SPRING FORCE

Spring Force and Hooke's Law

Variables		SI Unit
F_{sp}	spring force	N
Δx	displacement	m
k	spring constant	$\frac{N}{m}$

- A spring **changes length when a force is applied to both ends**. If the forces pull the ends away from each other the spring gets longer. If the forces push the ends together the spring gets shorter.
- In the real world there are different types of springs with different behaviors, and all materials actually behave similar to springs. But we usually start out by working with "ideal springs".
- An **ideal spring** is…
  - massless: the spring itself has no mass, no inertia and no weight
  - frictionless: there are no friction forces acting on or within the spring itself
  - linearly elastic / follows Hooke's law: the change in length is linearly proportional to the applied force

- **Hooke's Law** states that the magnitude of the force required to stretch or compress a spring by a displacement of Δx is linearly proportional to that the displacement, $F_{sp} = k\Delta x$, where k is the **spring constant** or **stiffness** of the spring.

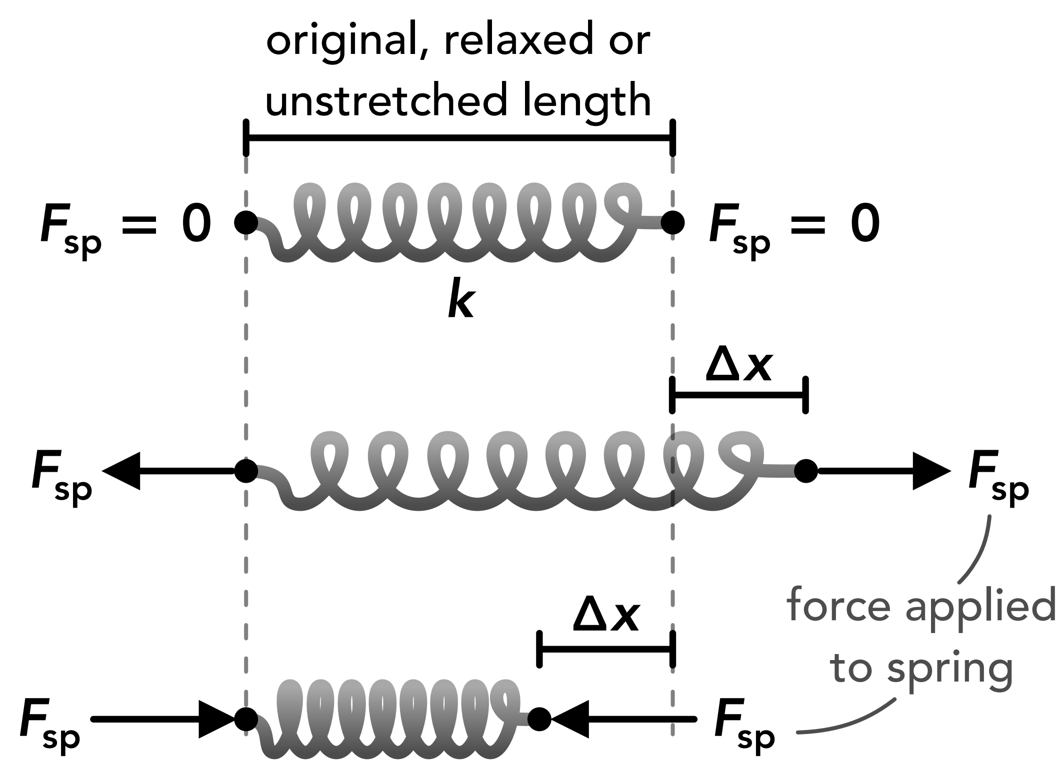
Spring force (Hooke's Law)

$$F_{sp} = k\Delta x$$

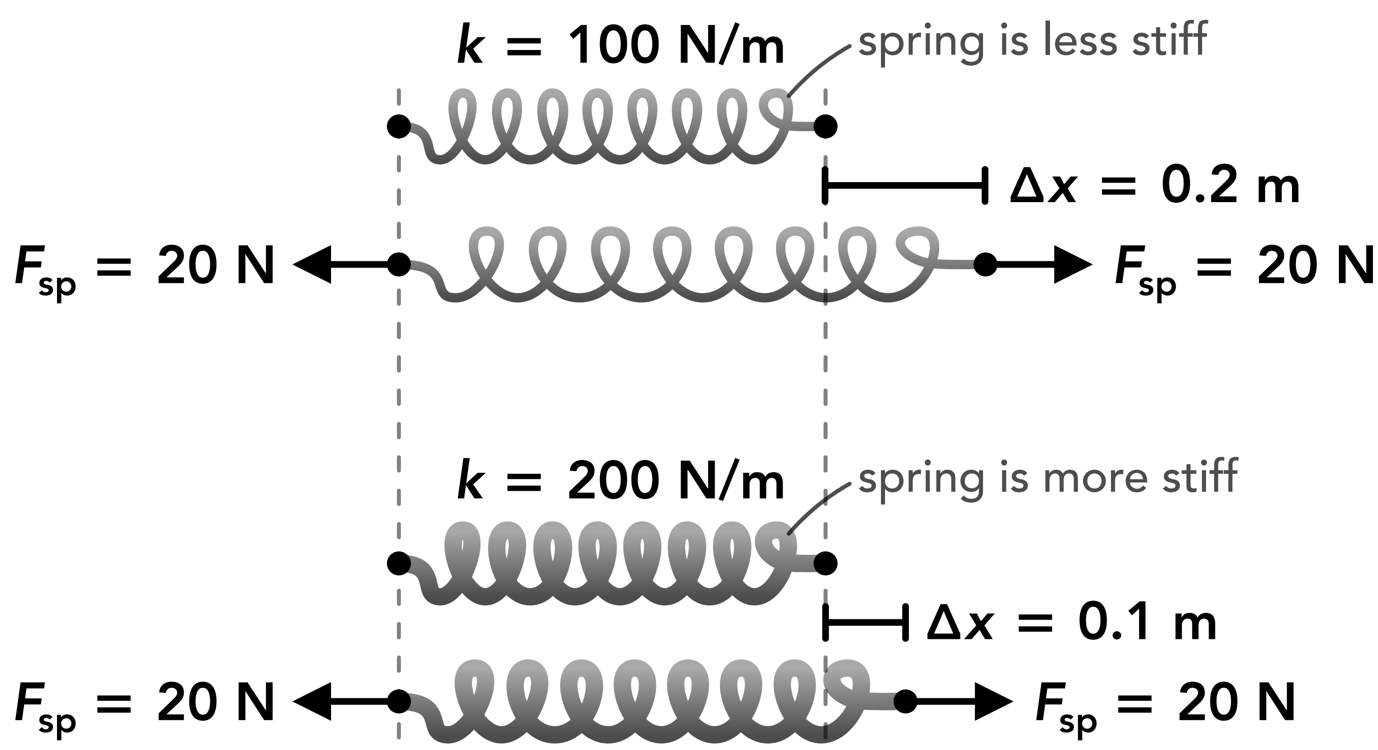
When no force is applied the spring is at its original length, relaxed length or unstretched length

When a tension (pulling) force is applied to both ends the spring gets longer by a change of Δx

When a compression (pushing) force is applied to both ends the spring gets shorter by a change of Δx



- The **spring constant *k*** is a value that represents the stiffness of a particular spring. A spring that is more stiff has a higher spring constant and requires more force to cause the same displacement as a spring that is less stiff and has a lower spring constant.
- The spring constant has a unit of Newtons/meter (N/m) given by units of force and displacement in Hooke's law.



