A-Level Pure Math Further Differentiation - Calculus

Cálculo (Universidad Privada del Norte)

04. FURTHER DIFFERENTIATION

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4.1. The Chain Rule

Supposing y is a function of t, and t itself is a function of x. If δy, δt *and* δx *are corresponding small increments in y, t, and x, then;*

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta t} \times \frac{\delta t}{\delta x}$$

As δy, δt *and* δx *tend to zero;*

$$\frac{\delta y}{\delta x} \to \frac{dy}{dx}, \quad \frac{\delta y}{\delta t} \to \frac{dy}{dt}, \quad \frac{\delta t}{\delta x} \to \frac{dt}{dx}$$

Hence $$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$ *is the* ***chain rule.***

The Chain rule provides us with a technique to differentiate composite functions; including expressions with indices without the need to first expand the expressions.

Example 1:

Differentiate:

(a) $(2x+3)^2$

Let $y = (2x+3)^2$

$y = 4x^2 + 12x + 9$

$\frac{dy}{dx} = 8x + 12$

$= 4(2x+3)$

Alternatively:

Let $y = (2x+3)^2$ *and* $t = 2x+3$; *then* $y = t^2$

$\frac{dt}{dx} = 2 \qquad \frac{dy}{dt} = 2t$

By the chain rule; $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dx} = 2t \times 2 = 4t$

$= 4(2x+3)$.

Previously, we expanded the expression, before differentiation.

Now, we introduce a substitute, t, for the expression in brackets.

Then we obtain dt/dx and dy/dt

We then use the chain-rule to relate dy/dt and dt/dx.

We obtain the same answer, but the chain rule is important especially when the power of the expression is high, and expansion is tedious.

$(b)\quad 2(3x+4)^4$

Let $y = 2(3x+4)^4$ and $t = 3x+4$, then $y = 2t^4$

$$\frac{dt}{dx} = 3 \qquad \frac{dy}{dt} = 8t^3$$

By the chain rule; $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$$\frac{dy}{dx} = 8t^3 \times 3$$
$$= 24t^3$$
$$= 24(3x+4)^3 \ .$$

Expanding the expression with power 4 would be tedious. We aim at substituting that term in brackets.

We obtain ${}^{dt}/_{dx}$ and ${}^{dy}/_{dt}$

We then use the chain-rule to relate ${}^{dy}/_{dt}$ and ${}^{dt}/_{dx}$.

We substitute back $t = 3x+4$.

$(c)\quad (3x-1)^{2/3}$

Let $y = (3x-1)^{2/3}$ and $t = 3x-1$; then $y = t^{2/3}$

$$\frac{dt}{dx} = 3 \qquad \frac{dy}{dt} = \tfrac{2}{3}t^{-1/3}$$

By the chain rule; $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$$= \tfrac{2}{3}t^{-1/3} \times 3$$
$$= 2t^{-1/3}$$
$$= 2(3x-1)^{-1/3} \ .$$

Fractional powers are treated in the same way, paying attention to the negative indices that arise.

We obtain ${}^{dt}/_{dx}$ and ${}^{dy}/_{dt}$

We then use the chain-rule to relate ${}^{dy}/_{dt}$ and ${}^{dt}/_{dx}$.

$(d)\quad (3-2x)^{-1/2}$

Let $y = (3-2x)^{-1/2}$ and $t = 3-2x$; then $y = t^{-1/2}$

$$\frac{dt}{dx} = -2 \qquad \frac{dy}{dt} = -\tfrac{1}{2}t^{-3/2}$$

By the chain rule; $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$$\frac{dy}{dx} = -\tfrac{1}{2}t^{-3/2} \times -2$$
$$= t^{-3/2}$$
$$= (3-2x)^{-3/2}$$

We obtain ${}^{dt}/_{dx}$ and ${}^{dy}/_{dt}$

We then use the chain-rule to relate ${}^{dy}/_{dt}$ and ${}^{dt}/_{dx}$.

