A confirmation of Euler's identity

Part I

Although there is no justification for this (we are not conducting a rigorous proof, but a confirmation), we begin by making the assumption that the function $\sin x$ can be written as an infinite polynomial (infinite series). Thus, we assume the following:

(1) $\sin x = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$

Substituting $x = 0$ gives $\sin 0 = a_0 + 0 + 0 + \cdots$; and since $\sin 0 = 0$ then $a_0 = 0$. Thus,

(2) $\sin x = a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$

We know the derivative of $\sin x$ and the general derivative of a power (power rule), i.e. $\frac{d}{dx}\left(ax^n\right) = a\frac{d}{dx}\left(x^n\right) = a\left(nx^{n-1}\right)$. Let's apply these and take the derivative of both sides of (2).

(3) $\frac{d}{dx}(\sin x) = \frac{d}{dx}\left(a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots\right)$

(4) $\cos x = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots$

Substituting $x = 0$ again gives $\cos 0 = a_1 + 0 + 0 + \cdots$ and since $\cos 0 = 1$ then $a_1 = 1$. Thus,

(5) $\sin x = x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$

Repeat the procedure by taking the derivative of both sides of (4) with $a_1 = 1$, giving

(6) $\frac{d}{dx}(\cos x) = \frac{d}{dx}\left(1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots\right)$

(7) $-\sin x = 2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + 5\cdot 4a_5x^3 + \cdots$

Substituting $x = 0$ again gives $-\sin 0 = 2a_2 + 0 + 0 + 0 + \cdots$ and since $\sin 0 = 0$ then $a_2 = 0$. Thus,

(8) $\sin x = x + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$

Repeat the procedure again – taking the derivative of both sides of (7) with $a_2 = 0$, giving

(9) $\frac{d}{dx}(-\sin x) = \frac{d}{dx}\left(0 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + 5\cdot 4a_5x^3 + \cdots\right)$

(10) $-\cos x = 3\cdot 2a_3 + 4\cdot 3\cdot 2a_4x + 5\cdot 4\cdot 3a_5x^2 + \cdots$

Substituting $x=0$ gives $-\cos 0 = 3\cdot 2a_3 + 0 + 0 + 0 + \cdots$ and since $-\cos 0 = -1$ then $a_3 = -\frac{1}{3\cdot 2}$. Thus,

(11) $\sin x = x - \frac{x^3}{3\cdot 2} + a_4x^4 + a_5x^5 + \cdots$

Repeat the procedure again – taking the derivative of both sides of (10) with $a_3 = -\frac{1}{6}$, giving

(12) $\frac{d}{dx}(-\cos x) = \frac{d}{dx}\left(-1 + 4\cdot 3\cdot 2a_4x + 5\cdot 4\cdot 3a_5x^2 + \cdots\right)$

(13) $\sin x = 4\cdot 3\cdot 2a_4 + 5\cdot 4\cdot 3\cdot 2a_5x + \cdots$

Substituting $x=0$ gives $\sin 0 = 4\cdot 3\cdot 2a_4 + 0 + 0 + \cdots$ and since $\sin 0 = 0$ then $a_4 = 0$. Thus,

(14) $\sin x = x - \frac{x^3}{3\cdot 2} + a_5x^5 + \cdots$

Repeat the procedure again – taking derivative of both sides of (13) with $a_4 = 0$, giving

(15) $\frac{d}{dx}(\sin x) = \frac{d}{dx}(5\cdot 4\cdot 3\cdot 2a_5x + \cdots)$

(16) $\cos x = 5\cdot 4\cdot 3\cdot 2a_5 + \cdots$

Substituting $x=0$ gives $\cos 0 = 5\cdot 4\cdot 3\cdot 2a_5 + 0 + 0 + \cdots$ and since $\cos 0 = 1$ then $a_5 = \frac{1}{5\cdot 4\cdot 3\cdot 2}$. Thus,

(17) $\sin x = x - \frac{x^3}{3\cdot 2} + \frac{x^5}{5\cdot 4\cdot 3\cdot 2} - \cdots$ or $\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

A clear pattern has been established. The signs of the terms alternate and the powers of x are consecutive odd numbers – and the denominator of each term is the factorial of the same consecutive odd numbers.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Therefore $\sin x$ is equivalent to an infinite polynomial, i.e. an infinite series. This infinite series can be represented using sigma notation as follows.