$F_{sp} = k\Delta x$

$(20\text{ N}) = (100\text{ N/m})\Delta x$

$\Delta x = 0.2\text{ m}$

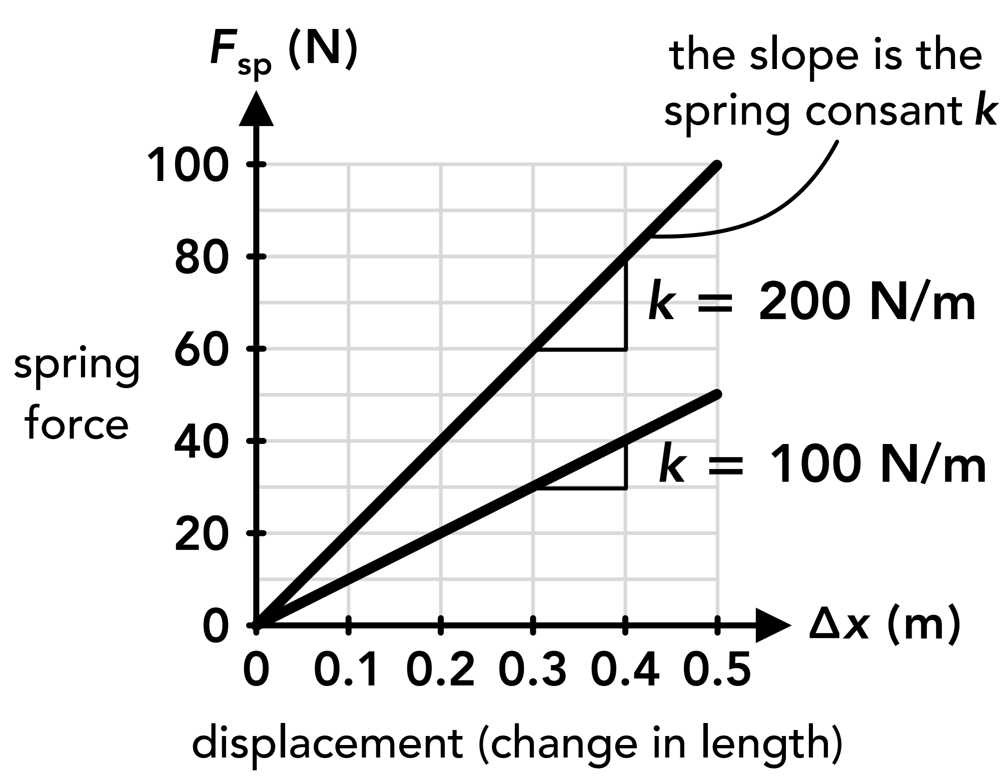
$F_{sp} = k\Delta x$

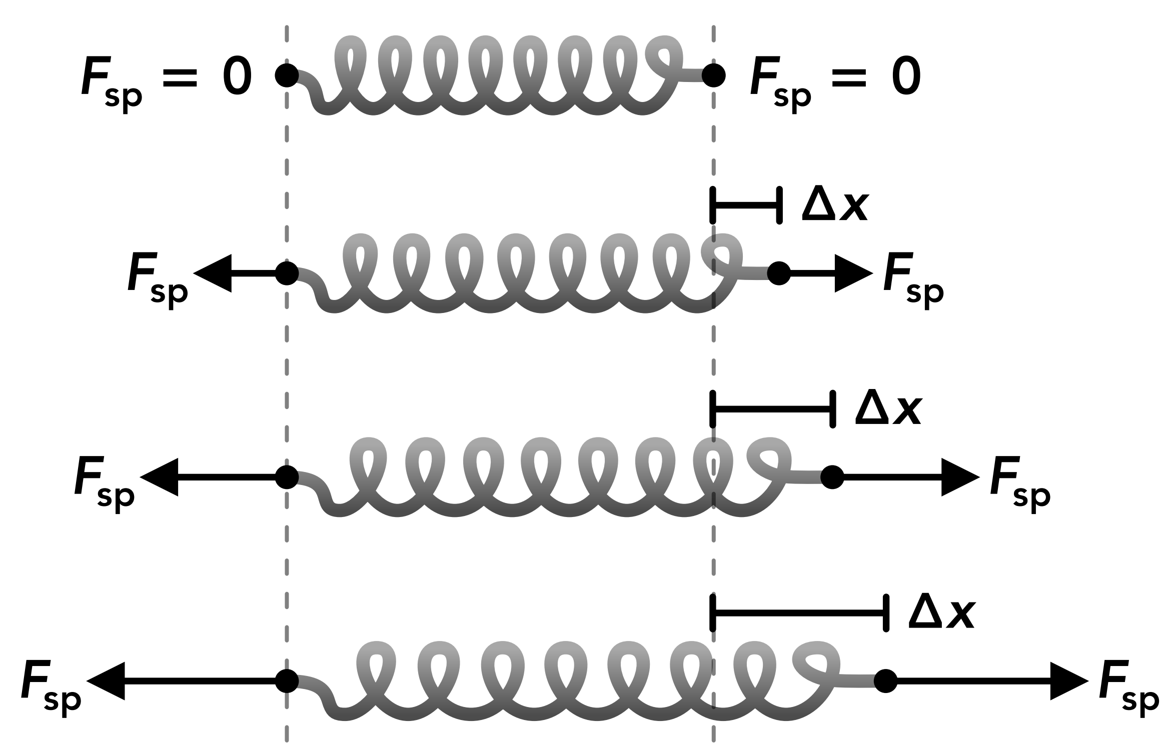
$(20\text{ N}) = (200\text{ N/m})\Delta x$

$\Delta x = 0.1\text{ m}$

- Since the change in length Δx is linearly proportional to the spring force F_{sp} , a graph of the spring force vs the displacement is a straight line.
- If the spring force is on the vertical axis and the displacement is on the horizontal axis, **the slope of the graph is the spring constant *k***. If the axes are flipped the slope is $1/k$.

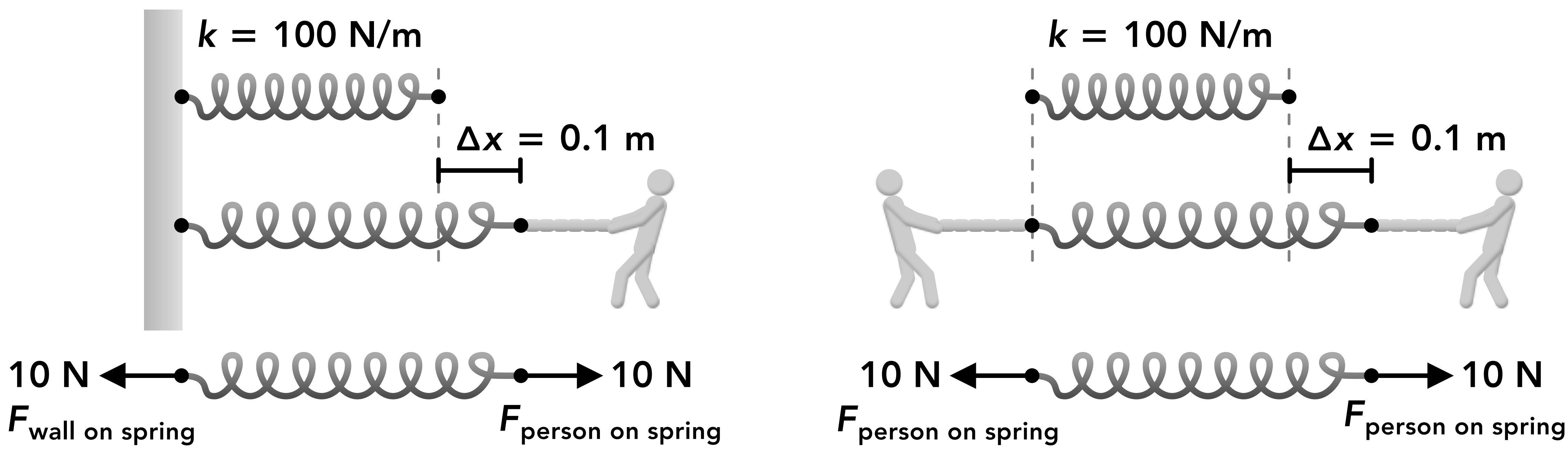
Hooke's Law: $F_{sp} = k\Delta x$





- A source of common confusion is the direction and magnitude of the spring force.
- A "spring force" is not a fundamental type of force like the gravitational force. When we use the term "spring force" we either mean a force exerted on the spring by an object, or the force exerted on an object by the spring.
- First, the forces acting on each end of a spring are equal in magnitude and opposite in direction. Not because they are a pair of equal and opposite forces as described in Newton's 3rd law of motion, but because we're treating the spring as ideal and **we're assuming the net force acting on the spring is zero**. In cases where the the spring is in static equilibrium and not moving (and therefore not accelerating) this must be true according to Newton's 2nd law of motion, $F_{net} = ma$. Even in cases where one or both ends of the spring are accelerating and the forces acting on the ends are changing, an ideal spring has no mass and **we assume it instantaneously transmits forces from one end to the other**. This is the same thing that happens for an ideal rope when working with tension forces, so you can think of the "spring force" on an object like a tension force acting on the object.
- Even when one end of the spring is fixed to a wall or a non-moving object, the wall still exerts a force on the spring just like if it were being pulled or pushed by a person or some other more "visible" force. If the wall was not exerting this force, the net force would not be zero and the spring would accelerate. Again, this is the same thing that happens with the tension force in a rope.

