Quiz1 - Complex Numbers ■ WORKED SOLUTIONS ■

1. Given that $\frac{2}{x+\mathrm{i}y}+\frac{1}{1-2\mathrm{i}}=\frac{2}{5}+\mathrm{i}$ where x and y are real, find the value of x and the value of y.

[6 marks]

1. $$\frac{2}{x+iy}\cdot\frac{x-iy}{x-iy}=\frac{2x-2iy}{x^2+y^2}=\frac{2x}{x^2+y^2}-\left(\frac{2y}{x^2+y^2}\right)i$$

$$\frac{1}{1-2i}\cdot\frac{1+2i}{1+2i}=\frac{1+2i}{1+4}=\frac{1}{5}+\frac{2}{5}i$$

$$\left(\frac{2x}{x^2+y^2}+\frac{1}{5}\right)+\left(-\frac{2y}{x^2+y^2}+\frac{2}{5}\right)i=\frac{2}{5}+i$$

real parts: $\frac{2x}{x^2+y^2}+\frac{1}{5}=\frac{2}{5} \Rightarrow \frac{2x}{x^2+y^2}=\frac{1}{5} \Rightarrow x^2+y^2=10x$

imaginary parts: $-\frac{2y}{x^2+y^2}+\frac{2}{5}=1 \Rightarrow \frac{2y}{x^2+y^2}=-\frac{3}{5} \Rightarrow x^2+y^2=-\frac{10}{3}y$

$\begin{cases} x^2+y^2=10x \\ x^2+y^2=-\frac{10}{3}y \end{cases}$ subtracting equations $\Rightarrow 10x+\frac{10}{3}y=0 \Rightarrow \frac{10}{3}y=-10x$

$y=-3x$

substituting: $x^2+(-3x)^2=-\frac{10}{3}(-3x) \Rightarrow x^2+9x^2=10x$

$10x^2-10x=0 \Rightarrow x(x-1)=0 \Rightarrow x=0$ or $x=1$

if $x=0$, then $y=0 \rightarrow$ not possible if $x=1$, then $y=-3(1)=-3$

therefore, $x=1$ and $y=-3$

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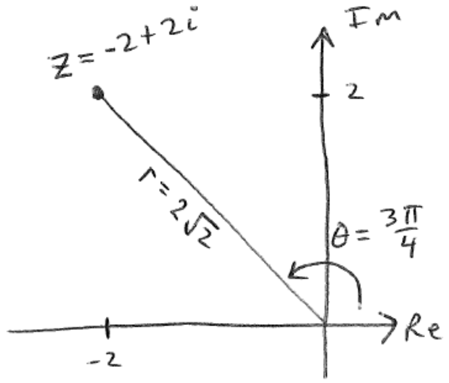
2. Find the three cube roots $-2+2\text{i}$ and express them in exponential form, $r\,\text{e}^{\text{i}\theta}$. **[9 marks]**

2. let $z = -2 + 2i$

$|z| = \sqrt{2^2 + 2^2}$
$= \sqrt{8} = 2\sqrt{2}$



$z = 2\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)$

$(a+bi)^3 = 2\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4} + k\cdot 2\pi\right),\ k \in \mathbb{Z}$

take cube root of both sides, i.e. raise to $\frac{1}{3}$ power

$a + bi = \sqrt{2}\ \text{cis}\left(\frac{\pi}{4} + k\cdot\frac{2\pi}{3}\right)$

$(\sqrt{8})^{\frac{1}{3}} = \left(8^{\frac{1}{2}}\right)^{\frac{1}{3}} = \left(2^3\right)^{\frac{1}{6}}$
$= 2^{\frac{1}{2}} = \sqrt{2}$

when $k=0$: $a + bi = \sqrt{2}\ \text{cis}\left(\frac{\pi}{4}\right)$

when $k=1$: $a + bi = \sqrt{2}\ \text{cis}\left(\frac{\pi}{4} + \frac{2\pi}{3}\right) = \sqrt{2}\ \text{cis}\left(\frac{11\pi}{12}\right)$

when $k=2$: $a + bi = \sqrt{2}\ \text{cis}\left(\frac{\pi}{4} + \frac{4\pi}{3}\right) = \sqrt{2}\ \text{cis}\left(\frac{19\pi}{12}\right) = \sqrt{2}\ \text{cis}\left(-\frac{5\pi}{12}\right)$

therefore, in exponential form the three cube roots of $-2+2i$

are: $\sqrt{2}\,e^{i\frac{\pi}{4}},\ \sqrt{2}\,e^{i\frac{11\pi}{12}},\ \sqrt{2}\,e^{i\left(-\frac{5\pi}{12}\right)}$

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3. Consider the following two complex numbers

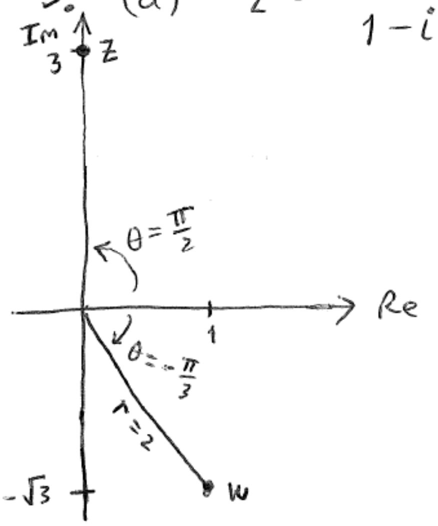
$$z = \frac{3+3\mathrm{i}}{1-\mathrm{i}} \quad \text{and} \quad w = \frac{4}{1+\mathrm{i}\sqrt{3}}$$

(a) Write each in modulus-argument form, $r\,\mathrm{cis}\,\theta$. **[6 marks]**

(b) Hence, find simplified expressions for zw and $\frac{z}{w}$ in modulus-argument form, $r\,\mathrm{cis}\,\theta$.

