 mL=(1/1000) L, which yields the conversion factor 1 L/1000 mL.

$$\begin{aligned}
11,725\,\text{mL} &= 11,725\,\text{mL}\cdot\frac{1\,\text{L}}{1000\,\text{mL}}\cdot\frac{1\,\text{daL}}{10\,\text{L}} && \text{Apply conversion factors.}\\
&= 11,725\,\cancel{\text{mL}}\cdot\frac{1\,\cancel{\text{L}}}{1000\,\cancel{\text{mL}}}\cdot\frac{1\,\text{daL}}{10\,\cancel{\text{L}}} && \text{Cancel common units.}\\
&= \frac{11,725\cdot 1\cdot 1}{1000\cdot 10}\,\text{daL} && \text{Multiply fractions.}\\
&= 1.1725\,\text{daL}
\end{aligned}$$

Alternately, we can set up our ordered list of units.

kL hL daL L dL cL mL

Note that we must move 4 places left to move from the millitres (mL) abbreviation to the dekalitres (daL) abbreviation. In like manner, we can convert 11,725 millilitres to dekalitres by moving the decimal 4 places to the left.

$$11,725.0\,\text{mL} = 1.1725\,0\,\text{daL} = 1.1725\,\text{daL}$$

Answer: 0.5763 dekalitres

❧ ❧ ❧ Exercises ☙ ☙ ☙

1. What is the meaning of the metric system prefix centi?

2. What is the meaning of the metric system prefix deka?

3. What is the meaning of the metric system prefix hecto?

4. What is the meaning of the metric system prefix kilo?

5. What is the meaning of the metric system prefix deci?

6. What is the meaning of the metric system prefix milli?

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7. What is the meaning of the metric system abbreviation mg?

8. What is the meaning of the metric system abbreviation g?

9. What is the meaning of the metric system abbreviation m?

10. What is the meaning of the metric system abbreviation km?

11. What is the meaning of the metric system abbreviation kL?

12. What is the meaning of the metric system abbreviation daL?

13. What is the meaning of the metric system abbreviation hm?

14. What is the meaning of the metric system abbreviation dm?

15. What is the meaning of the metric system abbreviation dam?

16. What is the meaning of the metric system abbreviation cm?

17. What is the meaning of the metric system abbreviation dL?

18. What is the meaning of the metric system abbreviation L?

19. What is the meaning of the metric system abbreviation hg?

20. What is the meaning of the metric system abbreviation kg?

21. What is the meaning of the metric system abbreviation dg?

22. What is the meaning of the metric system abbreviation dag?

23. What is the meaning of the metric system abbreviation hL?

24. What is the meaning of the metric system abbreviation cL?

---

25. Change 5,490 millimeters to meters.

26. Change 8,528 millimeters to meters.

27. Change 64 meters to millimeters.

28. Change 65 meters to millimeters.

29. Change 4,571 millimeters to meters.

30. Change 8,209 millimeters to meters.

31. Change 15 meters to centimeters.

32. Change 12 meters to centimeters.

33. Change 569 centimeters to meters.

34. Change 380 centimeters to meters.

35. Change 79 meters to centimeters.

36. Change 60 meters to centimeters.

37. Change 7.6 kilometers to meters.

38. Change 4.9 kilometers to meters.

39. Change 861 centimeters to meters.

40. Change 427 centimeters to meters.

41. Change 4,826 meters to kilometers.

42. Change 1,929 meters to kilometers.

43. Change 4,724 meters to kilometers.

44. Change 1,629 meters to kilometers.

45. Change 6.5 kilometers to meters.

46. Change 7.9 kilometers to meters.

47. Change 17 meters to millimeters.

48. Change 53 meters to millimeters.

---

49. Change 512 milligrams to centigrams.

50. Change 516 milligrams to centigrams.

51. Change 541 milligrams to centigrams.

52. Change 223 milligrams to centigrams.

53. Change 70 grams to centigrams.

54. Change 76 grams to centigrams.

55. Change 53 centigrams to milligrams.

56. Change 30 centigrams to milligrams.

57. Change 83 kilograms to grams.

58. Change 70 kilograms to grams.

59. Change 8,196 grams to kilograms.

60. Change 6,693 grams to kilograms.

61. Change 564 centigrams to grams.

62. Change 884 centigrams to grams.

63. Change 38 grams to centigrams.

64. Change 88 grams to centigrams.

65. Change 77 centigrams to milligrams.

66. Change 61 centigrams to milligrams.

67. Change 5,337 grams to kilograms.

68. Change 4,002 grams to kilograms.

69. Change 15 kilograms to grams.

70. Change 45 kilograms to grams.

71. Change 833 centigrams to grams.

72. Change 247 centigrams to grams.

---

73. Change 619,560 centilitres to kilolitres.

74. Change 678,962 centilitres to kilolitres.

75. Change 15.2 litres to millilitres.

76. Change 9.7 litres to millilitres.

77. Change 10,850 centilitres to litres.

78. Change 15,198 centilitres to litres.

79. Change 10.7 litres to millilitres.

80. Change 17.3 litres to millilitres.

81. Change 15,665 millilitres to litres.

82. Change 12,157 millilitres to litres.

83. Change 6.3 kilolitres to centilitres.

84. Change 8.3 kilolitres to centilitres.

85. Change 4.5 kilolitres to centilitres.

86. Change 6.2 kilolitres to centilitres.

87. Change 10.6 litres to centilitres.

88. Change 16.6 litres to centilitres.

89. Change 14,383 centilitres to litres.

90. Change 11,557 centilitres to litres.

91. Change 9.9 litres to centilitres.

92. Change 19.5 litres to centilitres.

93. Change 407,331 centilitres to kilolitres.

94. Change 827,348 centilitres to kilolitres.

95. Change 14,968 millilitres to litres.

96. Change 18,439 millilitres to litres.

Answers

1. 1/100

3. 100

5. 1/10

7. milligram

9. meter

11. kilolitre or kiloliter

13. hectometer

15. dekameter

17. decilitre or deciliter

19. hectogram

21. decigram

23. hectolitre or hectoliter

25. 5.49 meters

27. 64,000 millimeters

29. 4.571 meters

31. 1,500 centimeters

33. 5.69 meters

35. 7,900 centimeters

37. 7,600 meters

39. 8.61 meters

41. 4.826 kilometers

43. 4.724 kilometers

45. 6,500 meters

47. 17,000 millimeters

49. 51.2 centigrams

51. 54.1 centigrams

53. 7,000 centigrams

55. 530 milligrams

57. 83,000 grams

59. 8.196 kilograms

61. 5.64 grams

63. 3,800 centigrams

65. 770 milligrams

67. 5.337 kilograms

69. 15,000 grams

71. 8.33 grams

73. 6.1956 kilolitres

75. 15,200 millilitres

77. 108.5 litres

79. 10,700 millilitres

81. 15.665 litres

83. 630,000 centilitres

85. 450,000 centilitres

87. 1,060 centilitres

89. 143.83 litres

91. 990 centilitres

93. 4.07331 kilolitres

95. 14.968 litres

6.5 American Units to Metric Units and Vice-Versa

We often need to convert from the American system of units to the metric system of units or vice-versa (imagine traveling to a European country using the metric system). That will be our focus in this section.

Converting Units of Length

One meter is slightly longer than one yard. Indeed,

$$1\,\text{m} \approx 1.0936\,\text{yd}.$$

If we divide both sides of this equation by 1.0936, then

$$\begin{aligned} \frac{1\,\text{m}}{1.0936} &\approx \frac{1.0936\,\text{yd}}{1.0936} \\ 0.9144\,\text{m} &\approx 1\,\text{yd} \end{aligned}$$

Further conversions can be made. For example, to change meters to feet, we make the following conversions.

$$\begin{aligned} 1\,\text{m} &\approx 1\,\text{m} \cdot \frac{1\,\text{yd}}{0.9144\text{m}} \cdot \frac{3\,\text{ft}}{1\,\text{yd}} && \text{Apply conversion factors.} \\ &\approx 1\,\cancel{\text{m}} \cdot \frac{1\,\cancel{\text{yd}}}{0.9144\cancel{\text{m}}} \cdot \frac{3\,\text{ft}}{1\,\cancel{\text{yd}}} && \text{Cancel common units.} \\ &\approx \frac{1 \cdot 1 \cdot 3}{0.9144 \cdot 1}\,\text{ft} && \text{Multiply fractions.} \\ &\approx 3.2808\,\text{ft} \end{aligned}$$

Table 6.5 shows some of the most common conversions between American units of length and metric units of length.

American to Metric	**Metric to American**
$1\,\text{in} \approx 2.54\,\text{cm}$	$1\,\text{cm} \approx 0.3937\,\text{in}$
$1\,\text{ft} \approx 0.3048\,\text{m}$	$1\,\text{m} \approx 3.2808\,\text{ft}$
$1\,\text{yd} \approx 0.9144\,\text{m}$	$1\,\text{m} \approx 1.0936\,\text{yd}$
$1\,\text{mi} \approx 1.6093\,\text{km}$	$1\,\text{km} \approx 0.6214\,\text{mi}$

Table 6.5: Length conversions: American — Metric.

You Try It!

Change 227 miles to kilometers. Round to the nearest tenth of a kilometer.

EXAMPLE 1. A car's speedometer shows that a family has currently traveled 154 miles in route to their vacation destination. Convert this distance to kilometers.

Solution. Choose 1 mi = 1.6093 km from Table 6.5.

$$\begin{aligned}
154\,\text{mi} &\approx 154\,\text{mi} \cdot \frac{1.6093\,\text{km}}{1\,\text{mi}} && \text{Apply conversion factor.}\\
&\approx 154\,\cancel{\text{mi}} \cdot \frac{1.6093\,\text{km}}{1\,\cancel{\text{mi}}} && \text{Cancel common units.}\\
&\approx \frac{154 \cdot 1.6093}{1}\,\text{km} && \text{Multiply fractions.}\\
&\approx 247.8\,\text{km}
\end{aligned}$$

Hence, 154 miles is approximately 247.8 kilometers.

Alternate Solution. Note that this would work equally well if we chose 1 km ≈ 0.6214 mi from Table 6.5.

$$\begin{aligned}
154\,\text{mi} &\approx 154\,\text{mi} \cdot \frac{1\,\text{km}}{0.6214\,\text{mi}} && \text{Apply conversion factor.}\\
&\approx 154\,\cancel{\text{mi}} \cdot \frac{1\,\text{km}}{0.6214\,\cancel{\text{mi}}} && \text{Cancel common units.}\\
&\approx \frac{154 \cdot 1}{0.6214}\,\text{km} && \text{Multiply fractions.}\\
&\approx 247.8\,\text{km}
\end{aligned}$$

Answer: 365.3 kilometers

□

Converting Units of Weight and Mass

It is known that

$$1\,\text{kg} \approx 2.2\,\text{lb}.$$

Dividing both sides of this equation by 2.2,

$$\begin{aligned}
\frac{1\,\text{kg}}{2.2} &\approx \frac{2.2\,\text{lb}}{2.2}\\
0.454\,\text{kg} &\approx 1\,\text{lb}
\end{aligned}$$

A summary of the more common conversion factors regarding mass and weight are given in Table 6.6.

American to Metric	**Metric to American**
1 oz ≈ 28.35 g	1 g ≈ 0.035 oz
1 lb ≈ 0.454 kg	1 kg ≈ 2.2 lb

Table 6.6: Mass—Weight conversions: American — Metric.

You Try It!

Change 5.7 kilograms to ounces. Round to the nearest ounce.

EXAMPLE 2. Change 2.3 kilograms to ounces.

Solution. One kilogram weighs 2.2 pounds and there are 16 ounces in a pound.

$$\begin{aligned}
2.3\,\text{kg} &\approx 2.3\,\text{kg} \cdot \frac{2.2\,\text{lb}}{1\,\text{kg}} \cdot \frac{16\,\text{oz}}{1\,\text{lb}} && \text{Apply conversion factors.}\\
&\approx 2.3\,\cancel{\text{kg}} \cdot \frac{2.2\,\cancel{\text{lb}}}{1\,\cancel{\text{kg}}} \cdot \frac{16\,\text{oz}}{1\,\cancel{\text{lb}}} && \text{Cancel common units.}\\
&\approx \frac{2.3 \cdot 2.2 \cdot 16}{1 \cdot 1}\,\text{oz} && \text{Multiply fractions.}\\
&\approx 80.96\,\text{oz}
\end{aligned}$$

Hence, 2.3 kilograms weighs 80.96 ounces.

Alternate Solution. Another approach uses the facts that 1 kilogram equals 1000 grams and 1 ounce equals 28.35 grams.

$$\begin{aligned}
2.3\,\text{kg} &\approx 2.3\,\text{kg} \cdot \frac{1000\,\text{g}}{1\,\text{kg}} \cdot \frac{1\,\text{oz}}{28.35\,\text{g}} && \text{Apply conversion factors.}\\
&\approx 2.3\,\cancel{\text{kg}} \cdot \frac{1000\,\cancel{\text{g}}}{1\,\cancel{\text{kg}}} \cdot \frac{1\,\text{oz}}{28.35\,\cancel{\text{g}}} && \text{Cancel common units.}\\
&\approx \frac{2.3 \cdot 1000 \cdot 1}{1 \cdot 28.35}\,\text{oz} && \text{Multiply fractions.}\\
&\approx 81.13\,\text{oz}
\end{aligned}$$

Roundoff Error. Why the discrepancy in answers? This difference in approximations is due to something called *round-off error*. Indeed, in the first calculation, we used a conversion factor that is rounded to the nearest tenth; i.e., $1\,\text{kg} \approx 2.2\,\text{lb}$. If we use a more accurate conversion factor to change kilograms to pounds, namely $1\,\text{kg} \approx 2.2046\,\text{lb}$, we get the following result.

$$\begin{aligned}
2.3\,\text{kg} &\approx 2.3\,\text{kg} \cdot \frac{2.2046\,\text{lb}}{1\,\text{kg}} \cdot \frac{16\,\text{oz}}{1\,\text{lb}} && \text{Apply conversion factors.}\\
&\approx 2.3\,\cancel{\text{kg}} \cdot \frac{2.2046\,\cancel{\text{lb}}}{1\,\cancel{\text{kg}}} \cdot \frac{16\,\text{oz}}{1\,\cancel{\text{lb}}} && \text{Cancel common units.}\\
&\approx \frac{2.3 \cdot 2.2046 \cdot 16}{1 \cdot 1}\,\text{oz} && \text{Multiply fractions.}\\
&\approx 81.13\,\text{oz}
\end{aligned}$$

Note that this result is in better agreement with the second result above. □

Answer: 201 ounces

Important Observation. To obtain better approximations, you need to use a conversion factor that is more accurate. Any time you feel you need more accuracy, you might try an online conversion utility, such as the one at:

http://www.france-property-and-information.com/metric_conversion_table.htm

Converting Units of Volume

It is known that

$$1\,\text{qt} \approx 0.946\,\text{L}.$$

Dividing both sides of this equation by 0.946,

$$1\,\text{L} \approx 1.06\,\text{qt}.$$

Again, these conversion factors for volume have been well worked out by scientists. The more common conversion factors for volume are shown in Table 6.7.

American to Metric	**Metric to American**
$1\,\text{fl oz} \approx 0.030\,\text{L}$	$1\,\text{L} \approx 33.8\,\text{fl oz}$
$1\,\text{pt} \approx 0.473\,\text{L}$	$1\,\text{L} \approx 2.1\,\text{pt}$
$1\,\text{qt} \approx 0.946\,\text{L}$	$1\,\text{L} \approx 1.06\,\text{qt}$
$1\,\text{gal} \approx 3.785\,\text{L}$	$1\,\text{L} \approx 0.264\,\text{gal}$

Table 6.7: Volume conversions: American — Metric.

You Try It!

EXAMPLE 3. Change 2.5 dekalitres to gallons.

Solution. Recall that $1\,\text{daL} = 10\,\text{L}$ and $1\,\text{L} = 0.264\,\text{gal}$.

$$\begin{aligned}
2.5\,\text{daL} &\approx 2.5\,\text{daL} \cdot \frac{10\,\text{L}}{1\,\text{daL}} \cdot \frac{0.264\,\text{gal}}{1\,\text{L}} && \text{Apply conversion factors.}\\
&\approx 2.5\,\cancel{\text{daL}} \cdot \frac{10\,\cancel{\text{L}}}{1\,\cancel{\text{daL}}} \cdot \frac{0.264\,\text{gal}}{1\,\cancel{\text{L}}} && \text{Cancel common units.}\\
&\approx \frac{2.5 \cdot 10 \cdot 0.264}{1 \cdot 1}\,\text{gal} && \text{Multiply fractions.}\\
&\approx 6.6\,\text{gal}
\end{aligned}$$

Hence, 2.5 dekalitres is approximately equal to 6.6 gallons. □

Change 3.2 dekalitres to gallons.Round your answer to the nearest tenth of a gallon.

Answer: 0.8 gallons

Converting Units of Speed

Modern speedometers often show a car's speed in both miles per hour and kilometers per hour.

You Try It!

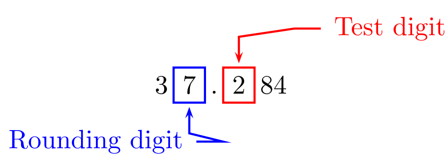
A car's speedometer registers 45 kilometers per hour. Change this speed to miles per hour. Round your answer to the nearest mile per hour.

EXAMPLE 4. A car's speedometer is showing it speeding along at 60 kilometers per hour. How fast is it traveling in miles per hour? Round your answer to the nearest mile per hour.

Solution. From Table 6.5, $1\,\text{km} \approx 0.6214\,\text{mi}$.

$$\begin{aligned}
60\,\frac{\text{km}}{\text{h}} &\approx 60\,\frac{\text{km}}{\text{h}} \cdot \frac{0.6214\,\text{mi}}{1\,\text{km}} && \text{Apply conversion factor.} \\
&\approx 60\,\frac{\cancel{\text{km}}}{\text{h}} \cdot \frac{0.6214\,\text{mi}}{1\,\cancel{\text{km}}} && \text{Cancel common units.} \\
&\approx \frac{60 \cdot 0.6214}{1}\,\frac{\text{mi}}{\text{h}} && \text{Multiply fractions.} \\
&\approx 37.284\,\frac{\text{mi}}{\text{h}}
\end{aligned}$$

To round to the nearest mile per hour, identify the rounding and test digits.



Because the test digit is less than 5, leave the rounding digit alone and truncate. Thus, to the nearest mile per hour, the speed is approximately 37 miles per hour.

Answer: 28 miles per hour

Exercises

1. Use the conversion $1\,\text{in} = 2.54\,\text{cm}$ to convert 68 inches to centimeters, rounded to the nearest tenth of a centimeter.

2. Use the conversion $1\,\text{in} = 2.54\,\text{cm}$ to convert 42 inches to centimeters, rounded to the nearest tenth of a centimeter.

3. Use the conversion $1\,\text{in} = 2.54\,\text{cm}$ to convert 44 centimeters to inches, rounded to the nearest tenth of a inch.

4. Use the conversion $1\,\text{in} = 2.54\,\text{cm}$ to convert 22 centimeters to inches, rounded to the nearest tenth of a inch.

5. Use the conversion $1\,\text{mi} = 1.6093\,\text{km}$ to convert 79 miles to kilometers, rounded to the nearest tenth of a kilometer.

6. Use the conversion $1\,\text{mi} = 1.6093\,\text{km}$ to convert 39 miles to kilometers, rounded to the nearest tenth of a kilometer.

7. Use the conversion $1\,\text{yd} = 0.9144\,\text{m}$ to convert 1489 centimeters to yards, rounded to the nearest tenth of a yard.

8. Use the conversion $1\,\text{yd} = 0.9144\,\text{m}$ to convert 1522 centimeters to yards, rounded to the nearest tenth of a yard.

9. Use the conversion $1\,\text{yd} = 0.9144\,\text{m}$ to convert 28 yards to centimeters, rounded to the nearest tenth of a centimeter.

10. Use the conversion $1\,\text{yd} = 0.9144\,\text{m}$ to convert 34 yards to centimeters, rounded to the nearest tenth of a centimeter.

11. Use the conversion $1\,\text{m} = 3.2808\,\text{ft}$ to convert 8.6 meters to inches, rounded to the nearest tenth of a inch.

12. Use the conversion $1\,\text{m} = 3.2808\,\text{ft}$ to convert 8.3 meters to inches, rounded to the nearest tenth of a inch.

13. Use the conversion $1\,\text{in} = 2.54\,\text{cm}$ to convert 60 inches to centimeters, rounded to the nearest tenth of a centimeter.

14. Use the conversion $1\,\text{in} = 2.54\,\text{cm}$ to convert 75 inches to centimeters, rounded to the nearest tenth of a centimeter.

15. Use the conversion $1\,\text{m} = 3.2808\,\text{ft}$ to convert 208 inches to meters, rounded to the nearest tenth of a inch.

16. Use the conversion $1\,\text{m} = 3.2808\,\text{ft}$ to convert 228 inches to meters, rounded to the nearest tenth of a inch.

17. Use the conversion $1\,\text{m} = 1.0936\,\text{yd}$ to convert 20 yards to meters, rounded to the nearest tenth of a meter.

18. Use the conversion $1\,\text{m} = 1.0936\,\text{yd}$ to convert 44 yards to meters, rounded to the nearest tenth of a meter.

19. Use the conversion $1\,\text{mi} = 1.6093\,\text{km}$ to convert 29 miles to kilometers, rounded to the nearest tenth of a kilometer.

20. Use the conversion $1\,\text{mi} = 1.6093\,\text{km}$ to convert 15 miles to kilometers, rounded to the nearest tenth of a kilometer.

21. Use the conversion $1\,\text{m} = 1.0936\,\text{yd}$ to convert 8.2 meters to yards, rounded to the nearest tenth of a yard.

22. Use the conversion $1\,\text{m} = 1.0936\,\text{yd}$ to convert 6.9 meters to yards, rounded to the nearest tenth of a yard.

23. Use the conversion $1\,\text{mi} = 1.6093\,\text{km}$ to convert 4.9 kilometers to miles, rounded to the nearest tenth of a mile.

24. Use the conversion $1\,\text{mi} = 1.6093\,\text{km}$ to convert 4.2 kilometers to miles, rounded to the nearest tenth of a mile.

25. Use the conversion 1 m = 1.0936 yd to convert 25 yards to meters, rounded to the nearest tenth of a meter.

26. Use the conversion 1 m = 1.0936 yd to convert 2 yards to meters, rounded to the nearest tenth of a meter.

27. Use the conversion 1 in = 2.54 cm to convert 47 centimeters to inches, rounded to the nearest tenth of a inch.

28. Use the conversion 1 in = 2.54 cm to convert 19 centimeters to inches, rounded to the nearest tenth of a inch.

29. Use the conversion 1 mi = 1.6093 km to convert 8.3 kilometers to miles, rounded to the nearest tenth of a mile.

30. Use the conversion 1 mi = 1.6093 km to convert 4.8 kilometers to miles, rounded to the nearest tenth of a mile.

31. Use the conversion 1 yd = 0.9144 m to convert 41 yards to centimeters, rounded to the nearest tenth of a centimeter.

32. Use the conversion 1 yd = 0.9144 m to convert 20 yards to centimeters, rounded to the nearest tenth of a centimeter.

33. Use the conversion 1 m = 3.2808 ft to convert 3.7 meters to inches, rounded to the nearest tenth of a inch.

34. Use the conversion 1 m = 3.2808 ft to convert 7.9 meters to inches, rounded to the nearest tenth of a inch.

35. Use the conversion 1 yd = 0.9144 m to convert 1323 centimeters to yards, rounded to the nearest tenth of a yard.

36. Use the conversion 1 yd = 0.9144 m to convert 1715 centimeters to yards, rounded to the nearest tenth of a yard.

37. Use the conversion 1 m = 1.0936 yd to convert 8.4 meters to yards, rounded to the nearest tenth of a yard.

38. Use the conversion 1 m = 1.0936 yd to convert 7.3 meters to yards, rounded to the nearest tenth of a yard.

39. Use the conversion 1 m = 3.2808 ft to convert 289 inches to meters, rounded to the nearest tenth of a inch.

40. Use the conversion 1 m = 3.2808 ft to convert 251 inches to meters, rounded to the nearest tenth of a inch.

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41. Use the conversion 1 kg = 2.2 lb to convert 15.8 kilograms to pounds, rounded to the nearest tenth of a pound.

42. Use the conversion 1 kg = 2.2 lb to convert 19.4 kilograms to pounds, rounded to the nearest tenth of a pound.

43. Use the conversion 1 oz = 28.35 g to convert 35 ounces to grams, rounded to the nearest tenth of a gram.

44. Use the conversion 1 oz = 28.35 g to convert 33 ounces to grams, rounded to the nearest tenth of a gram.

45. Use the conversion 1 kg = 2.2 lb to convert 2.48 kilograms to ounces, rounded to the nearest tenth of a ounce.

46. Use the conversion 1 kg = 2.2 lb to convert 3.74 kilograms to ounces, rounded to the nearest tenth of a ounce.

47. Use the conversion 1 kg = 2.2 lb to convert 2.35 kilograms to ounces, rounded to the nearest tenth of a ounce.

48. Use the conversion 1 kg = 2.2 lb to convert 3.57 kilograms to ounces, rounded to the nearest tenth of a ounce.

49. Use the conversion 1 kg = 2.2 lb to convert 15 pounds to kilograms, rounded to the nearest tenth of a kilogram.

50. Use the conversion 1 kg = 2.2 lb to convert 27 pounds to kilograms, rounded to the nearest tenth of a kilogram.

51. Use the conversion 1 kg = 2.2 lb to convert 10.4 kilograms to pounds, rounded to the nearest tenth of a pound.

52. Use the conversion 1 kg = 2.2 lb to convert 17.7 kilograms to pounds, rounded to the nearest tenth of a pound.

53. Use the conversion 1 kg = 2.2 lb to convert 352 ounces to kilograms, rounded to the nearest tenth of a kilogram.

54. Use the conversion 1 kg = 2.2 lb to convert 326 ounces to kilograms, rounded to the nearest tenth of a kilogram.

55. Use the conversion 1 oz = 28.35 g to convert 96 grams to ounces, rounded to the nearest tenth of a ounce.

56. Use the conversion 1 oz = 28.35 g to convert 100 grams to ounces, rounded to the nearest tenth of a ounce.

57. Use the conversion 1 oz = 28.35 g to convert 14 ounces to grams, rounded to the nearest tenth of a gram.

58. Use the conversion 1 oz = 28.35 g to convert 29 ounces to grams, rounded to the nearest tenth of a gram.

59. Use the conversion 1 kg = 2.2 lb to convert 54 pounds to kilograms, rounded to the nearest tenth of a kilogram.

60. Use the conversion 1 kg = 2.2 lb to convert 82 pounds to kilograms, rounded to the nearest tenth of a kilogram.

61. Use the conversion 1 oz = 28.35 g to convert 92 grams to ounces, rounded to the nearest tenth of a ounce.

62. Use the conversion 1 oz = 28.35 g to convert 103 grams to ounces, rounded to the nearest tenth of a ounce.

63. Use the conversion 1 kg = 2.2 lb to convert 388 ounces to kilograms, rounded to the nearest tenth of a kilogram.

64. Use the conversion 1 kg = 2.2 lb to convert 395 ounces to kilograms, rounded to the nearest tenth of a kilogram.

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65. Use the conversion 1 qt = 0.946 L to convert 55.1 litres to quarts, rounded to the nearest tenth of a quart.