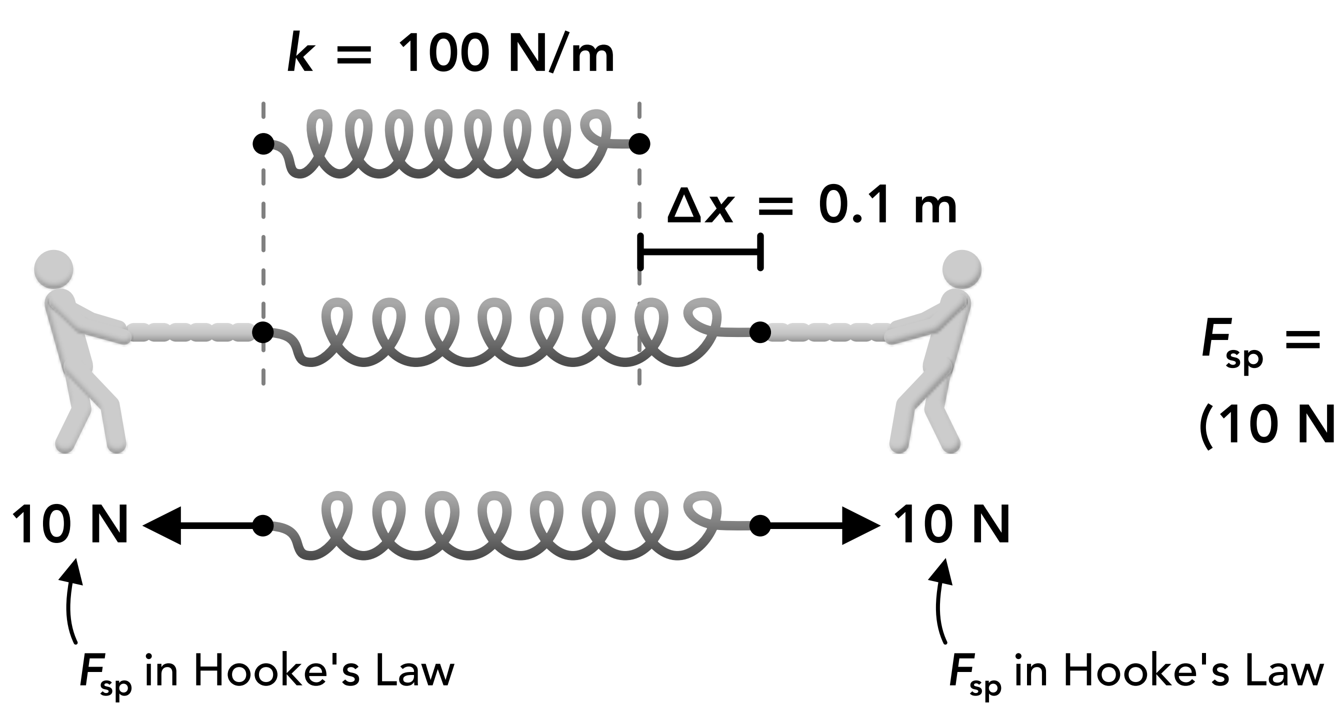
In both cases the spring is in static equilibrium (not moving) so the net force acting on the spring is zero.
The wall exerts a force on the spring just like if it were pulled or pushed by a person.



$$\sum F_x = ma_x$$
$$10\text{ N} - 10\text{ N} = m(0\text{ m/s}^2)$$

$$\sum F_x = ma_x$$
$$10\text{ N} - 10\text{ N} = m(0\text{ m/s}^2)$$

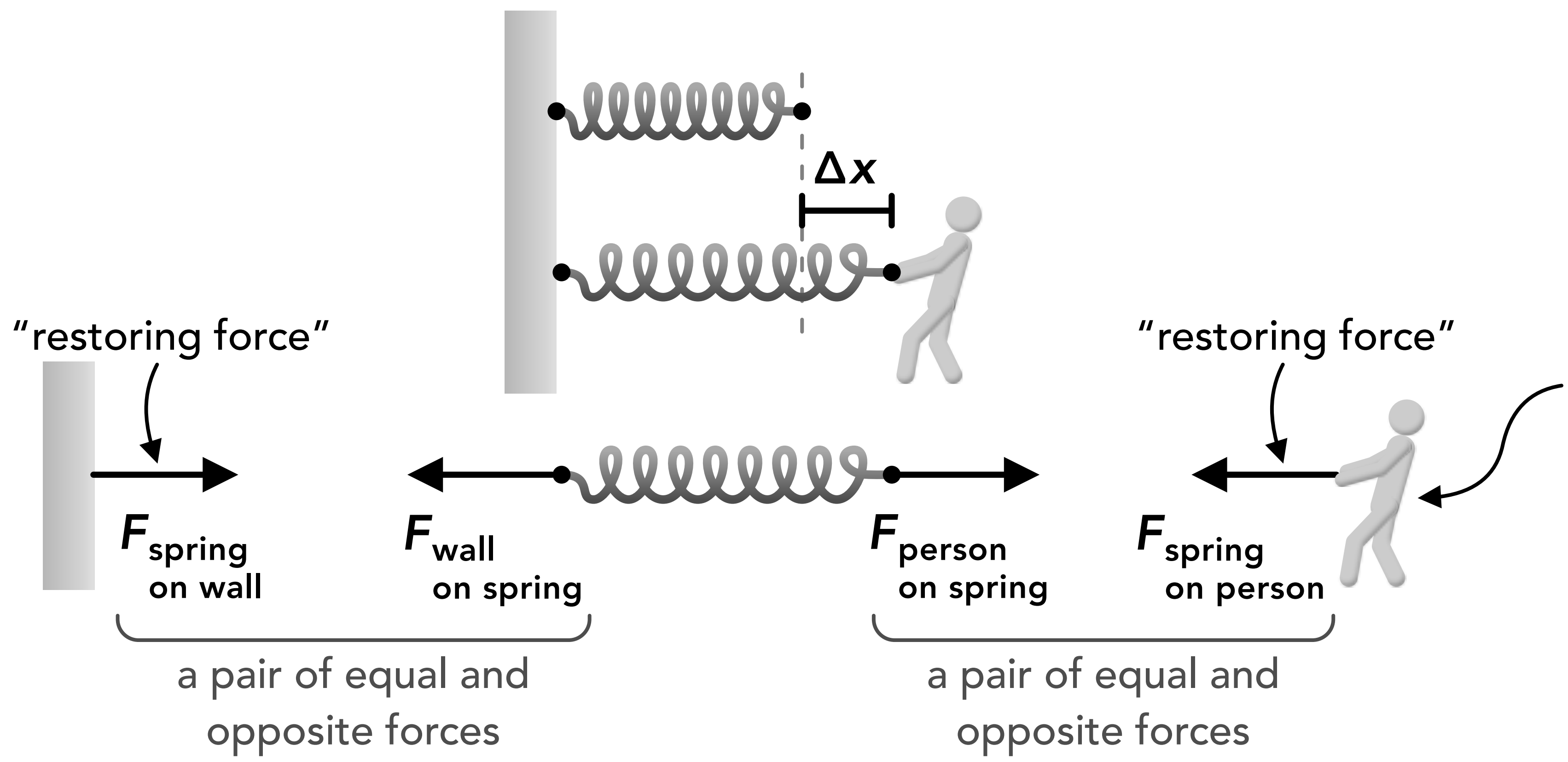
- Second, in the context of Hooke's Law the spring force F_{sp} refers to **the magnitude of the force acting on each end of a spring** (they're the same). We don't double the force or add the forces from each end together.



