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Pricing catastrophe bonds by an arbitrage approach

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Abstract

This paper develops a simple arbitrage approach to valuing insurance-linked securities, which accounts for catastrophic events and interest rate randomness, notwithstanding a framework of non-traded underlyings. It shows that holders of catastrophe bonds are in a short position on one-touch binary options based upon risk-tracking indexes that obey jump-diffusion processes. Using first-passage time distributions, this contribution provides a closed-form valuation expression in the context of pure crashes, while it resorts to numerical simulations in the case of mid-range catastrophes. Comparative statics results point out that the term structure of yield spreads of catastrophe bonds is hump-shaped as for corporate bonds.

JEL classification: C15, C63, G12, G13

Keywords: Catastrophe bonds; Path-dependent digital options; Jump-diffusion process; Incomplete markets; Monte-Carlo simulations

1. Introduction

Securitization of losses from catastrophic events such as hurricanes and earthquakes is gaining momentum. The insurance industry may face overwhelming environmental risks, as evidenced by Hurricane Andrew, which resulted in 30 billion US$ in 1992. On the other hand, daily fluctuations on worldwide financial markets reach tens of billion US$. Therefore, securitization is likely to offer a more effective mechanism for financing catastrophic losses than conventional insurance and reinsurance, as advocated for instance by Jaffee and Russell (1997).

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Index-linked catastrophic loss futures contracts were introduced on the Chicago Board of Trade (CBOT) in 1992. Call option spreads later superseded them, but were de-listed eventually because of low trading volume. New entrants include over-the-counter (OTC) products, primarily engineered by investment bankers. This trend has advanced in the recent years with the introduction of weather derivatives to the U.S. since 1997, with subsequent growth in Europe. For example, a European bank and an insurance company teamed up to launch a joint venture "Meteo transformer" meant to issue weather derivatives and insurance contracts in late 2000. Also, the stock exchange Euronext introduced electricity derivatives in 2001. Indeed, the needs for power and weather hazards are often related.

Key structures for insurance-risk transfer to capital markets are catastrophe bonds issued by industrial corporations and insurance–reinsurance companies. They enable the former to hedge natural risks by means of personalized contracts, and in so doing, to focus on their core business, and the latter to share their business risk with other market participants. The first successful catastrophe bond (catbond) was issued in 1997 by Swiss Re to cover earthquake losses; and the first catbond by a non-financial firm was issued in 1999 in order to cover earthquake losses in the Tokyo region for Oriental Land Company, Ltd., the owner of Tokyo Disneyland. More recently, Swiss reinsurer Zurich Re placed US$ 160 million of risk-linked securities in 2001, which provides it "fully collateralized protection against low-frequency, high-severity hurricane and earthquake exposures in the United States, as well as European windstorm." Transactions on the market for catastrophe risk are documented in Froot (2001).

As described in Fig. 1, the hedger (e.g., a corporation) pays an insurance premium in exchange for a pre-specified coverage if a catastrophic event occurs; and investors purchase an insurance-linked security for cash. The total amount (premium + cash proceeds) is directed to a tailor-made fund, called a special-purpose vehicle (SPV), which issues the catastrophe bonds to investors and purchases safe securities as Treasury bonds. Therefore, investors hold nature-linked assets whose cash flows—coupons and/or principal—are contingent on the risk occurrence. If the covered events happen during the risk-exposure period, the SPV compensates the firm and there is full or partial forgiveness of the repayment of principal and/or interest. If the defined events do not occur, the investors receive their principal plus interest equal to the risk-free rate plus a risk-premium.

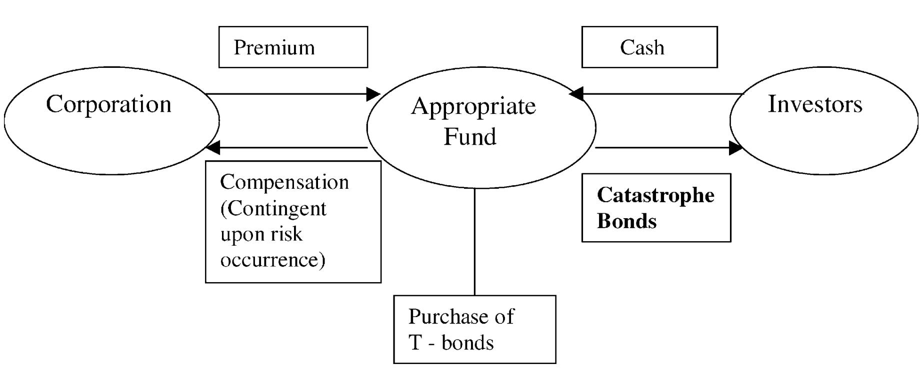

Fig. 1. Engineering of catastrophe bonds.