[6 marks]

3. (a) $z = \frac{3+3i}{1-i} \cdot \frac{1+i}{1+i} = \frac{3+6i+3i^2}{1-i^2} = \frac{0+6i}{2} = 3i$



$z = 3i \rightarrow z = 3\,\mathrm{cis}\left(\frac{\pi}{2}\right)$

$w = \frac{4}{1+i\sqrt{3}} \cdot \frac{1-i\sqrt{3}}{1-i\sqrt{3}} = \frac{4-4\sqrt{3}\,i}{1-3i^2} = \frac{4-4\sqrt{3}\,i}{4} = 1-i\sqrt{3}$

$w = 1-i\sqrt{3} \rightarrow w = 2\,\mathrm{cis}\left(-\frac{\pi}{3}\right)$

$|w| = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{4} = 2$

(b) $zw = \left[3\,\mathrm{cis}\left(\frac{\pi}{2}\right)\right]\left[2\,\mathrm{cis}\left(-\frac{\pi}{3}\right)\right]$

$= 6\,\mathrm{cis}\left(\frac{\pi}{2}-\frac{\pi}{3}\right) = 6\,\mathrm{cis}\left(\frac{3\pi}{6}-\frac{2\pi}{6}\right)$

$zw = 6\,\mathrm{cis}\left(\frac{\pi}{6}\right)$

$\frac{z}{w} = \frac{3\,\mathrm{cis}\left(\frac{\pi}{2}\right)}{2\,\mathrm{cis}\left(-\frac{\pi}{3}\right)} = \frac{3}{2}\,\mathrm{cis}\left(\frac{\pi}{2}-\left(-\frac{\pi}{3}\right)\right) = \frac{3}{2}\,\mathrm{cis}\left(\frac{3\pi}{6}+\frac{2\pi}{6}\right)$

$\frac{z}{w} = \frac{3}{2}\,\mathrm{cis}\left(\frac{5\pi}{6}\right)$

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4. (a) Find all roots for the equation $x^4 + 16 = 0$ given that $x \in \mathbb{C}$. **[8 marks]**

(b) Hence, express $x^4 + 16$ as the product of two quadratic polynomials with real coefficients. **[5 marks]**

4. (a) $x^4 + 16 = 0 \rightarrow x^4 = -16$

find four 4th roots of -16 ; let $z = -16 \rightarrow z = 16 \text{ cis}(\pi)$

$$(a+bi)^4 = 16 \text{ cis}(\pi + k \cdot 2\pi), \quad k \in \mathbb{Z}$$

$$a + bi = 2 \text{ cis}\left(\frac{\pi}{4}\right) = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \underline{\underline{\sqrt{2} + i\sqrt{2}}}$$

roots occur symmetrically about origin in complex plane, thus

other 3 roots are $\underline{\underline{-\sqrt{2} + i\sqrt{2}}}$, $\underline{\underline{-\sqrt{2} - i\sqrt{2}}}$ and $\underline{\underline{\sqrt{2} - i\sqrt{2}}}$

(b) $$x^4 + 16 = \left[x - (\sqrt{2} + i\sqrt{2})\right]\left[x - (\sqrt{2} - i\sqrt{2})\right]\left[x - (-\sqrt{2} + i\sqrt{2})\right]\left[x - (-\sqrt{2} - i\sqrt{2})\right]$$

$$= \left[(x - \sqrt{2}) - i\sqrt{2}\right]\left[(x - \sqrt{2}) + i\sqrt{2}\right]\left[(x + \sqrt{2}) - i\sqrt{2}\right]\left[(x + \sqrt{2}) + i\sqrt{2}\right]$$

$$= \left[(x - \sqrt{2})^2 - (i\sqrt{2})^2\right]\left[(x + \sqrt{2})^2 - (i\sqrt{2})^2\right]$$

$$= \left[x^2 - 2\sqrt{2}x + 2 + 2\right]\left[x^2 + 2\sqrt{2}x + 2 + 2\right]$$

$$= \left(x^2 - 2\sqrt{2}x + 4\right)\left(x^2 + 2\sqrt{2} + 4\right)$$

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Bonus: Show that $\dfrac{\cos 2\theta + i\sin 2\theta}{\cos 3\theta + i\sin 3\theta} = \cos\theta - i\sin\theta$ **[+4 marks]**

$$\frac{\cos 2\theta + i\sin 2\theta}{\cos 3\theta + i\sin 3\theta} = \frac{(\cos\theta + i\sin\theta)^2}{(\cos\theta + i\sin\theta)^3}$$

$$= \frac{1}{\cos\theta + i\sin\theta}$$

$$= (\cos\theta + i\sin\theta)^{-1}$$

$$= \cos(-\theta) + i\sin(-\theta)$$

$$= \cos\theta - i\sin\theta \qquad \text{Q.E.D.}$$