66. Use the conversion 1 qt = 0.946 L to convert 50.3 litres to quarts, rounded to the nearest tenth of a quart.

67. Use the conversion 1 L = 33.8 fl oz to convert 72073 fluid ounces to kilolitres, rounded to the nearest tenth of a kilolitre.

68. Use the conversion 1 L = 33.8 fl oz to convert 56279 fluid ounces to kilolitres, rounded to the nearest tenth of a kilolitre.

69. Use the conversion 1 L = 33.8 fl oz to convert 2.5 kilolitres to fluid ounces.

70. Use the conversion 1 L = 33.8 fl oz to convert 4.5 kilolitres to fluid ounces.

71. Use the conversion 1 qt = 0.946 L to convert 24 quarts to litres, rounded to the nearest tenth of a litre.

72. Use the conversion 1 qt = 0.946 L to convert 21 quarts to litres, rounded to the nearest tenth of a litre.

73. Use the conversion 1 qt = 0.946 L to convert 30 quarts to litres, rounded to the nearest tenth of a litre.

74. Use the conversion 1 qt = 0.946 L to convert 22 quarts to litres, rounded to the nearest tenth of a litre.

75. Use the conversion 1 gal = 3.785 L to convert 11.8 gallons to litres, rounded to the nearest tenth of a litre.

76. Use the conversion 1 gal = 3.785 L to convert 13.5 gallons to litres, rounded to the nearest tenth of a litre.

77. Use the conversion 1 gal = 3.785 L to convert 50.5 litres to gallons, rounded to the nearest tenth of a gallon.

78. Use the conversion 1 gal = 3.785 L to convert 55.9 litres to gallons, rounded to the nearest tenth of a gallon.

79. Use the conversion 1 L = 33.8 fl oz to convert 8.3 kilolitres to fluid ounces.

80. Use the conversion 1 L = 33.8 fl oz to convert 5.3 kilolitres to fluid ounces.

81. Use the conversion 1 qt = 0.946 L to convert 42.4 litres to quarts, rounded to the nearest tenth of a quart.

82. Use the conversion 1 qt = 0.946 L to convert 55.4 litres to quarts, rounded to the nearest tenth of a quart.

83. Use the conversion 1 gal = 3.785 L to convert 17.2 gallons to litres, rounded to the nearest tenth of a litre.

84. Use the conversion 1 gal = 3.785 L to convert 19.6 gallons to litres, rounded to the nearest tenth of a litre.

85. Use the conversion 1 L = 33.8 fl oz to convert 51274 fluid ounces to kilolitres, rounded to the nearest tenth of a kilolitre.

86. Use the conversion 1 L = 33.8 fl oz to convert 82164 fluid ounces to kilolitres, rounded to the nearest tenth of a kilolitre.

87. Use the conversion 1 gal = 3.785 L to convert 55.6 litres to gallons, rounded to the nearest tenth of a gallon.

88. Use the conversion 1 gal = 3.785 L to convert 59.2 litres to gallons, rounded to the nearest tenth of a gallon.

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89. Change 60 miles per hour to kilometers per hour. Round your answer to the nearest kilometer per hour.

90. Change 56 miles per hour to kilometers per hour. Round your answer to the nearest kilometer per hour.

91. Change 77 miles per hour to kilometers per hour. Round your answer to the nearest kilometer per hour.

92. Change 57 miles per hour to kilometers per hour. Round your answer to the nearest kilometer per hour.

93. Change 42 kilometers per hour to miles per hour. Round your answer to the nearest mile per hour.

94. Change 56 kilometers per hour to miles per hour. Round your answer to the nearest mile per hour.

95. Change 62 kilometers per hour to miles per hour. Round your answer to the nearest mile per hour.

96. Change 63 kilometers per hour to miles per hour. Round your answer to the nearest mile per hour.

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97. **Tallest tower.** The world's tallest tower in Dubai has 160 floors at a height of 2,717 feet. Convert the height of the tower to the nearest tenth of a meter. *Associated Press-Times-Standard 02/09/10 World's tallest tower closed a month after opening.*

98. **High peaks.** For the first time, foreigners will be allowed to climb nearly 100 high-altitude Himalayan peaks on the Indian side of Kashmir, peaks ranging from 9,840 feet to nearly 26,246 feet. Convert the highest of the peaks to the nearest tenth of a meter. *Associated Press-Times-Standard 04/11/10 India opens Himalayan peaks to foreigners.*

99. **Ancient find.** In the southern Egyptian town of Luxor, a 3,400-year-old 4-meter statue of Thoth, the ancient Egyptian god of Wisdom and Magic, was unearthed. Convert the height of the statue to the nearest tenth of a foot. *Associated Press-Times-Standard 03/17/10 3,400-year-old statues unearthed in Egypt.*

100. **Arctic wind.** Blizzard condition winds in the Arctic blew 80 miles per hour. Find the wind speed to the nearest kilometer per hour. *Associated Press-Times-Standard 12/31/09 After 562-mile ski trek, seven women reach the South Pole.*

101. **Solar plane.** The Solar Impulse lifted off from a military airport at a speed no faster than 28 miles per hour. Convert the speed of the solar-powered plane to the nearest kilometer per hour. *Associated Press-Times-Standard 04/09/10 Solar-powered plane makes successful maiden flight.*

Answers

1. 172.7 centimeters

3. 17.3 inches

5. 127.1 kilometers

7. 16.3 yards

9. 2560.3 centimeters

11. 338.6 inches

13. 152.4 centimeters

15. 5.3 meters

17. 18.3 meters

19. 46.7 kilometers

21. 9.0 yards

23. 3.0 miles

25. 22.9 meters

27. 18.5 inches

29. 5.2 miles

31. 3749.0 centimeters

33. 145.7 inches

35. 14.5 yards

37. 9.2 yards

39. 7.3 meters

41. 34.8 pounds

43. 992.3 grams

45. 87.3 ounces

47. 82.7 ounces

49. 6.8 kilograms

51. 22.9 pounds

53. 10.0 kilograms

55. 3.4 ounces

57. 396.9 grams

59. 24.5 kilograms

61. 3.2 ounces

63. 11.0 kilograms

65. 58.2 quarts

67. 2.1 kilolitres

69. 84500 ounces

71. 22.7 litres

73. 28.4 litres

75. 44.7 litres

77. 13.3 gallons

79. 280540 ounces

81. 44.8 quarts

83. 65.1 litres

85. 1.5 kilolitres

87. 14.7 gallons

89. 97 km/hr

91. 124 km/hr

93. 26 mi/hr

95. 39 mi/hr

97. 828.2 meters

99. 13.1 feet

101. 45 km/hr

Chapter 7

Percent

When one hears the word "percent," other words come immediately to mind, words such as "century," "cents," or "centimeters." A *century* equals 100 years. There are one hundred *cents* in a dollar and there are 100 *centimeters* in a meter. Thus, it should come as no surprise that *percent* means "parts per hundred."

In the world we live in we are constantly bombarded with phrases that contain the word "percent." The sales tax in California is 8.25%. An employee is asking his boss for a 5% raise. A union has seen a 6.25% increase in union dues. The population of a town is increasing at a rate of 2.25% per year.

In this chapter we introduce the concept of percent, first addressing how to facilitate writing percents in fraction or decimal form and also performing the reverse operations, changing fractions and decimals to percents. Next we use our expertise in solving equations to solve the more common forms that involve percents, then we apply this ability to solving common applications from the real world that use percents. We'll tackle applications of commission and sales tax, discount and marked price, percent increase or decrease, and simple interest.

Let's begin the journey.

7.1 Percent, Decimals, Fractions

In the square shown in Figure 7.1, a large square has been partitioned into ten rows of ten little squares in each row. I Figure 7.1, we've shaded 20 of 100 possible little squares, or 20% of the total number of little squares.

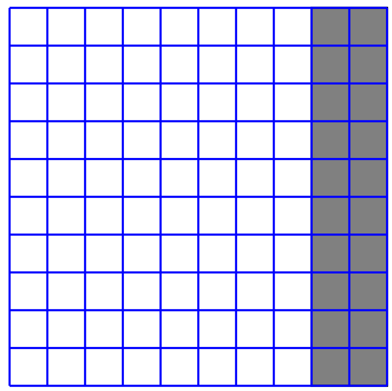

Figure 7.1: Shading 20 of 100 little squares, or 20% of the total number of little squares.

The Meaning of Percent. Percent means "parts per hundred."

In Figure 7.1, 80 out of a possible 100 squares are left unshaded. Thus, 80% of the little squares are unshaded. If instead we shaded 35 out of the 100 squares, then 35% of the little squares would be shaded. If we shaded all of the little squares, then 100% of the little squares would be shaded (100 out of 100).

So, when you hear the word "percent," think "parts per hundred."

Changing a Percent to a Fraction

Based on the discussion above, it is fairly straightforward to change a percent to a fraction.

Percent to Fraction. To change a percent to a fraction, drop the percent sign and put the number over 100.

You Try It!

EXAMPLE 1. Change 24% to a fraction.

Change 36% to a fraction reduced to lowest terms.

Solution. Drop the percent symbol and put 24 over 100.

$$\begin{aligned} 24\% &= \frac{24}{100} && \text{Percent: Parts per hundred.} \\ &= \frac{6}{25} && \text{Reduce.} \end{aligned}$$

Hence, 24% = 6/25.

Answer: 9/25

□

You Try It!

EXAMPLE 2. Change $14\frac{2}{7}\%$ to a fraction.

Change $11\frac{1}{9}\%$ to a fraction reduced to lowest terms.

Solution. Drop the percent symbol and put $14\frac{2}{7}$ over 100.

$$\begin{aligned} 14\frac{2}{7}\% &= \frac{14\frac{2}{7}}{100} && \text{Percent: Parts per hundred.} \\ &= \frac{\frac{100}{7}}{100} && \text{Mixed to improper fraction.} \\ &= \frac{100}{7} \cdot \frac{1}{100} && \text{Invert and multiply.} \\ &= \frac{\cancel{100}}{7} \cdot \frac{1}{\cancel{100}} && \text{Cancel.} \\ &= \frac{1}{7} \end{aligned}$$

Hence, $14\frac{2}{7}\% = 1/7$.

Answer: 1/9

□

You Try It!

EXAMPLE 3. Change 28.4% to a fraction.

Change 87.5% to a fraction reduced to lowest terms.

Solution. Drop the percent symbol and put 28.4 over 100.

$$\begin{aligned}
28.4\% &= \frac{28.4}{100} && \text{Percent: Parts per hundred.}\\
&= \frac{28.4 \cdot 10}{100 \cdot 10} && \text{Multiply numerator and denominator by 10.}\\
&= \frac{284}{1000} && \text{Multiplying by 10 moves decimal point one place right.}\\
&= \frac{71 \cdot 4}{250 \cdot 4} && \text{Factor.}\\
&= \frac{71}{250} && \text{Cancel common factor.}
\end{aligned}$$

Answer: 7/8

□

Changing a Percent to a Decimal

To change a percent to a decimal, we need only remember that percent means "parts per hundred."

You Try It!

Change 2.4% to a decimal.

EXAMPLE 4. Change 23.25% to a decimal.

Solution. Drop the percent symbol and put 23.25 over 100.

$$\begin{aligned}
23.25\% &= \frac{23.25}{100} && \text{Percent: Parts per hundred.}\\
&= 0.23\,25 && \text{Dividing by 100 moves decimal point 2 places left.}
\end{aligned}$$

Answer: 0.024

Therefore, 23.25% = 0.2325.

□

This last example motivates the following simple rule.

Changing a Percent to a Decimal. To change a percent to a decimal, drop the percent symbol and move the decimal point two places to the left.

You Try It!

Change $6\frac{3}{4}\%$ to a decimal.

EXAMPLE 5. Change $5\frac{1}{2}\%$ to a decimal.

Solution. Note that $1/2 = 0.5$, then move the decimal 2 places to the left.

$$5\frac{1}{2}\% = 5.5\% \qquad 1/2 = 0.5.$$

$$= 0.05\,5 \qquad \text{Drop \% symbol. Move decimal point 2 places left.}$$

$$= 0.055$$

Thus, $5\frac{1}{2}\% = 0.055$.

Answer: 0.0675

Changing a Decimal to a Percent

Changing a decimal to a percent is the exact opposite of changing a percent to a decimal. In the latter case, we drop the percent symbol and move the decimal point 2 places to the left. The following rule does just the opposite.

Changing a Decimal to a Percent. To change a decimal to a percent, move the decimal point two places to the right and add a percent symbol.

You Try It!

EXAMPLE 6. Change 0.0725 to a percent.

Change to 0.0375 to a percent.

Solution. Move the decimal point two places to the right and add a percent symbol.

$$0.0725 = 0\,07.25\%$$
$$= 7.25\%$$

Answer: 3.75%

You Try It!

EXAMPLE 7. Change 1.025 to a percent.

Change 0.525 to a percent.

Solution. Move the decimal point two places to the right and add a percent symbol.

$$1.025 = 1\,02.5\%$$
$$= 102.5\%$$

Answer: 52.5%

Changing a Fraction to a Percent

One way to proceed is to first change the fraction to a decimal, then change the resulting decimal to a percent.

Fractions to Percents: Technique #1. To change a fraction to a percent, follow these steps:

1. Divide numerator by the denominator to change the fraction to a decimal.
2. Move the decimal point in the result two places to the right and append a percent symbol.

You Try It!

Change 5/16 to a percent.

EXAMPLE 8. Use Technique #1 to change 5/8 to a percent.

Solution. Change 5/8 to a decimal, then change the decimal to a percent.

To change 5/8 to a decimal, divide 5 by 8. Since the denominator is a product of twos, the decimal should terminate.

$$\begin{array}{r} 0.625 \\ 8\overline{)5.000} \\ 4\,8 \\ \hline 20 \\ 16 \\ \hline 40 \\ 40 \\ \hline 0 \end{array}$$

To change 0.625 to a percent, move the decimal point 2 places to the right and append a percent symbol.

$$0.625 = 0\,62.5\% = 62.5\%$$

Answer: 31.35%

∎

A second technique is to create an equivalent fraction with a denominator of 100.

Fractions to Percents: Technique #2. To change a fraction to a percent, create an equivalent fraction with a denominator of 100.

You Try It!

Change 4/9 to a percent.

EXAMPLE 9. Use Technique #2 to change 5/8 to a percent.

Solution. Create an equivalent fraction for 5/8 with a denominator of 100.

$$\frac{5}{8} = \frac{x}{100}$$

Solve this proportion for x.

$$\begin{aligned} 8x &= 500 && \text{Cross multiply.} \\ \frac{8x}{8} &= \frac{500}{8} && \text{Divide both sides by 8.} \\ x &= \frac{125}{2} && \text{Reduce: Divide numerator and denominator by 4.} \\ x &= 62.5 && \text{Divide.} \end{aligned}$$

Thus,

$$\frac{5}{8} = \frac{62.5}{100} = 62.5\%.$$

Alternate Ending. We could also change 125/2 to a mixed fraction; i.e., $125/2 = 62\frac{1}{2}$. Then,

$$\frac{5}{8} = \frac{62\frac{1}{2}}{100} = 62\frac{1}{2}\%.$$

Same answer.

Answer: $44\frac{4}{9}\%$

Sometimes we will be content with an approximation.

You Try It!

EXAMPLE 10. Change 4/13 to a percent. Round your answer to the nearest tenth of a percent.

Change 4/17 to a percent. Round your answer to the nearest tenth of a percent.

Solution. We will use Technique #1.

To change 4/13 to a decimal, divide 4 by 13. Since the denominator has factors other than 2's and 5's, the decimal will repeat. However, we intend to round to the nearest tenth of a percent, so we will carry the division to four decimal places only. (*Four places are necessary because we will be moving the decimal point two places to the right.*)

```
      0.3076
  13)4.0000
     3 9
      100
       91
        90
        78
        12
```

To change the decimal to a percent, move the decimal point two places to the right.

$$0.3076 \approx 0\,30.76\% \approx 30.76\%$$

To round to the nearest tenth of a percent, identify the rounding and test digits.

$$30.\boxed{7}\,\boxed{6}\,\%$$

Test digit: 6

Rounding digit: 7

Because the test digit is greater than or equal to 5, add 1 to the rounding digit and truncate. Thus,

$$0.03076 \approx 30.8\%.$$

Answer: 23.5%

Exercises

In Exercises 1-18, convert the given percent to a fraction, and simplify the result.

1. $4\frac{7}{10}\%$

2. $7\frac{1}{4}\%$

3. $7\frac{2}{9}\%$

4. $4\frac{9}{10}\%$

5. 11.76%

6. 15.2%

7. 13.99%

8. 18.66%

9. $4\frac{1}{2}\%$

10. $8\frac{5}{8}\%$

11. 192%

12. 5%

13. 86%

14. 177%

15. 130%

16. 80%

17. 4.07%

18. 6.5%

In Exercises 19-34, convert the given percent to a decimal.

19. 124%

20. 4%

21. 0.6379%

22. 0.21%

23. 28%

24. 5.4%

25. 0.83%

26. 0.3344%

27. 8%

28. 3%

29. 59.84%

30. 0.17%

31. 155%

32. 7%

33. 36.5%

34. 39.7%

In Exercises 35-50, convert the given decimal to a percent.

35. 8.888

36. 5.1

37. 0.85

38. 0.08

39. 1.681

40. 3.372

41. 0.14

42. 4.89

43. 8.7

44. 8.78

45. 0.38

46. 1.67

47. 0.02

48. 0.07

49. 0.044

50. 0.29

In Exercises 51-68, convert the given fraction to a percent.

51. $\frac{1}{2}$

52. $\frac{29}{8}$

53. $\frac{5}{2}$

54. $\frac{4}{5}$

55. $\frac{8}{5}$

56. $\frac{7}{20}$

57. $\frac{14}{5}$

58. $\frac{3}{2}$

59. $\frac{9}{2}$

60. $\frac{18}{25}$

61. $\frac{9}{4}$

62. $\frac{7}{8}$

63. $\frac{7}{5}$

64. $\frac{4}{25}$

65. $\frac{6}{5}$

66. $\frac{23}{8}$

67. $\frac{12}{5}$

68. $\frac{13}{2}$

---

69. Convert 24/29 to a percent, and round your answer to the nearest hundredth of a percent.

70. Convert 5/3 to a percent, and round your answer to the nearest hundredth of a percent.

71. Convert 15/7 to a percent, and round your answer to the nearest tenth of a percent.

72. Convert 10/7 to a percent, and round your answer to the nearest tenth of a percent.

73. Convert 7/24 to a percent, and round your answer to the nearest hundredth of a percent.

74. Convert 5/6 to a percent, and round your answer to the nearest hundredth of a percent.

75. Convert 8/3 to a percent, and round your answer to the nearest tenth of a percent.

76. Convert 22/21 to a percent, and round your answer to the nearest tenth of a percent.

77. Convert 9/23 to a percent, and round your answer to the nearest tenth of a percent.

78. Convert 11/9 to a percent, and round your answer to the nearest tenth of a percent.

79. Convert 17/27 to a percent, and round your answer to the nearest hundredth of a percent.

80. Convert 22/27 to a percent, and round your answer to the nearest hundredth of a percent.

---

81. **Crime rates.** Preliminary crime rates for the first six months of 2009 compared to the same period in 2008 are shown below. *Associated Press-Times-Standard 12/22/09 Despite recession, the national crime rates keep falling.*

Murder	−10.0%
Forcible rape	−3.3%
Robbery	−6.5%
Aggravated assault	−3.2%
Burglary	−2.5%
Larceny-theft	−5.3%
Motor vehicle theft	−18.75%
Arson	−8.2%

Source: Federal Bureau of Investigation

i) What do the negative signs indicate?

ii) Which type of crime decreased the most?

iii) Which type of crime decreased the least?

82. **Major Hurricanes.** 5 of the 8 hurricanes in 2008 were categorized as major. Write the fractional number of major hurricanes in 2008 as a percent. *NOAA Associated Press 5/22/09*

83. **Chance of flood.** These excerpts are from the story *Corps: Dam work lessens Seattle-area flood chance* published in the *Times-Standard* on Nov. 6, 2009. Write all four of the odds of flooding as a percent chance. Round to the nearest tenth of a percent if necessary.

i) Col. Anthony Wright, from the U.S. Army Corps of Engineers, speaking of the repairs to the Green River Dam, reported there was now a 1-in-25 chance that a storm would force the corps to release enough water from the dam's reservoir to cause a flood downstream in the Green River Valley.

ii) The odds of widespread flooding in the valley improve to 1-in-32 when all the sandbagging and flood-protection efforts are factored in.

iii) Previously, the Corps of Engineers said the chance of widespread flooding was 1-in-4.

iv) When the dam operates at capacity, there is a 1-in-140 chance of flooding.

Answers

1. $\dfrac{47}{1000}$

3. $\dfrac{13}{180}$

5. $\frac{147}{1250}$

7. $\frac{1399}{10000}$

9. $\frac{9}{200}$

11. $\frac{48}{25}$

13. $\frac{43}{50}$

15. $\frac{13}{10}$

17. $\frac{407}{10000}$

19. 1.24

21. 0.006379

23. 0.28

25. 0.0083

27. 0.08

29. 0.5984

31. 1.55

33. 0.365

35. 888.8%

37. 85%

39. 168.1%

41. 14%

43. 870%

45. 38%

47. 2%

49. 4.4%

51. 50%

53. 250%

55. 160%

57. 280%

59. 450%

61. 225%

63. 140%

65. 120%

67. 240%

69. 82.76%

71. 214.3%

73. 29.17%

75. 266.7%

77. 39.1%

79. 62.96%

81.
i) The negative signs indicate the crime rate has decreased from previous measures.

ii) Motor vehicle theft decreased the most with an 18.75% decrease.

iii) Burglary decreased the least with a 2.5% decrease.

83.
i) 4% chance of flood

ii) 3.1% chance of flood

iii) 25% chance of flood

iv) 0.7% chance of flood

7.2 Solving Basic Percent Problems

There are three basic types of percent problems:

1. *Find a given percent of a given number.* For example, find 25% of 640.
2. *Find a percent given two numbers.* For example, 15 is what percent of 50?
3. *Find a number that is a given percent of another number.* For example, 10% of what number is 12?

Let's begin with the first of these types.

Find a Given Percent of a Given Number

Let's begin with our first example.

You Try It!

What number is 36% of 120?

EXAMPLE 1. What number is 25% of 640?

Solution. Let x represent the unknown number. Translate the words into an equation.

What number	is	25%	of	640
x	$=$	25%	$\cdot$	640

Now, solve the equation for x.

$$x = 25\% \cdot 640 \quad \text{Original equation.}$$
$$x = 0.25 \cdot 640 \quad \text{Change 25\% to a decimal: } 25\% = 0.25.$$
$$x = 160 \quad \text{Multiply: } 0.25 \cdot 640 = 160.$$

Thus, 25% of 640 is 160.

Alternate Solution. We could also change 25% to a fraction.

$$x = 25\% \cdot 640 \quad \text{Original equation.}$$
$$x = \frac{1}{4} \cdot 640 \quad \text{Change 25\% to a fraction: } 25\% = 25/100 = 1/4.$$
$$x = \frac{640}{4} \quad \text{Multiply numerators and denominators.}$$
$$x = 160 \quad \text{Divide: } 640/4 = 160.$$

Same answer.

Answer: 43.2

□

You Try It!

EXAMPLE 2. What number is $8\frac{1}{3}\%$ of 120?

What number is $4\frac{1}{6}\%$ of 1,200?

Solution. Let x represent the unknown number. Translate the words into an equation.

What number	is	$8\frac{1}{3}\%$	of	120
x	$=$	$8\frac{1}{3}\%$	$\cdot$	120

Now, solve the equation for x. Because

$$8\frac{1}{3}\% = 8.\overline{3}\% = 0.08\overline{3},$$

working with decimals requires that we work with a repeating decimal. To do so, we would have to truncate the decimal representation of the percent at some place and satisfy ourselves with an approximate answer. Instead, let's change the percent to a fraction and seek an exact answer.

$$\begin{aligned}
8\frac{1}{3}\% &= \frac{8\frac{1}{3}}{100} && \text{Percent: Parts per hundred.}\\
&= \frac{\frac{25}{3}}{100} && \text{Mixed to improper fraction.}\\
&= \frac{25}{3} \cdot \frac{1}{100} && \text{Invert and multiply.}\\
&= \frac{25}{300} && \text{Multiply numerators and denominators.}\\
&= \frac{1}{12} && \text{Reduce: Divide numerator and denominator by 25.}
\end{aligned}$$

Now we can solve our equation for x.

$$\begin{aligned}
x &= 8\frac{1}{3}\% \cdot 120 && \text{Original equation.}\\
x &= \frac{1}{12} \cdot 120 && 8\frac{1}{3}\% = 1/12.\\
x &= \frac{120}{12} && \text{Multiply numerators and denominators.}\\
x &= 10 && \text{Divide: } 120/12{=}10.
\end{aligned}$$

Thus, $8\frac{1}{3}\%$ of 120 is 10.