The pricing of these products must deal with incomplete markets and non-traded underlyings. Insurance-linked securities are primarily meant to transfer catastrophic nature risks. Natural catastrophes can be taken into account by resorting to mixed jump-diffusion processes for an underlying risk index. Financial markets are incomplete in this framework; thus, the methodology of replicating portfolios does not apply. Several lines can be followed along. Merton (1976) assumes that the risk associated with jumps can be diversified away. Therefore, the beta, in the sense of the Capital Asset Pricing Model, of portfolios that only incorporate this non-systematic risk is zero and their expected return is equal to the risk-free rate. Föllmer and Sondermann (1986), Föllmer and Schweizer (1991), and Schweizer (1992), in a series of articles, implement a variance-minimizing hedging approach that allows picking an equivalent martingale measure. Davis (1997) specifies a utility function for investors and can determine prices by solving an optimization problem under the historical probability measure. Other approaches try to capitalize on the completeness framework, either by assuming that the market is complete by restricting the problem to the case of non-random jump amplitudes (e.g., Cox & Ross, 1976) or by embedding the market in a complete market by adding another fictitious asset (e.g., Shirawaka, 1991). We will favor Merton's approach because it can be straightforwardly adapted to the scope of underlying state variables that are not investment assets. In turn, this non-tradability feature can be accounted for by introducing a market price of nature risk. What is more, to be fully consistent with the subject of bonds, interest rates must be modeled as evolving in a stochastic manner. On the other hand, to avoid technicalities, we will merely work with Gaussian spot interest rates rather than the whole term structure.

There is little academic research devoted to the pricing of catastrophe bonds, as opposed to works on their benefits, like in Anderson, Bendimerad, Canabarro, and Finkemeier (2000), and their management, including their combination with reinsurance, like in Croson and Kunreuther (2000). On the one hand, actuaries tend to price insurance contingent claims as random sums where weights are taken according to the historical probability. On the other hand, Froot (2001) provides evidence that catbonds generally trade at significant margins above the expected loss covered by the hedge. Consistently, practitioners can set the price of insurance to the expected value plus a function of the second moment of the distribution in order to take into account risk aversion; and with the corresponding probability of occurrence being obtained from a sample of historical observations. However, this pricing approach is reasonable only in the context of large diversified portfolios of identical and independent risks, so that the law of large numbers holds. Besides, in the process, rational option pricing is ignored. To remedy this deficiency in the existing literature, Poncet and Vaugirard (2002) derive a tractable formula within the more convincing arbitrage approach, in a framework of stochastic interest rates, but limit their analysis to non-catastrophic events. Compared with the paper of Poncet and Vaugirard (2002), which is our closest predecessor, our contribution lies in that we account for catastrophic events. Thus, to the best of our knowledge, this article develops the first valuation model of insurance-linked securities that deals with catastrophic events and interest rate randomness within an arbitrage approach. Moreover, this article, which works on OTC derivatives, must be contrasted with related papers coping with exchange-traded insurance derivatives, notably, CBOT derivatives by Cummins and Geman (1995).

We first vindicate the existence of well-defined arbitrage prices for catastrophe bonds, notwithstanding a framework of incomplete markets and non-traded underlying state variables. Then, we show that their valuation comes down to computing first-passage time distributions, since bondholders are showed to be in a short position on one-touch digital options based on risk-tracking indices that follow jump-diffusion processes in a Gaussian interest rate framework. Finally, we perform a comparative statics analysis that pinpoint the sensitivity of insurance bond prices to the exposure to nature risk and that show that the term structure of yield spreads is hump-shaped as for corporate bonds.

The remainder of the paper is organized as follows. Section 2 presents the framework. Section 3 derives solutions. Section 4 is devoted to comparative statics. Section 5 summarizes the article and suggests a possible extension. For ease of exposition, most proofs are in Appendix A.

2. The valuation framework

In this section, we underscore the binary structure of catastrophe bonds and we extend and adapt the jump-diffusion model of Merton (1976) to develop a valuation framework that allows for catastrophic events, interest rate uncertainty, and non-traded underlying state variables.

2.1. Product structure and economy

The catastrophe-bond payoff has a binary structure. An insurance-linked security can be thought of as a corporate bond with insurance risk instead of default risk. The bondholder expects to lose interest or a fraction of the principal if a natural risk index, whose value at date t is denoted I_t, hits a pre-specified threshold K. More specifically, if the index does not reach the threshold during a risk-exposure period T, the bondholder is paid the face value F. Otherwise, he receives the face value minus a write-down coefficient in percentage w. We allow the bond maturity T' to be longer than the risk-exposure period T to account for possible lags in the risk-index assessment at expiration. We focus on the case where the risk index starts below the barrier. This boundary is supposed to be fixed.

Let $(\Omega, \Im, P)$ define a probability space, where Ω is the set of states of the world, $\Im$ is a σ-algebra of subsets of Ω and P is a probability measure on $\Im$. Processes are defined on this probability space and on a trading horizon $[0, T']$. $\{W_t : 0 \leq t \leq T'\}$ and $\{W_{2t} : 0 \leq t \leq T'\}$ are two standard Brownian motion. $\{N_t : 0 \leq t \leq T'\}$ is a Poisson process with an intensity parameter λ_{p}. $\{U_j : j \geq 1\}$ is a sequence of identically and independently distributed (i.i.d.) random variables with values in $]-1; +\infty[$; U_j occurs at time τ_j defined by (N_t), that is, $\tau_j = \inf\{0 \leq t \leq T', N_t = j\}$. The σ-fields generated by (W_t), (W_{2t}), (N_t), and (U_j) are supposed to be independent. For all t in $[0, T']$, let F_t be the σ-field generated by the random variables W_s, W_{2s}, N_s for $s \leq t$, and $U_j \mathbf{1}_{\{j \leq N_t\}}$ for $j \geq 1$. The filtration $\{F_t : 0 \leq t \leq T'\}$ represents the information flow reaching market players. (F_t) is further augmented to encompass all P-null events. The four sources of randomness (W_t), (N_t), (U_j), and (W_{2t}) account for non-catastrophic nature risk, the occurrences of catastrophes, the size of catastrophes, and the uncertainty of interest rates.

