♦ UNIT 2: *COMPLEX NUMBERS*, ANALYSIS AND MATRICES

MODULE 1: *COMPLEX NUMBERS AND CALCULUS II*

GENERAL OBJECTIVES

On completion of this Module, students should:

1. *develop the ability to represent and deal with objects in the plane through the use of complex numbers;*

2. *be confident in using the techniques of differentiation and integration;*

3. *develop* the ability to use concepts to model and solve real-world problems.

SPECIFIC OBJECTIVES

(A) Complex Numbers

Students should be able to:

1. recognise the need to use complex numbers to find the roots of the general quadratic equation $ax^2 + bx + c = 0$, when $b^2 - 4ac < 0$;

2. *use the concept that complex roots of equations with constant coefficients occur in conjugate pairs*;

3. write the roots of the equation in that case and relate the sums and products to a, b and c;

4. calculate the square root of a complex number;

5. express complex numbers in the form $a + b\text{i}$ where a, b are real numbers, and identify the real and imaginary parts;

6. add, subtract, multiply and divide complex numbers in the form $a + b\text{i}$, where a and b are real numbers;

7. find the principal value of the argument θ of a non-zero complex number, where $-\pi < \theta \leq \pi$;

8. find the modulus and conjugate of a given complex number;

9. interpret modulus and argument of complex numbers on the Argand diagram;

10. represent complex numbers, their sums, differences and products on an Argand diagram;

11. find the set of all points z (locus of z) on the Argand Diagram such that z satisfies given properties;

12. apply De Moivre's theorem for integral values of n;

13. *use* $e^{ix} = \cos x + i \sin x$, *for real* x.

CONTENT

(A) Complex Numbers

1. Nature of roots of a quadratic equation, sums and products of roots.

2. *Conjugate pairs of complex roots.*

3. Addition, subtraction, multiplication and division of complex numbers in the form $a + b\mathrm{i}$ where a, b are the real and imaginary parts, respectively, of the complex number.

4. The modulus, argument and conjugate of a complex number.

5. Representation of complex numbers on an Argand diagram.

6. Locus of a point.

7. De Moivre's theorem for integral n.

8. *Polar-argument and exponential forms of complex numbers*.

SPECIFIC OBJECTIVES

(B) Differentiation II

Students should be able to:

1. find the derivative of $e^{f(x)}$, where $f(x)$ is a differentiable function of x;

2. find the derivative of $\ln f(x)$ (to include functions of x – polynomials or trigonometric);

3. apply the chain rule to obtain gradients *and equations* of tangents and normals to curves given by their parametric equations;

4. use the concept of implicit differentiation, with the assumption that one of the variables is a function of the other;

5. differentiate any combinations of polynomials, trigonometric, exponential and logarithmic functions;

6. differentiate inverse trigonometric functions;

7. obtain second derivatives, $f''(x)$, of the functions in 3, 4, 5 above;

8. *find the first and second partial derivatives of u = f (x, y).*

CONTENT

(B) Differentiation II

1. Application of the chain rule to differentiation.

2. Chain rule and differentiation of composite functions.

3. Gradients of tangents and normals.

4. Implicit differentiation.

5. First derivative of a function which is defined parametrically.

6. Differentiation of inverse trigonometric functions.

7. Differentiation of combinations of functions.

8. Second derivative, that is, $f''(x)$.

9. *First partial derivative.*

10. *Second partial derivative.*

SPECIFIC OBJECTIVES

(C) Integration II

Students should be able to:

1. express a rational function ***(proper and improper)*** in partial fractions in the cases where the denominators are:

   (a) distinct linear factors,

   (b) repeated linear factors,

(c) quadratic factors,

(d) repeated quadratic factors,

(e) combinations of (a) to (d) above (repeated factors will not exceed power 2);

2. express an improper rational function as a sum of a polynomial and partial fractions;

3. integrate rational functions in Specific Objectives 1 and 2 above;

4. integrate trigonometric functions using appropriate trigonometric identities;

5. integrate exponential functions and logarithmic functions;

6. find integrals of the form $\int \frac{f'(x)}{f(x)}\,dx$;

7. use substitutions to integrate functions (the substitution will be given in all but the most simple cases);

8. use integration by parts for combinations of functions;

9. integrate inverse trigonometric functions;

10. derive and use reduction formulae to obtain integrals;

11. use the trapezium rule as an approximation method for evaluating the area under the graph of the function.