$$F_{sp} = k\,\Delta x$$
$$(10\text{ N}) = (100\text{ N/m})(0.1\text{ m})$$

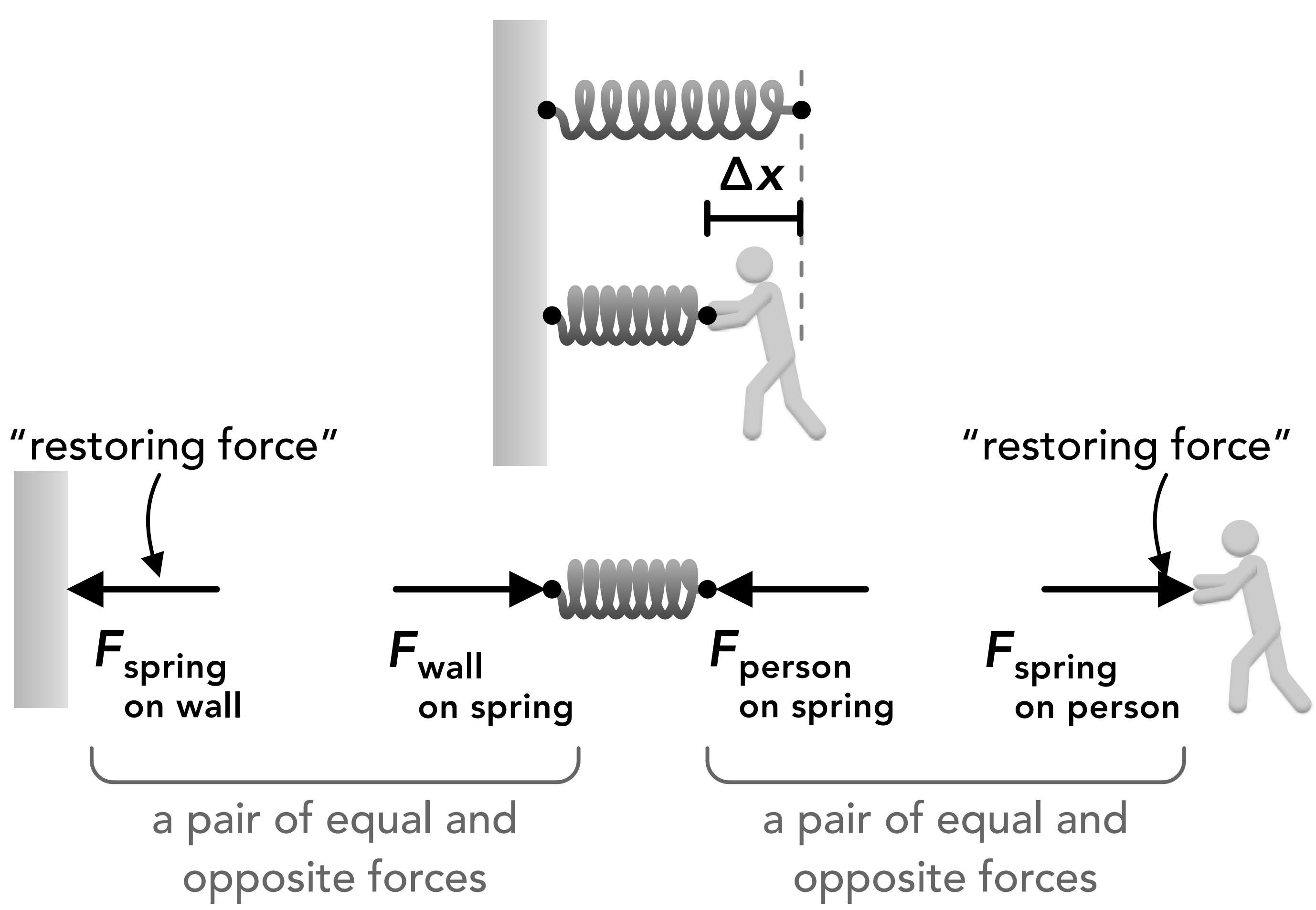
- Third, when we use the term "spring force" we need to be specific about which object the force is exerted on and which object is causing the force. When a spring is attached to an object the spring and the object exert contact forces on each other. The force exerted on the spring by the object is equal and opposite to the force exerted on the object by the spring (these are a pair of forces as described in Newton's 3rd law of motion).
- When a spring changes length, **it also exerts a force on the objects it's in contact with** (again this is just from Newton's 3rd law of motion). The force exerted by the spring on an object is called the **restoring force** because this force is trying to restore the spring to its original length.
- In most cases, we're focused on an object that is in contact with a spring, not the spring itself. In those scenarios we usually call the force exerted on the object by the spring the "spring force" F_{sp}.
- It's also important to clearly label the forces in a free body diagram so we know what a force is acting on and what is causing the force. Remember, the free body diagram for an object only shows the forces acting on that object.

Free body diagrams of the wall, the spring and the person when the spring is stretched

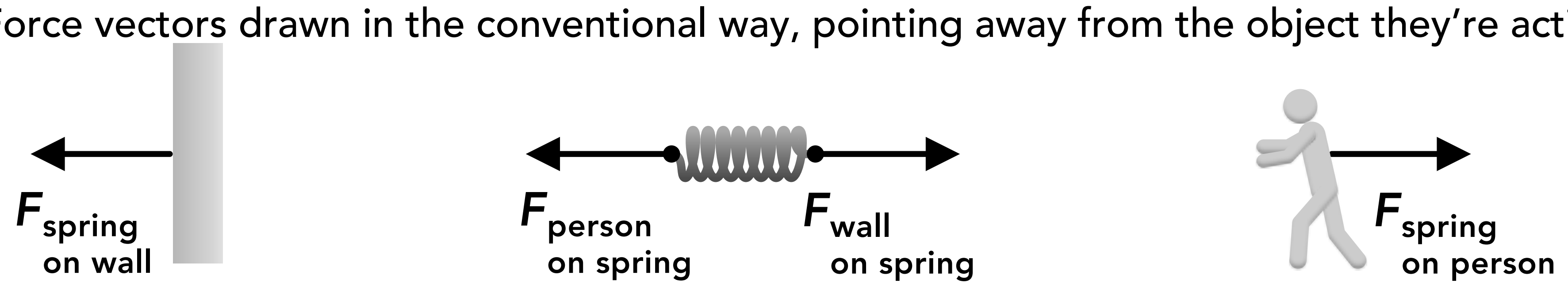


We're usually focused on the object in contact with the spring, not the spring itself. A free body diagram of the object shows the "restoring force" exerted on the object by the spring, which is equal and opposite to the force exerted on the spring by the object, which is the force in Hooke's Law.

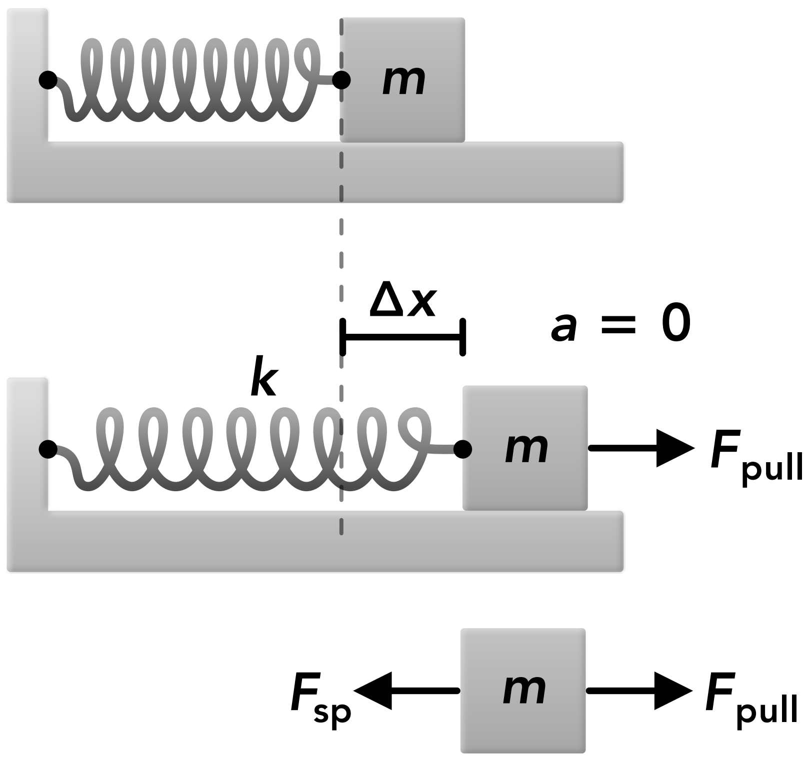
Free body diagrams of the wall, the spring and the person when the spring is compressed



