Lagrangian Mechanics: Problems & Practice

Worksheet 4

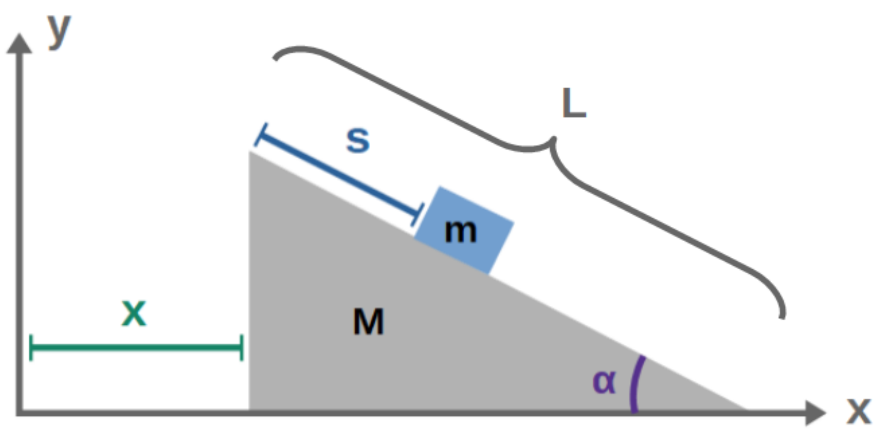
This worksheet is associated with **Chapter 4** of the book *Lagrangian Mechanics For The Non-Physicist*.

Problems:

(4.1) The Moving Inclined Plane

A block of mass m is placed on an inclined ramp of mass M under a gravitational acceleration of magnitude g downwards. The plane is tilted by an angle α with respect to the ground, such that the block begins sliding down the ramp. The total length of the ramp is L. In addition, the ramp itself is allowed to move in the horizontal direction. Friction can be neglected.

The system is placed in a Cartesian coordinate system with the ramp being initially at the origin. Choose the x-coordinate of the ramp and the displacement s of the block measured from the top of the ramp as your generalized coordinates (see the picture below). Find the equations of motion for the coordinates x and s and solve them as a function of time. You can assume the initial conditions $x(0) = 0$ and $s(0) = 0$ and that both the block and the ramp are at rest initially.



Hint: The Lagrangian for the system consists of the kinetic and potential energies of both the ramp and the block. Since the ramp is fixed at the $y = 0$ level, its potential energy is zero. You need to find the kinetic energy of the ramp well as the kinetic and potential energy of the block and express these in terms of x and s.

(4.2) Central Potentials

A particle of mass m is moving in a central force field, which can be described by a general potential energy $V(r)$. A central force field simply means that the particle is under a force that only depends on its radial distance to the center (r).

Find the equations of motion for the particle in spherical coordinates (r, φ, θ). Also calculate the generalized momenta associated with each of the coordinates. Are any of them conserved?

Hint: Begin from constructing the kinetic energy of the particle first in Cartesian coordinates. Then, use the following relations between the Cartesian and spherical coordinates:

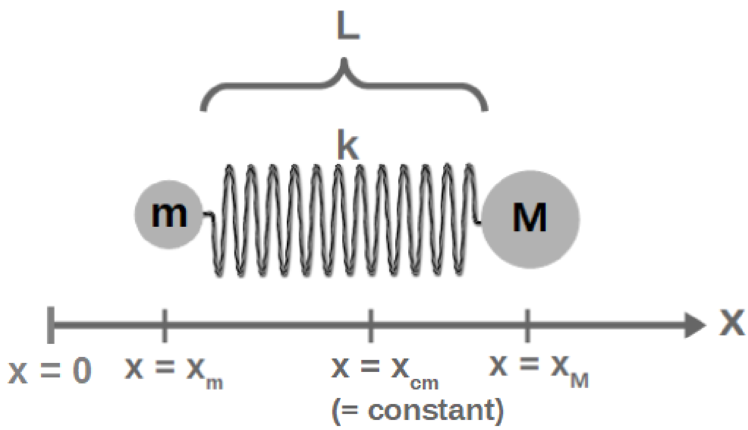
$$x = r\cos\theta\sin\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\varphi$$

(4.3a) Coupled Oscillations: Equation of Motion

Consider two masses m and M in one dimension connected by a spring with spring constant k and equilibrium length L. The only force present in the system is the spring force and the whole system is taken to be at rest initially. This kind of a system could be used to model, for example, the vibrations of a linear diatomic molecule.