2.2. Assumptions

Financial markets are frictionless. There are no transaction costs or differential taxes, trading takes place continuously in time, borrowing and short selling are allowed without restriction and with full proceeds available, and borrowing and lending rates are equal.

Assumption 1. Interest rates obey a mean-reverting process.

The risk-free spot interest rate r follows the process, under the historical probability P:

$$\mathrm{d}r(t) = a(b - r(t))\,\mathrm{d}t + \sigma_r\,\mathrm{d}W_{2t}, \tag{1}$$

where a, b, and σ_r are constants.

The formula of Vasicek (1977) is available for the corresponding risk-free zero-coupon bond.

Besides, it will be convenient to consider r constant in an early stage of this paper: $\sigma_r = 0$; $r(t) = r(0) = b$ for every t in $[0, T']$.

Assumption 2. The risk index is driven by a Poisson jump-diffusion process.

$(I_t)_{t\geq 0}$ is right-continuous and satisfies under the historical probability:

$$\frac{\mathrm{d}I_t}{I_{t^-}} = \mu(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W_t + J_t\,\mathrm{d}N_t, \tag{2}$$

where I_{t^-} stands for the index value just before t; $\mu(\cdot)$ is the drift parameter and can be stochastic; $\sigma(\cdot)$ is a deterministic volatility parameter of the Brownian component of the process; (N_t) is a Poisson process accounting for the expected number of jumps per time unit; (J_t) depicts the stochastic size of the jumps:

$$J_t = \sum_{n=1,+\infty} U_n \mathbf{1}_{]\tau_{n-1},\tau_n]}(t),$$

where (U_j) and (τ_j) are defined in the previous subsection, that is, at time τ_j, the jump of I_t is given by

$$\Delta I_{\tau_j} = I_{\tau_j} - I_{\tau_j^-} = I_{\tau_j^-} \times U_j, \quad \text{or} \quad I_{\tau_j} = I_{\tau_j^-}(1 + U_j).$$

$J_t\,\mathrm{d}N_t$ is then a handy notation to designate a compounded Poisson process.

In addition, $(1 + U_j)$'s are log-normally i.i.d.

Changes in the risk-tracking index comprise three components: the expected instantaneous index change conditional on no occurrences of catastrophes, the unanticipated instantaneous index change, which is the reflection of causes that have a marginal impact on the gauge, and the instantaneous change due to the arrival of a catastrophe. We focus on physical indexes rather than reported loss claims. Cummins and Geman (1995) evaluated the latter. Scientists may explain the drift μ. For instance, in the case of a temperature or a precipitation index, global warming and the greenhouse effect are likely relevant. The volatility parameter $\sigma(\cdot)$ is assumed to evolve deterministically. Therefore, no additional sources of randomness are introduced and the valuation process is similar to the case where it is constant. From now on, we will refer to it as σ. The assumption of log-normality of $(1 + U_j)$ must not be seen as restrictive. In effect, several distributions could be taken into consideration since we resort to simulations. A mixed jump-mean-reverting process for the risk index would be also a natural candidate to cope with cyclical meteorological phenomena. Yet, to keep things tractable, we only consider the case where the no-jump component is a geometric Brownian motion. We stress also that we do not attempt to calibrate the parameters. Indeed, practitioners have the relevant information to assess those, whereas we are interested in performing simulations, which by nature involve several sets of parameters.

Assumption 3. Investors are neutral toward nature jump risk (A31) and non-catastrophic changes in the risk index can be replicated by existing quoted securities, as for changes in interest rates (A32).

We split attitude toward risk to deal with two issues: a non-traded risk index and market incompleteness when accounting for jumps. Assumption (A31) means that investors acknowledge that natural catastrophe risk can be diversified away when it comes to pricing contingent claims upon environmental risk. The rationale underlying this stance is that natural catastrophes are barely correlated to financial storms, which is supported for example by the empirical study of Hoyt and McCullough (1999), with a qualification due to the low number of transactions though. Therefore, merely holding other usual financial assets enables investors to diversify nature jump risk. In other words, catbonds provide a valuable new source of diversification for investors because catastrophic losses are "zero-beta events" in the sense of the Capital Asset Pricing Model, as emphasized for example by Litzenberger, Beaglehole, and Reynolds (1996) or Canter, Cole, and Sandor (1997). This assumption is a reformulation of the stance of Merton (1976): "jump risk is not systematic." Since the risk index is not traded, we avoid using this vernacular, which, strictly speaking, makes sense in the scope of capital asset pricing models. Besides, the main drawback of the Merton assumption, which lies in that it implies that the market itself is not subject to jumps, is irrelevant here. As for Assumption (A32), while we can assume routinely that changes in domestic interest rates are replicable due to the existence of risk-free bonds, the analogous assumption relative to non-catastrophic changes in the risk index warrants some justification. Though nature derivatives were withdrawn from the CBOT, we may for instance argue that continuous changes in the risk index can be mimicked by instruments such as energy and power derivatives, weather derivatives or contingent claims on several commodities.