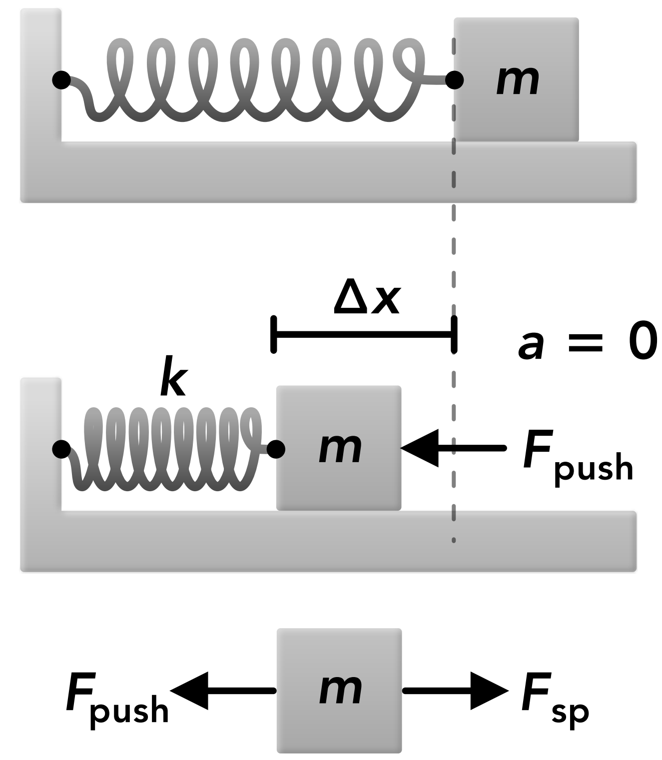
Force vectors drawn in the conventional way, pointing away from the object they're acting on:



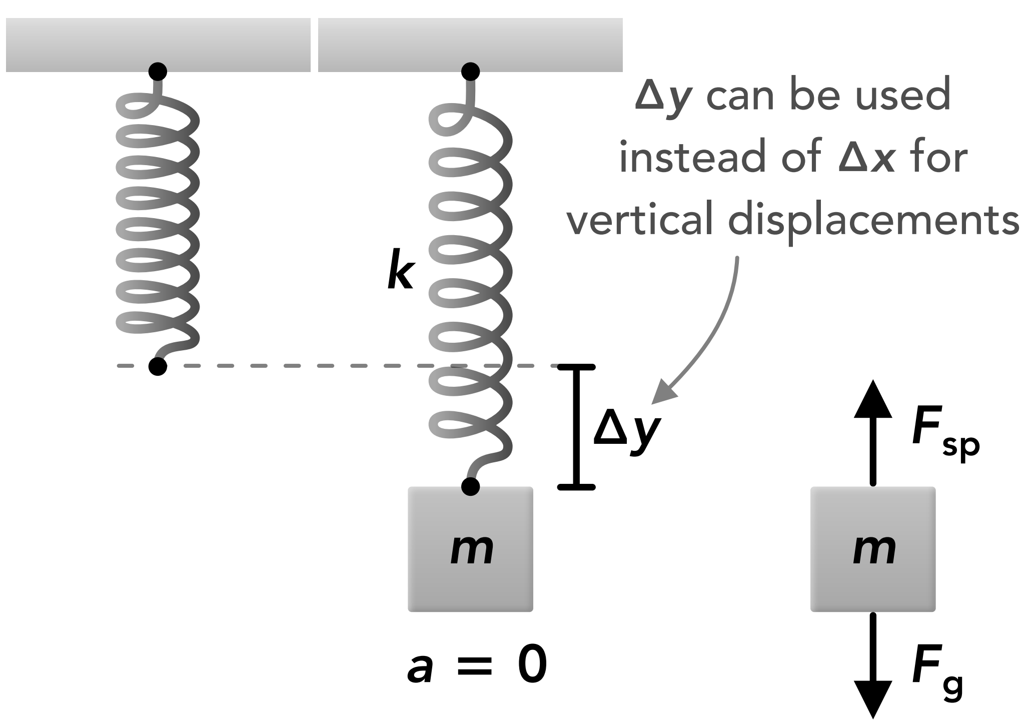
Examples of free body diagrams and Newton's 2nd law involving spring forces



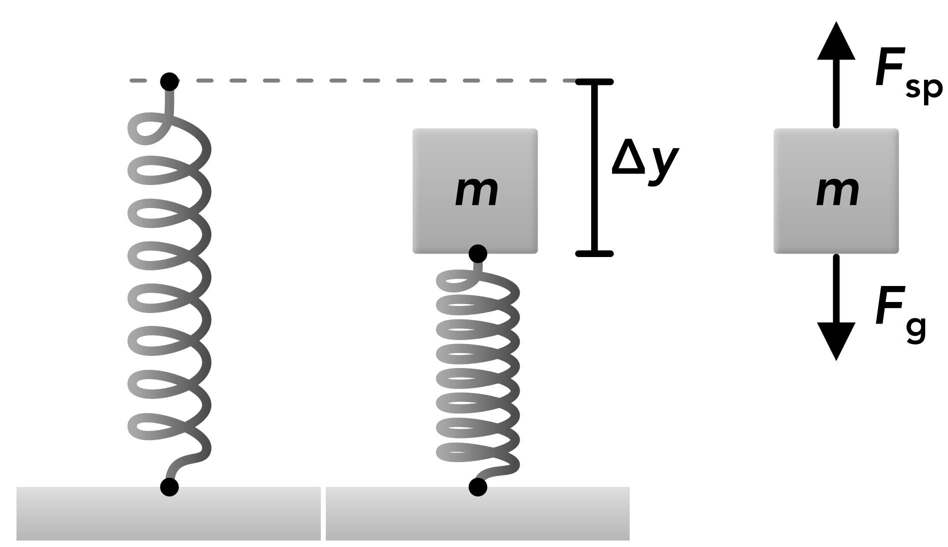
$$\sum F_x = ma_x$$
$$F_{pull} - F_{sp} = m(0)$$
$F_{sp} = k\,\Delta x$ $\quad F_{pull} - (k\Delta x) = 0$



$$\sum F_x = ma_x$$
$$F_{sp} - F_{push} = m(0)$$
$F_{sp} = k\,\Delta x$ $\quad (k\Delta x) - F_{push} = 0$



$$\sum F_y = ma_y$$
$$F_{sp} - F_g = m(0)$$
$F_{sp} = k\,\Delta y$ $\quad (k\Delta y) - (mg) = 0$
$F_g = mg$

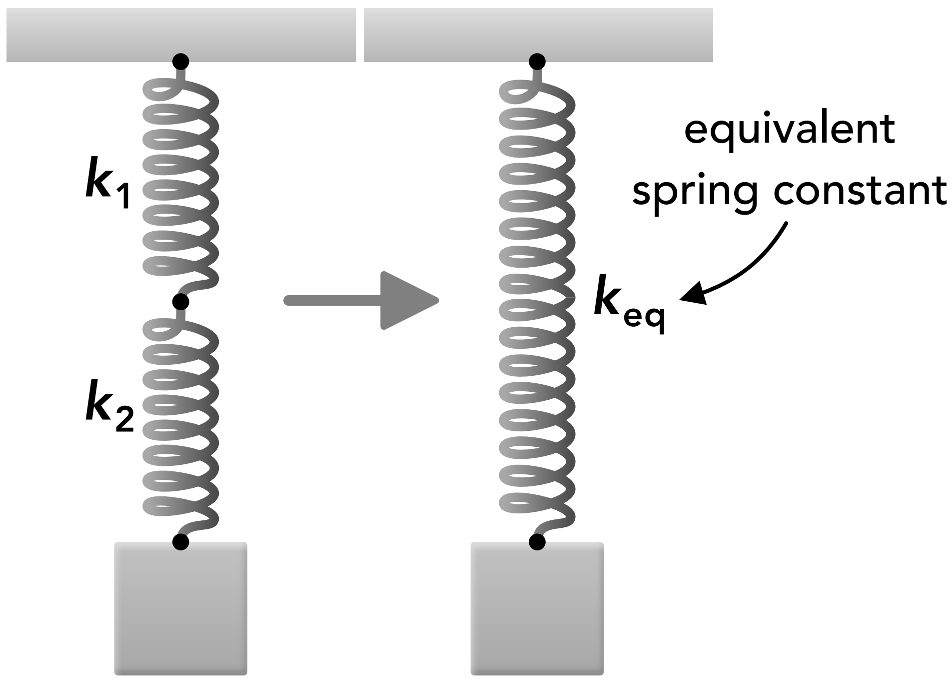


$$\sum F_y = ma_y$$
$$F_{sp} - F_g = m(0)$$
$F_{sp} = k\,\Delta y$ $\quad (k\Delta y) - (mg) = 0$
$F_g = mg$