Answer: 50

□

You Try It!

EXAMPLE 3. What number is $105\frac{1}{4}\%$ of 18.2?

What number is $105\frac{3}{4}\%$ of 222?

Solution. Let x represent the unknown number. Translate the words into an equation.

What number	is	$105\frac{1}{4}\%$	of	18.2
x	$=$	$105\frac{1}{4}\%$	$\cdot$	18.2

In this case, the fraction terminates as $1/4 = 0.25$, so

$$105\frac{1}{4}\% = 105.25\% = 1.0525.$$

Now we can solve our equation for x.

$$x = 105\frac{1}{4}\% \cdot 18.2 \quad \text{Original equation.}$$

$$x = 1.0525 \cdot 18.2 \quad 5\frac{1}{4}\% = 1.0525.$$

$$x = 19.1555 \quad \text{Multiply.}$$

Answer: 234.765

Thus, $105\frac{1}{4}\%$ of 18.2 is 19.1555.

□

Find a Percent Given Two Numbers

Now we'll address our second item on the list at the beginning of the section.

You Try It!

14 is what percent of 25?

EXAMPLE 4. 15 is what percent of 50?

Solution. Let x represent the unknown percent. Translate the words into an equation.

15	is	what percent	of	50
15	$=$	x	$\cdot$	50

The commutative property of multiplication allows us to change the order of multiplication on the right-hand side of this equation.

$$15 = 50x.$$

Now we can solve our equation for x.

$$15 = 50x \quad \text{Original equation.}$$

$$\frac{15}{50} = \frac{50x}{50} \quad \text{Divide both sides by 50.}$$

$$\frac{15}{50} = x \quad \text{Simplify right-hand side.}$$

$$x = 0.30 \quad \text{Divide: } 15/50 = 0.30.$$

But we must express our answer as a percent. To do this, move the decimal two places to the right and append a percent symbol.

$$0.30 = 0\,30.\% = 30\%$$

Thus, 15 is 30% of 50.

Alternative Conversion. At the third step of the equation solution, we had

$$x = \frac{15}{50}.$$

We can convert this to an equivalent fraction with a denominator of 100.

$$x = \frac{15 \cdot 2}{50 \cdot 2} = \frac{30}{100}$$

Thus, $15/50 = 30/100 = 30\%$.

Answer: 56%

You Try It!

EXAMPLE 5. 10 is what percent of 80?

10 is what percent of 200?

Solution. Let x represent the unknown percent. Translate the words into an equation.

10	is	what percent	of	80
10	$=$	x	$\cdot$	80

The commutative property of multiplication allows us to write the right-hand side as

$$10 = 80x.$$

Now we can solve our equation for x.

$10 = 80x$	Original equation.
$\frac{10}{80} = \frac{80x}{80}$	Divide both sides by 80.
$\frac{1}{8} = x$	Reduce: $10/80 = 1/8$.
$0.125 = x$	Divide: $1/8 = 0.125$.

But we must express our answer as a percent. To do this, move the decimal two places to the right and append a percent symbol.

$$0.125 = 0\,12.5\% = 12.5\%$$

Thus, 10 is 12.5% of 80.

Alternative Conversion. At the third step of the equation solution, we had

$$x = \frac{1}{8}.$$

We can convert this to an equivalent fraction with a denominator of 100 by setting up the proportion

$$\frac{1}{8} = \frac{n}{100}$$

Cross multiply and solve for n.

$$8n = 100 \qquad \text{Cross multiply.}$$

$$\frac{8n}{8} = \frac{100}{8} \qquad \text{Divide both sides by 8.}$$

$$n = \frac{25}{2} \qquad \text{Reduce: Divide numerator and denominator by 4.}$$

$$n = 12\frac{1}{2} \qquad \text{Change 25/2 to mixed fraction.}$$

Hence,

$$\frac{1}{8} = \frac{12\frac{1}{2}}{100} = 12\frac{1}{2}\%.$$

Same answer.

Answer: 5%

Find a Number that is a Given Percent of Another Number

Let's address the third item on the list at the beginning of the section.

You Try It!

20% of what number is 45?

EXAMPLE 6. 10% of what number is 12?

Solution. Let x represent the unknown number. Translate the words into an equation.

10%	of	what number	is	12
10%	$\cdot$	x	$=$	12

Change 10% to a fraction: 10% = 10/100 = 1/10.

$$\frac{1}{10}x = 12$$

Now we can solve our equation for x.

$$10\left(\frac{1}{10}x\right) = 10(12) \qquad \text{Multiply both sides by 10.}$$

$$x = 120 \qquad \text{Simplify.}$$

Thus, 10% of 120 is 12.

Alternative Solution. We can also change 10% to a decimal: 10% = 0.10. Then our equation becomes

$$0.10x = 12$$

Now we can divide both sides of the equation by 0.10.

$$\frac{0.10x}{0.10} = \frac{12}{0.10} \qquad \text{Divide both sides by 0.10.}$$

$$x = 120 \qquad \text{Divide: } 12/0.10 = 120.$$

Same answer.

Answer: 225

You Try It!

$12\frac{2}{3}\%$ of what number is 760?

EXAMPLE 7. $11\frac{1}{9}\%$ of what number is 20?

Solution. Let x represent the unknown number. Translate the words into an equation.

$11\frac{1}{9}\%$	of	what number	is	20
$11\frac{1}{9}\%$	$\cdot$	x	$=$	20

Change $11\frac{1}{9}\%$ to a fraction.

$$11\frac{1}{9}\% = \frac{11\frac{1}{9}}{100} \qquad \text{Percent: Parts per hundred.}$$

$$= \frac{\frac{100}{9}}{100} \qquad \text{Mixed to improper: } 11\frac{1}{9} = 100/9.$$

$$= \frac{100}{9} \cdot \frac{1}{100} \qquad \text{Invert and multiply.}$$

$$= \frac{\cancel{100}}{9} \cdot \frac{1}{\cancel{100}} \qquad \text{Cancel.}$$

$$= \frac{1}{9} \qquad \text{Simplify.}$$

Replace $11\frac{1}{9}\%$ with 1/9 in the equation and solve for x.

$$\frac{1}{9}x = 20 \qquad 11\frac{1}{9}\% = 1/9.$$

$$9\left(\frac{1}{9}x\right) = 9(20) \qquad \text{Multiply both sides by 9.}$$

$$x = 180$$

Thus, $11\frac{1}{9}\%$ of 180 is 20.

Answer: 6,000

❧ ❧ ❧ Exercises ☙ ☙ ☙

1. What number is 22.4% of 125?

2. What number is 159.2% of 125?

3. 60% of what number is 90?

4. 25% of what number is 40?

5. 200% of what number is 132?

6. 200% of what number is 208?

7. 162.5% of what number is 195?

8. 187.5% of what number is 90?

9. 126.4% of what number is 158?

10. 132.5% of what number is 159?

11. 27 is what percent of 45?

12. 9 is what percent of 50?

13. 37.5% of what number is 57?

14. 162.5% of what number is 286?

15. What number is 85% of 100?

16. What number is 10% of 70?

17. What number is 200% of 15?

18. What number is 50% of 84?

19. 50% of what number is 58?

20. 132% of what number is 198?

21. 5.6 is what percent of 40?

22. 7.7 is what percent of 35?

23. What number is 18.4% of 125?

24. What number is 11.2% of 125?

25. 30.8 is what percent of 40?

26. 6.3 is what percent of 15?

27. 7.2 is what percent of 16?

28. 55.8 is what percent of 60?

29. What number is 89.6% of 125?

30. What number is 86.4% of 125?

31. 60 is what percent of 80?

32. 16 is what percent of 8?

33. What number is 200% of 11?

34. What number is 150% of 66?

35. 27 is what percent of 18?

36. 9 is what percent of 15?

37. $133\frac{1}{3}\%$ of what number is 80?

38. $121\frac{2}{3}\%$ of what number is 73?

39. What number is $54\frac{1}{3}\%$ of 6?

40. What number is $82\frac{2}{5}\%$ of 5?

41. What number is $62\frac{1}{2}\%$ of 32?

42. What number is $118\frac{3}{4}\%$ of 32?

43. $77\frac{1}{7}\%$ of what number is 27?

44. $82\frac{2}{3}\%$ of what number is 62?

45. What number is $142\frac{6}{7}\%$ of 77?

46. What number is $116\frac{2}{3}\%$ of 84?

47. $143\frac{1}{2}\%$ of what number is 5.74?

48. $77\frac{1}{2}\%$ of what number is 6.2?

49. $141\frac{2}{3}\%$ of what number is 68?

50. $108\frac{1}{3}\%$ of what number is 78?

51. What number is $66\frac{2}{3}\%$ of 96?

52. What number is $79\frac{1}{6}\%$ of 48?

53. $59\frac{1}{2}\%$ of what number is 2.38?

54. $140\frac{1}{5}\%$ of what number is 35.05?

55. $78\frac{1}{2}\%$ of what number is 7.85?

56. $73\frac{1}{2}\%$ of what number is 4.41?

57. What number is $56\frac{2}{3}\%$ of 51?

58. What number is $64\frac{1}{2}\%$ of 4?

59. What number is $87\frac{1}{2}\%$ of 70?

60. What number is $146\frac{1}{4}\%$ of 4?

61. It was reported that 80% of the retail price of milk was for packaging and distribution. The remaining 20% was paid to the dairy farmer. If a gallon of milk cost $3.80, how much of the retail price did the farmer receive?

62. At $1.689 per gallon of gas the cost is distributed as follows:

Crude oil supplies	$0.95
Oil Companies	$0.23
State and City taxes	$0.23
Federal tax	$0.19
Service Station	$0.10

Data is from Money, March 2009 p. 22, based on U. S. averages in December 2008. Answer the following questions rounded to the nearest whole percent.

a) What % of the cost is paid for crude oil supplies?

b) What % of the cost is paid to the service station?

❧ ❧ ❧ Answers ☙ ☙ ☙

1. 28

3. 150

5. 66

7. 120

9. 125

11. 60

13. 152

15. 85

17. 30

19. 116

21. 14

23. 23

25. 77

27. 45

29. 112

31. 75

33. 22

35. 150

37. 60

39. 3.26

41. 20

43. 35

45. 110

47. 4

49. 48

51. 64

53. 4

55. 10

57. 28.9

59. 61.25

61. $0.76

7.3 General Applications of Percent

In this section we will look at an assortment of practical problems involving percent.

You Try It!

EXAMPLE 1. Myrna notes that 20% of her class is absent.If the class has 45 students, how many students are absent?

Aaron notes that 15% of his class is absent. If the class has 80 students, how many students are absent?

Solution. Let n represent the number of students that are absent. Then we can translate the problem statement into words and symbols.

Number absent	is	20%	of	total number of students in the class
n	$=$	20%	$\cdot$	45

Because $20\% = 0.20$,

$$\begin{aligned} n &= 0.20 \cdot 45 && 20\% = 0.20 \\ n &= 9 && \text{Multiply: } 0.20 \cdot 45 = 9. \end{aligned}$$

Therefore, 9 students are absent.

Answer: 12

You Try It!

EXAMPLE 2. Misty answered 90% of the questions on her mathematics examination correctly. If Misty had 27 correct answers, how many questions were on the exam?

Erin asnwered 85% of the questions on her english examination correctly. If she had 34 correct answers, how many questions were on her exam?

Solution. Let N represent the number of questions on the examination.

Number of correct answers	is	90%	of	total number of questions
27	$=$	90%	$\cdot$	N

Because $90\% = 0.90$, this last equation can be written as

$$27 = 0.90N.$$

Solve for N.

$$\begin{aligned} \frac{27}{0.90} &= \frac{0.90N}{0.90} && \text{Divide both sides by 0.90.} \\ 30 &= N && \text{Divide: } 27/0.90 = 30. \end{aligned}$$

Hence, there were 30 questions on the examination.

Answer: 40

You Try It!

Alphonso answered 19 of 25 questions on his biology test correctly. What percent of the questions did he mark correctly?

EXAMPLE 3. Misty answered 30 of 40 possible questions on her sociology examination correctly. What percent of the total number of questions did Misty mark correctly?

Solution. Let p represent the percent of the total number of questions marked correctly. Then we can translate the problem statement into words and symbols.

Number of correct answers	is	what percent	of	total number of questions
30	=	p	·	40

Because multiplication is commutative, we can write the last equation in the form

$$30 = 40p.$$

Solve for p.

$$\frac{30}{40} = \frac{40p}{40} \quad \text{Divide both sides by 40.}$$

$$\frac{3}{4} = p \quad \text{Reduce: } 30/40 = 3/4.$$

We need to change $p = 3/4$ to a percent. There are two ways to do this:

- We can divide 3 by 4 to get

$$\begin{aligned} p &= \frac{3}{4} \\ &= 0.75 && \text{Divide: } 3/4 = 0.75. \\ &= 75\% && \text{Move decimal point 2 places right.} \end{aligned}$$

- We can create an equivalent fraction with a denominator of 100; i.e.,

$$\begin{aligned} p &= \frac{3}{4} \\ &= \frac{3 \cdot 25}{4 \cdot 25} && \text{Multiply numerator and denominator by 25.} \\ &= \frac{75}{100} && \text{Simplify numerator and denominator.} \\ &= 75\%. && \text{Percent means parts per hundred.} \end{aligned}$$

Either way, Misty got 75% of the questions on her sociology examination correct.

Answer: 76%

□

You Try It!

25 millilitres of a 40 millilitre solution is sulfuric acid. What percent of the solution is sulfuric acid?

EXAMPLE 4. 35 millilitres of a 60 millilitre solution is hydrochloric acid. What percent of the solution is hydrochloric acid?

Solution. Let p represent the percent of the percent of the solution that is hydrochloric acid. Then we can translate the problem statement into words and symbols.

Amount of hydrochloric acid	is	what percent	of	the total amount of solution
35	=	p	·	60

Because multiplication is commutative, we can write the right-hand side of the last equation as follows.

$$35 = 60p$$

Now we can solve for p.

$$\frac{35}{60} = \frac{60p}{60} \quad \text{Divide both sides by 60.}$$

$$\frac{7}{12} = p \quad \text{Reduce: Divide numerator and denominator by 5.}$$

Now we must change p to a percent. We can do this exactly by creating an equivalent fraction with a denominator of 100.

$$\frac{7}{12} = \frac{n}{100}$$

Solve for n.

$$12n = 700 \quad \text{Cross multiply.}$$

$$\frac{12n}{12} = \frac{700}{12} \quad \text{Divide both sides by 12.}$$

$$n = \frac{175}{3} \quad \text{Reduce: Divide numerator and denominator by 4.}$$

$$n = 58\frac{1}{3} \quad \text{Change improper to mixed fraction.}$$

Hence,

$$p = \frac{7}{12} = \frac{58\frac{1}{3}}{100} = 58\frac{1}{3}\%.$$

Thus, $58\frac{1}{3}\%$ of the solution is hydrochloric acid.

Approximate Solution. If all that is needed is an approximate answer, say correct to the nearest tenth of a percent, then we would take a different approach starting with the line from above that has

$$\frac{35}{60} = p.$$

We would divide 35 by 60 to get

$$p \approx 0.5833.$$

Move the decimal two places to the right and append a percent symbol.

$$p \approx 0.5833 \approx 0\,58.33\% \approx 58.33\%.$$

Round to the nearest tenth of a percent.

Test digit

$$p \approx 58.\boxed{3}\,\boxed{3}\,\%$$

Rounding digit

Because the test digit is less than 5, leave the rounding digit alone and truncate. Thus, correct to the nearest tenth of a percent,

$$p \approx 58.3\%.$$

Answer: 62.5%

Note that $p \approx 58.3\%$ is approximate, but $p = 58\frac{1}{3}\%$ is exact.

□

Exercises

1. 31 millilitres of a 250 millilitre solution is sulphuric acid. What percent of the solution is sulphuric acid? Round your answer to the nearest tenth of a percent.

2. 34 millilitres of a 211 millilitre solution is phosphoric acid. What percent of the solution is phosphoric acid? Round your answer to the nearest tenth of a percent.

3. A family has completed 186 miles of a planned 346 mile trip. Find the percentage of the planned trip already traveled. Round your answer to the nearest percent.

4. A family has completed 153 miles of a planned 431 mile trip. Find the percentage of the planned trip already traveled. Round your answer to the nearest percent.

5. Erin takes roll in her fifth grade class and finds that 19 out of 34 total students on her roster are present. Find the percentage of the class that is present, correct to the nearest percent.

6. Barbara takes roll in her fifth grade class and finds that 15 out of 38 total students on her roster are present. Find the percentage of the class that is present, correct to the nearest percent.

7. Raven answered 135 of 150 possible questions on the meteorology examination correctly. What percent of the total number of questions did Raven mark correctly?

8. Liz answered 30 of 50 possible questions on the algebra examination correctly. What percent of the total number of questions did Liz mark correctly?

9. A family has traveled 114 miles of a planned trip. This is 37% of the total distance they must travel on the trip. Find, correct to the nearest mile, the total distance they will travel on their trip.

10. A family has traveled 102 miles of a planned trip. This is 23% of the total distance they must travel on the trip. Find, correct to the nearest mile, the total distance they will travel on their trip.

11. Trudy takes roll in her class at the university and finds that 65 students are present. If this is 50% of the total class enrollment, how many students are in the class?

12. Sandra takes roll in her class at the university and finds that 104 students are present. If this is 80% of the total class enrollment, how many students are in the class?

13. Bill earns a commission on all sales he makes. He sells a bed for $591 and earns a commission of $43. Find the percent commission, rounded to the nearest tenth of a percent.

14. Ira earns a commission on all sales he makes. He sells a sofa for $408 and earns a commission of $39. Find the percent commission, rounded to the nearest tenth of a percent.

15. Tami answered 70% of the questions on the physics examination correctly. If Tami had 98 correct answers, how many questions were on the exam?

16. Trinity answered 90% of the questions on the chemistry examination correctly. If Trinity had 99 correct answers, how many questions were on the exam?

17. A state charges 8% sales tax on all sales. If the sales tax on a computer is $20, find the sales price of the computer, correct to the nearest dollar.

18. A state charges 6.5% sales tax on all sales. If the sales tax on a bed is $33, find the sales price of the bed, correct to the nearest dollar.

19. Kenon earns 6% commission all his sales. If the sale of a computer earns him a $37 commission, find the sales price of the computer, correct to the nearest dollar.

20. Donald earns 4.5% commission all his sales. If the sale of a dryer earns him a $24 commission, find the sales price of the dryer, correct to the nearest dollar.

21. A 23% nitric acid solution contains 59 millilitres of nitric acid. How many total millilitres of solution are present? Round your answer to the nearest millilitre.

22. A 27% sulphuric acid solution contains 67 millilitres of sulphuric acid. How many total millilitres of solution are present? Round your answer to the nearest millilitre.

23. In a state, a television sold for $428 is assessed a sales tax of $45. Find the sales tax rate, rounded to the nearest tenth of a percent.

24. In a state, a refrigerator sold for $503 is assessed a sales tax of $44. Find the sales tax rate, rounded to the nearest tenth of a percent.

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25. **Mars gravity.** The force of gravity on Mars is only 38% of the force of gravity on earth. If you weigh 150 pounds on earth, how much will you weigh on Mars?

26. **Wiretaps.** In 2008, there were a total of 1,891 applications to federal and state judges to authorize the interception of wire, oral, or electronic communications. If 94% of all wiretap applications were for a portable device such as a cell phone or pager, how many applications were made to tap mobile devices? Round-off to the nearest application. *Associated Press Times-Standard 4/28/09*

27. **Seniors.** 13% of Humboldt County's population is age 65 and older, about 2% more than the state's average. If the population of Humboldt County is approximately 130,000, how many people in Humboldt County are age 65 and older? *Times-Standard 6/10/2009*

28. **Antibiotics.** "The U.S. used about 35 million pounds of antibiotics last year. 70 percent of the drugs went to pigs, chickens, and cows." How many million pounds of antibiotics went to the pigs, chickens, and cows? *Associated Press-Times-Standard 12/29/09 Pressure rises to stop antibiotics in agriculture.*

29. **Grow faster.** "Approximately 28 million pounds of antibiotics were fed to farm animals in the US during 2008. Thirteen percent of that was fed to healthy animals to make them grow faster." How many pounds of antibiotics were fed to healthy animals? *Associated Press-Times-Standard 12/29/09 Pressure rises to stop antibiotics in agriculture.*

30. **CO2 emissions.** The accord agreed to by the US at the Copenhagen climate talks had greenhouse gas emissions held to 3.5% of 1990 levels. If 1990 levels were 5022 MMT (millions of metric tons), how many millions of metric tons might greenhouse emissions be held to? Round the result to the nearest MMT. *Associated Press-Times-Standard 12/19/09 Elements of new Copenhagen accord.*

31. **Water supply.** A new water desalination plant, the largest in the Western hemisphere, could come online by 2012 in Carlsbad, California, providing 50 million gallons of drinking water per day, or 10% of the supply for San Diego County. What is the total amount of drinking water supplied to San Diego County daily? *Associated Press-Times-Standard*

32. **Earthquake damage.** After the recent earthquake in Chile, an estimated 33 million gallons of Chilean wine, or 13% of annual production, was lost. Estimate the total annual production of Chilean wine rounded to the nearest millions of gallons. *Associated Press-Times Standard 03/24/10 Hemorrhaging cabernet: Earthquake hits winemakers in Chile.*

33. **Snowpack.** At a meadow near Echo Summit in the northern Sierra Nevada, water officials measured the snow at 65.7 inches. The water content was 25.9 inches, which is 92% of the average for this time of year. Determine the average water content for this time of year rounded to the nearest tenth of an inch. *Associated Press-Times Standard 04/02/10 California's Sierra snowpack slightly above normal.*

34. **Storefronts.** According to the Times-Standard, as of April 2008 the Bayshore Mall had 55 occupied storefronts and 17 vacant storefronts. What percent of total storefronts are vacant? Round your answer to the nearest whole number. *Times-Standard 4/19/09*

35. **Recovered.** In Humboldt County, California, 427 of the 499 vehicles stolen between August 2008 and August 2009 were recovered. What percent of the stolen vehicles were recovered? Round your result to the nearest tenth of a percent. *Times-Standard CHP offers tips on avoiding vehicle theft.*

36. **Freshman admissions.** Stanford University sent acceptance letters to 2, 300 of 32, 022 freshman applicants. What percent of freshman applicants got acceptance letters, rounded to the nearest percent? *Associated Press-Times-Standard 03/30/10 Stanford U. reports record-low admission rate.*

37. **Reduce.** Each year, Americans throw out an average of about 1, 600 pounds of waste per person. Arcata, CA resident Michael Winkler only uses one trash bag every year – totaling at most 40 pounds. Find the percent of average annual waste per person Mr. Winkler throws out to a tenth of a percent. *Times-Standard Allison White 12/26/09 Waste not...*

38. **Population decrease.** The table below shows the population of Detroit, Michigan. *Associated Press-Times-Standard 03/09/10 Detroit wants to save itself by shrinking.*

Year	Population
1950	1,849,568
1990	1,027,974
2005	890,963

What is the population of Detroit in 2005 as a percent of the population in 1950? Round your result to the nearest percent.

❧ ❧ ❧ Answers ☙ ☙ ☙

1. 12.4

3. 54

5. 56

7. 90

9. 308 mi

11. 130 students

13. 7.3

15. 140

17. $250

19. $617

21. 257 ml

23. 10.5

25. 57 pounds

27. 16,900

29. 3.84 million pounds

31. 500 million gallons

33. The average water content is 28.2 inches.

35. 85.6% of the stolen vehicles were recovered.

37. Mr. Winkler throws out 2.5% of the average American's waste.

7.4 Percent Increase or Decrease

A person's salary can increase by a percentage. A town's population can decrease by a percentage. A clothing firm can discount its apparel. These are the types of applications we will investigate in this section.

Percent Increase

EXAMPLE 1. A salesperson is granted a 5% salary increase. If the salesperson's current salary is $4,000 per month, what will be his new salary?

Solution. Let x represent the salesperson's salary increase. Then we can translate the problem into words and symbols.

Salary increase	is	5%	of	original salary
x	$=$	5%	$\cdot$	4000

Solve for x.

$$x = 0.05 \cdot 4000 \quad 5\% = 0.05.$$
$$x = 200 \quad \text{Multiply: } 0.05 \cdot 4000 = 200.$$

Therefore, the salary increase is $200. To compute the new salary N, we must add this increase to the original salary.

New salary	is	original salary	plus	increase
N	$=$	4000	$+$	200

Thus, the new salary is $N = \$4,200$ per month.

Alternative Solution. If the salesperson is to receive a 5% increase in his salary, then his new salary will be 105% of his original salary. Let N represent his new monthly salary. Then,

New salary	is	105%	of	original salary
N	$=$	105%	$\cdot$	4000

Solve for N.

$$N = 1.05 \cdot 4000 \quad 105\% = 1.05.$$
$$N = 4200 \quad \text{Multiply: } 1.05 \cdot 4000 = 4200.$$

Same answer. □

You Try It!

A computer technician is granted a 4% salary increase. If the salesperson's current salary is $2,800 per month, what will be his new salary?

Answer: $2,912

You Try It!

A statistician making a salary of $3,200 per month has his salary increased to $3,368 per month. What is the percent increase?

EXAMPLE 2. A salesperson making a salary of $4,500 per month has his salary increased to $5,000 per month. What is the percent increase?

Solution. To find the increase in salary, first subtract the original salary from the new salary.

$$\begin{aligned} \text{Salary increase} &= \text{new salary} - \text{original salary} \\ &= 5000 - 4500 \\ &= 500 \end{aligned}$$

Hence, the salesperson sees an increase in salary of $500.