Proposition 1. *There exists a well-defined arbitrage price for quoted contingent claims upon the risk index.*

More specifically, let C_I be such a contingent claim. Then,

$$C_I(t) = \boldsymbol{E}^Q(D(t, T')C_I(T')/F_t), \tag{3}$$

where Q is the unique restriction to the σ-field generated by W and W_2 of any equivalent martingale measure, with the stochastic discount factor given by $D(t, T') = \exp\left(-\int_{t,T'} r(u)\,\mathrm{d}u\right)$, and where the dynamics of I and r under Q are described by

$$\frac{\mathrm{d}I_t}{I_{t^-}} = (\mu(t) - \lambda(t)\sigma(t))\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W'_t + J_t\,\mathrm{d}N_t, \tag{4}$$

and

$$\mathrm{d}r(t) = a(b^* - r(t))\,\mathrm{d}t + \sigma_r\,\mathrm{d}W'_{2t}, \tag{5}$$

where $\lambda(\cdot)$ *is the market price of nature risk, and* W' *and* W'_2 *are the* Q*-standard Brownian motion that correspond to the* P*-standard Brownian motion* W *and* W_2*, and are obtained by the Girsanov theorem. Last,* $b^* = b - \lambda_r\sigma_r/a$*, with* $(-\lambda_r)$ *the risk premium for risk-free bonds.*

Proof. See Appendix A. □

3. Solutions

In this section, we derive a generic valuation expression for pure discount catastrophe bonds, from which we will either run simulations or derive a closed-form formula in a particular situation. Then, we show how to cope with coupon bonds and payments at hit.

3.1. Generic formula

Proposition 2. *Let* $\mathrm{IB}(t)$ *be the price of a zero-coupon insurance bond at time* t *and* $T_{I,K}$ *the first passage time of* I *through* K*. We suppose that all cash payments are done at date of maturity* T'*. Then,*

$$\mathrm{IB}(t) = FP(t, T')\{1 - w\boldsymbol{E}^Q(\mathbf{1}_{T_{I,K}\le T}/F_t)\}, \tag{6}$$

where Q *is the risk-adjusted probability measure defined univocally earlier, the dynamics of* I *and* r *under* Q *were given in* Proposition 1*, and where,*

$$P(t, T') = \exp[-(T' - t)R(T' - t, r(t))],$$

with

$$R(\theta, r) = R_\infty - [1/(a\theta)]\{(R_\infty - r)(1 - \mathrm{e}^{-a\theta}) - [\sigma_r^2/(4a^2)](1 - \mathrm{e}^{-a\theta})^2\},$$

and

$$R_\infty = b^* - [\sigma_r^2/(2a^2)].$$

Proof. Bondholders receive at T':

$$F\mathbf{1}_{T_{I,K}>T} + (1 - w)F\mathbf{1}_{T_{I,K}\le T} = F - wF\mathbf{1}_{T_{I,K}\le T},$$

where F is the bond face value and $\mathbf{1}_A$ stands for the indicator function of set A. Therefore, bondholders are in a short position on a one-touch up-and-in digital option on the risk-tracking index. Now, we extend Proposition 1 to over-the-counter contingent claims and we obtain Eq. (6). $P(t, T')$ is further made explicit by the Vasicek formula. We refer to Appendix A for details.

Henceforth, we give prices at time $t = 0$ for the sake of simplicity and without loss of generality. □

3.2. Crash case: closed-form formula

The risk index and the barrier are such that the latter will be broken at the first catastrophe.

Proposition 3. *In the case of a crash, and when interest rates are assumed constant*:

$$\mathrm{IB} = F\,\mathrm{e}^{-rT'}(1 - w\boldsymbol{E}^{Q}(\mathbf{1}_{T_{I,K}\le T})),$$

with

$$\boldsymbol{E}^{Q}(\mathbf{1}_{T_{I,K}\le T}) = \mathrm{e}^{-\lambda_{\mathrm{p}}T}\left\{N(d_1) + \left(\frac{I_0}{K}\right)^{1-2\delta/\sigma^2} N(d_2)\right\} + (1 - \mathrm{e}^{-\lambda_{\mathrm{p}}T}), \tag{7}$$

where

$$d_1 = \frac{\ln(I_0/K) + (\delta - \sigma^2/2)T}{\sigma T^{1/2}}, \quad \text{and} \quad d_2 = \frac{\ln(I_0/K) - (\delta - \sigma^2/2)T}{\sigma T^{1/2}},$$

with

$$\delta = \mu - \lambda\sigma.$$

Proof. See Appendix A. □

3.3. Mid-range catastrophe case: Monte-Carlo simulations

Here, the nature of compounded Poisson processes is truly at work, with mid-range catastrophes that occur according to Poisson arrival times.

We start from the generic Formula (6) and we focus on $\boldsymbol{E}^{Q}(\mathbf{1}_{T_{I,K}\le T})$ since the Vasicek closed-form formula is available for $P(0, T')$.