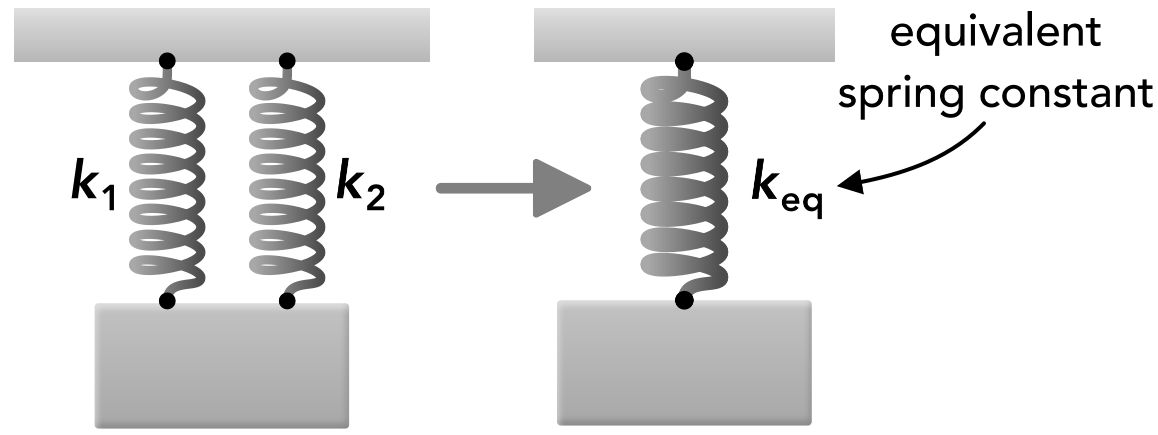
Combining Springs in Series and Parallel

- Multiple springs can be combined together **in series** or **in parallel**.
- Together, the group of springs can be treated as **a single spring with an equivalent spring constant** k_{eq} . The equivalent spring constant is calculated in a different way for springs in series and springs in parallel.

Two springs added in series



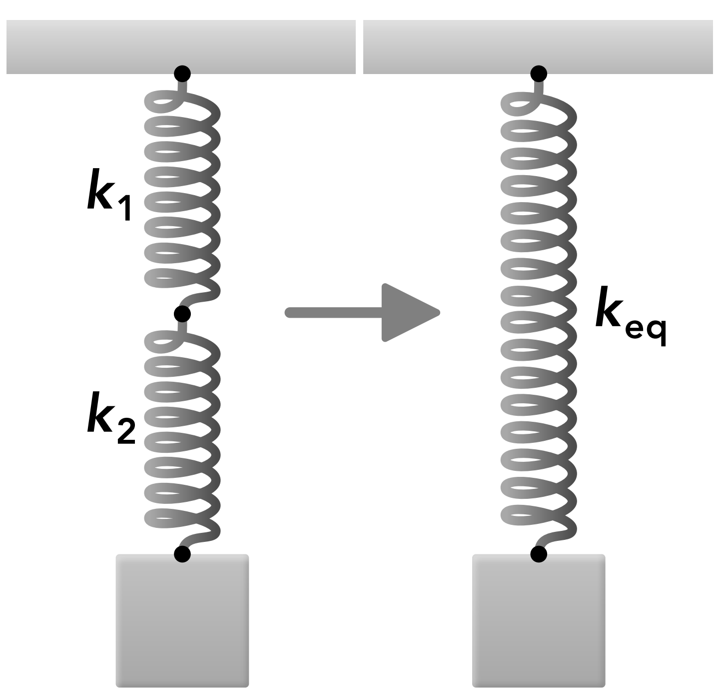
Two springs added in parallel



- Springs added **in series** are connected end-to-end.
- Adding an additional spring in series always **decreases** the equivalent spring constant or stiffness.
- The original lengths of the springs are added together.
- The force applied to the end of a series of springs is the **same force** applied to each individual spring.
- The displacements of each spring are added together.

Equivalent spring constant for springs in series

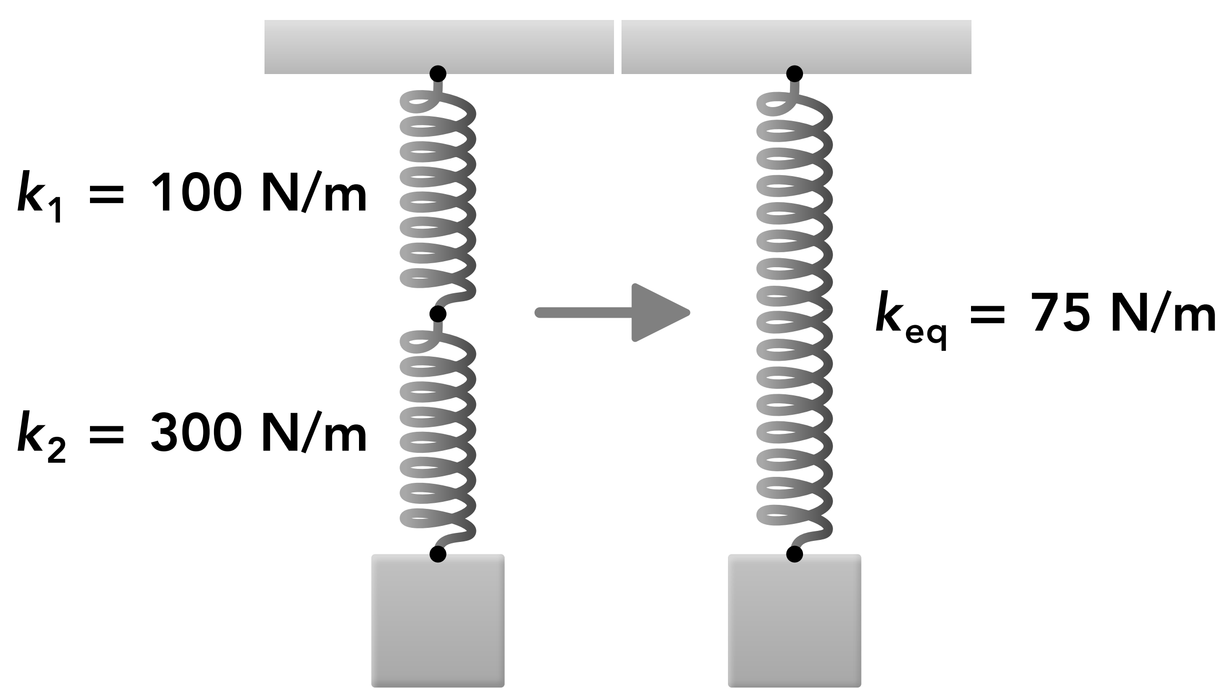
$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots$$



Example:

$$\frac{1}{k_{eq}} = \frac{1}{100} + \frac{1}{300}$$

$$k_{eq} = 75 \text{ N/m}$$



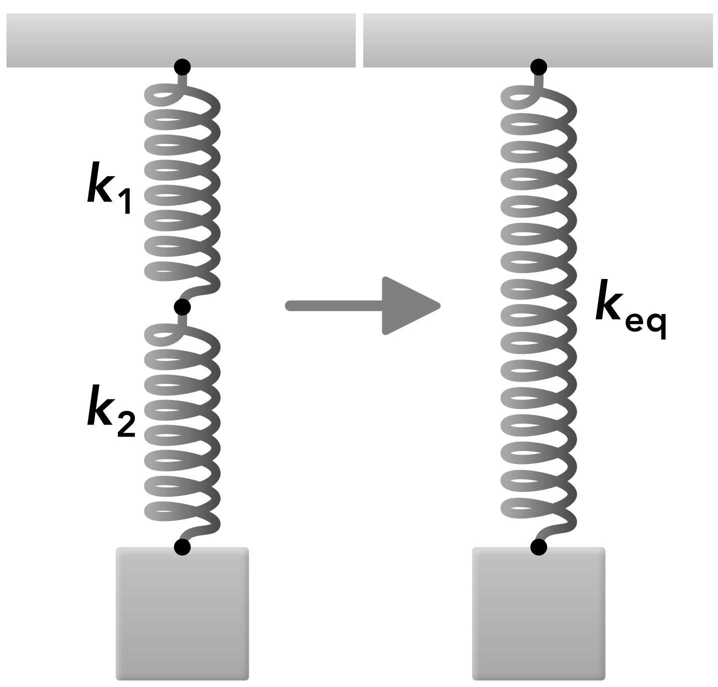
Combining Springs in Series and Parallel

- A group of springs can be added together and treated as **a single spring with an equivalent spring constant k_{eq}**.