Next, let p represent the salesperson's percent salary increase. Then we can translate the problem into words and symbols.

Salary increase	is	what percent	of	original salary
500	$=$	p	$\cdot$	4500

The commutative property of multiplication allows us to change the order of multiplication on the right-hand side of this last equation.

$$500 = 4500p$$

Solve for p.

$$\frac{500}{4500} = \frac{4500p}{4500} \quad \text{Divide both sides by 4500.}$$

$$\frac{1}{9} = p \quad \text{Reduce by dividing numerator and denominator of 500/4500 by 500.}$$

We need to change $p = 1/9$ to a percent. We can find an exact answer by creating an equivalent fraction with a denominator of 100.

$$\frac{1}{9} = \frac{n}{100} \quad \text{Make equivalent fraction.}$$

$$9n = 100 \quad \text{Cross multiply.}$$

$$\frac{9n}{9} = \frac{100}{9} \quad \text{Divide both sides by 9.}$$

$$n = 11\frac{1}{9} \quad \text{Convert 100/9 to mixed fraction.}$$

Hence, the percent increase is

$$p = \frac{1}{9} = \frac{11\frac{1}{9}}{100} = 11\frac{1}{9}\%.$$

Alternative Solution. An alternative approach is to ask what percent of the original salary equals the new salary. In this approach, let p represent the percent of the original salary that equals the new salary.

New salary	is	what percent	of	original salary
5000	=	p	$\cdot$	4500

Solve for p.

$$5000 = 4500p \quad \text{Change the order of multiplication.}$$

$$\frac{5000}{4500} = \frac{4500p}{4500} \quad \text{Divide both sides by 4500.}$$

$$\frac{10}{9} = p \quad \text{Reduce: Divide numerator and denominator of 5000/4500 by 500.}$$

We need to change 10/9 to a percent. Again, create an equivalent fraction with a denominator of 100.

$$\frac{10}{9} = \frac{n}{100} \quad \text{Make equivalent fraction.}$$

$$9n = 1000 \quad \text{Cross multiply.}$$

$$\frac{9n}{9} = \frac{1000}{9} \quad \text{Divide both sides by 9.}$$

$$n = 111\,\frac{1}{9} \quad \text{Convert 1000/9 to a mixed fraction.}$$

Thus,

$$p = \frac{10}{9} = \frac{111\frac{1}{9}}{100} = 111\,\frac{1}{9}\%.$$

Hence, the new salary is $111\,\frac{1}{9}\%$ of the original salary. To find the percent increase, subtract 100% from $111\,\frac{1}{9}\%$.

$$111\,\frac{1}{9}\% - 100\% = 11\,\frac{1}{9}\%$$

This represents an $11\,\frac{1}{9}\%$ increase in salary, which is the same answer garnered by the first solution technique.

Answer: $5\frac{1}{4}\%$

Percent Decrease

You Try It!

Several retail stores close and the population of Athens decreases by 7.2% as a result. If the original population was 12,500, what is the new population?

EXAMPLE 3. Due to a mill closure, the population of Silvertown decreases by 8.5%. If the original population was 10,200 hardy souls, what is the new population?

Solution. Let x represent the population decrease. Then we can translate the problem into words and symbols.

Population decrease	is	8.5%	of	original population
x	$=$	8.5%	$\cdot$	10200

Solve for x.

$$x = 0.085 \cdot 10200 \quad 8.5\% = 0.085.$$
$$x = 867 \quad \text{Multiply: } 0.085 \cdot 10200 = 867.$$

Therefore, the population decrease is 867. To compute the new population P, we must subtract this decrease from the original population.

New population	is	original population	minus	population decrease
P	$=$	10200	$-$	867

Thus, the new population is $P = 9,333$ hardy souls.

Alternative Solution. Subtract 8.5% from 100% to obtain

$$100\% - 8.5\% = 91.5\%.$$

Thus, if 8.5% of the population leaves town, then 91.5% of the population stays. Thus, the new population P is calculated from the original as follows:

New population	is	91.5%	of	original population
P	$=$	91.5%	$\cdot$	10200

Solve for P.

$$P = 0.915 \cdot 10200 \quad 91.5\% = 0.915.$$
$$P = 9333 \quad \text{Multiply: } 0.915 \cdot 10200 = 9333.$$

Answer: 11,600

Same answer.

□

You Try It!

A textile mill closure results in the population of the adjacent town decreasing from 8,956 to 7,800. What is the percent decrease in the population, rounded to the nearest tenth of a percent?

EXAMPLE 4. Millertown falls on hard times and its population decreases from 11,256 to 10,923 in the space of one year. What is the percent decrease, rounded to the nearest hundredth of a percent?

Solution. To find the decrease in population, first subtract the current population from the original population.

$$\begin{aligned} \text{Population decrease} &= \text{original population} - \text{current population} \\ &= 11256 - 10923 \\ &= 333 \end{aligned}$$

Hence, the population has decreased by 333 people.

Next, let p represent the percent population decrease. Then we can translate the problem into words and symbols.

Population decrease	is	what percent	of	original population
333	$=$	p	$\cdot$	11256

Solve for p.

$$\begin{aligned} \frac{333}{11256} &= \frac{11256p}{11256} && \text{Divide both sides by 11256.} \\ 0.02958 &\approx p && \text{Divide: } 333/11256 \approx 0.02958. \end{aligned}$$

To change p to a percent, move the decimal point two places to the right and append a percent symbol.

$$p \approx 0.02958 \approx 0\,02.958\% \approx 2.958\%.$$

We are asked to round to the nearest hundredth of a percent.

Test digit

$$p \approx 2.9\boxed{5}\,\boxed{8}\,\%$$

Rounding digit

Because the test digit is greater than or equal to 5, add 1 to the rounding digit and truncate. That is,

$$p \approx 2.96\%.$$

Thus, the population of Millertown decreases approximately 2.96%.

Alternative Solution. An alternative approach is to ask what percent of the original population equals the new population.

New population	is	what percent	of	original population
10923	$=$	p	$\cdot$	11256

Solve for p.

$$\begin{aligned} 10923 &= 11256p && \text{Change the order of multiplication.} \\ \frac{10923}{11256} &= \frac{11256p}{11256} && \text{Divide both sides by 11256.} \\ 0.97041 &\approx p && \text{Divide: } 10923/11256 \approx 0.97041. \end{aligned}$$

To change p to a percent, move the decimal two places to the right and append a percent symbol.

$$p \approx 0.97041 \approx 0\,97.041\% \approx 97.041\%.$$

We are asked to round to the nearest hundredth of a percent.

Test digit

$$p \approx 97.0\boxed{4}\,\boxed{1}\,\%$$

Rounding digit

Because the test digit is less than 5, leave the rounding digit alone and truncate. That is,

$$p \approx 97.04\%.$$

Thus, 97.04% of the Millertown population remains. To find the percent decrease (the percent who left), subtract 97.04% from 100%.

$$100\% - 97.04\% = 2.96\%$$

Hence, the population of Millertown decreases by 2.96%. Same answer. □

Answer: 12.9%

Discount

Another important application of percent is the discounting of goods.

You Try It!

A pair of hiking boots is marked at \$200. During a sale, the boots are discounted by 8%. What is the new price of the boots?

EXAMPLE 5. A pair of skis is marked at \$310. However, a sign in the shop indicates that skis are being discounted at 15%. What will be the new selling price of the skis?

Solution. Let D represent the discount (in dollars) given for our pair of skis. Then, in words and symbols:

Discount	is	15%	of	original marked price
D	$=$	15%	$\cdot$	310

Solve for D.

$$D = 0.15 \cdot 310 \qquad 15\% = 0.15.$$

$$D = 46.50 \qquad \text{Multiply: } 0.15 \cdot 310 = 46.50.$$

Hence, the discount is \$46.50. To find the new selling price, subtract this discount from the original selling price.

$$\begin{aligned} \text{New selling price} &= \text{original selling price} - \text{discount} \\ &= 310 - 46.50 \\ &= 263.50 \end{aligned}$$

Hence, the new selling price is \$263.50.

Alternate Solution. Subtract 15% from 100% to obtain

$$100\% - 15\% = 85\%.$$

That is, if an item is discounted 15%, then its new selling price S is 85% of its original marked price.

New selling price	is	85%	of	original marked price
S	$=$	85%	$\cdot$	310

Solve for S.

$$S = 0.85 \cdot 310 \quad 85\% = 0.85.$$
$$S = 263.50 \quad \text{Multiply: } 0.85 \cdot 310 = 263.50.$$

Thus, the new selling price is \$263.50. Same answer.

Answer: \$184

You Try It!

EXAMPLE 6. A pair of ski boots marked at \$210 is sold for \$180. Find the percent discount, correct to the nearest tenth of a percent.

A computer marked at \$2,000 is sold at a discount for \$1,850. Find the percent discount, correct to the nearest tenth of a percent.

Solution. We can find the discount (in dollars) by subtracting the sale price from the original marked price.

$$\begin{aligned}\text{Discount} &= \text{original marked price} - \text{sale price}\\ &= 210 - 180\\ &= 30\end{aligned}$$

Hence, the boots are discounted \$30.

Let p represent the percent discount. Then, in words and symbols:

Discount	is	percent discount	of	original marked price
30	$=$	p	$\cdot$	210

Solve for p.

$$30 = 210p \quad \text{Change order of multiplication.}$$
$$\frac{30}{210} = \frac{210p}{210} \quad \text{Divide both sides by 210.}$$
$$\frac{1}{7} = p \quad \text{Reduce: Divide numerator and denominator of } 30/210 \text{ by 30.}$$
$$p \approx 0.1428 \quad \text{Divide: } 1/7 \approx 0.1428.$$

To change p to a percent, move the decimal point two places to the right and append a percent symbol.

$$p \approx 0.1428 \approx 0\,14.28\% \approx 14.28\%.$$

Round to the nearest tenth of a percent.

Test digit

$$p \approx 14.\boxed{2}\,\boxed{8}\,\%$$

Rounding digit

Because the test digit is greater than or equal to 5, add 1 to the rounding digit and truncate. Thus, correct to the nearest tenth of a percent, the percent discount is

$$p \approx 14.3\%.$$

Alternate Solution. An alternative approach is to ask what percent p of the original marked price equals the selling price.

New selling price	is	what percent	of	original marked price
180	=	p	$\cdot$	210

Solve for p.

$180 = 210p$	Change the order of multiplication.
$\dfrac{180}{210} = \dfrac{210p}{210}$	Divide both sides by 210.
$\dfrac{6}{7} = p$	Reduce: Divide numerator and denominator of 180/210 by 30.
$p \approx 0.8571$	Divide: $6/7 \approx 0.8571$.

To change p to a percent, move the decimal point two places to the right and append a percent symbol.

$$p \approx 0.8571 \approx 0\,85.71\% \approx 85.71\%.$$

Round to the nearest tenth of a percent.

Test digit

$$p \approx 85.\boxed{7}\,\boxed{1}\,\%$$

Rounding digit

Because the test digit is less than 5, do nothing to the rounding digit and truncate. Thus, correct to the nearest tenth of a percent,

$$p \approx 85.7\%.$$

Thus, the new selling price is 85.7% of the original marked price. Subtract 85.7% from 100%.

$$100\% - 85.7\% = 14.3\%.$$

That is, if the new selling price is 85.7% of the original price, then the percent discount is 14.3%. This is the same answer found with the first method.

Answer: 7.5%

□

Exercises

1. A television set is marked at $447. However, a sign in the shop indicates that the television set is being discounted at 20.5%. What will be the new selling price of the television set? Round your answer to the nearest penny.

2. A stereo set is marked at $380. However, a sign in the shop indicates that the stereo set is being discounted at 7.5%. What will be the new selling price of the stereo set? Round your answer to the nearest penny.

3. Due to a ball bearing plant closure, Anselm falls on hard times and its population decreases from 10,794 to 8,925 in the space of one year. What is the percent decrease, rounded to the nearest hundredth of a percent?

4. Due to a logging mill closure, Carlytown falls on hard times and its population decreases from 12,113 to 10,833 in the space of one year. What is the percent decrease, rounded to the nearest hundredth of a percent?

5. A car rack is marked at $500. However, a sign in the shop indicates that the car rack is being discounted at 3.5%. What will be the new selling price of the car rack? Round your answer to the nearest penny.

6. A car rack is marked at $405. However, a sign in the shop indicates that the car rack is being discounted at 17.5%. What will be the new selling price of the car rack? Round your answer to the nearest penny.

7. Due to a textile mill closure, the population of Silvertown decreases by 4.1%. If the original population was 14,678 hardy souls, what is the new population, correct to the nearest person?

8. Due to a department store closure, the population of Petroria decreases by 5.3%. If the original population was 14,034 hardy souls, what is the new population, correct to the nearest person?

9. A bartender is granted a 4.6% salary increase. If the bartender's current salary is $2,500 per month, find the bartender's new monthly salary, rounded to the nearest dollar.

10. A bartender is granted a 5.5% salary increase. If the bartender's current salary is $2,900 per month, find the bartender's new monthly salary, rounded to the nearest dollar.

11. A car rack marked at $358 is sold for $292. Find the percent discount, correct to the nearest tenth of a percent.

12. A bicycle marked at $328 is sold for $264. Find the percent discount, correct to the nearest tenth of a percent.

13. Due to a auto manufacturing plant closure, Carlytown falls on hard times and its population decreases from 14,393 to 12,623 in the space of one year. What is the percent decrease, rounded to the nearest hundredth of a percent?

14. Due to a ball bearing plant closure, Mayville falls on hard times and its population decreases from 8,494 to 6,609 in the space of one year. What is the percent decrease, rounded to the nearest hundredth of a percent?

15. Due to a auto manufacturing plant closure, the population of Silvertown decreases by 2.4%. If the original population was 8,780 hardy souls, what is the new population, correct to the nearest person?

16. Due to a textile mill closure, the population of Ghosttown decreases by 6.1%. If the original population was 14,320 hardy souls, what is the new population, correct to the nearest person?

17. A clerk making a salary of $2,600 per month has her salary increased to $2,950 per month. Find the percent increase correct to the nearest tenth of a percent.

18. A clerk making a salary of $3,600 per month has her salary increased to $4,100 per month. Find the percent increase correct to the nearest tenth of a percent.

19. A bartender making a salary of $4,200 per month has her salary increased to $4,300 per month. Find the percent increase correct to the nearest tenth of a percent.

20. A bartender making a salary of $3,200 per month has her salary increased to $3,550 per month. Find the percent increase correct to the nearest tenth of a percent.

21. A gardener is granted a 5.1% salary increase. If the gardener's current salary is $3,200 per month, find the gardener's new monthly salary, rounded to the nearest dollar.

22. A secretary is granted a 2.8% salary increase. If the secretary's current salary is $3,600 per month, find the secretary's new monthly salary, rounded to the nearest dollar.

23. A television set marked at $437 is sold for $347. Find the percent discount, correct to the nearest tenth of a percent.

24. A camera marked at $390 is sold for $328. Find the percent discount, correct to the nearest tenth of a percent.

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25. Suppose that the price of an 8-ounce can of tomato sauce increased from $0.20 to $0.28.

a) What was the amount of increase?

b) What was the percent increase?

26. The following table summarizes summertime gasoline prices in San Francisco, CA. The price is the number of dollars required to purchase one gallon of unleaded gasoline. *Data from gasbuddy.com.*

Year	Price per gallon
2003	1.80
2004	2.28
2005	2.57
2006	3.20
2007	3.28
2008	4.61
2009	3.01

What is the percent increase or decrease from 2003 to 2005? Round your answer to the nearest whole percent.

27. Refer to the table of gas prices in Exercise 26. What is the percent increase or decrease from 2005 to 2007? Round your answer to the nearest whole percent.

28. Refer to the table of gas prices in Exercise 26. What is the percent increase or decrease from 2007 to 2009? Round your answer to the nearest whole percent.

29. **Rate hike.** A family paying \$858 monthly for health coverage is faced with a 39% hike in rates. What will be their new monthly cost after the increase? *Associated Press-Times-Standard 02/09/10 HHS secretary asks Anthem Blue Cross to justify rate hike.*

30. **Parking fine.** San Francisco's Metropolitan Transportation Agency was expected to consider raising fines for the use of fake, stolen, or borrowed disabled parking tags from \$100 to \$825. What is the percent increase for this fine? *Associated Press-Times-Standard 01/06/10 Fines for fake disabled parking tags may go up in San Francisco.*

31. **Industrial move.** Regulations in California have caused factory farmers to move out of state. Idaho's industrial dairy went from 180,000 cows in 1990 to 530,000 cows in 2009. What is the percent increase for Idaho industrial dairy cows, rounded to the nearest percent? *Associated Press-Times-Standard 02/09/10 Idaho, others prepare for California egg farm exodus.*

32. **Whooping crane.** Whooping crane populations made a remarkable comeback from just 15 birds in 1941 to about 400 birds worldwide in 2010. What is the percent increase for the whooping crane population over the past seventy years, rounded to the nearest percent? *Associated Press-Times-Standard 03/17/10 Plucky whooping crane gives wildlife experts hope.*

33. **Underwater.** The loss of Arctic sea ice will allow for an underwater fiber optic cable that will cut the time it takes to send a message from London to Tokyo from a current 140 milliseconds down to 88 milliseconds. *Associated Press-Times-Standard 01/22/10 Global warming opens up Northwest Passage for underwater fiber optic cable.*

a) What is the estimated percent decrease in communication time?

b) At a cost of \$1.2 billion, what is the cost per millisecond of saving (rounded to the nearest dollar)?

34. **Chinook salmon.** During the 2009 season in the Sacramento river basin, a record low 39,500 chinook salmon were recorded, way down from the more than 750,000 counted in 2002. What is the percent decrease in the salmon count from 2002 to 2009, rounded to the nearest percent? *Associated Press-Times-Standard 02/25/10 Feds predict better year for California salmon.*

35. **Daylight hours.** In middle latitudes, summer days can have as many as 14 hours of daylight, while winter days can have a few as 10 hours of daylight. What percent more daylight is there in summer than in winter?

36. **Cyber-experts.** Defense Secretary Robert Gates said the Pentagon will increase the number of cyber-experts it can train each year from 80 to 250 by 2011. What percent increase is this? Round your answer to the nearest percent. *Associated Press Times-Standard 4/19/09*

37. **Home prices.** Real estate data for Humboldt County, California, is given below. *Associated Press-Times-Standard 01/17/10 How is our local real estate market compared to other regions?*

Year	Number of homes sold	Average home price
2000	1,358	\$152,257
2005	1,432	\$344,500
2009	833	\$285,000

a) What percent change in average home price occurred between 2000 and 2009?

b) What percent change in homes sold occurred between 2000 and 2009?

❧ ❧ ❧ Answers ☙ ☙ ☙

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1. \$355.36

3. 17.32%

5. \$482.50

7. 14,076 people

9. \$2,615

11. 18.4%

13. 12.30%

15. 8,569

17. 13.5%

19. 2.4%

21. \$3,363

23. 20.6%

25.
a) \$0.08
b) 40%

27. 28% increase

29. \$1,193

31. 194% increase

33.
a) 37%
b) About \$23,076,923 per millisecond

35. 40% more daylight

37.
a) 87% increase in average home price
b) 39% decrease in home sales

7.5 Interest

One way of awarding interest is called *simple interest*. Before we provide the formula used in calculating simple interest, let's first define some basic terms.

Balance. The balance is the current amount in an account or the current amount owed on a loan.

Principal. The principal is the initial amount invested or borrowed.

Rate. This is the interest rate, usually given as a percent per year.

Time. This is the time duration of the loan or investment. If the interest rate is per year, then the time must be measured in years.

To calculate the simple interest on an account or loan, use the following formula.

Simple Interest. Simple interest is calculated with the formula

$$I = Prt,$$

where I is the interest, P is the principal, r is the interest rate, and t is the time.

You Try It!

How much simple interest is earned if \$2,500 is invested at 5% per year for 8 years?

EXAMPLE 1. How much simple interest is earned if \$1,200 is invested at 4% per year for 5 years?

Solution. Set up the formula for simple interest.

$$I = Prt$$

The principal is $P = \$1200$, the interest rate is $r = 4\% = 0.04$ per year, and the time or duration of the investment is $t = 5$ years. Substitute each of these numbers into the simple interest formula $I = Prt$.

$$\begin{aligned} I &= (1200)(0.04)(5) && \text{Substitute 1200 for } P\text{, 0.04 for } r\text{, and 5 for } t. \\ &= 240 && \text{Multiply.} \end{aligned}$$

Hence, the interest earned in 5 years is \$240.

Answer: \$1,000

□

To find the balance, we must add the interest to the principal.

Calculating the Balance. To find the balance, add the interest to the principal. That is,

$$\text{Balance} = \text{Principal} + \text{Interest}.$$

You Try It!

EXAMPLE 2. A contractor borrows \$5,000 at 4.5% per year. The interest accrued is simple interest. The duration of the loan is 6 months. How much will the contractor have to pay back at the end of the 6-month loan period?

An accountant borrows \$8,000 at 5.5% per year. The interest accrued is simple interest. The duration of the loan is 3 months. How much will the aacountant have to pay back at the end of the 3-month loan period?

Solution. Set up the formula for simple interest.

$$I = Prt$$

The principal is $P = \$5000$, the interest rate is $r = 4.5\% = 0.045$ per year, and the time or duration of the loan is $t = 6$ months. Because the interest rate is per year, the time must be changed to years. That is,

$$\begin{aligned}
6\,\text{months} &= 6\,\text{months} \cdot \frac{1\,\text{yr}}{12\,\text{months}} && \text{Apply conversion factor.}\\
&= 6\,\cancel{\text{months}} \cdot \frac{1\,\text{yr}}{12\,\cancel{\text{months}}} && \text{Cancel common unit.}\\
&= \frac{6}{12}\,\text{yr} && \text{Multiply numerators; multiply denominators.}\\
&= \frac{1}{2}\,\text{yr} && \text{Reduce.}
\end{aligned}$$

Substitute these numbers into the simple interest formula $I = Prt$.

$$\begin{aligned}
I &= (5000)(0.045)\left(\frac{1}{2}\right) && \text{Substitute 5000 for } P\text{, 0.045 for } r\text{, and 1/2 for } t.\\
&= 112.50 && \text{Multiply.}
\end{aligned}$$

Hence, the interest accrued in 6 months is \$112.50. Therefore,

$$\begin{aligned}
\text{Amount owed} &= \text{Principal} + \text{Interest}\\
&= 5000 + 112.50\\
&= 5112.50
\end{aligned}$$

That is, the amount owed at the end of the 6-month loan period is \$5,112.50. □

Answer: \$8,110

You Try It!

The owner of Alioto Motors takes out a 8-month loan at 4% per year simple interest. At the end of the 8-month loan period, the interest owed is \$80. What was the principal amount borrowed?

EXAMPLE 3. A business owner takes out a 4-month loan at 5.4% per year simple interest. At the end of the 4-month loan period, the interest owed is \$90. What was the principal amount borrowed?

Solution. Set up the formula for simple interest.

$$I = Prt$$

The interest owed is $I = \$90$, the interest rate is $r = 5.4\% = 0.054$ per year, and the time or duration of the loan is $t = 4$ months. Because the interest rate is per year, the time must be changed to years. That is,

$$\begin{aligned}
4\,\text{months} &= 4\,\text{months} \cdot \frac{1\,\text{yr}}{12\,\text{months}} && \text{Apply conversion factor.}\\
&= 4\,\cancel{\text{months}} \cdot \frac{1\,\text{yr}}{12\,\cancel{\text{months}}} && \text{Cancel common unit.}\\
&= \frac{4}{12}\,\text{yr} && \text{Multiply numerators; multiply denominators.}\\
&= \frac{1}{3}\,\text{yr} && \text{Reduce.}
\end{aligned}$$

Substitute these numbers into the simple interest formula $I = Prt$.

$$\begin{aligned}
90 &= P(0.054)\left(\frac{1}{3}\right) && \text{Substitute 90 for } I,\ 0.054 \text{ for } r, \text{ and } 1/3 \text{ for } t.\\
90 &= \frac{0.054}{3}P && \text{Rearrange order of multiplication.}\\
90 &= 0.018P && \text{Divide: } 0.054/3 = 0.018.
\end{aligned}$$

Solve the equation for P.

$$\begin{aligned}
\frac{90}{0.018} &= \frac{0.018P}{0.018} && \text{Divide both sides by 0.018.}\\
5000 &= P && \text{Divide: } 90/0.018 = 5000.
\end{aligned}$$

Thus, the principal amount borrowed was \$5,000. □

Answer: \$3,000

You Try It!

A manufacturer borrows \$10,000 for 4 years. At the end of the 4-year loan period, the interest owed is \$3,200. What was the simple interest rate?

EXAMPLE 4. A pet shop owner borrows \$8,000 for 6 months. At the end of the 6-month loan period, the interest owed is \$200. What was the simple interest rate?

Solution. Set up the formula for simple interest.

$$I = Prt$$

The principal is $P = \$8,000$, the interest owed is $I = \$200$, and the duration of the loan is $t = 6$ months. As we saw in Example 2, 6 months equals 1/2 year. Substitute these numbers into the simple interest formula $I = Prt$.

$$200 = (8000)(r)\left(\frac{1}{2}\right) \quad \text{Substitute 8000 for } P\text{, 200 for } I\text{, and 1/2 for } t.$$

$$200 = \frac{8000}{2}r \quad \text{Rearrange order of multiplication.}$$

$$200 = 4000r \quad \text{Divide: } 8000/2 = 4000.$$

Solve this last equation for r.

$$\frac{200}{4000} = \frac{4000r}{4000} \quad \text{Divide both sides by 4000.}$$

$$\frac{1}{20} = r \quad \text{Reduce: Divide numerator and denominator by 200.}$$

We need to change r to a percent. This is easily accomplished by creating an equivalent fraction with a denominator of 100.

$$\begin{aligned} \frac{1}{20} &= \frac{1 \cdot 5}{20 \cdot 5} \\ &= \frac{5}{100} \\ &= 5\% \end{aligned}$$

Thus, the simple interest rate is 5%. Answer: 8%

□

Extending the Simple Interest Formula

In Example 2, we had to add the interest to the principal to discover the balance owed at the end of the loan. That is,

$$\text{Balance} = \text{Principal} + \text{Interest},$$

or in symbols,

$$A = P + I,$$

where A is the balance, P is the principal, and I is the simple interest. Because $I = Prt$, we substitute Prt for I in the last equation to get

$$A = P + Prt.$$

Use the distributive property to factor P from each term on the right.

$$\begin{aligned} A &= P \cdot 1 + P \cdot rt \\ A &= P(1 + rt). \end{aligned}$$

Balance Formula Using Simple Interest. If simple interest is applied, then the balance is given by the formula

$$A = P(1 + rt),$$

where A is the balance, P is the principal, r is the simple interest rate, and t is the duration of the loan or investment.