Note: we simply use the substitution (t) to simplify our working and should not be left in the final

Example 2:

Differentiate:

(a) $\dfrac{1}{\sqrt{(3x+1)}}$

Let $y = (3x+1)^{-\frac{1}{2}}$ and $t = 3x+1$; then $y = t^{-\frac{1}{2}}$

$$\frac{dt}{dx} = 3 \qquad \frac{dy}{dt} = -\tfrac{1}{2}t^{-\frac{3}{2}}$$

$$\frac{dy}{dx} = -\tfrac{1}{2}t^{-\frac{3}{2}} \times 3$$

$$= -\frac{3}{2}(3x+1)^{-3/2} .$$

The square root in the denominator becomes index $\left(-\frac{1}{2}\right)$.

We find $^{dt}/_{dx}$ and $^{dy}/_{dt}$ and relate them using the chain rule.

We substitute back $(3x+1)$ for t.

(b) $\left(3\sqrt{x} - 2x\right)^3$

Let $y = \left(3\sqrt{x} - 2x\right)^3$ and $t = 3x^{1/2} - 2x$; then $y = t^3$

$$\frac{dt}{dx} = \tfrac{3}{2}x^{-\frac{1}{2}} - 2 \qquad \frac{dy}{dt} = 3t^2$$

$$\frac{dy}{dx} = 3t^2 \times \left(\tfrac{3}{2}x^{-\frac{1}{2}} - 2\right)$$

$$= 3\left(3\sqrt{x} - 2x\right)^2\left(\frac{3}{2\sqrt{x}} - 2\right) .$$

We find $^{dt}/_{dx}$ and $^{dy}/_{dt}$ and relate them using the chain rule.

The index $\left(-\frac{1}{2}\right)$ can be written as a square root in the denominator.

(c) $\left(2x^2 - \frac{3}{x^2}\right)^{1/3}$

Let $y = \left(2x^2 - \frac{3}{x^2}\right)^{1/3}$ and $t = 2x^2 - 3x^{-2}$; then $y = t^{1/3}$

$$\frac{dt}{dx} = 4x + 6x^{-3} \qquad \frac{dy}{dt} = \tfrac{1}{3}t^{-2/3}$$

$$\frac{dy}{dx} = \tfrac{1}{3}t^{-\frac{2}{3}} \times (4x + 6x^{-3})$$

$$= \tfrac{1}{3}\left(2x^2 - \tfrac{3}{x^2}\right)^{-\frac{2}{3}}\left(4x + \tfrac{6}{x^3}\right) .$$

We find $^{dt}/_{dx}$ and $^{dy}/_{dt}$ and relate them using the chain rule.

One may consider moving negative indices into the denominator where desirable.

$(d)\ \dfrac{1}{\sqrt{(1-x^2)}}$

$$\text{Let } y = (1-x^2)^{-1/2} \text{ and } t = 1-x^2; \text{ then } y = t^{-1/2}$$

$$\frac{dt}{dx} = -2x \qquad \frac{dy}{dt} = -\tfrac{1}{2}t^{-3/2}$$

$$\frac{dy}{dx} = -\tfrac{1}{2}t^{-\frac{3}{2}} \times -2x$$

$$= x(1-x^2)^{-3/2}$$

$$= \frac{x}{\sqrt{(1-x^2)^3}} \ .$$

We find ${}^{dt}/_{dx}$ and ${}^{dy}/_{dt}$ and relate them using the chain rule.

Power $\left(-\frac{3}{2}\right)$ may be written as $\left(3 \times -\frac{1}{2}\right)$ where $\left(-\frac{1}{2}\right)$ may be written as a square root in the denominator.

At this point it is worth mentioning that the chain rule can be carried out in the "background", i.e. without showing the $\boldsymbol{t-substitution}$ as illustrated below:

Remember to consider the term in bracket as a single term $\boldsymbol{t}$ of the chain rule.

Example 3:

Differentiate:

$(a)\ (3-4x)^{-3}$

$$\text{Let } y = (3-4x)^{-3}$$

$$\frac{dy}{dx} = [-3(3-4x)^{-4}] \times \left\{\frac{d}{dx}(3-4x)\right\}$$

$$= -3(3-4x)^{-4} \times -4$$

$$= 12(3-4x)^{-4} \ .$$

We multiply by the power of the term (-3); and rewrite the term with its power reduced by 1;

Then multiply by the derivative of the term.