$$\sin x = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} \quad \text{or} \quad \sin x = \sum_{n=0}^{\infty}(-1)^{n}\frac{x^{2n+1}}{(2n+1)!}$$

Part II

Assuming that the function $\cos x$ can also be expressed as an infinite polynomial, we could carry out a similar procedure to find it. But a more efficient method is to consider that if two functions are equal then it must logically follow that their derivatives are also equal, i.e. if $f(x)=g(x)$ then $f'(x)=g'(x)$. Thus, since $\frac{d}{dx}(\sin x)=\cos x$ then the derivative of the infinite polynomial equivalent to $\sin x$ found above will be equal to the infinite series for $\cos x$.

$$\cos x=\frac{d}{dx}(\sin x)=\frac{d}{dx}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right)=1-\frac{3x^2}{6}+\frac{5x^4}{120}-\frac{7x^6}{5040}+\cdots=1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots$$

Thus, $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$

This infinite series for $\cos x$ can be represented with sigma notation as follows.

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

Part III

Recall the following definition for the number e.

$$\mathrm{e}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Let's use the binomial theorem to expand $\left(1+\frac{1}{n}\right)^n$.

binomial theorem: $(a+b)^n=\sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$ where $\binom{n}{r}=\frac{n!}{r!(n-r)!}$

$$\left(1+\frac{1}{n}\right)^n=1+n\left(\frac{1}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3+\cdots$$

Thus, $\mathrm{e}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\lim_{n\to\infty}\left(1+n\left(\frac{1}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3+\cdots\right)$

$$=\lim_{n\to\infty}\left(1+\frac{n}{n}+\frac{n^2-n}{n^2}\cdot\frac{1}{2!}+\frac{n^3-3n^2+2n}{n^3}\cdot\frac{1}{3!}+\cdots\right)$$

As n goes to infinity, the limit of each of the expressions $\frac{n}{n}, \frac{n^2-n}{n^2}, \frac{n^3-3n^2+2n}{n^3}$ is 1.

Thus, $\mathrm{e} = \lim_{n\to\infty}\left(1+\frac{n}{n}+\frac{n^2-n}{n^2}\cdot\frac{1}{2!}+\frac{n^3-3n^2+2n}{n^3}\cdot\frac{1}{3!}+\cdots\right)$

$$\mathrm{e} = 1+1+1\cdot\frac{1}{2!}+1\cdot\frac{1}{3!}+\cdots = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots$$

But, we want the infinite series for e^x. Consider $\mathrm{e}^x = \lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\right)^x = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx}$

In order to use our result above from the binomial theorem, we need to re-write the expression $\left(1+\frac{1}{n}\right)^{nx}$ in the form $\left(1+\frac{a}{b}\right)^b$. Letting $m = nx$, it follows that $\frac{1}{n}=\frac{x}{m}$; and if $n\to\infty$ then $m\to\infty$.

Substituting gives $\mathrm{e}^x = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx} = \lim_{m\to\infty}\left(1+\frac{x}{m}\right)^m$. Using the result from the binomial theorem

$$\mathrm{e}^x = \lim_{m\to\infty}\left(1+\frac{x}{m}\right)^m = \lim_{m\to\infty}\left(1+m\left(\frac{x}{m}\right)+\frac{m(m-1)}{2!}\left(\frac{x}{m}\right)^2+\frac{m(m-1)(m-2)}{3!}\left(\frac{x}{m}\right)^3+\cdots\right)$$

$$=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$$

In sigma notation, $\mathrm{e}^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

Part IV

Now, we're ready to put the pieces together for Euler's identity. Here are the results we've obtained:

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

$$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$$

$$\mathrm{e}^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\frac{x^7}{7!}+\cdots$$

To work our way towards confirming Euler's identity, $e^{i\theta} = \cos\theta + i\sin\theta$, let's replace x with θ in the series for $\sin x$ and $\cos x$, and replace x with $i\theta$ in the series for e^x.

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$i\sin\theta = i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \cdots$$

Simplifying powers of the imaginary number i produces the following

$$e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \frac{i^7\theta^7}{7!} + \cdots = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \cdots$$

and grouping together real and imaginary terms gives

$$e^{i\theta} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) + \cdots$$

Thus, confirming that $e^{i\theta} = \cos\theta + i\sin\theta$.

Most 'beautiful' equation – special case of Euler's identity

For $e^{i\theta} = \cos\theta + i\sin\theta$, replacing θ with π gives

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 + i\cdot 0$$

Therefore, $e^{i\pi} + 1 = 0$