You Try It!

If $8,000 is invested at 4.25% simple interest, what will be the balance after 4 years?

EXAMPLE 5. If $4,000 is invested at 6.25% simple interest, what will be the balance after 2 years?

Solution. Start with the balance formula for simple interest.

$$A = P(1 + rt)$$

The principal is $P = \$4,000$, the rate is $r = 6.25\% = 0.0625$ per year, and the time is $t = 2$ years. Substitute these numbers in the balance formula $A = P(1 + rt)$.

$A = 4000\big(1 + (0.0625)(2)\big)$	Substitute 4000 for P, 0.0625 for r, and 2 for t.
$A = 4000(1 + 0.125)$	Order of Ops: $0.0625 \cdot 2 = 0.125$.
$A = 4000(1.125)$	Order of Ops: $1 + 0.125 = 1.125$.
$A = 4500$	Multiply: $4000 \cdot 1.125 = 4500$.

Hence, the balance at the end of two years is $A = \$4,500$.

$9,360

You Try It!

The balance due on a 4-year loan is $6,300. If the simple interest rate is 10%, what was the principal borrowed?

EXAMPLE 6. The balance due on a 2-year loan is $3,360. If the simple interest rate is 6%, what was the principal borrowed?

Solution. Start with the balance formula for simple interest.

$$A = P(1 + rt)$$

The balance is $A = \$3360$, the rate is $r = 6\% = 0.06$ per year, and the time is $t = 2$ years. Substitute these numbers in the balance formula $A = P(1 + rt)$.

$3360 = P\big(1 + (0.06)(2)\big)$	Substitute 3360 for A, 0.06 for r, and 2 for t.
$3360 = P(1 + 0.12)$	Order of Ops: $0.06 \cdot 2 = 0.12$.
$3360 = P(1.12)$	Order of Ops: $1 + 0.12 = 1.12$.
$3360 = 1.12P$	Change order of multiplication.

Solve this last equation for P.

$$\frac{3360}{1.12} = \frac{1.12P}{1.12}$$ Divide both sides by 1.12.

$$3000 = P$$ Divide: $3360/1.12 = 3000$.

Hence, the principal borrowed was $P = \$3,000$.

Answer: $4,500

You Try It!

EXAMPLE 7. The balance due on a 2-year loan is $2,200. If the principal borrowed was $2,000, what was the rate of simple interest?

The balance due on a 2-year loan is $4,640. If the principal borrowed was $4,000, what was the rate of simple interest?

Solution. Start with the balance formula for simple interest.

$$A = P(1 + rt)$$

The balance is $A = \$2,200$, the principal is $P = \$2,000$, and the time is $t = 2$ years. Substitute these numbers in the balance formula $A = P(1 + rt)$.

$$2200 = 2000(1 + (r)(2))$$ Substitute 2200 for A, 2000 for P, and $t = 2$.

$$2200 = 2000(1 + 2r)$$ Change the order of muliplication.

Solve this last equation for r.

$$2200 = 2000 + 4000r$$ Distribute 2000.

$$2200 - 2000 = 2000 + 4000r - 2000$$ Subtract 2000 from both sides.

$$200 = 4000r$$ Simplify both sides.

$$\frac{200}{4000} = \frac{4000r}{4000}$$ Divide both sides by 4000.

$$\frac{1}{20} = r$$ Reduce: $200/4000 = 1/20$.

Of course, r must be changed to a percent. In Example 4, we encountered this same fraction.

$$r = \frac{1}{20} = \frac{5}{100} = 5\%$$

Hence, the rate of simple interest is $r = 5\%$.

Answer: 8%

Exercises

1. How much simple interest is earned if \$7,600 is invested at 8% per year for 7 years?

2. How much simple interest is earned if \$2,500 is invested at 5% per year for 6 years?

3. How much simple interest is earned if \$5,800 is invested at 3.25% per year for 4 years?

4. How much simple interest is earned if \$2,000 is invested at 8.5% per year for 6 years?

5. How much simple interest is earned if \$2,400 is invested at 8.25% per year for 5 years?

6. How much simple interest is earned if \$4,000 is invested at 6.5% per year for 6 years?

7. How much simple interest is earned if \$4,000 is invested at 7.25% per year for 6 years?

8. How much simple interest is earned if \$8,200 is invested at 8% per year for 6 years?

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9. A business owner borrows \$3,600 for 2 months at a 4.5% per year simple interest rate. At the end of the 2-month loan period, how much interest is owed?

10. A business owner borrows \$3,200 for 4 months at a 9% per year simple interest rate. At the end of the 4-month loan period, how much interest is owed?

11. A business owner borrows \$2,400 for 6 months at a 2% per year simple interest rate. At the end of the 6-month loan period, how much interest is owed?

12. A business owner borrows \$2,200 for 4 months at a 3% per year simple interest rate. At the end of the 4-month loan period, how much interest is owed?

13. A business owner takes out a 6-month loan at a 8% per year simple interest rate. At the end of the 6-month loan period, the interest owed is \$68. What was the principal amount borrowed?

14. A business owner takes out a 4-month loan at a 6% per year simple interest rate. At the end of the 4-month loan period, the interest owed is \$194. What was the principal amount borrowed?

15. A business owner borrows \$3,600 for 3 months at a 8% per year simple interest rate. At the end of the 3-month loan period, how much interest is owed?

16. A business owner borrows \$2,400 for 4 months at a 8.25% per year simple interest rate. At the end of the 4-month loan period, how much interest is owed?

17. A business owner takes out a 2-month loan at a 8.5% per year simple interest rate. At the end of the 2-month loan period, the interest owed is \$85. What was the principal amount borrowed?

18. A business owner takes out a 3-month loan at a 2% per year simple interest rate. At the end of the 3-month loan period, the interest owed is \$45. What was the principal amount borrowed?

19. A business owner borrows \$4,000 for 3 months. At the end of the 3-month loan period, the interest owed is \$35. What was the simple yearly interest rate (as a percent)?

20. A business owner borrows \$4,200 for 4 months. At the end of the 4-month loan period, the interest owed is \$63. What was the simple yearly interest rate (as a percent)?

21. A business owner takes out a 6-month loan at a 7% per year simple interest rate. At the end of the 6-month loan period, the interest owed is \$287. What was the principal amount borrowed?

22. A business owner takes out a 6-month loan at a 2% per year simple interest rate. At the end of the 6-month loan period, the interest owed is \$40. What was the principal amount borrowed?

23. A business owner borrows \$7,300 for 2 months. At the end of the 2-month loan period, the interest owed is \$73. What was the simple yearly interest rate (as a percent)?

24. A business owner borrows \$5,600 for 6 months. At the end of the 6-month loan period, the interest owed is \$182. What was the simple yearly interest rate (as a percent)?

25. A business owner borrows \$3,200 for 6 months. At the end of the 6-month loan period, the interest owed is \$96. What was the simple yearly interest rate (as a percent)?

26. A business owner borrows \$5,700 for 4 months. At the end of the 4-month loan period, the interest owed is \$133. What was the simple yearly interest rate (as a percent)?

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27. Suppose that \$6,700 is invested at 9% simple interest per year. What will the balance be after 4 years?

28. Suppose that \$5,200 is invested at 3.5% simple interest per year. What will the balance be after 2 years?

29. Suppose that \$1,600 is invested at 2% simple interest per year. What will the balance be after 3 years?

30. Suppose that \$8,100 is invested at 8.25% simple interest per year. What will the balance be after 4 years?

31. Suppose that \$8,900 is invested at 2.5% simple interest per year. What will the balance be after 2 years?

32. Suppose that \$9,800 is invested at 2.75% simple interest per year. What will the balance be after 6 years?

33. Suppose that \$5,400 is invested at 4.25% simple interest per year. What will the balance be after 2 years?

34. Suppose that \$8,400 is invested at 4.5% simple interest per year. What will the balance be after 4 years?

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35. The balance on a 6-year loan is $10,222. If the principal borrowed was $7,600, what was the simple interest rate (as a percent)?

36. The balance on a 8-year loan is $12,264. If the principal borrowed was $8,400, what was the simple interest rate (as a percent)?

37. The balance on a 5-year loan is $4,640. If the simple interest rate is 9% per year, what was the principal borrowed?

38. The balance on a 6-year loan is $6,838. If the simple interest rate is 5.25% per year, what was the principal borrowed?

39. The balance on a 9-year loan is $9,593. If the simple interest rate is 9% per year, what was the principal borrowed?

40. The balance on a 8-year loan is $10,032. If the simple interest rate is 4% per year, what was the principal borrowed?

41. The balance on a 3-year loan is $5,941. If the principal borrowed was $5,200, what was the simple interest rate (as a percent)?

42. The balance on a 2-year loan is $9,589. If the principal borrowed was $8,600, what was the simple interest rate (as a percent)?

43. The balance on a 5-year loan is $5,400. If the principal borrowed was $4,000, what was the simple interest rate (as a percent)?

44. The balance on a 6-year loan is $12,635. If the principal borrowed was $9,500, what was the simple interest rate (as a percent)?

45. The balance on a 5-year loan is $11,550. If the simple interest rate is 7.5% per year, what was the principal borrowed?

46. The balance on a 8-year loan is $3,160. If the simple interest rate is 7.25% per year, what was the principal borrowed?

47. The balance on a 4-year loan is $5,720. If the principal borrowed was $4,400, what was the simple interest rate (as a percent)?

48. The balance on a 8-year loan is $4,422. If the principal borrowed was $3,300, what was the simple interest rate (as a percent)?

49. The balance on a 8-year loan is $9,768. If the simple interest rate is 4% per year, what was the principal borrowed?

50. The balance on a 2-year loan is $8,322. If the simple interest rate is 7% per year, what was the principal borrowed?

Answers

1. $4,256

3. $754

5. $990

7. $1,740

9. $27

11. $24

13. $1,700

15. $72

17. $6,000

19. 3.5%

21. \$8, 200

23. 6%

25. 6%

27. \$9, 112

29. \$1, 696

31. \$9, 345

33. \$5, 859

35. 5.75%

37. \$3, 200

39. \$5, 300

41. 4.75%

43. 7%

45. \$8, 400

47. 7.5%

49. \$7, 400

7.6 Pie Charts

In this section we turn our attention to pie charts, but before we do, we need to establish some fundamentals regarding measurement of angles.

If you take a circle and divide it into 360 equal increments, then each increment is called *one degree* (1°). See Figure 7.2.

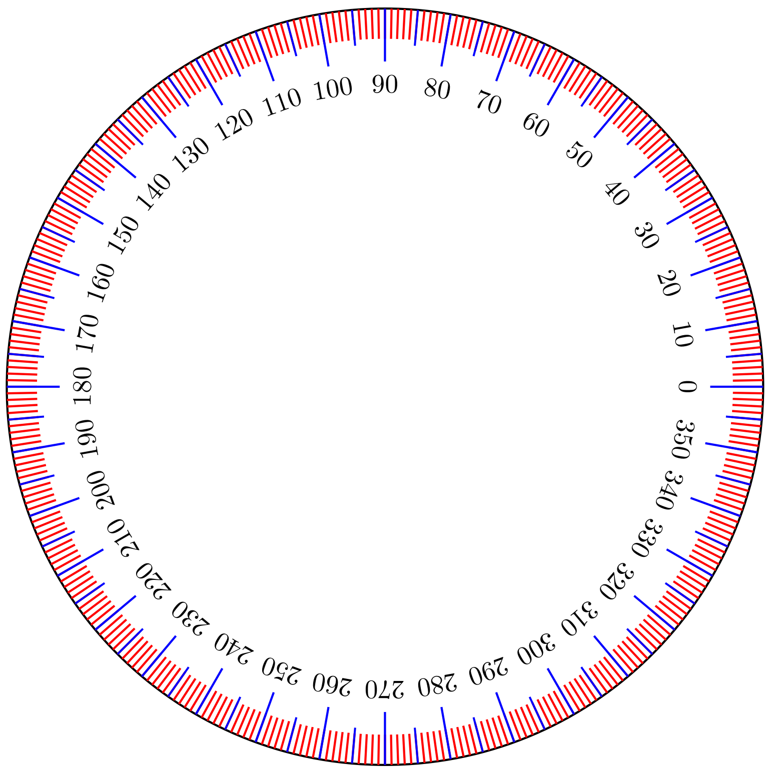


Figure 7.2: There are 360 degrees (360°) in a cricle.

A *ray* is a line that starts at a point and then extends indefinitely in one direction. The starting point of the ray is called its *vertex*.

Figure 7.3: A ray with vertex V extends indefinitely in one direction.

If two rays have a common vertex, they form what is called an *angle*. In Figure 7.4 we've labeled the first ray as the "Initial Side" of the angle, and the second as the "Terminal Side" of the angle.

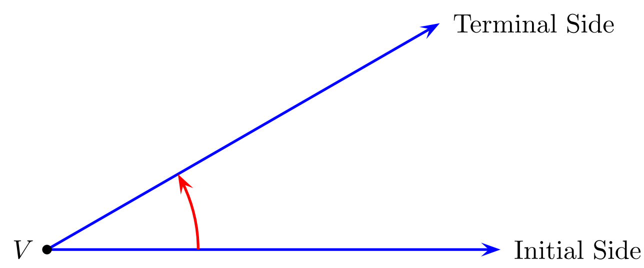


Figure 7.4: Two rays with a common vertex V form an angle.

We can find the degree measure of the angle by using a device called a *protractor*. Align the notch in the center of the base of the protractor with the vertex of the angle, then align the base of the protractor with the initial side of the angle. The terminal side of the angle will intersect the protractor edge where we can read the degree measure of the angle (see Figure 7.5). In Figure 7.5, note that the terminal side of the angle passes through the tick mark at the number 30, indicating that the degree measure of this angle is 30°.

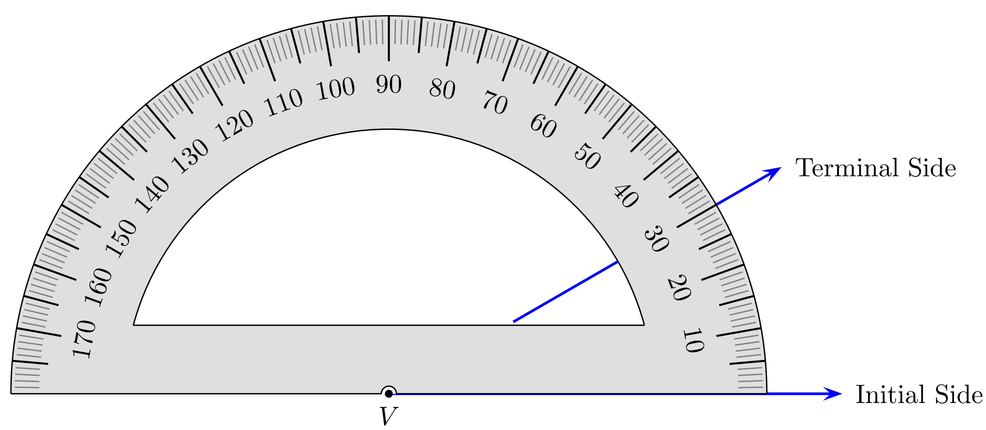


Figure 7.5: The degree measure is 30°.

Pie Charts

Now that we can measure angles, we can turn our attention to constructing *pie charts*.

Pie Chart. A pie chart is a circular chart that is divided into sectors, each sector representing a particular quantity. The area of each sector is a percentage of the area of the whole circle and is determined by calculating the quantity it represents as a percentage of the whole.

You Try It!

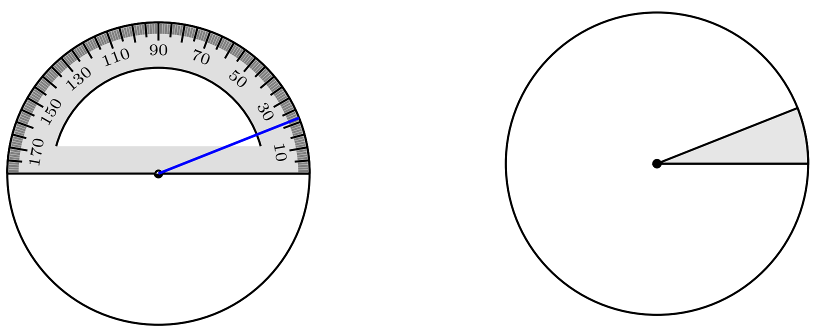
In a recent Gallup poll, 50% of the American public say it is the federal government's responsibility to make sure all Americans have health care, 47% disagree, and 3% were undecided. Create a pie chart that displays these percentages.

EXAMPLE 1. In a recent Gallup poll, 66% of those polled approved of the President's job performance, 28% disapproved, and 6% were undecided. Create a pie chart that displays these percentages.

Solution. Let's begin with 6% undecided. We want to create a sector that is 6% of the area of the whole circle. There are 360 degrees in a full circle, so 6% of this number is

$$\begin{aligned} 6\% \cdot 360^\circ &= 0.06 \cdot 360^\circ \\ &= 21.6^\circ. \end{aligned}$$

Start with a circle, set the baseline notch of the protractor on the center of the circle, then mark an angle of 21.6°, as seen in Figure 7.6(a). Shade the resulting region as shown in Figure 7.6(b), called a *sector*, which represents 6% of the total area of the circle.



(a) Mark an angle of 21.6°.

(b) Shaded sector is 6% of total circular area.

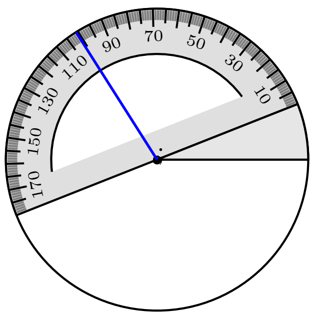
Figure 7.6: Sector with central angle 21.6° represents the 6% of the polling sample that were undecided about the president's performance.

Next, 28% disapproved of the President's job performance. Thus,

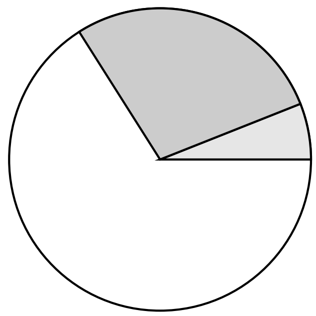
$$\begin{aligned} 28\% \cdot 360^\circ &= 0.28 \cdot 360^\circ \\ &= 100.8^\circ. \end{aligned}$$

Therefore, a sector with a central angle of 100.8° will represent 28% job disapproval.

Place the notch on the baseline of your protractor on the center of the circle, then align the baseline of the protractor with the terminal side of the first angle, as shown in Figure 7.7(a). Mark a central angle of 100.8°, as shown in Figure 7.7(a). Shade the resulting second sector with a darker shade of gray, as shown in Figure 7.7(b). This sector contains 28% of the total area of the circle and represents the portion of the polling sample that disapproved of the president's job performance.



(a) Mark the next angle at 100.8°.



(b) Second shaded sector is 28% of total circular area.

Figure 7.7: Sector with central angle 100.8° represents the 28% of the polling sample that disapproved of the president's performance.

Finally, as we've shaded the sectors representing 6% and 28% of the polling data in Figure 7.7(b), the remaining sector in Figure 7.7(b), shaded in white, represents the 66% of the polling sample who approved of the president's job performance (and 66% of the area of the whole circle).

Once you have computed and plotted the correct central angles for each of the sectors, you will want to label your pie chart. One possible annotation method is shown in Figure 7.8.

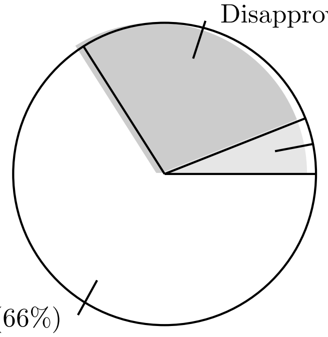
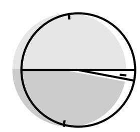


Figure 7.8: Annotating the final pie chart.

Answer:



You Try It!

Two hundred people were asked whether they vote "Yes" or "No" on Proposition 8. There were 150 "Yes" votes and 50 "No" votes. Create a pie chart showing the distribution of these responses.

EXAMPLE 2. One thousand people were polled with the question "Where does your dog sleep during the night?" The responses are shown in the following table.

Location	**Number**
Outside	30
Another Room	220
On the Bedroom Floor	330
On the Bed	420
Totals	1000

Create a pie chart showing the distribution of these responses.

Solution. The first step is to express the number in each location as a percentage of the totals. For example,

Outside	is	what percent	of	total
30	=	p	$\cdot$	1000

Solving for p,

$$\frac{1000p}{1000} = \frac{30}{1000} \qquad \text{Divide both sides by 1000.}$$

$$p = 0.03 \qquad \text{Divide: } 30/1000 = 0.03.$$

Thus, $p = 3\%$. In similar fashion, divide the number in each location by 1000 to find the following percents.

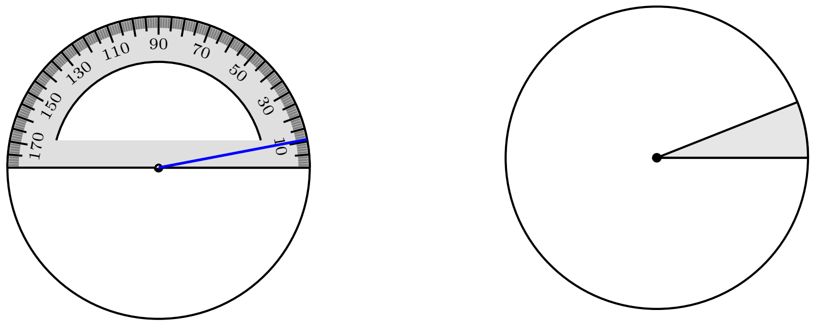
Location	**Number**	**Percent**
Outside	30	3%
Another Room	220	22%
On the Bedroom Floor	330	33%
On the Bed	420	42%
Totals	1000	100%

Note that the individual percents must total 100%.

Let's begin with the fact that 3% of the dog owners have their dogs sleep outside. To find the portion of the full circle that represents 3%, we take 3% of 360°.

$$\begin{aligned} 3\% \cdot 360° &= 0.03 \cdot 360° \\ &= 10.8° \end{aligned}$$

Start with a circle, set the baseline notch of the protractor on the center of the circle, then mark an angle of 10.8°, as seen in Figure 7.9(a). Shade the



(a) Mark an angle of 10.8°.

(b) Shaded sector is 3% of total circular area.

Figure 7.9: Sector with central angle 10.8° represents the 3% of the polling sample that have their dogs sleep outside.

resulting sector as shown in Figure 7.9(b), which represents 3% of the total area of the circle.

Next, 22% have their dog sleep in another room.

$$\begin{aligned} 22\% \cdot 360° &= 0.22 \cdot 360° \\ &= 79.2°. \end{aligned}$$

Therefore, a sector with a central angle of 79.2° will represent the fact that 22% of the dog owners have their dog sleep in another room.

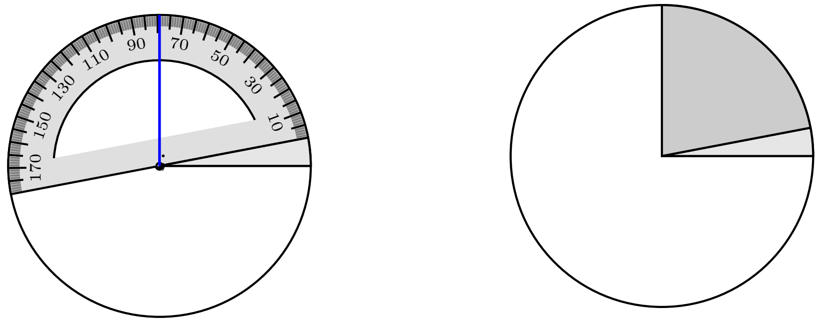
Place the notch on the baseline of your protractor on the center of the circle, then align the baseline of the protractor with the terminal side of the first angle, as shown in Figure 7.10(a). Mark a central angle of 79.2°, as shown in Figure 7.7(a). Shade the resulting second sector with a darker shade of gray, as shown in Figure 7.10(b). This sector contains 22% of the total area of the circle and represents the portion of the polling sample whose dog sleeps in another room.

Next, 33% allow their dog sleep on the bedroom floor.

$$\begin{aligned} 33\% \cdot 360° &= 0.33 \cdot 360° \\ &= 118.8°. \end{aligned}$$

Therefore, a sector with a central angle of 118.8° will represent the fact that 33% of the dog owners allow their dog to sleep on the bedroom floor.

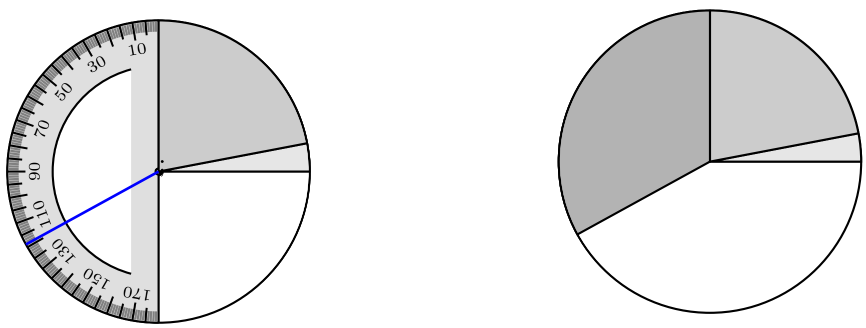
Place the notch on the baseline of your protractor on the center of the circle, then align the baseline of the protractor with the terminal side of the second sector, as shown in Figure 7.11(a). Mark a central angle of 118.8°, as shown in Figure 7.11(a). Shade the resulting second sector with a darker shade of gray, as shown in Figure 7.11(b). This sector contains 33% of the total area of the circle and represents the portion of the polling sample whose dog sleeps on the bedroom floor.