Proposition 4. *The core subroutine to assess* $\boldsymbol{E}^{Q}(\mathbf{1}_{T_{I,K}\le T})$ *is*:

$$I_{n+1} = I_n\left\{\left[1 + \left(\mu\left(\frac{Tn}{N}\right) - \lambda\left(\frac{Tn}{N}\right)\sigma\left(\frac{Tn}{N}\right)\right)\right]\Delta t + \sigma\left(\frac{T_n}{N}\right)g(0,1)\sqrt{\Delta t} + \sum_{j=1,N(\lambda_{\mathrm{p}}\,\Delta t)}[\exp(g_j(kv,\delta)) - 1]\right\} \tag{8}$$

and

$$r_{n+1} = a(b^*)\,\Delta t + (1 - a\,\Delta t)r_n + \sigma_r g_2(0,1)\sqrt{\Delta t}, \tag{9}$$

where N *is the number of steps,* $\Delta t = T/N$, $kv = \ln(1 + \boldsymbol{E}(U_1))$, $\delta^2 = \mathrm{var}(\ln(1 + U_1))$, *and where* $g(0, 1)$, $g_2(0, 1)$, *and* $g_j(kv, \delta)$ *follow independent normal distributions with respective parameters* $(0, 1)$ *and* (kv, δ). *In addition,* $N(\lambda_p\,\Delta t)$ *is simulated by using that, if* N_θ *is a Poisson random variable with intensity* θ, *then,*

$$N_\theta = \sum_{n\ge 1} n\mathbf{1}_{\{UF_1UF_2\cdots UF_nUF_{n+1}\le \mathrm{e}^{-\theta}\le UF_1UF_2\cdots UF_n\}}, \tag{10}$$

where $(UF_i)_{i\geq 1}$ *are independently and uniformly distributed on* $[0, 1]$.

Then, we track if the index hits the barrier during the risk-exposure period by incrementing by 1 *if* I *breaks* K *along a given sample path, and we obtain* $\boldsymbol{E}^Q(\mathbf{1}_{T_{I,K}\leq T})$ *by averaging over the number of trajectories. It is noteworthy that simulating digital options exhibits a high level of instability around their thresholds, and this issue is tackled in* Vaugirard (2001).

Proof. The whole scheme comes straightforwardly from the Q-dynamics determined in Proposition 1 and we do not elaborate. □

3.4. Extensions

3.4.1. Pricing coupon bonds by a value-additivity feature

We first notice that only breaking the barrier triggers the default on the bond. In particular, coupon payments bear no influence on the contingency of default. Consequently, an insurance coupon bond can be seen as a portfolio of pure discount bonds whose weights match coupon rates. It is noteworthy that we are in a more favorable situation than for corporate bonds. In the latter case indeed, the fact that coupon payments impinge on the solvency status of the indebted firm rules out this convenient value-additivity feature. This is the reason why most models of contingent claims on corporate bonds deal with zero-coupon bonds only. In the remainder of this article, without loss of generality our bonds will be meant to be pure discount bonds, unless otherwise stated.

3.4.2. Payments at hit

The payoff schedule of the catastrophe bond is now:

$$F\mathbf{1}_{T_{I,K}>T,} \quad \text{paid at } T',$$

and

$$(1-w)F\mathbf{1}_{T_{I,K}\leq T}, \quad \text{paid at } T_{I,K}.$$

Therefore,

$$\mathrm{IB} = FP(0,T')\boldsymbol{E}^Q(\mathbf{1}_{T_{I,K}>T}) + (1-w)F\boldsymbol{E}^Q(D(0,T_{I,K})\mathbf{1}_{T_{I,K}\leq T}), \tag{11}$$

where $D(0,T_{I,K}) = \exp\left(-\int_{0,T_{I,K}} r(u)\,\mathrm{d}u\right)$. Then, both expectations are simulated like in the previous subsection. The additional difficulty due to assessing simultaneously an integral in the second expectation operator is merely incremental.

4. Comparative statics

In this section, we assess the impact of the exposure to nature risk and we show that the term structure of catbond yield spreads is hump-shaped.

Table 1
Catastrophe bond price and nature risk

	Intensity λ_p			
	0	0.5	1	2
$X = 0.5$	765	730	700	640
$X = 0.8$	370	335	320	280
$\boldsymbol{E}(U_1) = 0.1$	765	730	700	640
$\boldsymbol{E}(U_1) = 0.2$	765	710	650	550
$\sigma = 0.2$	895	860	815	735
$\sigma = 0.5$	765	730	700	640
$T = 0.5$	860	845	825	795
$T = 1$	765	730	700	640
Risk-free bond	905	905	905	905

Default parameters: $X = I/K = 0.5$; $F = 1{,}000$; $w = 0.9$; $T = T' = 1$ year; $\sigma = 0.5$; $\boldsymbol{E}(U_1) = 0.1$; $\delta = 0.2$; $r = 0.1$; $a = 0.1$; $b^* = 0.1$; $\sigma_r = 0.03$; $\mu = 0.2$; $\lambda = 0.1$. These results exhibit the variation of catastrophe-bond prices according to the risk exposure: Poisson intensity; index–barrier distance; jump size; index volatility (diffusion component); risk-exposure period. The number of simulations is 5,000, and prices are set at the center of 95% confidence intervals.