CONTENT

(C) Integration II

1. Partial fractions.

2. Integration of rational functions, using partial fractions.

3. Integration by substitution.

4. Integration by parts.

5. Integration of inverse trigonometric functions.

6. Integration by reduction formula.

7. Area under the graph of a continuous function (Trapezium Rule).

Suggested Teaching and Learning Activities

To facilitate students' attainment of the objectives of this Module, teachers are advised to engage students in the teaching and learning activities listed below.

Principal Argument of a Complex Number

The representation of the complex number $z = 1 + \mathrm{i}$ *on the Argand diagram may be used to introduce this topic. Encourage students to indicate and evaluate the argument of* z. *The students' answers should be displayed on the chalkboard.*

Indicate that the location of z *on the Argand diagram is unique, and therefore only one value of the argument is needed to position* z. *That argument is called the principal argument, arg* z, *where:*

$$-\pi < \text{principal argument} \leq \pi.$$

Students should be encouraged to calculate the principal argument by either solving:

(a) *the simultaneous equations*

$$\cos\theta = \frac{\mathrm{Re}(z)}{|z|} \text{ and } \sin\theta = \frac{\mathrm{Im}(z)}{|z|}, \text{ with} -\pi < \theta \leq \pi\,;$$

or,

(b) *the equation*

$$\tan\theta = \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} \text{ for } \mathrm{Re}(z) \neq 0 \text{ and } -\pi < \theta \leq \pi,$$

together with the representation of z on the Argand diagram.

(c) *Students should be encouraged to find the loci of z-satisfying equations such as:*

(i) $|z - a| = k$;
(ii) $|z - c| = |z - b|$;
(iii) $\arg(z - a) = \alpha$.

RESOURCES

Bostock, L. and Chandler, S. — *Core Mathematics for A-Levels*, United Kingdom: Stanley Thornes Publishing Limited, 1997.

Bradie, B. — *Rate of Change of Exponential Functions*: A Precalculus Perspective, Mathematics Teacher Vol. 91(3), p. 224 – 237.

Campbell, E.	*Pure Mathematics for CAPE, Vol. 2, Jamaica:* LMH Publishing Limited, 2007.
Caribbean Examinations Council	*The Exponential and Logarithmic Functions – An Investigation, Barbados:* 1998*.*
Martin, A., Brown, K., Rigby, P. and Ridley, S.	Pure Mathematics, *Cheltenham, United Kingdom*: *Stanley Thornes (Publishers) Limited, 2000.*

UNIT 2

MODULE 2: SEQUENCES, SERIES AND APPROXIMATIONS

GENERAL OBJECTIVES

On completion of this Module, students should:

1. *understand* the concept of a sequence as a function from the natural numbers to the real numbers;

2. *understand* the difference between sequences and series;

3. distinguish between convergence and/or divergence of some standard series or sequences;

4. apply successive approximations to roots of equations;

5. *develop* the ability to use concept to model and solve real-world problems.

SPECIFIC OBJECTIVES

(A) Sequences

Students should be able to:

1. define the concept of a sequence $\{a_n\}$ of terms a_n as a function from the positive integers to the real numbers;

2. write a specific term from the formula for the n^{th} term, or from a recurrence relation;

3. describe the behaviour of convergent and divergent sequences, through simple examples;

4. apply mathematical induction to establish properties of sequences.