We can describe the system by placing it on a one-dimensional coordinate axis x. The positions of the two masses are described by coordinates x_m and x_M.

Since there are no external forces acting on the system and the system is initially at rest (so the total momentum is zero), the center of mass of the system, given by the formula $x_{cm} = \dfrac{mx_m + Mx_M}{m + M}$, must be constant at all times.



The spring is then stretched from equilibrium by an amount δ, so that it begins oscillating. As the spring oscillates, the displacement δ changes with time, so $\delta = \delta(t)$.

Find the Lagrangian and the equation of motion for the system using the displacement of the spring $\delta(t)$ as your generalized coordinate (this is the only generalized coordinate you need to describe the entire system with).

Hint: The fact that the center of mass of the system, x_{cm}, is always constant is a constraint on the system, which you can use to express one of the coordinates in terms of the others.

(4.3b) Coupled Oscillations: Solution

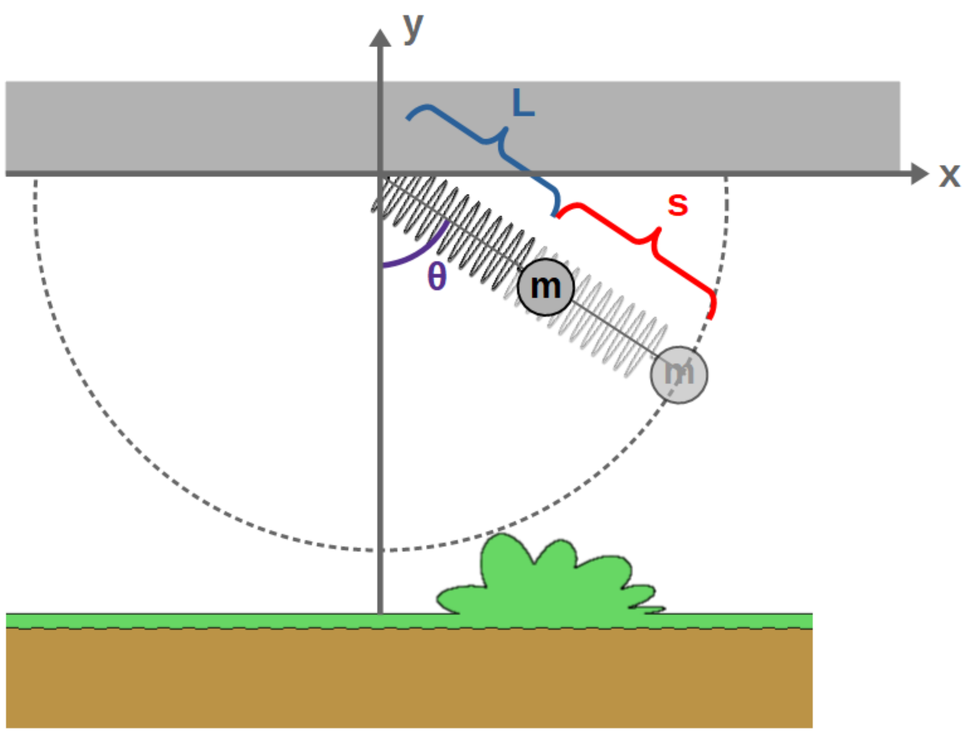
Solve the equation of motion for $\delta(t)$. You can assume that $\delta(0) = 0$. Find the frequency ω and period T of oscillation. What happens to these in the case that $M \gg m$, i.e. one of the masses is much larger than the other?

Hint: Check out the "Example: Harmonic Spring Motion" -section in Chapter 1 if you need help with finding the solution to the equation of motion.

(4.4a) The Elastic Pendulum: Equations of Motion

Consider again a pendulum swinging in a two-dimensional plane, but instead of the pendulum bob being attached to a rigid rod, it is attached to the end of a spring that can stretch. The mass of the pendulum bob is m, the equilibrium (unstretched) length of the spring is L and the spring constant is k. You may assume the spring to be massless so that gravity only acts on the pendulum bob.