4.1. Exposure to nature risk

Prices of catastrophe bonds decrease if nature risk increases, as indicated in Table 1. This first set of results follows immediately from the observation that the bondholders are short a digital up-and-in option. More specifically, the longer the distance between the actual index level and the barrier the higher the catbond price; in other words, the further the risk-tracking index from the insurance trigger point the smaller the discount for nature risk. Moreover, if the index volatility rises, then the catbond price decreases. Indeed, the risk-adjusted probability that the risk-tracking index reaches the threshold increases. Further, other things being equal, the catbond is less valuable with a longer risk exposure streak. Indeed, the price of the corresponding risk-free bond is unaffected—since the time to maturity is unchanged—while the odds that the index hits the barrier are higher with a longer risk-exposure period. Finally, prices of catbonds decrease if the parameters of the Poisson process are sharper, since the probability of I hitting the threshold rises.

4.2. Yield spread

The yield of a zero-coupon bond B with maturity T' is $-\ln(B)/T'$, so the yield spread of a catastrophe bond of maturity T' and with risk-exposure period T is: $-\ln(1-w\boldsymbol{E}^Q(T_{I,K\leq T})/T'$. The yield spread increases if the risk-exposure period lengthens and decreases if the time to maturity increases. Therefore, when further imposing that the two time variables match $(T = T')$, the overall sensitivity depends on which effect prevails.

As depicted in Fig. 2, the term structure of yield spreads is hump-shaped. This result is familiar for defaultable bonds, as in Longstaff and Schwartz (1995). In our different context

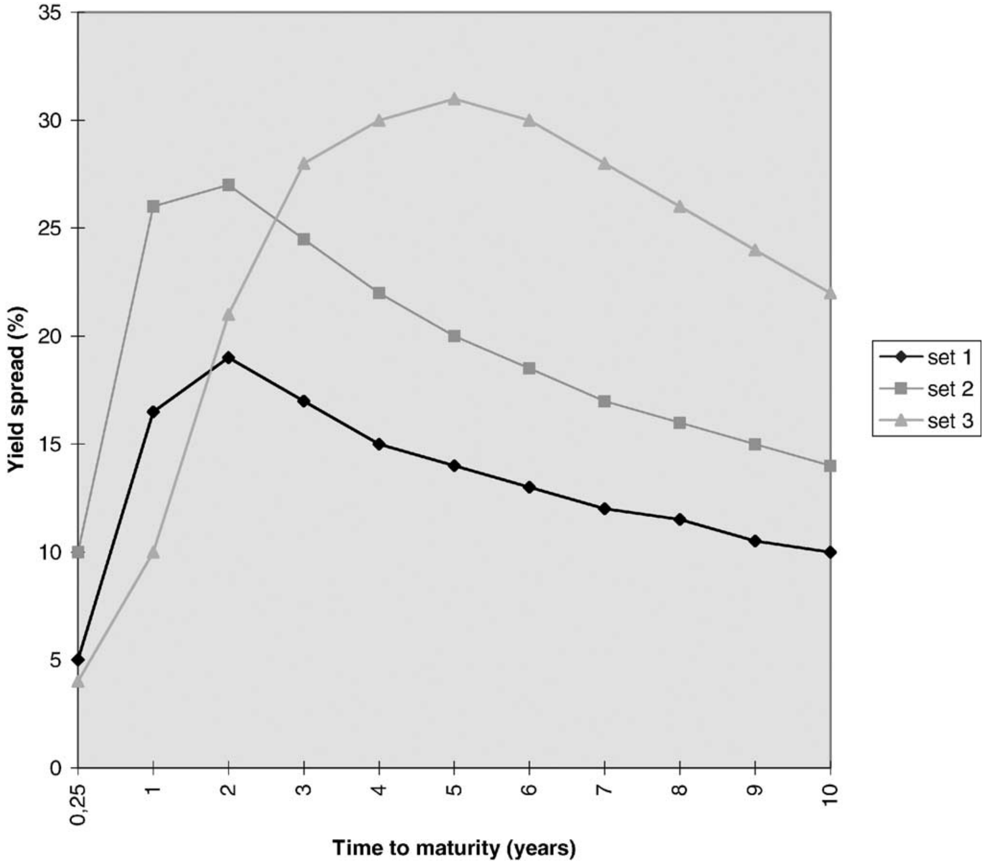

Fig. 2. Yield spread is hump-shaped. Yield spreads of catastrophe bonds as a function of time to maturity. Default parameters: $K = 100$; $I = 50$; $w = 0.9$; $\boldsymbol{E}(U_1) = 0.1$; $\delta = 0.2$; $r = 0.1$; $a = 0.1$; $b^* = 0.1$; $\sigma_r = 0.03$; $\mu = 0.2$; $\lambda = 0.1$. Three sets of parameters are used: set 1 ($\lambda_p = 0$, $\sigma = 0.5$); set 2 ($\lambda_p = 1$, $\sigma = 0.5$); set 3 ($\lambda_p = 0$, $\sigma = 0.2$).