CONTENT

(A) Sequences

1. Definition, convergence, divergence *and* limit of a sequence.

2. Sequences defined by recurrence relations.

3. Application of mathematical induction to sequences.

SPECIFIC OBJECTIVES

(B) Series

Students should be able to:

1. use the summation (Σ) notation;
2. define a series, as the sum of the terms of a sequence;
3. identify the n^{th} term of a series, in the summation notation;
4. define the m^{th} partial sum S_m as the sum of the first m terms of the sequence, that is,

   $S_m = \sum_{r=1}^{m} a_r$;
5. apply mathematical induction to establish properties of series;
6. find the sum to infinity of a convergent series;
7. apply the method of differences to appropriate series, and find their sums;
8. use the Maclaurin theorem for the expansion of series;
9. *use the Taylor theorem for the expansion of series*.

CONTENT

(B) Series

1. Summation notation (Σ).
2. *Series as the sum of terms of a sequence.*
3. *Convergence and/or divergence of series to which the method of differences can be applied.*
4. *The Maclaurin series.*
5. *The Taylor series.*
6. Applications of mathematical induction to series.

SPECIFIC OBJECTIVES

(C) The Binomial Theorem

Students should be able to:

1. explain the meaning and use simple properties of $n!$ and $\binom{n}{r}$, that is, ${}^{n}C_{r}$, where $n, r \in \mathbb{Z}$;

2. *recognise that ${}^{n}C_{r}$ that is, $\binom{n}{r}$, is the number of ways in which r objects may be chosen from n distinct objects;*

3. expand $(a+b)^n$ for $n \in \mathbb{Q}$;

4. apply the Binomial Theorem to real-world problems, for example, in mathematics of finance, science.

CONTENT

(C) The Binomial Theorem

1. Factorials and Binomial coefficients; their interpretation and properties.

2. The Binomial Theorem.

3. Applications of the Binomial Theorem.

SPECIFIC OBJECTIVES

(D) Roots of Equations

Students should be able to:

1. test for the existence of a root of $f(x) = 0$ where f is continuous using the Intermediate Value Theorem;

2. *use interval bisection to find an approximation for a root in a given interval;*

3. *use linear interpolation to find an approximation for a root in a given interval;*

4. explain, in geometrical terms, the working of the Newton-Raphson method;

UNIT 2
MODULE 2: SEQUENCES, SERIES AND APPROXIMATIONS (cont'd)

5. use the Newton-Raphson method to find successive approximations to the roots of $f(x) = 0$, where f is differentiable;

6. *use a given iteration to determine a root of an equation to a specified degree of accuracy.*

CONTENT

(D) Roots of Equations

Finding successive approximations to roots of equations using:

1. *Intermediate Value Theorem;*

2. *Interval Bisection;*

3. *Linear Interpolation;*

4. *Newton - Raphson Method;*

5. *Iteration.*

__Suggested Teaching and Learning Activities__

To facilitate students' attainment of the objectives of this Module, teachers are advised to engage students in the learning activities listed below.

1. The Binomial Theorem

Students may be motivated to do this topic by having successive expansions of $(a + x)^n$ and then investigating the coefficients obtained when expansions are carried out.

$$(a + b)^1 = a + b$$

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + a^4$$

and so on.

By extracting the coefficients of each term made up of powers of a, x or a and x.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Students should be encouraged to use the emerging pattern to generate further expansions of $(a + x)^n$. This can be done by generating the coefficients from Pascal's Triangle and then investigating other patterns. For example, by looking at the powers of a and x (powers of x increase from 0 to n, while powers of a decrease from n to 0; powers of a and x add up to n).

In discussing the need to find a more efficient method of doing the expansions, the Binomial Theorem may be introduced. However, this can only be done after the students are exposed to principles of counting, with particular reference to the process of <u>selecting</u>. In so doing, teachers will need to guide students through appropriate examples involving the selection of r objects, say, from a group of n unlike objects. This activity can lead to defining ${}^nC_r$ as the number of ways of selecting r objects from a group of n unlike objects.

In teaching this principle, enough examples should be presented before ${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$ formula is developed.

The binomial theorem may then be established by using the expansion of $(1 + x)^n$ as a starting point. A suggested approach is given below:

Consider $(1 + x)^n$.

