Mixing problems

Mixing problems are an application of separable differential equations. They're word problems that require us to create a separable differential equation based on the concentration of a substance in a tank.

Usually we'll have a substance like salt that's being added to a tank of water at a specific rate. At the same time, the salt water mixture is being emptied from the tank at a specific rate. We usually that the contents of the tank are always perfectly mixed, and we're asked to model the concentration in the tank at a certain time. The formula we use to model concentration is

$$\frac{dy}{dt} = C_1 r_1 - C_2 r_2$$

where

C_1 is the concentration of the substance being added

r_1 is the rate at which the substance is added

C_2 is the concentration of the substance being removed

r_2 is the rate at which the substance is removed

Once we've plugged everything into the mixing problem formula, we'll need to treat it as a separable differential equation, which means that we'll separate variables, integrate both sides of the equation, and then try to find a general solution.

Example

A tank contains $1{,}500$ L of water and 20 kg of dissolved salt. Fresh water is entering the tank at 15 L/min (the solution stays perfectly mixed), and the solution drains at a rate of 10 L/min. How much salt is in the tank at t minutes and at 10 minutes?

We'll start with the mixing problem formula

$$\frac{dy}{dt} = C_1 r_1 - C_2 r_2$$

In this problem, we're interested in the concentration of salt in the tank.

$C_1 = 0$ kg/L because no salt is being added into the tank.

$r_1 = 15$ L/min because this is the rate at which water is entering the tank

$C_2 = \frac{y}{1{,}500}$ kg/L because we're not sure how much salt is leaving the tank, but we know the initial amount of water is $1{,}500$ L

$r_2 = 10$ L/min because this is the rate at which the solution is leaving the tank

If we plug all these values into the formula, we get

$$\frac{dy}{dt} = (0)(15) - \left(\frac{y}{1{,}500}\right)(10)$$

$$\frac{dy}{dt} = -\frac{y}{150}$$

Now we'll separate the variables.

$$dy = -\frac{y}{150}\ dt$$

$$\frac{1}{y}\ dy = -\frac{1}{150}\ dt$$

With the variables separated, we'll integrate both sides of the equation.

$$\int \frac{1}{y}\ dy = \int -\frac{1}{150}\ dt$$

$$\ln|y| + C_1 = -\frac{1}{150}t + C_2$$

Collect and simplify the constants.

$$\ln|y| = -\frac{1}{150}t + C_2 - C_1$$

$$\ln|y| = -\frac{1}{150}t + C$$

Raise both sides to the base e in order to eliminate the natural log.

$$e^{\ln|y|} = e^{-\frac{1}{150}t+C}$$

$$|y| = e^{-\frac{1}{150}t}e^{C}$$

e^C is a constant, so it can simplify to just C. And we can remove the absolute value by adding $\pm$ to the other side of the equation.

$$|y| = Ce^{-\frac{1}{150}t}$$

$$y = \pm Ce^{-\frac{1}{150}t}$$

The $\pm$ gets absorbed into the constant C and so the explicit equation for y is

$$y = Ce^{-\frac{1}{150}t}$$

We were told that initially 20 kg of dissolved salt existed in the tank. This is essentially the initial condition $y(0) = 20$.

$$20 = Ce^{-\frac{1}{150}(0)}$$

$$20 = C(1)$$

$$C = 20$$

Plugging this back into the general solution, we get

$$y(t) = 20e^{-\frac{1}{150}t}$$

This is the equation that models the amount of salt in the tank at t minutes. If we want to figure out how much salt is in the tank after 5 minutes, we just plug 5 in for t. If we want to figure out how much salt is in the tank after 20 minutes, we just plug 20 in for t.

We've also been asked in this problem to find the amount of salt in the tank after 10 minutes. Plugging 10 in for t, we get

$$y(10) = 20e^{-\frac{1}{150}(10)}$$

$$y(10) = 20e^{-\frac{1}{15}}$$

$$y(10) = 18.7$$

After 10 minutes, there's 18.7 kg of salt in the tank.