Table 2
Yield spread and risk exposure

$X = I/K$			μ		λ_p			$\boldsymbol{E}(U_1)$	
0.2	0.5	0.8	0.2	0.5	0	1	2	0.1	0.2
8	26	103	26	50	16.5	26	35	26	33

These results exhibit the catastrophe-bond yield spread (in percentage) for different sets of nature risk exposure: proximity to barrier, index historical drift, Poisson intensity and jump size. λ_p: intensity parameter of Poisson process; $X = I/K$: index–barrier ratio; μ: index historical drift; $\boldsymbol{E}(U_1)$: jump size. Default parameters: $K = 100$; $I = 50$; $\sigma = 0.5$; $w = 0.9$; $\boldsymbol{E}(U_1) = 0.1$; $\delta = 0.2$; $r = 0.1$; $a = 0.1$; $b^* = 0.1$; $\sigma_r = 0.03$; $\mu = 0.2$; $\lambda = 0.1$; $T = T' = 1$ year.

of natural risks, this humped shape can be explained as follows. The yield spread first worsens since a longer risk-exposure period increases the probability of natural event occurrences. Along the way, the time to maturity reaches a value such that the index is expected to break the barrier. From that moment on, a longer risk-exposure period raises the chances that the index drifts away from the trigger point.

Yield spreads widen with the risk exposure, as indicated in Table 2. Higher risk exposures, either due to a shorter distance between the index and the barrier or to sharper parameters for the compounded Poisson process, induce wider yield spreads, since the barrier is more likely to be broken for a given time to maturity.

5. Conclusion

This paper vindicated the existence of a well-defined arbitrage price for catastrophe bonds, notwithstanding a framework of incomplete markets and non-traded underlying state variables. Then, it showed that their valuation comes down to computing first-passage time distributions, since bondholders were showed to be in a short position on one-touch digital options based upon risk-tracking indexes that obey Poisson jump-diffusion processes in a Gaussian interest rate framework. It provided a closed-form valuation expression when working in the context of pure crashes. In the case of mid-range catastrophes, numerical simulations were resorted to. Comparative statics results pinpointed the sensitivity of insurance bond prices to the exposure to nature risk and showed that the term structure of yield spreads is hump-shaped as for corporate bonds. Moreover, yield spreads widen with the exposure to nature risk. As for the issue of hedging catastrophe bonds, we did not elaborate since hedging one-touch digital options is known to be complicated, and this problem is compounded here by the risk-index non-tradability. Moreover, this stance is consistent with our assumption of neutrality toward nature jump risk. Indeed, as often with exotic options or other structured products, actual diversification merely occurs through holding and monitoring an entire book of options in a more or less continuous manner.

A follow-up is worth considering. It would be useful to account for foreign exchange risk. Indeed, to the extent that natural risks are world-widely spread, whereas catastrophe bonds are almost exclusively denominated in US$ for liquidity reasons, many catastrophe bondholders face currency risk in addition to nature and interest rate risks.

Appendix A

Proof of Proposition 1. Let Q be the unique martingale measure, equivalent to the objective probability measure P, with complete financial markets. We now define W' and W_2' the Q-standard Brownian motion that are routinely derived from the P-standard Brownian motion W and W_2 by means of the Girsanov theorem, which is legitimate due to Assumption (A32).

To fix ideas, let I' be the process which dynamics are described by the geometric-Brownian component of the jump-diffusion process set by Formula (2). Then, any attainable contingent claim $C_{I'}$ has a well-defined price given by $C_{I'}(t) = \boldsymbol{E}^Q(D(t, T')C_{I'}(T')/F_t)$. Now, due to Assumption (A31), investors use the same arbitrage price when it comes to contingent claims upon I, that is, the risk-adjusted probability measure Q is used to evaluate non-attainable contingent claims C_I upon I, since they are neutral toward nature jump risk: $C_I(t) = \boldsymbol{E}^Q(D(t, T')C_I(T')/F_t)$; which yields Formula (3).

We now only have to account for the non-tradability to identify the dynamics of I under Q. We introduce the market price of nature risk λ associated to I and we obtain Formula (4). We refer to Hull (2000, Section 19.1) for a textbook treatment of this point. Formula (5) for interest rates is usual. □

Proof of Proposition 2. Bondholders receive at T':

$$F\mathbf{1}_{T_{I,K}>T} + (1-w)F\mathbf{1}_{T_{I,K}\leq T} = F - wF\mathbf{1}_{T_{I,K}\leq T}.$$

Now, extending Proposition 1 to over-the-counter contingent claims, we can take the Q-expectation conditioned on the set of information $(F_t)_t$ generated by I and r to price the bond:

$$\begin{aligned}\mathrm{IB}(t) &= \boldsymbol{E}^Q(D(t,T') \times \text{Payoff}/F_t) = \boldsymbol{E}^Q(D(t,T')F(1 - w\mathbf{1}_{T_{I,K}\leq T})/F_t)\\ &= F\boldsymbol{E}^Q(D(t,T')/F_t) - wF\boldsymbol{E}^Q(D(t,T')\mathbf{1}_{T_{I,K}\leq T}/F_t).\end{aligned}$$

By definition, $P(t,T') = \boldsymbol{E}^Q(D(t,T')/F_t))$. In addition, since W' and W_2' are independent under Q, and since r does not appear in the risk-adjusted drift of I, then $D(\cdot, T')$ and $\mathbf{1}_{T_{I,K}\leq T}$ are independent under Q. So, we have:

$$\boldsymbol{E}^Q(D(t,T')\mathbf{1}_{T_{I,K}\leq T}/F_t) = \boldsymbol{E}^Q(D(t,T')/F_t)\boldsymbol{E}^Q(\mathbf{1}_{T_{I,K}\leq T}/F_t),$$

and we obtain:

$$\mathrm{IB}(t) = FP(t,T')\{1 - w\boldsymbol{E}^Q(\mathbf{1}_{T_{I,K}\leq T}/F_t)\}.$$