To expand, the student is expected to multiply $(1 + x)$ by itself n times, that is,
$(1 + x)^n = (1 + x)(1 + x)(1 + x) \ldots (1 + x)$.

The result of the expansion is found as given below:

The constant term is obtained by multiplying all the 1's. The result is therefore 1.

The term in x is obtained by multiplying $(n - 1)$ 1's and one x. This x, however, may be chosen from any of the n brackets. That is, we need to choose one x out of n different brackets. This can be done in ${}^nC_1$ ways. Hence, the coefficient of x is ${}^nC_1$.

Similarly, the term in x^2 may be obtained by choosing two x's and $(n – 2)$ 1's. The x's may be chosen from any two of the n brackets. This can be done in ${}^{n}C_{2}$ ways. The coefficient of x^2 is therefore ${}^{n}C_{2}$.

This process continues and the expansion is obtained:

$$(1+x)^n = 1 + {}^{n}C_{1}x + {}^{n}C_{2}x^2 + {}^{n}C_{3}x^3 + \ldots + x^n.$$

This is known as the binomial theorem. The theorem may be written as

$$(1+x)^n = \sum_{r=0}^{n} {}^{n}C_{r}x^r.$$

The generalisation of this could be done as a class activity where students are asked to show that:

$$(a+b)^n = a^n + {}^{n}C_{1}a^{n-1}b + {}^{n}C_{2}a^{n-2}b^2 + {}^{n}C_{3}a^{n-3}b^3 + \ldots + b^n.$$

This is the binomial expansion of $(a + b)^n$ for positive integral values of n. The expansion terminates after $(n + 1)$ terms.

2. The Intermediate Value Theorem

(a) Motivate with an example.

Example: A taxi is travelling at 5 km/h at 8:00 a.m. Fifteen minutes later the speed is 100 km/h. Since the speed varies continuously, clearly at some time between 8:00 a.m. and 8:15 a.m. the taxi was travelling at 75 km/h.

Note that the taxi could have traveled at 75 km/h at more than one time between 8:00 a.m. and 8:15 a.m.

(b) Use examples of continuous functions to illustrate the Intermediate Value Theorem.

Example: $f(x) = x^2 – x – 6$ examined on the intervals (3.5, 5) and (0, 4).

3. Existence of Roots

Introduce the existence of the root of a continuous function $f(x)$ between given values a and b as an application of the Intermediate Value Theorem.

Emphasis should be placed on the fact that:

(a) f must be continuous between a and b;

(b) The product of f (a) and f (b) is less than zero, that is, f (a) and f (b) must have opposite signs.

4. ***Interval Bisection***

Initially students should be able to determine an interval in which a real root lies. If f *(a) and* f (b) *are of opposite signs, and* f *is continuous, then* $a < x < b$, *for the equation* f (x) = 0.

Students may be asked to investigate $x = \frac{\alpha + \beta}{2}$ *and note the resulting sign to determine which side of* $\frac{\alpha + \beta}{2}$ *the root lies. This method can be repeated until same answer to the desired degree of accuracy is obtained.*

5. ***Linear Interpolation***

Given the points (x_0, y_0) and (x_1, y_1) *on a continuous curve* y = f (x), *students can establish that for* f (x_0) *and* f (x_1) *with opposite signs and that* f *is continuous, then* $x_0 < x < x_1$, *for the equation* f (x) = 0. *If* $|\mathrm{f}(x_0)| < |\mathrm{f}(x_1)|$ *say, students can be introduced to the concept of similar triangles to find successive approximations, holding* f (x_1) *constant. This intuitive approach is formalised in* ***linear interpolation***, *where the two points* (x_0, y_0) and (x_1, y_1) *can be joined by a straight line and the* x*-value of the point on this line is calculated. A first approximation for* x *can be found using*

$$\frac{x}{\mathrm{f}(x_0)} = \frac{x_1}{\mathrm{f}(x_1)}\ .$$

Successive approximations can be found with this approach until the same answer to the desired degree of accuracy is obtained.