- Springs added **in series** are connected end-to-end.
- Adding an additional spring in series always **decreases** the equivalent spring constant or stiffness.
- The original lengths of the springs are added together.
- The force applied to the end of a series of springs is the **same force** applied to each individual spring.
- The displacements of each spring are added together.

Equivalent spring constant for springs in series

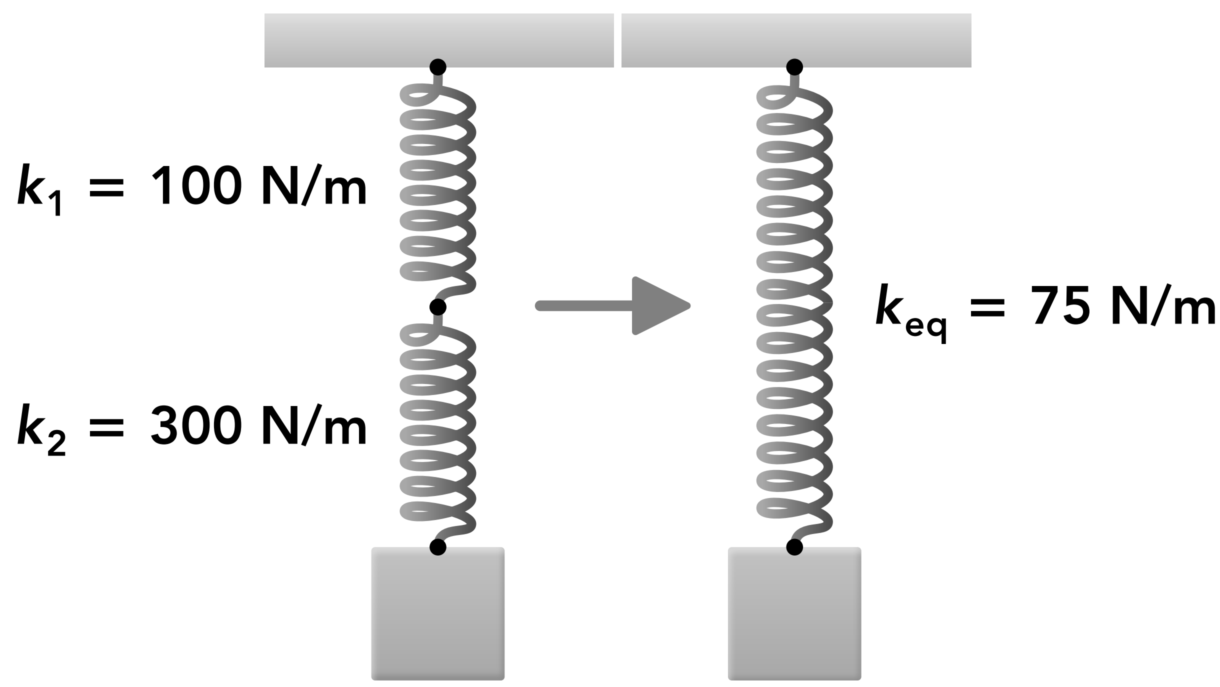
$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots$$



Example:

$$\frac{1}{k_{eq}} = \frac{1}{100} + \frac{1}{300}$$

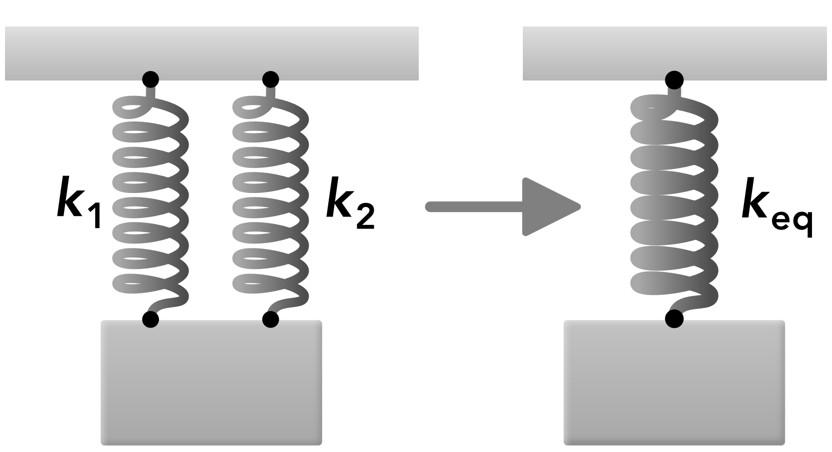
$$k_{eq} = 75 \text{ N/m}$$



- Springs added **in parallel** are all connected to the same two objects or surfaces at each end.
- The equivalent spring constant is just the sum of the individual spring constants.
- Adding an additional spring in parallel always **increases** the equivalent spring constant or stiffness.
- Each spring has the **same displacement** but a **different amount of force**.
- When adding springs in parallel like this, they must have the same original length and we're assuming the object translates (moves linearly) but doesn't rotate, even though there are different forces acting on it at different points.

Equivalent spring constant for springs in parallel

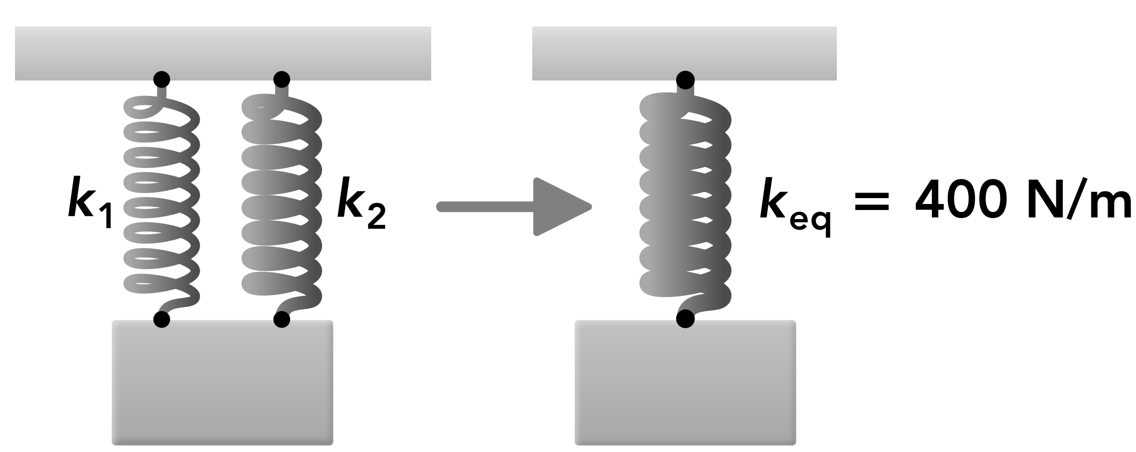
$$k_{eq} = k_1 + k_2 + \ldots$$



Example:

$$k_{eq} = 100 + 300$$

$$k_{eq} = 400 \text{ N/m}$$



$k_1 = 100$ N/m

$k_2 = 300$ N/m

- Groups of springs can be in series and in parallel with other groups of springs. First, the springs within a group are added together (in series or parallel) and then the groups can be added together (in series or parallel).

  1. Springs 1 and 3 are added in series to get an equivalent spring constant k_{1+3}
  2. Spring "1+3" and spring 2 are added in parallel to get a final equivalent spring constant k_{1+2+3}