□

Proof of Proposition 3. With obvious notation,

$$\begin{aligned}\boldsymbol{E}^Q(\mathbf{1}_{T_{I,K}\leq T}) = {} & \boldsymbol{E}^Q(\mathbf{1}_{T_{I,K}\leq T}/\text{no jumps}) \times \boldsymbol{P}^Q(\text{no jumps}) + \boldsymbol{E}^Q(\mathbf{1}_{T_{I,K}\leq T}/\text{jumps})\\ & \times \boldsymbol{P}^Q(\text{jumps}).\end{aligned}$$

Now,

$$\boldsymbol{P}^Q(\text{no jumps}) = \mathrm{e}^{-\lambda_{\mathrm{p}}T},$$

and

$$\boldsymbol{E}^Q(\mathbf{1}_{T_{I,K}\leq T}/\text{jumps}) = I \text{ by hypothesis (crash)},$$

and

$$\boldsymbol{E}^Q(\mathbf{1}_{T_{I,K}\leq T}/\text{no jumps}) = \boldsymbol{E}^Q(\mathbf{1}_{T_{I,K}\leq T}/I \text{ geometric Brownian}).$$

The latter expression can be computed by applying a standard result on the one-sided first-passage time distribution for a drifted Brownian motion starting from zero (e.g., see Harrison, 1985, Chapter 1) to:

$$\left(\frac{1}{\sigma}\right)\ln\left(\frac{I_t}{I_0}\right) = W'_t + \left(\frac{\delta}{\sigma} - \frac{\sigma}{2}\right)t \quad \text{through the barrier} \quad \left(\frac{1}{\sigma}\right)\ln\left(\frac{K}{I_0}\right). \qquad \square$$

References

Anderson, R. R., Bendimerad, F., Canabarro, E., & Finkemeier, M. (2000). Analyzing insurance-linked securities. *Journal of Risk Finance*, *1*(2), 49–78.

Canter, M., Cole, J. B., & Sandor, R. L. (1997). Insurance derivatives: A new asset class for the capital markets and a new hedging tool for the insurance industry. *Journal of Applied Corporate Finance*, *10*(3), 69–83.

Cox, J. C., & Ross, S. A. (1976). The valuation of options for alternative stochastic processes. *Journal of Financial Economics*, *3*, 145–166.

Croson, D. C., & Kunreuther, H. C. (2000). Customizing indemnity contracts and indexed cat bonds for natural hazard risks. *Journal of Risk Finance*, *1*(3), 24–42.

Cummins, J. D., & Geman, H. (1995). Pricing catastrophe insurance futures and call spreads: an arbitrage approach. *Journal of Fixed Income*, *1*, 46–57.

Davis, M. H. A. (1997). Option pricing in incomplete markets. In M. A. H. Dempster & S. R. Pliska (Eds.). *Mathematics of derivative securities* (pp. 216–226). Cambridge: Cambridge University Press.

Föllmer, H., & Schweizer, M. (1991). Hedging of contingent claims under incomplete information. In M. H. A. Davis & R. J. Elliott (Eds.). *Applied stochastic analysis* (vol. 5, pp. 389–414). New York: Stochastic monographs, Gordon and Breach.

Föllmer, H., & Sondermann, D. (1986). Hedging of non-redundant contingent claims under incomplete information. In W. Hildenbrand & A. Mas-Colell (Eds.). *Contribution to mathematical economics* (pp. 205–223). Amsterdam: North-Holland.

Froot, K. A. (2001). The market for catastrophe risk: A clinical examination. *Journal of Financial Economics*, *60*, 529–571.

Harrison, J. M. (1985). *Brownian motion and stochastic flow systems*. New York: Wiley.

Hoyt, R. E., & McCullough, K. A. (1999). Catastrophe insurance options: Are they zero-beta assets? *The Journal of Insurance Issues*, *22*(2), 147–163.

Hull, J. C. (2000). *Options, futures and other derivatives*. New York: Prentice Hall.

Jaffee, D. M., & Russell, T. (1997). Catastrophe insurance, capital markets and uninsurable risks. *Journal of Risk and Insurance*, *64*, 205–230.

Litzenberger, R. H., Beaglehole, D. R., & Reynolds, C. E. (1996). Assessing catastrophe reinsurance-linked securities as a new asset class. *Journal of Portfolio Management*, *13*, 76–86.

Longstaff, F., & Schwartz, E. (1995). A simple approach to valuing risky fixed and floating rate debt and determining swap spreads. *The Journal of Finance*, *50*, 789–819.

Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, *3*, 125–144.

Poncet, P., & Vaugirard, V. E. (2002). The pricing of insurance-linked securities under interest rate uncertainty. *Journal of Risk Finance*, *3*(3), 48–59.

Schweizer, M. (1992). Mean-variance hedging for general claims. *Annals of Applied Probabilities*, *2*, 171–179.

Shirawaka, H. (1991). Interest-rate option pricing with Poisson–Gaussian forward rate curve processes. *Mathematical Finance*, *1*, 77–94.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics*, *5*, 177–188.

Vaugirard, V. E. (2001). Monte Carlo applied to exotic digital options. *Applied Mathematical Finance*, *8*(3), 183–196.