(a) Mark the next angle at 79.2°.

(b) Second shaded sector is 22% of total circular area.

Figure 7.10: Sector with central angle 79.2° represents the 22% of the polling sample whose dogs sleep in another room.



(a) Mark the next angle at 118.8°.

(b) Third shaded sector is 33% of total circular area.

Figure 7.11: Sector with central angle 118.8° represents the 33% of the polling sample whose dogs sleep on the bedroom floor.

Because the first three sectors, shaded in various levels of gray, represent 3%, 22%, and 33% of the total circular area, respectively, the remaining sector (shaded in white) automatically represents

$$100\% - (3\% + 22\% + 33\%) = 42\%$$

of the total circular area. This region represents the percent of dog owners who allow their dogs to sleep on the bed. The final result, with annotations, is shown in Figure 7.12.

Answer:

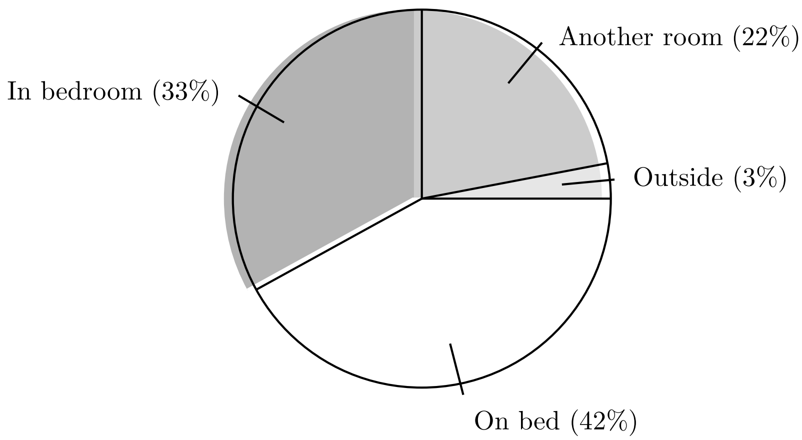
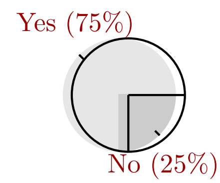


Figure 7.12: Annotating the final pie chart.



□

❧ ❧ ❧ Exercises ☙ ☙ ☙

1. In an election for class president, Raven received 21% of the votes, Anita received 27%, Jamal received 24% of the votes, and 28% of the votes were received by other candidates. If a pie chart is created for this data, what will be the degree measure of the central angle of the sector representing Raven's share of the vote? Round your answer to the nearest degree.

2. In an election for class president, Fernando received 26% of the votes, Luisa received 26%, Ali received 26% of the votes, and 22% of the votes were received by other candidates. If a pie chart is created for this data, what will be the degree measure of the central angle of the sector representing Fernando's share of the vote? Round your answer to the nearest degree.

3. In an election for class president, Akbar received 23% of the votes, Ali received 27%, Juanita received 30% of the votes, and 20% of the votes were received by other candidates. If a pie chart is created for this data, what will be the degree measure of the central angle of the sector representing Akbar's share of the vote? Round your answer to the nearest degree.

4. In an election for class president, Kamili received 21% of the votes, Bernardo received 22%, Fernando received 30% of the votes, and 27% of the votes were received by other candidates. If a pie chart is created for this data, what will be the degree measure of the central angle of the sector representing Kamili's share of the vote? Round your answer to the nearest degree.

5. In an election for class president, Jamal received 30% of the votes, Luisa received 20%, Kamili received 28% of the votes, and 22% of the votes were received by other candidates. If a pie chart is created for this data, what will be the degree measure of the central angle of the sector representing Jamal's share of the vote? Round your answer to the nearest degree.

6. In an election for class president, Juanita received 30% of the votes, Ali received 24%, Estevan received 24% of the votes, and 22% of the votes were received by other candidates. If a pie chart is created for this data, what will be the degree measure of the central angle of the sector representing Juanita's share of the vote? Round your answer to the nearest degree.

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7. In an election for class president, Chin received 5 votes, Mabel received 13 votes, and Juanita received the remaining 32 votes cast. If a pie chart is created for this voting data, what will be the degree measure of the central angle of the sector representing Chin's share of the vote? Round your answer to the nearest degree.

8. In an election for class president, Anita received 11 votes, Jose received 9 votes, and Bernardo received the remaining 30 votes cast. If a pie chart is created for this voting data, what will be the degree measure of the central angle of the sector representing Anita's share of the vote? Round your answer to the nearest degree.

9. In an election for class president, Kamili received 14 votes, Jamal received 9 votes, and Jose received the remaining 27 votes cast. If a pie chart is created for this voting data, what will be the degree measure of the central angle of the sector representing Kamili's share of the vote? Round your answer to the nearest degree.

10. In an election for class president, Jun received 13 votes, Abdul received 15 votes, and Raven received the remaining 22 votes cast. If a pie chart is created for this voting data, what will be the degree measure of the central angle of the sector representing Jun's share of the vote? Round your answer to the nearest degree.

11. In an election for class president, Hue received 13 votes, Ali received 6 votes, and Henry received the remaining 31 votes cast. If a pie chart is created for this voting data, what will be the degree measure of the central angle of the sector representing Hue's share of the vote? Round your answer to the nearest degree.

12. In an election for class president, Mercy received 9 votes, Bernardo received 7 votes, and Hans received the remaining 34 votes cast. If a pie chart is created for this voting data, what will be the degree measure of the central angle of the sector representing Mercy's share of the vote? Round your answer to the nearest degree.

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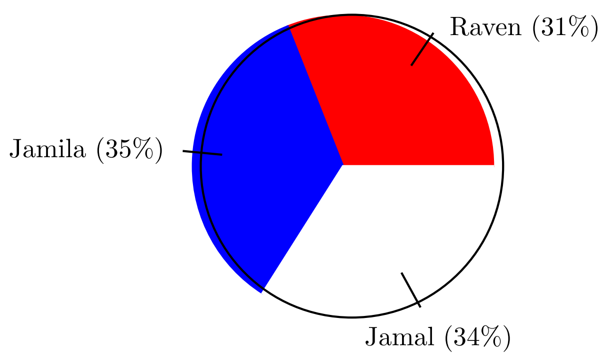
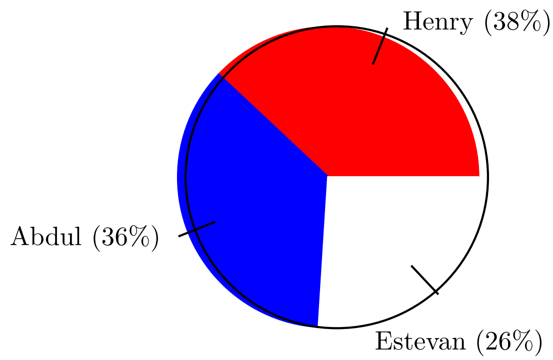
13. In an election for class president, the vote distribution among three candidates is shown in the following pie chart.



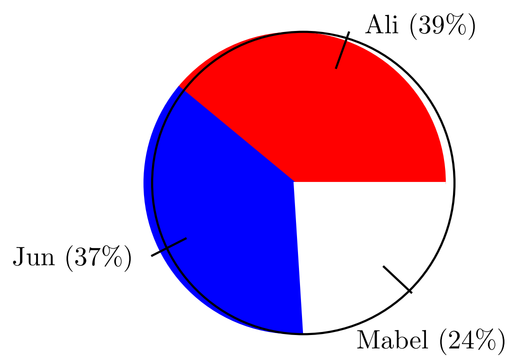
If there were a total of 95 votes cast in the election, find the number of votes that Raven received, correct to the nearest vote.

14. In an election for class president, the vote distribution among three candidates is shown in the following pie chart.



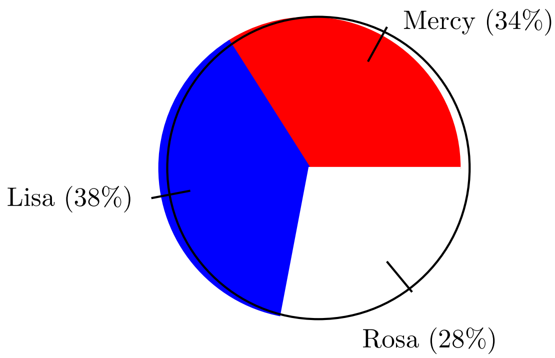
If there were a total of 79 votes cast in the election, find the number of votes that Henry received, correct to the nearest vote.

15. In an election for class president, the vote distribution among three candidates is shown in the following pie chart.



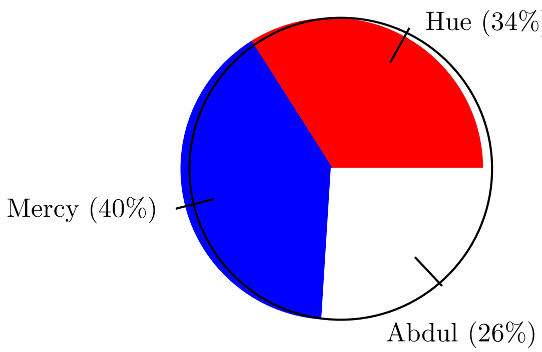
If there were a total of 58 votes cast in the election, find the number of votes that Ali received, correct to the nearest vote.

16. In an election for class president, the vote distribution among three candidates is shown in the following pie chart.



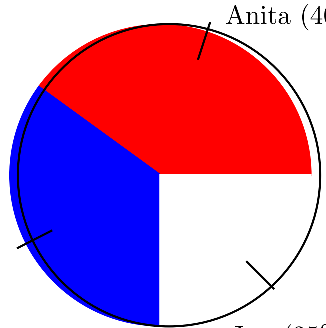
If there were a total of 65 votes cast in the election, find the number of votes that Mercy received, correct to the nearest vote.

17. In an election for class president, the vote distribution among three candidates is shown in the following pie chart.



If there were a total of 95 votes cast in the election, find the number of votes that Hue received, correct to the nearest vote.

18. In an election for class president, the vote distribution among three candidates is shown in the following pie chart.



If there were a total of 75 votes cast in the election, find the number of votes that Anita received, correct to the nearest vote.

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19. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Ali	45
Jamal	34
Jun	52

Use a protractor to help create a pie chart showing the distribution of votes.

20. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Aisha	39
Akbar	31
Fernando	36

Use a protractor to help create a pie chart showing the distribution of votes.

21. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Bernardo	44
Rosa	40
Abdul	58

Use a protractor to help create a pie chart showing the distribution of votes.

22. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Estevan	46
Ali	58
Henry	49

Use a protractor to help create a pie chart showing the distribution of votes.

23. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Mercy	56
Hans	53
Lisa	41

Use a protractor to help create a pie chart showing the distribution of votes.

24. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Estevan	60
Hue	33
Aisha	31

Use a protractor to help create a pie chart showing the distribution of votes.

25. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Raven	43
Mabel	40
Bernardo	52

Use a protractor to help create a pie chart showing the distribution of votes.

26. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Hue	48
Lisa	48
Akbar	47

Use a protractor to help create a pie chart showing the distribution of votes.

27. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Jun	57
Lisa	30
Aisha	58

Use a protractor to help create a pie chart showing the distribution of votes.

28. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Bernardo	54
Mabel	38
Henry	49

Use a protractor to help create a pie chart showing the distribution of votes.

29. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Henry	35
Bernardo	32
Estevan	47

Use a protractor to help create a pie chart showing the distribution of votes.

30. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Bernardo	38
Fernando	49
Aisha	44

Use a protractor to help create a pie chart showing the distribution of votes.

31. **Guard deployment.** The table shows the number of guard troop services since Sept. 11, 2001 (as of Dec. 2008; some troops have activated multiple times). *Associated Press-Times-Standard 02/18/10 Guard troops wait for promised pay.*

Mission	Troops
Operation Iraqi Freedom	193,598
Operation Enduring Freedom (Afghanistan)	29,212
Other missions	35,849

Use a protractor to help create a pie chart showing the distribution of National Guard troops.

❧ ❧ ❧ Answers ☙ ☙ ☙

1. 76°

3. 83°

5. 108°

7. 36°

9. 101°

11. 94°

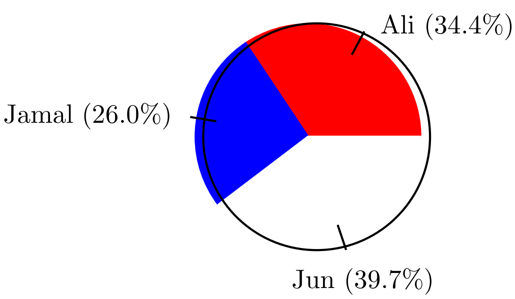
13. 29 votes

15. 23 votes

17. 32 votes

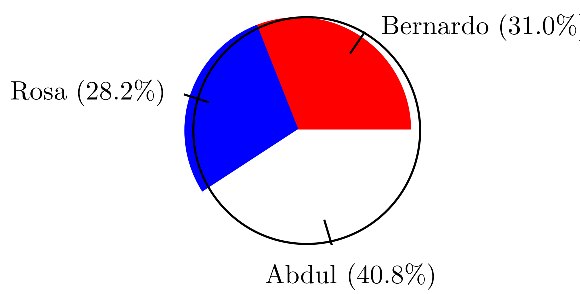
19.

Candidate	Votes	Percent	Degrees
Ali	45	34.4%	123.84°
Jamal	34	26.0%	93.6°
Jun	52	39.7%	142.92°



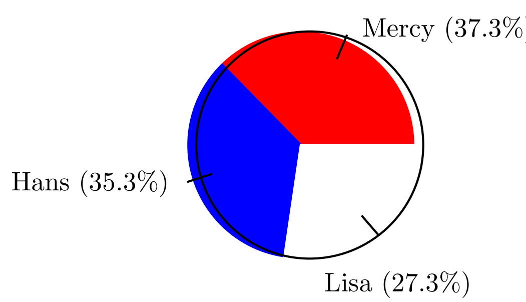
21.

Candidate	Votes	Percent	Degrees
Bernardo	44	31.0%	111.6°
Rosa	40	28.2%	101.52°
Abdul	58	40.8%	146.88°



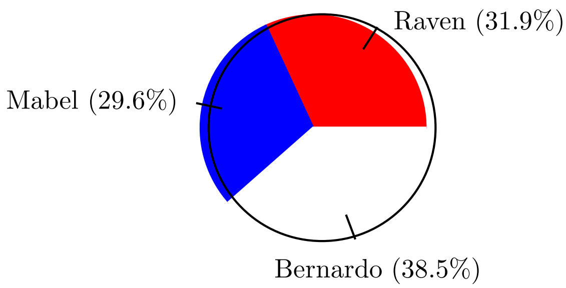
23.

Candidate	Votes	Percent	Degrees
Mercy	56	37.3%	134.28°
Hans	53	35.3%	127.08°
Lisa	41	27.3%	98.28°



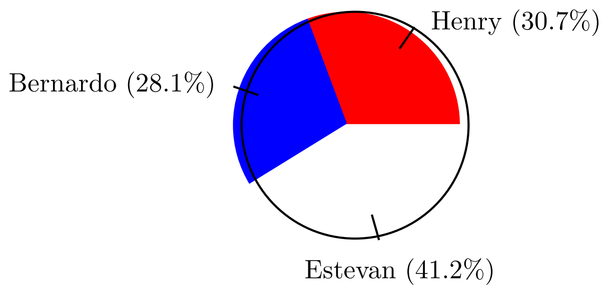
25.

Candidate	Votes	Percent	Degrees
Raven	43	31.9%	114.84°
Mabel	40	29.6%	106.56°
Bernardo	52	38.5%	138.6°



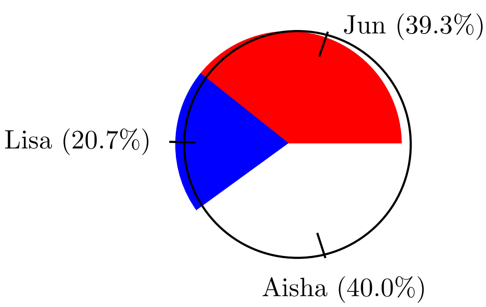
29.

Candidate	Votes	Percent	Degrees
Henry	35	30.7%	110.52°
Bernardo	32	28.1%	101.16°
Estevan	47	41.2%	148.32°



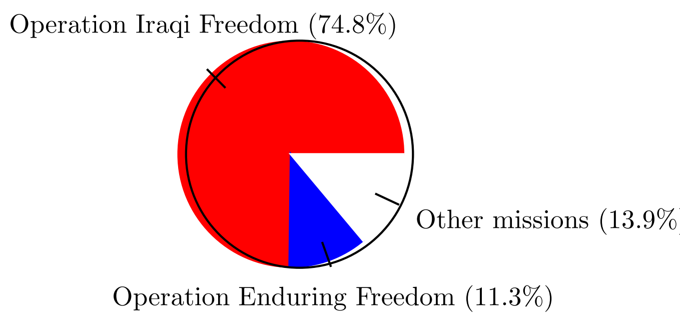
27.

Candidate	Votes	Percent	Degrees
Jun	57	39.3%	141.48°
Lisa	30	20.7%	74.52°
Aisha	58	40.0%	144°



31.

Mission	Troops	Percent	Degrees
Operation Iraqi Freedom	193,598	74.8%	269.28°
Operation Enduring Freedom	29,212	11.3%	40.68°
Other missions	35,849	13.9%	50.04°



Chapter 8

Graphing

René Descartes (1596-1650) was a French philosopher and mathematician. As a philosopher, he is famous for the saying "Cogito ergo sum" ("I think, therefore I am"), and his writings led many to consider him the *Father of Modern Philosophy*. Even today, a number of his writings are standard faire in university philosophy departments.

However, it is Descartes' work in mathematics that form the basis for this chapter, particularly his invention of the *Cartesian Coordinate System* which bears his name. Descartes' invention of the coordinate system created an entirely new branch of mathematics called *analytic geometry*, which established a permanent link between the plane and solid geometry of the ancient Greeks and the algebra and analysis of modern mathematics. As a result of his work, mathematicians were able to describe curves with equations, unheard of before Descartes' invention of the coordinate system. Rather than describing a circle as the "locus of all points equidistant from a given point," mathematicians were now able to refer to a circle centered at the point $(0, 0)$ with radius r as the graph of the equation $x^2 + y^2 = r^2$.

The bridge created between geometry and analysis as a result of Descartes' methods laid the groundwork for the discovery of the calculus by Newton and Leibniz. For his efforts, mathematicians often refer to Descartes as the *Father of Analytic Geometry*.

In this chapter we will introduce readers to the Cartesian coordinate system and explain the correspondence between points in the plane and ordered pairs of numbers. Once an understanding of the coordinate system is sufficiently developed, we will develop the concept of the graph of an equation. In particular, we will address the graphs of a class of equations called *linear equations*.

8.1 The Cartesian Coordinate System

Let's begin with the concept of an *ordered pair* of whole numbers.

Ordered Pairs of Whole Numbers. The construct (x, y), where x and y are whole numbers, is called an ordered pair of whole numbers.

Examples of ordered pairs of whole numbers are $(0, 0)$, $(2, 3)$, $(5, 1)$, and $(4, 9)$.

Order Matters. Pay particular attention to the phrase "ordered pairs." Order matters. Consequently, the ordered pair (x, y) is not the same as the ordered pair (y, x), because the numbers are presented in a different order.

We've seen how to plot whole numbers on a number line. For example, in Figure 8.1, we've plotted the whole numbers 2, 5, and 7 as shaded "dots" on the number line.

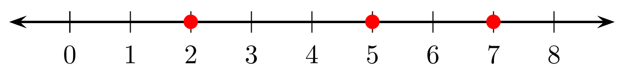
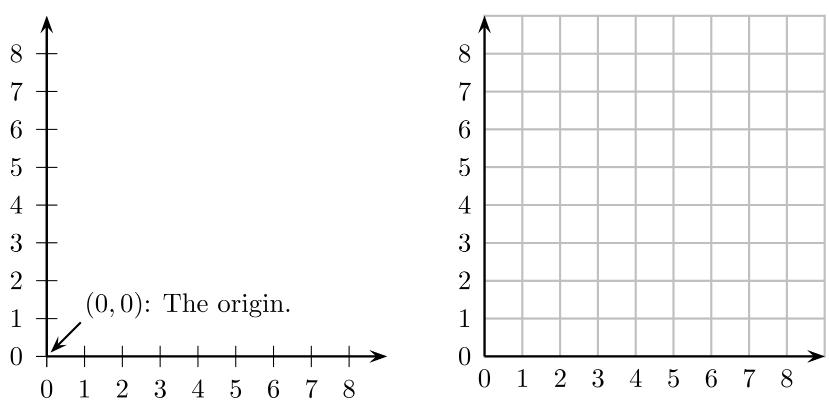


Figure 8.1: Plotting the whole numbers 2, 5, and 7 on a number line.

To plot ordered pairs, we need two number lines, called the *horizontal and vertical axes*, that intersect at the zero location of each line and are at right angles to one another, as shown in Figure 8.2(a). The point where the zero locations touch is called the *origin* of the coordinate system and has coordinates $(0, 0)$. In Figure 8.2(b), we've added a grid. The resulting construct is an example of a *Cartesian Coordinate System.*

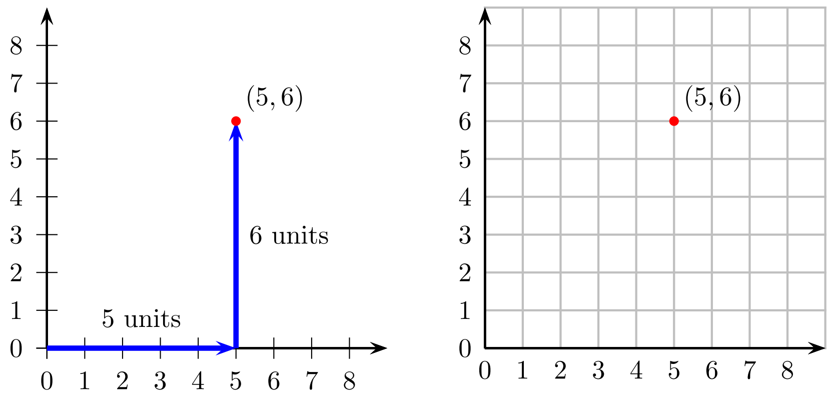


(a) Horizontal and vertical axes.

(b) Adding a grid.

Figure 8.2: A Cartesian coordinate system.

Now, consider the ordered pair of whole numbers $(5, 6)$. To plot this point on the "coordinate system" in Figure 8.3(a), start at the origin $(0, 0)$, then move 5 units in the horizontal direction, then 6 units in the vertical direction, then plot a point. The result is shown in Figure 8.3(a). Adding a grid of horizontal and vertical lines at each whole number makes plotting the point $(5, 6)$ much clearer, as shown in Figure 8.3(b).



(a) 5 right and 6 up. (b) Plotting on a grid.

Figure 8.3: Plotting the Point $(5, 6)$ in a Cartesian Coordinate System.

The numbers in the ordered pair $(5, 6)$ are called the *coordinates* of the plotted point in Figure 8.3(b). The first number of the ordered pair is called the *abscissa* and measures the horizontal distance to the plotted point. The second number is called the *ordinate* and measures the vertical distance to the plotted point. The combination of axes and grid in Figure 8.3(b) is called a *coordinate system*.

The grid in Figure 8.3(b) is a visualization that greatly eases the plotting of ordered pairs. However, you don't have to draw these gridlines yourself. Instead, you should work on graph paper.

Graph Paper Requirement. All plotting should be done on graph paper.

You Try It!

Plot the following ordered pairs of whole numbers: $(2, 2)$, $(5, 5)$, and $(7, 4)$.

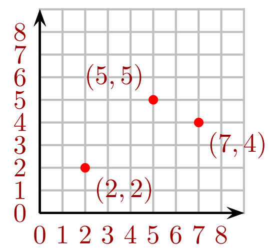
EXAMPLE 1. Plot the following ordered pairs of whole numbers: $(3, 2)$, $(8, 6)$, and $(2, 7)$.

Solution. Create a Cartesian coordinate system on graph paper, then:

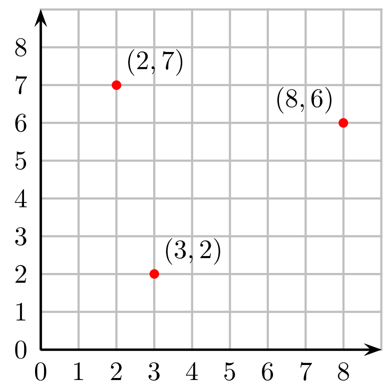
- To plot the ordered pair $(3, 2)$, start at the origin, then move 3 units to the right and 2 units up.

- To plot the ordered pair $(8, 6)$, start at the origin, then move 8 units to the right and 6 units up.
- To plot the ordered pair $(2, 7)$, start at the origin, then move 2 units to the right and 7 units up.

Answer:



The results are shown on the following Cartesian coordinate system.



Allowing for Negative Numbers

Again, we've seen how to plot both positive and negative numbers on a number line. For example, in Figure 8.4, we've plotted the numbers -4, $-3/2$, 2.2 and 4.