$$\frac{1}{k_{1+3}} = \frac{1}{k_1} + \frac{1}{k_3}$$

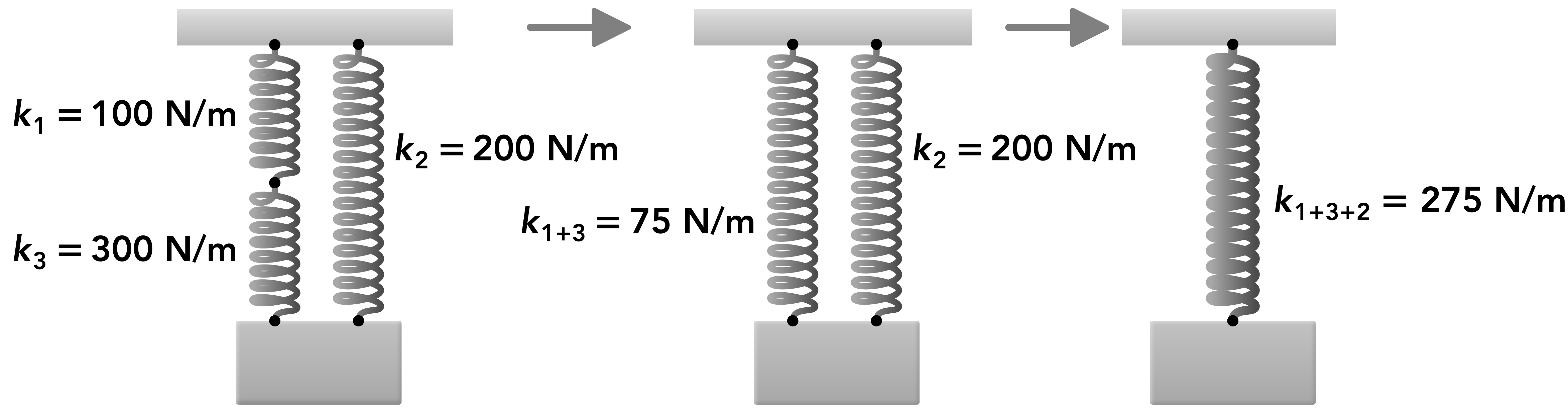
$$\frac{1}{k_{1+3}} = \frac{1}{100} + \frac{1}{300}$$

$$k_{1+3} = 75 \text{ N/m}$$

$$k_{1+3+2} = k_{1+3} + k_2$$

$$k_{1+3+2} = 75 + 200$$

$$k_{1+3+2} = 275 \text{ N/m}$$



  1. Springs 1 and 2 are added in parallel to get an equivalent spring constant k_{1+2}
  2. Springs 3 and 4 are added in parallel to get an equivalent spring constant k_{3+4}
  3. Spring "1+2" and spring "3+4" are added in series to get a final equivalent spring constant $k_{1+2+3+4}$

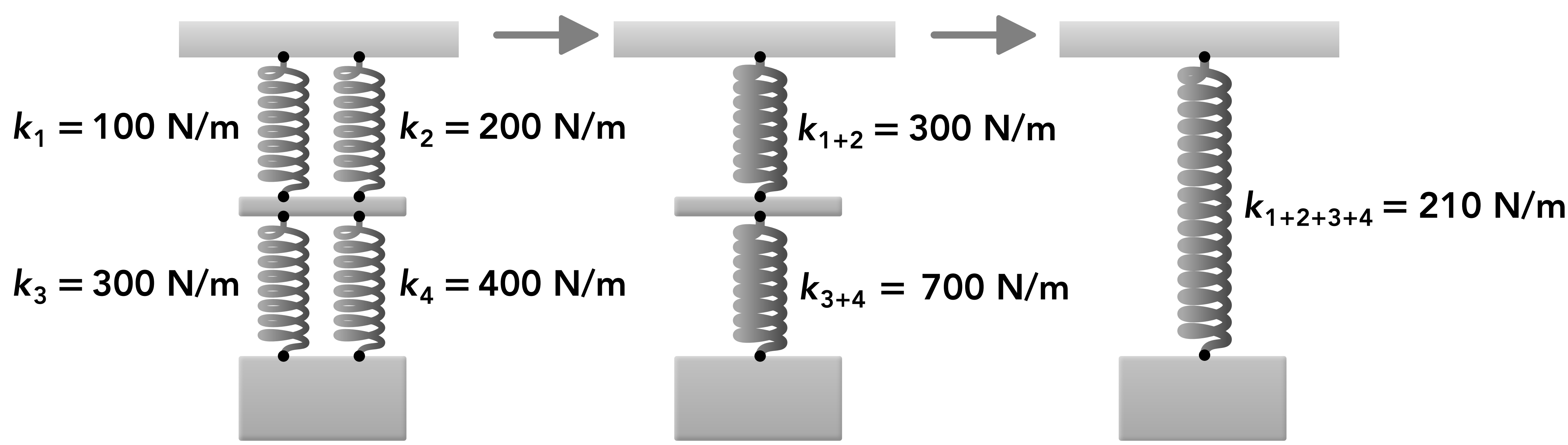
$$k_{1+2} = k_1 + k_2 = 100 + 200 = 300$$

$$k_{3+4} = k_3 + k_4 = 300 + 400 = 700$$

$$\frac{1}{k_{1+2+3+4}} = \frac{1}{k_{1+2}} + \frac{1}{k_{3+4}}$$

$$\frac{1}{k_{1+2+3+4}} = \frac{1}{300} + \frac{1}{700}$$

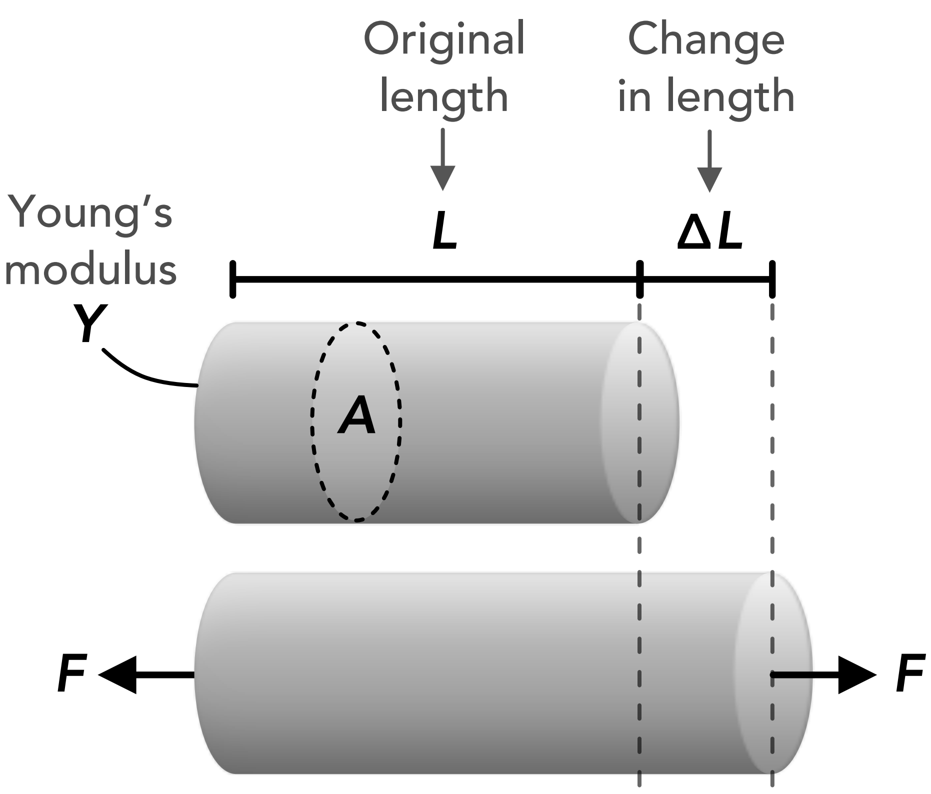
$$k_{1+2+3+4} = 210 \text{ N/m}$$



Elasticity of Materials

- The elastic behavior of materials is complex and depend on many factors such as material properties, the shape and dimensions of the object, the directions of the applied forces, the change in length itself, and more.
- A basic model can be used to describe the elastic behavior of a material in a way that's similar to a spring. Note that this model is only accurate up to a certain amount of strain (percent change in length), after which the material no longer behaves "elastically" and will begin to permanently deform and eventually break.

Variables		SI Unit
F	force	N
k	spring constant	$\frac{\mathrm{N}}{\mathrm{m}}$
Y , E	Young's modulus	$\frac{\mathrm{N}}{\mathrm{m}^2}$
A	cross-sectional area	m^2
L	length	m



"Spring constant" for a material

$$k = \frac{YA}{L}$$

Elastic Force

$$F = \frac{YA}{L}\Delta L$$

Stress

$$\frac{F}{A} = Y\frac{\Delta L}{L}$$

$\frac{\Delta L}{L}$ is called "strain"

- Young's modulus Y (sometimes referred to as the elastic modulus E) is a property of the material that describes its stiffness. This is a material property that does not depend on the size or shape of the object, and is not the same as a spring constant k. However, an equivalent "spring constant" k can be found using the object's Young's modulus, cross sectional area and original length as shown in the equation above.
- Like a spring, the force applied to the object is proportional to its change in length, now represented as ΔL.
- Because objects are different shapes and sizes, it's often more useful to work with a concept called **stress** which is the amount of force applied per unit of area. It's also more useful to describe the percent change in length, known as **strain** ($\Delta L / L$) instead of the absolute change in length.