$(b)\ (3x^2+5)^3$

$$\text{Let } y = (3x^2+5)^3$$

$$\frac{dy}{dx} = [3(3x^2+5)^2] \times \left\{\frac{d}{dx}(3x^2+5)\right\}$$

$$= 3(3x^2+5)^2 \times 6x$$

$$= 18x(3x^2+5) \ .$$

We multiply by the power of the term (3); and rewrite the term with its power reduced by 1;

Then we multiply by the derivative of the term.

$(c)\ \ (2x^2 - 4x)^2$

$$\text{Let } y = (2x^2 - 4x)^2$$

$$\frac{dy}{dx} = 2(2x^2 - 4x)^1 \times (4x - 4)$$

$$= 16(x^2 - 2x)(x - 1)\ .$$

We multiply by the power of the term (2); and rewrite the term with its power reduced by 1.

Then we multiply by the derivative of the term.

$(d)\ \ (3x^2 - 5x)^{-2/3}$

$$\text{Let } y = (3x^2 - 5x)^{-2/3}$$

$$\frac{dy}{dx} = -\tfrac{2}{3}(3x^2 - 5x)^{-5/3} \times (6x - 5)$$

$$= -\tfrac{2}{3}(6x - 5)(3x^2 - 5x)^{-5/3}$$

$$= -\frac{2(6x - 5)}{3(3x^2 - 5x)^{5/3}}$$

We multiply by the power of the term $\left(-\frac{2}{3}\right)$; and rewrite the term with its power reduced by 1.

Then we multiply by its derivative.

One may bring the negative power into the denominator or simply leave it as above.

$(e)\ \ \sqrt{\left(x^2 - \frac{1}{x^2}\right)}$

$$\text{Let } y = (x^2 - x^{-2})^{1/2}$$

$$\frac{dy}{dx} = \tfrac{1}{2}(x^2 - x^{-2})^{-1/2}\,(2x + 2x^{-3})$$

$$= \tfrac{1}{2}(x^2 - x^{-2})^{-1/2} \times 2(x + x^{-3})$$

$$= (x^2 - x^{-2})^{-1/2}(x + x^{-3})$$

$$= \frac{\left(x + \frac{1}{x^3}\right)}{\sqrt{\left(x^2 - \frac{1}{x^2}\right)}}\ .$$

We get rid of the square root by writing it as "to power $\frac{1}{2}$".

We multiply by the power of the term $\left(\frac{1}{2}\right)$; and rewrite the term with its power reduced by 1.

Then we multiply by its derivative.

Exercise 4A:

1. Differentiate:

(a) $(x + 1)^2$

(b) $(2x + 3)^2$

(c) $(3x+2)^3$

(d) $2(4x-1)^3$

(e) $\frac{1}{8}(5-8x)^4$

(f) $\frac{3}{16}(4x+3)^4$

(g) $(5x+2)^{-2}$

(h) $-\frac{1}{6}(2x+3)^{-3}$

(i) $(2x+1)^{\frac{1}{2}}$

(j) $8(3x+5)^{\frac{1}{4}}$

(k) $(6x+5)^{\frac{3}{2}}$

(l) $(3-8x)^{-\frac{3}{2}}$

2. Integrate:

(a) $3(x+1)^2$

(b) $8(2x+3)^3$

(c) $6(4-3x)^{-3}$

(d) $(2x+3)^{\frac{1}{2}}$

(e) $(5-4x)^{-\frac{1}{2}}$

(f) $(3x+4)^{-\frac{3}{2}}$

3. Differentiate:

(a) $\frac{1}{(2x+3)}$

(b) $\frac{1}{(3x+4)^2}$

(c) $\frac{1}{\sqrt{(4x-1)}}$

(d) $\frac{1}{\sqrt[3]{(6x+1)}}$

(e) $\frac{1}{\sqrt[3]{(12x+5)^2}}$

(f) $\frac{1}{\sqrt[4]{(3x-4)^3}}$

4. Integrate:

(a) $\frac{1}{(x+2)^2}$

(b) $\frac{1}{(2x+1)^3}$

(c) $\frac{1}{\sqrt{(2x+3)}}$

(d) $\frac{1}{\sqrt[3]{(3x+4)}}$

(e) $\frac{1}{\sqrt{(4-6x)^3}}$

(f) $\frac{1}{\sqrt[3]{(12x+1)^2}}$

5. Differentiate:

(a) $(2x^2+1)^2$

(b) $(3x^2+2)^3$

(c) $(x^3+4)^2$

(d) $(2x^4+3)^3$

(e) $(3x^2+x)^2$

(f) $(2x^3-3x^2)^3$

(g) $(5x^2-4x^3)^{-1}$

(h) $(6x-x^4)^{-\frac{2}{3}}$

6. Differentiate:

(a) $\frac{1}{(x^2+1)}$

(b) $\frac{1}{\sqrt{(2x^2+3)}}$

(c) $\frac{1}{\sqrt{(x^3+x)}}$

(d) $\frac{1}{\sqrt[3]{(2x^2-3x^3)}}$

(e) $\frac{1}{\sqrt[3]{(2x^3+x^2)^2}}$

(f) $\frac{1}{\sqrt[4]{(x^2+x)}}$

7. Differentiate with respect to x and simplify:

(a) $\sqrt{\left(\frac{x+1}{x+2}\right)}$

(b) $\sqrt{\left\{\frac{(x+2)^3}{x-1}\right\}}$

(c) $\sqrt{\left(\frac{x^2+1}{x^2-1}\right)}$

(d) $\frac{(1-\sqrt{x})^2}{\sqrt{(x^2-1)}}$

4.2. Rates of Change

We can also use the chain rule to investigate related rates of change:

Example 4:

The area of the surface of a sphere is $4\pi r^2$, r being the radius. Find the rate of change of the area in $square\ cm\ per\ second$ when $r = 2cm$, given that the radius increases at a rate of $1\ cm/s$.

$$Given;\quad A = 4\pi r^2 \quad and \quad \frac{dr}{dt} = 1\,cm/s, \quad \frac{dA}{dt} = ?$$

$$\frac{dA}{dr} = 8\pi r$$

$$by\ the\ chain\ rule;\quad \frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

$$\frac{dA}{dt} = 8\pi r \times 1$$

$$= 8\pi r \quad cm^2/s$$

$$at\quad r = 2\ cm;\quad \frac{dA}{dt} = 8\pi \times 2$$

$$= 16\pi \quad cm^2/s\ .$$

In this case, the chain rule lets us relate rates of change.
We can find the rate of change of area, ${}^{dA}/_{dt}$ if we have the rate of change of radius ${}^{dr}/_{dt}$, and ${}^{dA}/_{dr}$ obtained by simply differentiating the formula for Area.

We put $r = 2$ to find the rate at that particular time.

Example 5:

An ink blot on a piece of paper spreads at a rate of ½ cm^2/s. Find the rate of increase of the radius of the circular blot when the radius is ½ cm.

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r \qquad \frac{dA}{dt} = \frac{1}{2}cm^2/s \qquad \frac{dr}{dt} = ?$$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

$$\therefore\quad \frac{dr}{dt} = \frac{dA}{dt}\Big/\frac{dA}{dr}$$

$$\frac{dr}{dt} = {}^{\frac{1}{2}}\!/_{2\pi r} = \frac{1}{4\pi r}$$

$$At\quad r = \frac{1}{2};\quad \frac{dr}{dt} = \frac{1}{4\pi \times \frac{1}{2}}$$

$$= \frac{1}{2\pi} \quad cm/s$$

Rate of ink spread is change in area, ${}^{dA}/_{dt} = \frac{1}{2}\ cm^2/s$ (given above).