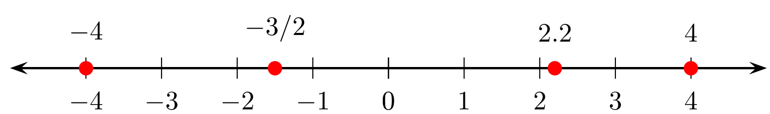
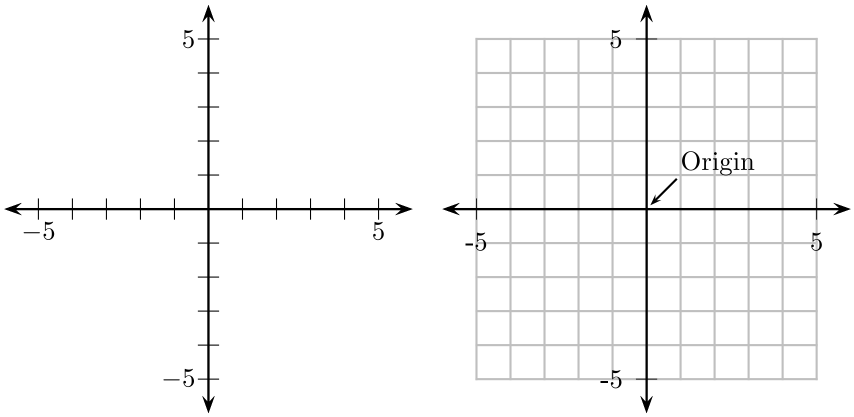


Figure 8.4: Plotting the numbers -4, $-3/2$, 2.2, and 4.

Note that the positive direction is to the right, the negative to the left. That is, to plot the number 2.2, we move 2.2 units to the right on the line, but to plot the number $-3/2$, we move $3/2$ units to the left.

To plot ordered pairs having both positive and negative numbers, we need two such number lines that intersect at the zero location of each line and are at right angles to one another, as shown in Figure 8.5(a). As before, adding a grid of horizontal and vertical lines at each integer will be extremely helpful when plotting points (see Figure 8.5(b)). The system of axes and grid in Figure 8.5(b) is called the *Cartesian Coordinate System*, named after its inventor, Renè Descartes.



(a) Horizontal and vertical axes.

(b) Adding a grid.

Figure 8.5: The Cartesian coordinate system.

Plotting Points in the Cartesian Coordinate System. On the horizontal axis, the positive direction is to the right, negative is to the left. On the vertical axis, the positive direction is up, negative is down. The point $(0, 0)$ is called the *origin* of the coordinate system, and is the starting point for all point plotting.

You Try It!

EXAMPLE 2. Sketch the points $(4, 3)$, $(-3, 2)$, $(-2, -4)$, and $(3, -3)$ on a Cartesian coordinate system.

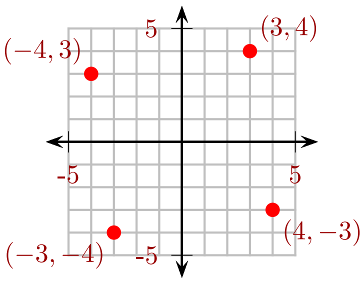
Sketch the points $(3, 4)$, $(-4, 3)$, $(-3, -4)$, and $(4, -3)$ on a Cartesian coordinate system.

Solution. Set up a Cartesian coordinate system on graph paper.

- To plot the point $(4, 3)$, start at the origin, move 4 units to the right, then 3 units up.
- To plot the point $(-3, 2)$, start at the origin, move 3 units to the left, then 2 units up.
- To plot the point $(-2, -4)$, start at the origin, move 2 units to the left, then 4 units down.
- To plot the point $(3, -3)$, start at the origin, move 3 units to the right, then 3 units down.

These points are plotted and shown in Figure 8.6.

Answer:



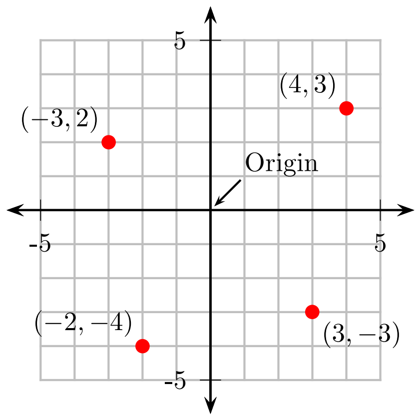
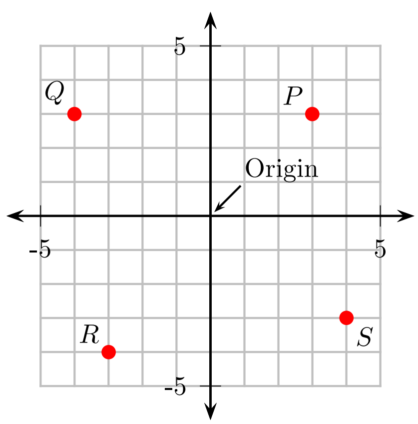


Figure 8.6: Plotting points in the Cartesian coordinate system.

□

You Try It!

EXAMPLE 3. What are the coordinates of the points P, Q, R, and S in the Cartesian coordinate system that follows?

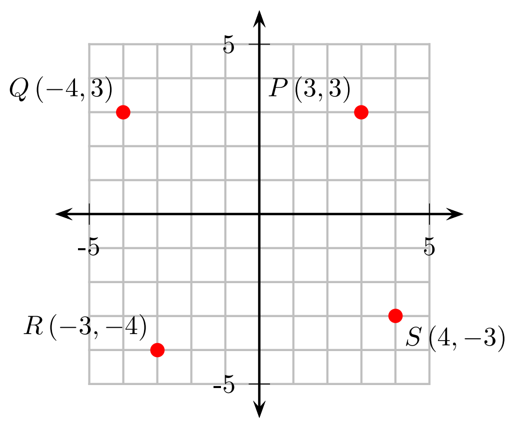


Solution. Make all measurements from the origin.

- To obtain the coordinates of point P, start at the origin, move 3 units to the right, then 3 units up. Hence, the coordinates of the point P are $(3, 3)$.

- To obtain the coordinates of point Q, start at the origin, move 4 units to the left, then 3 units up. Hence, the coordinates of the point Q are $(-4, 3)$.
- To obtain the coordinates of point R, start at the origin, move 3 units to the left, then 4 units down. Hence, the coordinates of the point R are $(-3, -4)$.
- To obtain the coordinates of point S, start at the origin, move 4 units to the right, then 3 units down. Hence, the coordinates of the point S are $(4, -3)$.

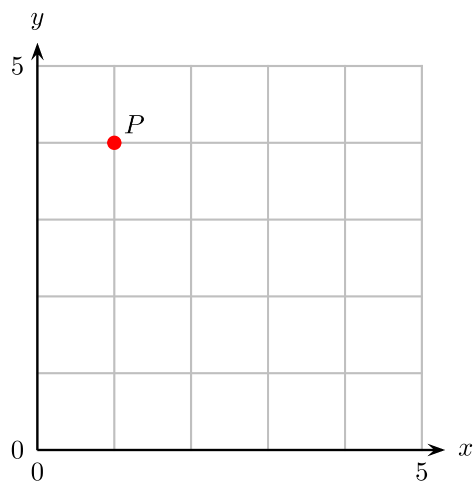
These results are shown on the following Cartesian coordinate system.



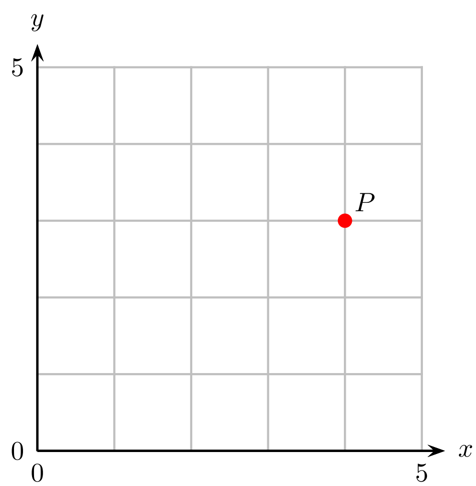
□

❧ ❧ ❧ Exercises ☙ ☙ ☙

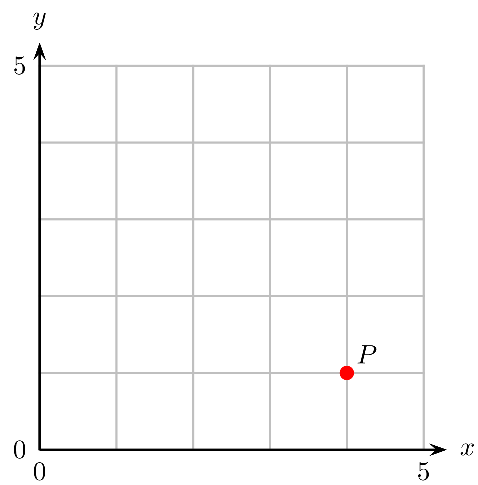
1. Identify the coordinates of the point P.



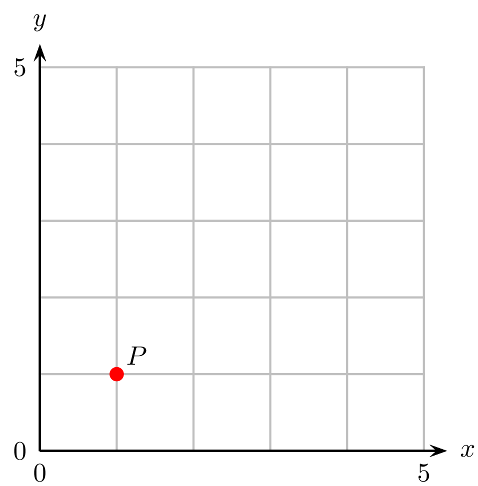
2. Identify the coordinates of the point P.



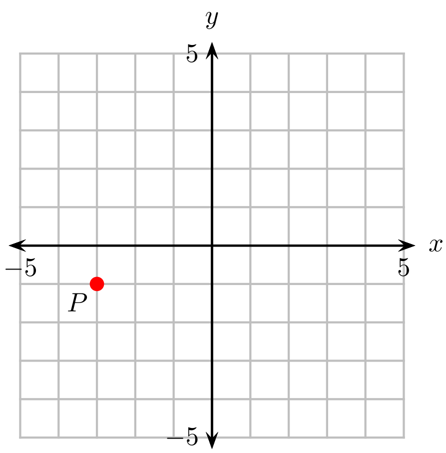
3. Identify the coordinates of the point P.



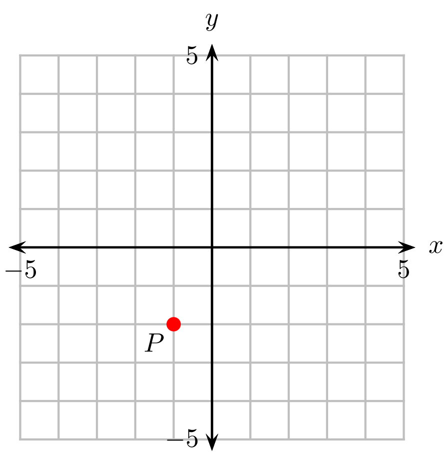
4. Identify the coordinates of the point P.



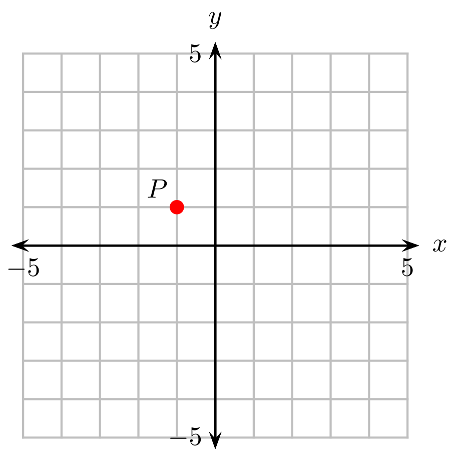
5. Identify the coordinates of the point P.



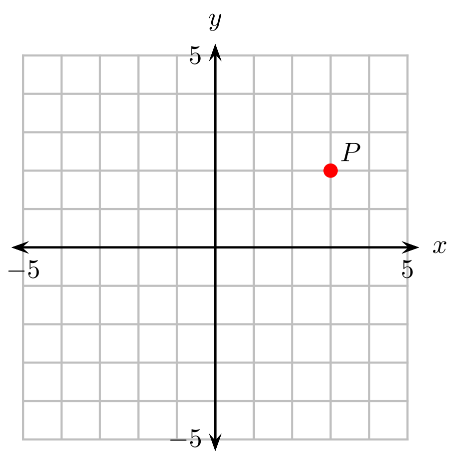
6. Identify the coordinates of the point P.



7. Identify the coordinates of the point P.



8. Identify the coordinates of the point P.



9. The points $A(-1, 1)$, $B(1, 1)$, $C(1, 2)$, and $D(-1, 2)$ are the vertices of a rectangle. Plot these points, draw the rectangle $ABCD$, then compute the area of rectangle $ABCD$.

10. The points $A(-3, -4)$, $B(4, -4)$, $C(4, -1)$, and $D(-3, -1)$ are the vertices of a rectangle. Plot these points, draw the rectangle $ABCD$, then compute the area of rectangle $ABCD$.

11. The points $A(-2,-1)$, $B(3,-1)$, $C(3,3)$, and $D(-2,3)$ are the vertices of a rectangle. Plot these points, draw the rectangle $ABCD$, then compute the area of rectangle $ABCD$.

12. The points $A(-3,-1)$, $B(2,-1)$, $C(2,2)$, and $D(-3,2)$ are the vertices of a rectangle. Plot these points, draw the rectangle $ABCD$, then compute the area of rectangle $ABCD$.

---

13. The points $A(-4,-2)$, $B(1,-2)$, $C(1,1)$, and $D(-4,1)$ are the vertices of a rectangle. Plot these points, draw the rectangle $ABCD$, then compute the perimeter of rectangle $ABCD$.

14. The points $A(-4,-4)$, $B(1,-4)$, $C(1,-3)$, and $D(-4,-3)$ are the vertices of a rectangle. Plot these points, draw the rectangle $ABCD$, then compute the perimeter of rectangle $ABCD$.

15. The points $A(-1,2)$, $B(3,2)$, $C(3,3)$, and $D(-1,3)$ are the vertices of a rectangle. Plot these points, draw the rectangle $ABCD$, then compute the perimeter of rectangle $ABCD$.

16. The points $A(-4,2)$, $B(3,2)$, $C(3,4)$, and $D(-4,4)$ are the vertices of a rectangle. Plot these points, draw the rectangle $ABCD$, then compute the perimeter of rectangle $ABCD$.

---

17. The points $A(-3,-1)$, $B(1,-1)$, and $C(-3,0)$ are the vertices of a triangle. Plot these points, draw the triangle ABC, then compute the area of the triangle ABC.

18. The points $A(-3,-2)$, $B(1,-2)$, and $C(-3,2)$ are the vertices of a triangle. Plot these points, draw the triangle ABC, then compute the area of the triangle ABC.

19. The points $A(-1,-2)$, $B(0,-2)$, and $C(-1,0)$ are the vertices of a triangle. Plot these points, draw the triangle ABC, then compute the area of the triangle ABC.

20. The points $A(-2,-3)$, $B(-1,-3)$, and $C(-2,1)$ are the vertices of a triangle. Plot these points, draw the triangle ABC, then compute the area of the triangle ABC.

---

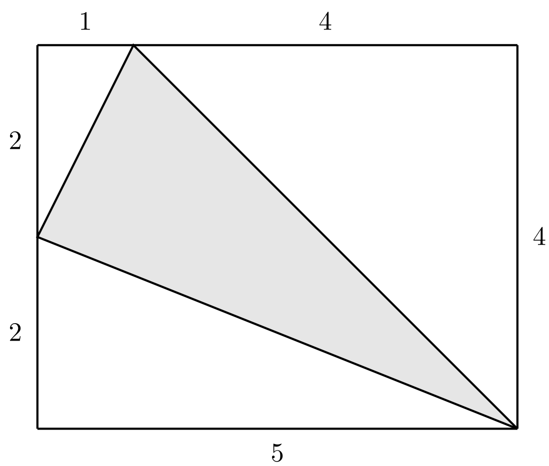
21. Plot the points $A(-3,-3)$ and $B(0,0)$ and find the straight-line distance between the two points. *Hint: Create a right triangle, then use the Pythagorean Theorem.*

22. Plot the points $A(-2,-3)$ and $B(1,2)$ and find the straight-line distance between the two points. *Hint: Create a right triangle, then use the Pythagorean Theorem.*

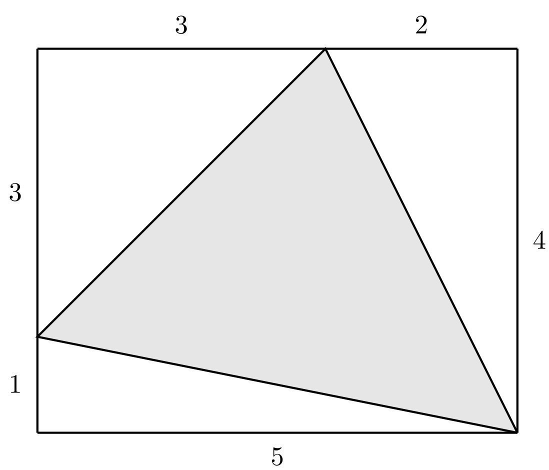
23. Plot the points $A(-2,-3)$ and $B(0,0)$ and find the straight-line distance between the two points. *Hint: Create a right triangle, then use the Pythagorean Theorem.*

24. Plot the points $A(-3,-2)$ and $B(2,2)$ and find the straight-line distance between the two points. *Hint: Create a right triangle, then use the Pythagorean Theorem.*

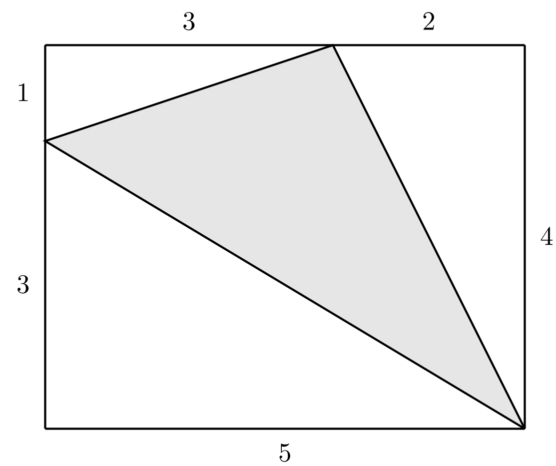
25. Find the area of the shaded triangle.



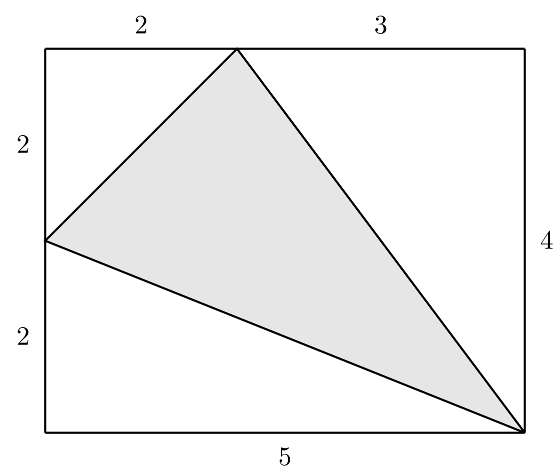
26. Find the area of the shaded triangle.



27. Find the area of the shaded triangle.



28. Find the area of the shaded triangle.



Hint: In Exercises 29-32, surround the triangle with a rectangle, like those shown in Exercises 25-28.

29. Find the area of the triangle with vertices at $A(-4,-1)$, $B(4,-2)$, and $C(1,3)$.

30. Find the area of the triangle with vertices at $A(-4,2)$, $B(3,0)$, and $C(0,4)$.

31. Find the area of the triangle with vertices at $A(-3,1)$, $B(3,-3)$, and $C(1,4)$.

32. Find the area of the triangle with vertices at $A(1,2)$, $B(3,0)$, and $C(2,4)$.

❧ ❧ ❧ Answers ☙ ☙ ☙

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1. $(1,4)$

3. $(4,1)$

5. $(-3,-1)$

7. $(-1,1)$

9. 2 square units

11. 20 square units

13. 16 units

15. 10 units

17. 2 square units

19. 1 square units

21. $\sqrt{18}$

23. $\sqrt{13}$

25. 6

27. 7

29. $\dfrac{37}{2}$

31. 17

8.2 Graphing Linear Equations

Consider $y = x + 1$ an *equation in two variables*. If we substitute the ordered pair $(x, y) = (1, 2)$ into the equation $y = x + 1$, that is, if we replace x with 1 and y with 2, we get a true statement.

$y = x + 1$	Original equation.
$2 = 1 + 1$	Substitute: 1 for x and 2 for y.
$2 = 2$	Simplify.

We say that the ordered pair $(1, 2)$ is a *solution* of the equation $y = x + 1$.

Solution of an Equation in Two Variables. If substituting the ordered pair $(x, y) = (a, b)$ into an equation (replace x with a and y with b) produces a true statement, then the ordered pair (a, b) is called a *solution* of the equation and is said to "satisfy the equation."

You Try It!

Which of the ordered pairs $(1, 7)$ and $(2, 9)$ are solution of the equation $y = 3x + 4$?

EXAMPLE 1. Which of the ordered pairs are solutions of the equation $y = 2x + 5$: (a) $(-3, -2)$, or (b) $(5, 15)$?

Solution. Substitute the points into the equation to determine which are solutions.

a) To determine if $(-3, -2)$ is a solution of $y = 2x + 5$, substitute -3 for x and -2 for y in the equation $y = 2x + 5$.

$y = 2x + 5$	Original equation.
$-2 = 2(-3) + 5$	Substitute: -3 for x and -2 for y.
$-2 = -6 + 5$	Multiply first: $2(-3) = -6$
$-2 = -1$	Add: $-6 + 5 = -1$.

Because the resulting statement is false, the ordered pair $(-3, -2)$ does **not** satisfy the equation. The ordered pair $(-3, -2)$ is **not** a solution of $y = 2x + 5$.

a) To determine if $(5, 15)$ is a solution of $y = 2x + 5$, substitute 5 for x and 15 for y in the equation $y = 2x + 5$.

$y = 2x + 5$	Original equation.
$15 = 2(5) + 5$	Substitute: 5 for x and 15 for y.
$15 = 10 + 5$	Multiply first: $2(5) = 10$
$15 = 15$	Add: $10 + 5 = 15$.

The resulting statement is true. The ordered pair $(5, 15)$ **does** satisfy the equation. Hence, $(5, 15)$ **is** a solution of $y = 2x + 5$.

Answer: $(1, 7)$

□

The Graph of an Equation

We turn our attention to the *graph* of an equation.

The Graph of an Equation. The graph of an equation is the set of all ordered pairs that are solutions of the equation.

In the equation $y = 2x + 5$, the variable y depends on the value of the variable x. For this reason, we call y the *dependent* variable and x the *independent* variable. We're free to make choices for x, but the value of y will depend upon our choice for x.

We will also assign the horizontal axis to the independent variable x and the vertical axis to the dependent variable y (see Figure 8.7).

The graph of $y = 2x+5$ consists of all ordered pairs that are solutions of the equation $y = 2x+5$. So, our first task is to find ordered pairs that are solutions of $y = 2x + 5$. This is easily accomplished by selecting an arbitrary number of values, substituting them for x in the equation $y = 2x + 5$, then calculating the resulting values of y. With this thought in mind, we pick arbitrary integers $-7, -6, \ldots, 2$, substitute them for x in the equation $y = 2x + 5$, calculate the resulting value of y, and store the results in a table.

$$\begin{aligned}
y &= 2(-7) + 5 = -9\\
y &= 2(-6) + 5 = -7\\
y &= 2(-5) + 5 = -5\\
y &= 2(-4) + 5 = -3\\
y &= 2(-3) + 5 = -1\\
y &= 2(-2) + 5 = 1\\
y &= 2(-1) + 5 = 3\\
y &= 2(0) + 5 = 5\\
y &= 2(1) + 5 = 7\\
y &= 2(2) + 5 = 9
\end{aligned}$$

$y = 2x + 5$		
x	y	(x, y)
-7	-9	$(-7, -9)$
-6	-7	$(-6, -7)$
-5	-5	$(-5, -5)$
-4	-3	$(-4, -3)$
-3	-1	$(-3, -1)$
-2	1	$(-2, 1)$
-1	3	$(-1, 3)$
0	5	$(0, 5)$
1	7	$(1, 7)$
2	9	$(2, 9)$

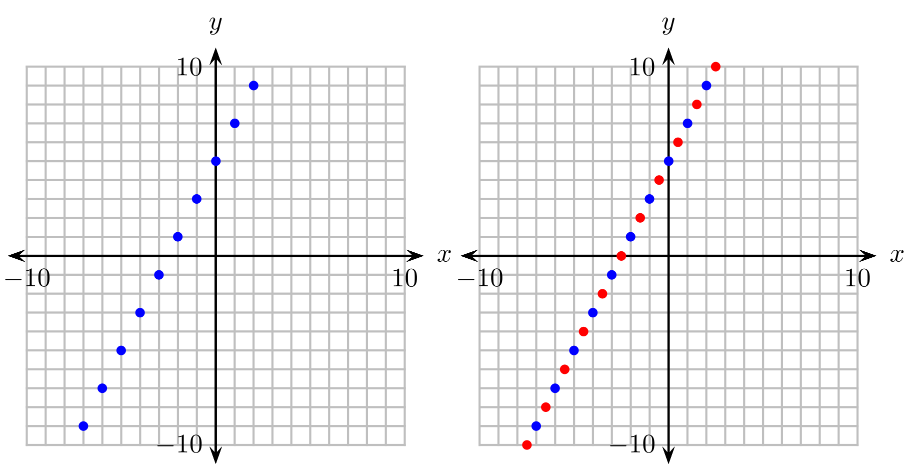
The result is 10 ordered pairs that satisfy the equation $y = 2x + 5$. Therefore, we have 10 ordered pairs that belong to the graph of $y = 2x + 5$. They are plotted in Figure 8.7(a).