RESOURCE

Bostock, L. and Chandler, S. — *Core Mathematics for A-Levels,* United Kingdom*:* Stanley Thornes Publishing Limited, 1997.

Campbell, E. — *Pure Mathematics for CAPE, Vol. 2, Jamaica:* LMH Publishing Limited, 2007.

UNIT 2
MODULE 3: COUNTING, MATRICES AND *DIFFERENTIAL EQUATIONS*

GENERAL OBJECTIVES

On completion of this Module, students should:

1. *develop* the ability to analyse and solve simple problems dealing with choices and arrangements;

2. *develop an understanding of* the algebra of matrices;

3. *develop* the ability to analyse and solve systems of linear equations;

4. *develop skills to model some real-world phenomena by means of differential equations, and solve these;*

5. *develop* the ability to use concepts to model and solve real-world problems.

SPECIFIC OBJECTIVES

(A) Counting

Students should be able to:

1. state the principles of counting;

2. find the number of ways of arranging n distinct objects;

3. find the number of ways of arranging n objects some of which are identical;

4. find the number of ways of choosing r distinct objects from a set of n distinct objects;

5. identify a sample space*;*

6. *identify* the numbers of possible outcomes in a given sample space*;*

7. *use Venn diagrams to illustrate the principles of counting;*

8. *use possibility space diagram to identify a sample space*;

9. *define* and calculate P(A), the probability of an event A occurring as the number of possible ways in which A can occur divided by the total number of possible ways in which all equally likely outcomes, including A, occur;

10. use the fact that $0 \leq \mathrm{P}(A) \leq 1$;

11. demonstrate and use the property that the total probability for all possible outcomes in the sample space is 1;

12. use the property that P(A') $= 1 - \mathrm{P}(A)$ is the probability that event A does not occur*;*

13. use the property P($A \cup B$) = P (A) + P (B) – P($A \cap B$) for event A and B;

14. use the property P($A \cap B$) = 0 or P ($A \cup B$) = P (A) + P (B), where A and B are mutually exclusive events;

15. use the property P($A \cap B$) = P(A) × P(B), where A and B are independent events;

16. *use the property* P($A|B$) = $\dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$ *where* P(B) $\neq$ *0.*

17. *use a tree diagram to list all possible outcomes for conditional probability.*

CONTENT

(A) Counting

1. Principles of counting.

2. Arrangements with and without repetitions.

3. Selections.

4. Venn diagram.

5. Possibility space diagram.

6. Concept of probability and elementary applications.

7. *Tree diagram*.

SPECIFIC OBJECTIVES

(B) Matrices and Systems of Linear Equations

Students should be able to:

1. operate with conformable matrices, carry out simple operations and manipulate matrices using their properties;

2. evaluate the determinants of $n \times n$ matrices, $1 \leq n \leq 3$;

UNIT 2

MODULE 3: COUNTING, MATRICES AND *DIFFERENTIAL EQUATIONS* (cont'd)

3. reduce a system of linear equations to echelon form;

4. row-reduce the augmented matrix of an $n \times n$ system of linear equations, $n = 2, 3$;

5. determine whether the system is consistent, and if so, how many solutions it has;

6. find all solutions of a consistent system;

7. invert a non-singular 3×3 matrix;

8. solve a 3×3 system of linear equations, having a non-singular coefficient matrix, by using its inverse.

CONTENT

(B) Matrices and Systems of Linear Equations

1. $m \times n$ matrices, for $1 \leq m \leq 3$, and $1 \leq n \leq 3$, and equality of matrices.

2. Addition of conformable matrices, zero matrix and additive inverse, associativity, commutativity, distributivity, transposes.

3. Multiplication of a matrix by a scalar.

4. Multiplication of conformable matrices.

5. Square matrices, singular and non-singular matrices, unit matrix and multiplicative inverse.

6. $n \times n$ determinants, $1 \leq n \leq 3$.

7. $n \times n$ systems of linear equations, consistency of the systems, equivalence of the systems, solution by reduction to row echelon form, $n = 2, 3$.

8. $n \times n$ systems of linear equations by row reduction of an augmented matrix, $n = 2, 3$.