As your generalized coordinates describing the pendulum bob, choose the angle of the pendulum with respect to the vertical (θ) as well as the displacement of the spring from its equilibrium length (s).



Find the Lagrangian and the equations of motion for the pendulum bob using the above mentioned generalized coordinates. Check that the equations of motion reduce to those of the simple pendulum in the limit that the spring constant $k \to \infty$ (in other words, the spring becomes completely rigid).

Hint: Begin by expressing the Cartesian coordinates x and y of the pendulum bob in terms of the generalized coordinates θ and s, similarly as in the simple pendulum example covered in Chapter 4. However, for this elastic pendulum system, the radial distance r is NOT constant and you instead need to express it using the displacement of the spring, the coordinate s.

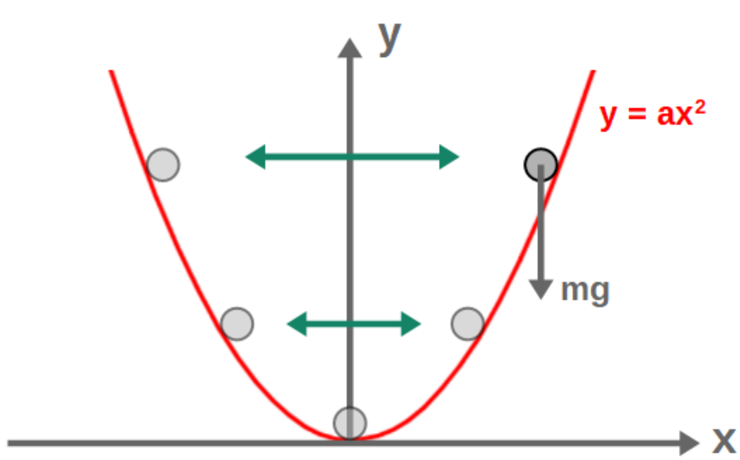
(4.4b) The Elastic Pendulum: Special Solutions

For a simple pendulum, there exists a possible *static solution* (a solution where the pendulum stays completely still) to the equations of motion at $\theta = 0$. What happens to the equations of motion for the elastic pendulum if you set $\theta = 0$ for all times? Solve the resulting equations of motion for $s(t)$ corresponding to this special case. You may assume that $s(0) = 0$.

(4.5a) Particle In a Parabolic Potential: The Equation of Motion

Consider a particle of mass m placed inside a parabola-shaped bowl. The particle rolls back and forth inside the bowl under the effects of gravity, such that it remains confined to a parabolic trajectory at all times.

We can describe the system in a Cartesian coordinate system, such that the shape of the bowl and also the particle's trajectory are described by the curve $y = ax^2$, where a is some constant (with units of inverse length).



In this problem, you have one object in two dimensions with one additional constraint (the particle must stay on the parabola), so you need $1 \cdot 2 - 1 = 1$ generalized coordinate to describe the system with.

Identify a suitable generalized coordinate to describe the motion of the particle with.

Then, **find the Lagrangian as well as the equation of motion** for the particle such that the constraints in the system are implicitly encoded in the Lagrangian and thus, also in the equation of motion. Does the particle undergo simple harmonic motion as it rolls back and forth on the parabola?

Hint: The constraint here is that the particle's Cartesian coordinates are related by the equation of the parabola $y = ax^2$ at all times.

(4.5b) Particle In a Parabolic Potential: Constraint Forces

Find the constraint forces that keep the particle on its parabolic trajectory. In other words, repeat the same calculation as in problem (4.5a), but now using an unconstrained Lagrangian and the Euler-Lagrange equation with constraints:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \lambda \frac{\partial g}{\partial q_i}$$

Hint: The constraint equation $y = ax^2$ can be written as $y - ax^2 = 0$. From this, you can identify the constraint function g as $g(x, y) = y - ax^2$ (reminder; the constraint function is defined in terms of the constraint equation as $g = 0$). If you need more help, take a look at the strategies we used to find constraint forces in Chapter 4.