However, we're not finished, because the graph of the equation $y = 2x + 5$ is the set of **all** points that satisfy the equation and we've only plotted 10 such points. Let's plot some more points. Select some more x-values, compute the corresponding y-value, and record the results in a table.

$$y = 2(-7.5) + 5 = -10$$
$$y = 2(-6.5) + 5 = -8$$
$$y = 2(-5.5) + 5 = -6$$
$$y = 2(-4.5) + 5 = -4$$
$$y = 2(-3.5) + 5 = -2$$
$$y = 2(-2.5) + 5 = 0$$
$$y = 2(-1.5) + 5 = 2$$
$$y = 2(-0.5) + 5 = 4$$
$$y = 2(0.5) + 5 = 6$$
$$y = 2(1.5) + 5 = 8$$
$$y = 2(2.5) + 5 = 10$$

$y = 2x + 5$		
x	y	(x, y)
-7.5	-10	$(-7.5, -10)$
-6.5	-8	$(-6.5, -8)$
-5.5	-6	$(-5.5, -6)$
-4.5	-4	$(-4.5, -4)$
-3.5	-2	$(-3.5, -2)$
-2.5	0	$(-2.5, 0)$
-1.5	2	$(-1.5, 2)$
-0.5	4	$(-0.5, 4)$
0.5	6	$(0.5, 6)$
1.5	8	$(1.5, 8)$
2.5	10	$(2.5, 10)$

That's 11 additional points that we add to the graph in Figure 8.7(b).



(a) Ten points that satisfy the equation $y = 2x + 5$.

(b) Eleven additional points that satisfy the equation $y = 2x + 5$.

Figure 8.7: Plotting points that satisfy the equation $y = 2x + 5$.

Note that we can continue indefinitely in this manner, adding points to the table and plotting them. However, sooner or later, we have to make a leap of faith, and imagine what the final graph will look like when all of the points that satisfy the equation $y = 2x + 5$ are plotted. We do so in Figure 8.8, where the final graph takes on the appearance of a line.

Ruler Use. All lines must be drawn with a ruler. This includes the x- and y-axes.

Important Observation. When we use a ruler to draw a line through the plotted points in Figure 8.7(b), arriving at the final result in Figure 8.8, we must understand that this is a shortcut technique for plotting all of the remaining ordered pairs that satisfy the equation. We're not really drawing a line through the plotted points. Rather, we're shading all of the ordered pairs that satisfy the equation $y = 2x + 5$.

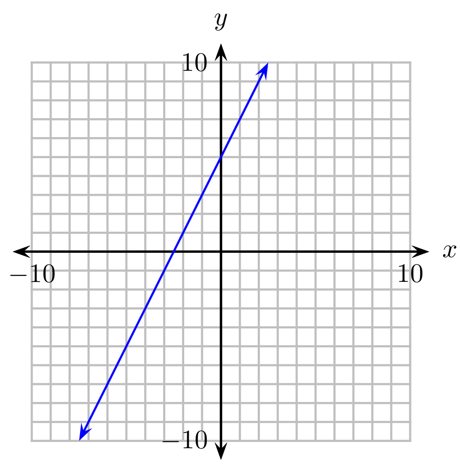


Figure 8.8: The graph of the equation $y = 2x + 5$.

The Result. The graph of the equation $y = 2x + 5$, pictured in Figure 8.8, is a line. Actually, the graph is an infinite collection of points satisfying the equation $y = 2x + 5$ that takes the shape of a line, but it's all right to say the graph of $y = 2x + 5$ is a line.

Ordered Pairs and the Graph. Because the graph of an equation is the collection of all ordered pairs that satisfy the equation, we have two important results:

1. If an ordered pair satisfies an equation, then the point in the Cartesian plane represented by the ordered pair is on the graph of the equation.
2. If a point is on the graph of an equation, then the ordered pair representation of that point satisfies the equation.

You Try It!

EXAMPLE 2. Find the value of k so that the point $(2, k)$ is on the graph of the equation $y = 3x - 2$.

Solution. If the point $(2, k)$ is on the graph of $y = 3x - 2$, then it must satisfy the equation $y = 3x - 2$.

$y = 3x - 2$	Original equation.
$k = 3(2) - 2$	The point $(2, k)$ is on the graph. Substitute 2 for x and k for y in $y = 3x - 2$.
$k = 6 - 2$	Multiply: $3(2) = 6$.
$k = 4$	Subtract: $6 - 2 = 4$.

Thus, $k = 4$.

Find the value of k so that the point $(k, -3)$ is on the graph of the equation $y = 4x + 2$.

Answer: $k = -5/4$

□

Linear Equations

Let's plot the graph of another equation.

You Try It!

EXAMPLE 3. Sketch the graph of $y = -2x + 1$.

Solution. Select arbitrary values of x: $-4, -3, \ldots, 5$. Substitute these values into the equation $y = -2x + 1$, calculate the resulting value of y, then arrange your results in a table.

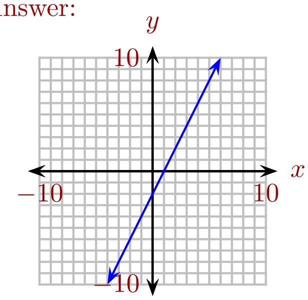
$$y = -2(-4) + 1 = 9$$
$$y = -2(-3) + 1 = 7$$
$$y = -2(-2) + 1 = 5$$
$$y = -2(-1) + 1 = 3$$
$$y = -2(0) + 1 = 1$$
$$y = -2(1) + 1 = -1$$
$$y = -2(2) + 1 = -3$$
$$y = -2(3) + 1 = -5$$
$$y = -2(4) + 1 = -7$$
$$y = -2(5) + 1 = -9$$

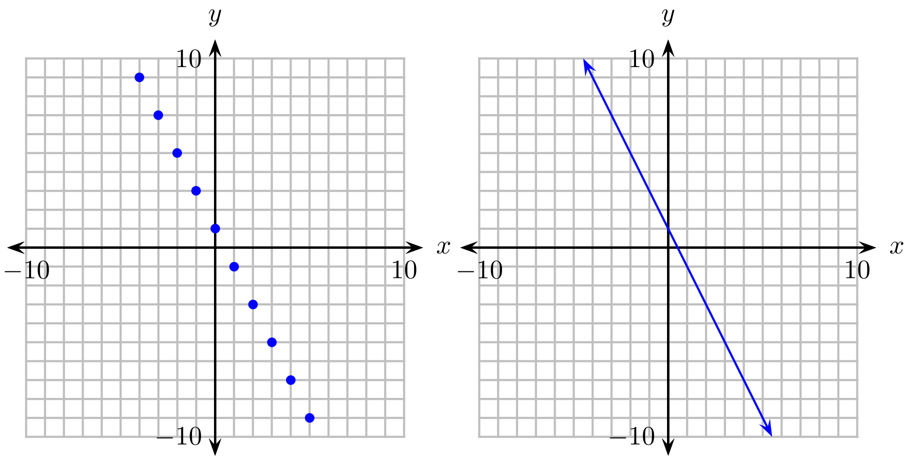
$y = -2x + 1$		
x	y	(x, y)
-4	9	$(-4, 9)$
-3	7	$(-3, 7)$
-2	5	$(-2, 5)$
-1	3	$(-1, 3)$
0	1	$(0, 1)$
1	-1	$(1, -1)$
2	-3	$(2, -3)$
3	-5	$(3, -5)$
4	-7	$(4, -7)$
5	-9	$(5, -9)$

Sketch the graph of $y = 2x - 2$.

We've plotted the points in the table in Figure 8.9(a). There is enough evidence in Figure 8.9(a) to imagine that if we plotted all of the points that satisfied the equation $y = -2x + 1$, the result would be the line shown in Figure 8.9(b).

Answer:





(a) Ten points that satisfy the equation $y = -2x + 1$.

(b) Plotting all points that satisfy the equation $y = -2x + 1$.

Figure 8.9: The graph of the equation $y = -2x + 1$ is a line.

The graph of $y = 2x + 5$ in Figure 8.8 is a line. The graph of $y = -2x + 1$ in Figure 8.9(b) is also a line. This would lead one to suspect that the graph of the equation $y = mx + b$, where m and b are constants, will always be a line. Indeed, this is always the case.

Linear Equations. The graph of $y = mx + b$, where m and b are constants, will always be a line. For this reason, the equation $y = mx + b$ is called a *linear equation.*

You Try It!

Which of the following equations is a linear equation?

a) $y = 2x^3 + 5$

b) $y = -3x - 5$

EXAMPLE 4. Which of the following equations is a linear equation? (a) $y = -3x + 4$, (b) $y = \frac{2}{3}x + 3$, and (c) $y = 2x^2 + 4$.

Solution. Compare each equation with the general form of a linear equation, $y = mx + b$.

a) Note that $y = -3x + 4$ has the form $y = mx + b$, where $m = -3$ and $b = 4$. Hence, $y = -3x + 4$ is a linear equation. Its graph is a line.

b) Note that $y = \frac{2}{3}x + 3$ has the form $y = mx + b$, where $m = 2/3$ and $b = 3$. Hence, $y = \frac{2}{3}x + 3$ is a linear equation. Its graph is a line.

c) The equation $y = 2x^2 + 4$ does **not** have the form $y = mx + b$. The exponent of 2 on the x prevents this equation from being linear. This is a nonlinear equation. Its graph is not a line.

Answer: $y = -3x - 5$

The fact that $y = mx + b$ is a linear equation enables us to quickly sketch its graph.

You Try It!

EXAMPLE 5. Sketch the graph of $y = -\frac{3}{2}x + 4$.

Sketch the graph of $y = -\frac{1}{2}x + 2$.

Solution. The equation $y = -\frac{3}{2}x + 4$ has the form $y = mx + b$. Therefore, the equation is linear and the graph will be a line. Because two points determine a line, we need only find two points that satisfy the equation $y = -\frac{3}{2}x + 4$, plot them, then draw a line through them with a ruler. We choose $x = -2$ and $x = 2$, calculate y, and record the results in a table.

$$y = -\frac{3}{2}(-2) + 4 = 3 + 4 = 7$$
$$y = -\frac{3}{2}(2) + 4 = -3 + 4 = 1$$

$y = -\frac{3}{2}x + 4$		
x	y	(x, y)
-2	7	$(-2, 7)$
2	1	$(2, 1)$

Plot the points $(-2, 7)$ and $(2, 1)$ and draw a line through them. The result is shown in Figure 8.10.

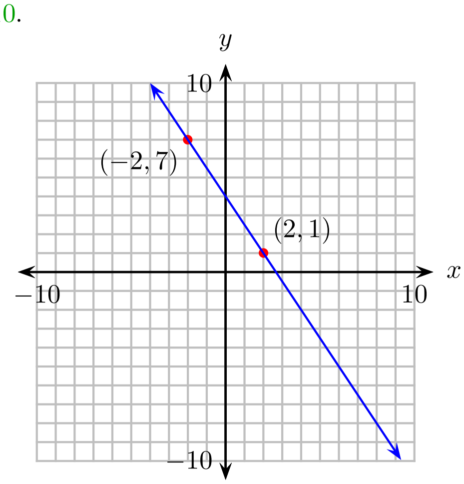
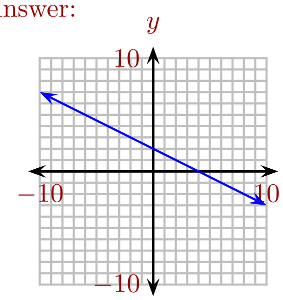


Figure 8.10: The graph of $y = -\frac{3}{2}x + 4$ is a line.

Answer:



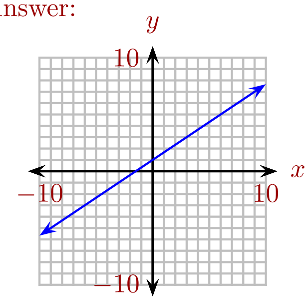
You may have noted in Example 5 that are choices of -2 and 2 for x eased the calculation of the corresponding y-values because of the resulting cancellation.

Choosing Strategic Values. When plotting a linear equation, it is a good strategy to choose values of x that simplify the calculation of the corresponding y-values.

You Try It!

Sketch the graph of $y = \frac{2}{3}x + 1$.

Answer:



EXAMPLE 6. Sketch the graph of $y = \frac{1}{3}x + 3$.

Solution. The equation $y = \frac{1}{3}x + 3$ has the form $y = mx + b$. Therefore, the equation is linear and the graph will be a line. Because two points determine a line, we need only find two points that satisfy the equation $y = \frac{1}{3}x + 3$, plot them, then draw a line through them with a ruler. We choose $x = -6$ and $x = 6$, calculate y, and record the results in a table.

$$y = \tfrac{1}{3}(-6) + 3 = -2 + 3 = 1$$
$$y = \tfrac{1}{3}(6) + 3 = 2 + 3 = 5$$

$y = \frac{1}{3}x + 3$		
x	y	(x, y)
-6	1	$(-6, 1)$
6	5	$(6, 5)$

Plot the points $(-6, 1)$ and $(6, 5)$ and draw a line through them. The result is shown in Figure 8.11.

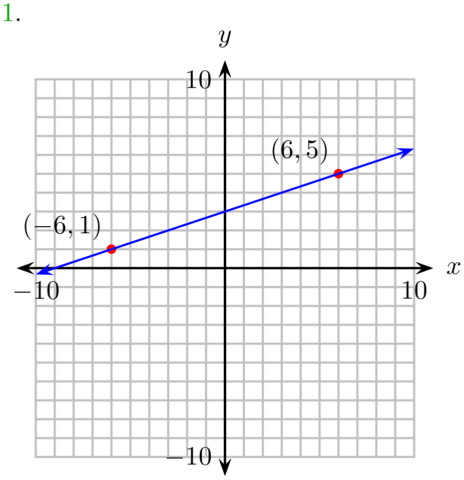


Figure 8.11: The graph of $y = \frac{1}{3}x + 3$ is a line.

□

Exercises

1. Which of the points $(2, -14)$, $(-1, -6)$, $(-8, 11)$, and $(3, -13)$ is a solution of the equation $y = -2x - 8$?

2. Which of the points $(1, -2)$, $(8, 23)$, $(-3, -23)$, and $(8, 24)$ is a solution of the equation $y = 4x - 9$?

3. Which of the points $(1, -1)$, $(-2, 20)$, $(-4, 31)$, and $(-9, 64)$ is a solution of the equation $y = -6x + 7$?

4. Which of the points $(-8, -61)$, $(4, 42)$, $(-3, -18)$, and $(-6, -46)$ is a solution of the equation $y = 9x + 8$?

5. Which of the points $(2, 15)$, $(-8, -74)$, $(2, 18)$, and $(5, 40)$ is a solution of the equation $y = 9x - 3$?

6. Which of the points $(-9, -52)$, $(-8, -44)$, $(-7, -37)$, and $(8, 35)$ is a solution of the equation $y = 5x - 5$?

7. Which of the points $(-2, 12)$, $(-1, 12)$, $(3, -10)$, and $(-2, 14)$ is a solution of the equation $y = -5x + 4$?

8. Which of the points $(6, 25)$, $(-8, -14)$, $(8, 33)$, and $(-7, -9)$ is a solution of the equation $y = 3x + 9$?

9. Determine k so that the point $(9, k)$ is a solution of $y = -6x + 1$.

10. Determine k so that the point $(-9, k)$ is a solution of $y = 2x + 3$.

11. Determine k so that the point $(k, 7)$ is a solution of $y = -4x + 1$.

12. Determine k so that the point $(k, -4)$ is a solution of $y = 8x + 3$.

13. Determine k so that the point $(k, 1)$ is a solution of $y = 4x + 8$.

14. Determine k so that the point $(k, -7)$ is a solution of $y = -7x + 5$.

15. Determine k so that the point $(-1, k)$ is a solution of $y = -5x + 3$.

16. Determine k so that the point $(-3, k)$ is a solution of $y = 3x + 3$.

In Exercises 17-24, which of the given equations is a linear equation?

17. $y = 6x^2 + 4$, $y = x^2 + 6x + 4$, $y = 6x + 4$, $y = \sqrt{6x + 4}$

18. $y = -2x + 1$, $y = x^2 - 2x + 1$, $y = \sqrt{-2x + 1}$, $y = -2x^2 + 1$

19. $y = x + 7$, $y = \sqrt{x + 7}$, $y = x^2 + 7$, $y = x^2 + x + 7$

20. $y = x^2 + 5x + 1$, $y = 5x^2 + 1$, $y = \sqrt{5x + 1}$, $y = 5x + 1$

21. $y = x^2 - 2x - 2$, $y = -2x^2 - 2$, $y = \sqrt{-2x - 2}$, $y = -2x - 2$

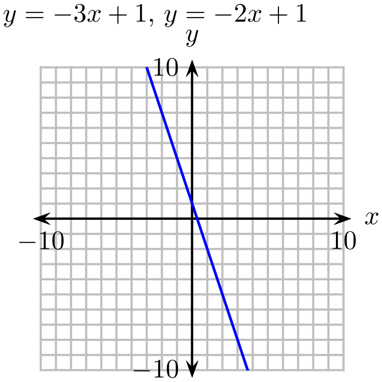
22. $y = x^2 + 5x - 8$, $y = 5x^2 - 8$, $y = \sqrt{5x - 8}$, $y = 5x - 8$

23. $y = x^2 + 7x - 3$, $y = 7x^2 - 3$, $y = 7x - 3$, $y = \sqrt{7x - 3}$

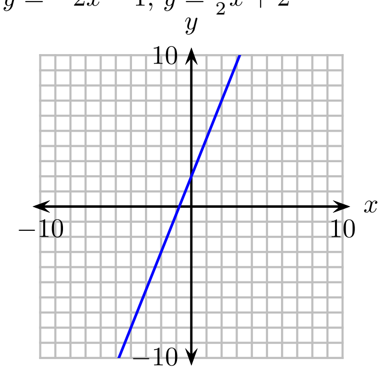
24. $y = \sqrt{-4x - 3}$, $y = x^2 - 4x - 3$, $y = -4x - 3$, $y = -4x^2 - 3$

In Exercises 25-28, which of the given equations has the given graph?

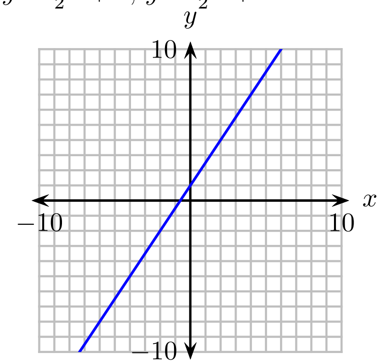
25. $y = -\frac{3}{2}x + 2$, $y = \frac{3}{2}x - 3$,
$y = -3x + 1$, $y = -2x + 1$



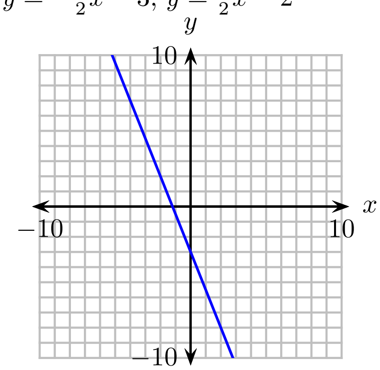
26. $y = -3x - 2$, $y = \frac{3}{2}x + 1$,
$y = -2x - 1$, $y = \frac{5}{2}x + 2$



27. $y = \frac{5}{2}x - 2$, $y = 3x + 3$,
$y = \frac{3}{2}x + 1$, $y = \frac{1}{2}x + 1$



28. $y = 3x + 1$, $y = \frac{5}{2}x - 1$,
$y = -\frac{5}{2}x - 3$, $y = \frac{3}{2}x - 2$



In Exercises 29-44, on graph paper, sketch the graph of the given equation.

29. $y = 3x - 2$

30. $y = \frac{5}{2}x + 1$

31. $y = -2x - 1$

32. $y = \frac{5}{2}x + 2$

33. $y = -2x + 2$

34. $y = -\frac{5}{2}x - 2$

35. $y = -2x - 2$

36. $y = -\frac{5}{2}x + 1$

37. $y = 2x - 2$

38. $y = \frac{5}{2}x - 1$

39. $y = \frac{3}{2}x + 1$

40. $y = 2x + 2$

41. $y = 2x - 3$

42. $y = -\frac{5}{2}x - 1$

43. $y = \frac{3}{2}x + 3$

44. $y = 3x + 1$

---

45. Sketch the lines $y = \frac{1}{2}x - 1$ and $y = \frac{5}{2}x - 2$ on graph paper. As you sweep your eyes from left to right, which line rises more quickly?

46. Sketch the lines $y = \frac{5}{2}x + 1$ and $y = 3x + 1$ on graph paper. As you sweep your eyes from left to right, which line rises more quickly?

47. Sketch the line $y = -\frac{1}{2}x + 1$ and $y = -3x + 3$. As you sweep your eyes from left to right, which line falls more quickly?

48. Sketch the line $y = -3x - 1$ and $y = -\frac{5}{2}x - 2$. As you sweep your eyes from left to right, which line falls more quickly?

49. Sketch the line $y = -3x - 1$ and $y = -\frac{1}{2}x - 2$. As you sweep your eyes from left to right, which line falls more quickly?

50. Sketch the line $y = -3x - 1$ and $y = -\frac{1}{2}x + 1$. As you sweep your eyes from left to right, which line falls more quickly?

51. Sketch the lines $y = \frac{3}{2}x - 2$ and $y = 3x + 1$ on graph paper. As you sweep your eyes from left to right, which line rises more quickly?

52. Sketch the lines $y = \frac{1}{2}x + 3$ and $y = \frac{5}{2}x + 1$ on graph paper. As you sweep your eyes from left to right, which line rises more quickly?

Answers

1. $(-1, -6)$

3. $(-4, 31)$

5. $(2, 15)$

7. $(-2, 14)$

9. $k = -53$

11. $k = -\dfrac{3}{2}$

13. $k = -\dfrac{7}{4}$

15. $k = 8$

17. $y = 6x + 4$

19. $y = x + 7$

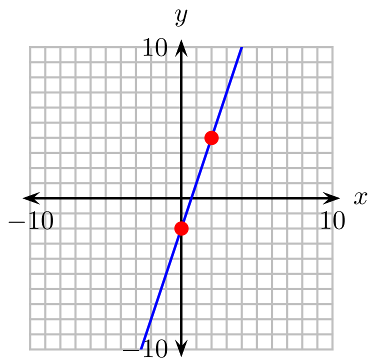
21. $y = -2x - 2$

23. $y = 7x - 3$

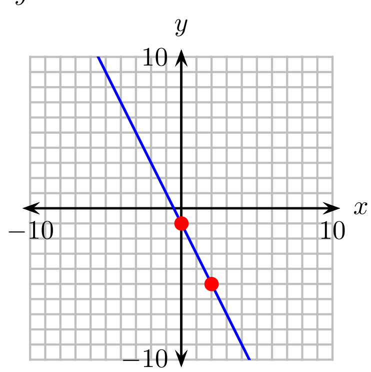
25. $y = -3x + 1$

27. $y = \frac{3}{2}x + 1$

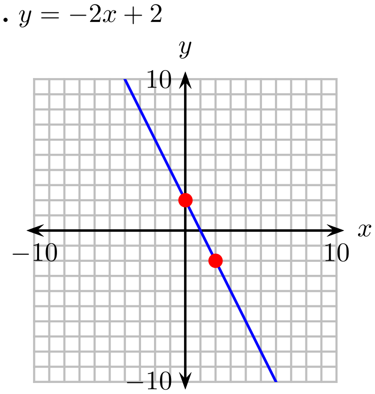
29. $y = 3x - 2$



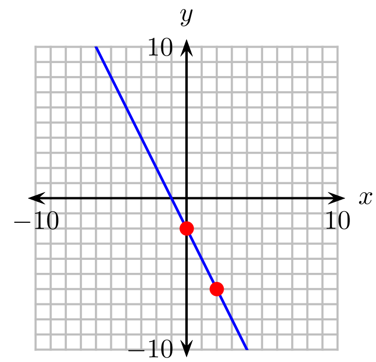
31. $y = -2x - 1$



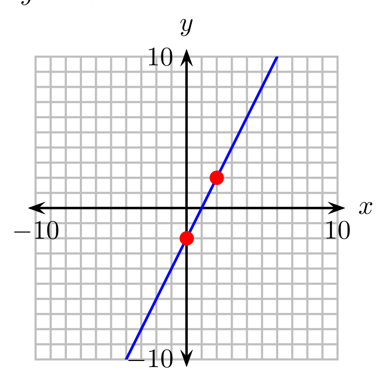
33. $y = -2x + 2$



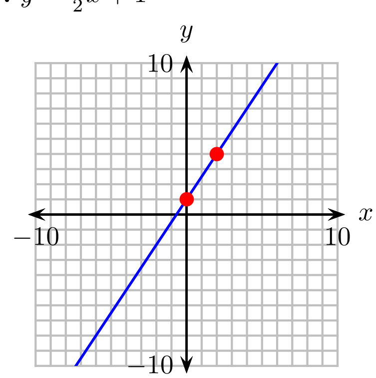
35. $y = -2x - 2$



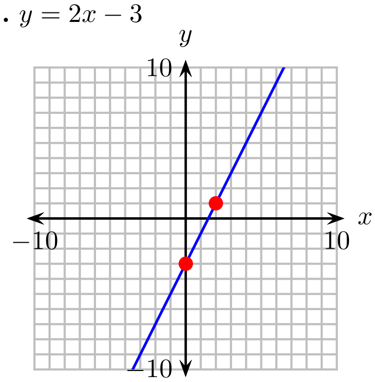
37. $y = 2x - 2$



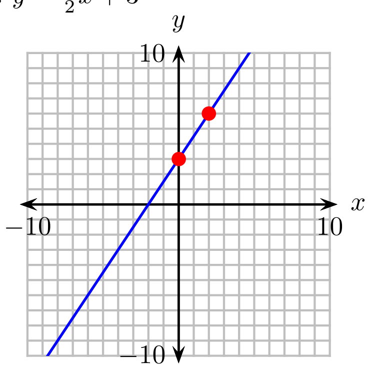
39. $y = \frac{3}{2}x + 1$



41. $y = 2x - 3$



43. $y = \frac{3}{2}x + 3$



45. The graph of $y = \frac{5}{2}x - 2$ rises more quickly.

47. The graph of $y = -3x + 3$ falls more quickly.

49. The graph of $y = -3x - 1$ falls more quickly.

51. The graph of $y = 3x + 1$ rises more quickly.

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