SPECIFIC OBJECTIVES

(C) *Differential Equations and Modeling*

Students should be able to:

1. *solve first order linear differential equations* $y' - ky = \mathrm{f}(x)$ *using an integrating factor, given that* k *is a real constant or a function of* x*, and* f *is a function;*

2. *solve first order linear differential equations given boundary conditions;*

3. *solve second order ordinary differential equations with constant coefficients of the form*

$$ay'' + by' + cy = 0 = \mathrm{f}(x), \text{ where } a, b, c \in \mathbb{R} \text{ and } \mathrm{f}(x) \textit{ is:}$$

   (a) *a polynomial,*
   (b) *an exponential function,*
   (c) *a trigonometric function;*

   and the complementary function may consist of

   (a) *2 real and distinct root,*
   (b) *2 equal roots,*
   (c) *2 complex roots*;

4. *solve second order ordinary differential equation given boundary conditions;*

5. *use substitution to reduce a second order ordinary differential equation to a suitable form.*

CONTENT

(C) ***Differential Equations and Modeling***

1. *Formulation and solution of differential equations of the form* $y' - ky = \mathrm{f}(x)$*, where* k *is a real constant or a function of* x*, and* f *is a function.*

2. *Second order ordinary differential equations.*

<u>Suggested Teaching and Learning Activities</u>

To facilitate students' attainment of the objectives of this Module, teachers are advised to engage students in the teaching and learning activities listed below.

1. Counting

Consider the three scenarios given below.

(a) Throw two dice. Find the probability that the sum of the dots on the uppermost faces of the dice is 6.

(b) An insurance salesman visits a household. What is the probability that he will be successful in selling a policy?

(c) A hurricane is situated 500km east of Barbados. What is the probability that it will hit the island?

These three scenarios are very different for the calculation of probability. In 'a', the probability is calculated as the number of successful outcomes divided by the total possible number of outcomes. In this classical approach, the probability assignments are based on equally likely outcomes and the entire sample space is known from the start.

The situation in 'b' is no longer as well determined as in 'a'. It is necessary to obtain historical data for the salesman in question and estimate the required probability by dividing the number of successful sales by the total number of households visited. This frequency approach still relies on the existence of data and its applications are more realistic than those of the classical methodology.

For 'c' it is very unclear that a probability can be assigned. Historical data is most likely unavailable or insufficient for the frequency approach. The statistician might have to revert to informed educated guesses. This is quite permissible and reflects the analyst's prior opinion. This approach lends itself to a Bayesian methodology.

One should note that the rules and results of probability theory remain exactly the same regardless of the method used to estimate the probability of events.

2. Systems of Linear Equations in Two Unknowns

(a) In order to give a geometric interpretation, students should be asked to plot on graph paper the pair of straight lines represented by a given pair of linear equations in two unknowns, and to examine the relationship between the pair of straight lines in the cases where the system of equations has been shown to have:

(i) one solution;

(ii) many solutions;

(iii) no solutions.

(b) Given a system of equations with a unique solution, there exist equivalent systems, obtained by row-reduction, having the same solution. To demonstrate this, students should be asked to plot on the same piece of graph paper all the straight lines represented by the successive pairs of linear equations which result from each of the row operations used to obtain the solution.

UNIT 2
MODULE 3: *COUNTING, MATRICES AND DIFFERENTIAL EQUATIONS (cont'd)*
RESOURCES

Bolt, B. and Hobbs, D.	*101 Mathematical Projects: A Resource Book*, United Kingdom: Cambridge University Press, 1994.
Bostock, L. and Chandler, S.	*Core Mathematics for A-Levels*, United Kingdom: Stanley Thornes Publishing Limited, 1997.
Campbell, E.	*Pure Mathematics for CAPE, Vol. 2, Jamaica:* LMH Publishing Limited, 2007.
Crawshaw, J. and Chambers, J.	*A Concise Course in A-Level Statistics*, Cheltenham, United Kingdom: Stanley Thornes (Publishers) Limited, 1999.