We can relate ${}^{dA}/_{dt}$ as shown and obtain ${}^{dr}/_{dt}$ from it by making ${}^{dr}/_{dt}$ the subject. OR

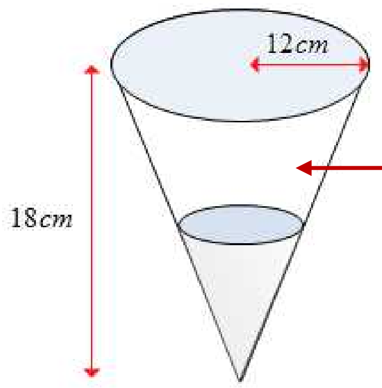
$$\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt}$$

$$and \quad \frac{dr}{dA} = {}^{1}\Big/_{\frac{dA}{dr}}$$

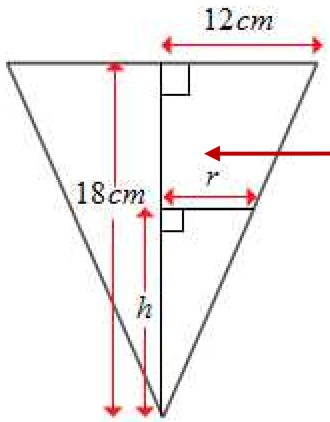
We enter ${}^{dA}/_{dt}$ and ${}^{dA}/_{dr}$. We put $\left(r = \frac{1}{2}\right)$ to find ${}^{dr}/_{dt}$ at that particular time.

Example 6:

A hollow circular cone is held vertex downwards beneath a tap leaking at a rate of $2\ cm^3/s$.
Find the rate of rise of the water level when the depth is $6cm$ given that the height of the cone is $18cm$ and radius $12cm$.



Since we are finding the rate of rise of water level, $^{dh}/_{dt}$, we shall use the dimensions of the cone to eliminate r such that the only variable left is h.



We have extracted a longitudinal section of the cone shown. And we shall use the concept of similar triangles to find an expression for r in terms of h.

$$v = \frac{1}{3}\pi r^2 h \qquad \frac{dv}{dt} = 2\ cm^3/s \qquad \frac{dh}{dt} = ?$$

Since we're interested in $\frac{dh}{dt}$ *we express* r *in terms of* h.

by similar Δs;

$$\frac{18}{h} = \frac{12}{r}$$

$$r = \tfrac{12}{18}h = \tfrac{2}{3}h$$

$$Now\ \ v = \tfrac{1}{3}\pi \times \left(\tfrac{2}{3}h\right)^2 \times h$$

$$v = \tfrac{4}{27}\pi h^3$$

$$hence\ \ \frac{dv}{dh} = \tfrac{4}{9}\pi h^2$$

Rate at which water is leaking is $^{dv}/_{dt} = 2\,cm^3/s$ (given above).

By the concept of similar triangle;

$$\frac{h_1}{h_2} = \frac{r_1}{r_2}$$

where h_1 and r_1 are the height and base of the bigger outer triangle; and h_2 and r_2 for the smaller inner triangle.

We express r in terms of h and eliminate it from the formula for v, leaving one variable h.

By the chain rule; $\frac{dv}{dt} = \frac{dv}{dh} \times \frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{dv}{dt} \Big/ \frac{dv}{dh}$

$\frac{dh}{dt} = 2 \Big/ \tfrac{4}{9}\pi h^2$

$= \frac{9}{2\pi h^2}$

At $h = 6\ cm$; $\frac{dh}{dt} = \frac{9}{2\pi \times 6^2}$

$= \frac{1}{8\pi}\ cm/s$.

We can relate $^{dv}/_{dt}$ as shown and obtain $^{dh}/_{dt}$ from it by making $^{dh}/_{dt}$ the subject. OR

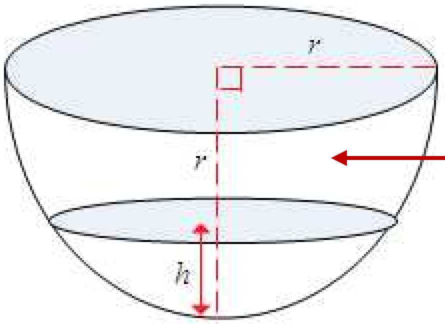
$\frac{dh}{dt} = \frac{dh}{dv} \times \frac{dv}{dt}$

and $\frac{dh}{dv} = 1 \Big/ \frac{dv}{dh}$

We put $h = 6$ to find the rate at that particular time.

Example 7:

A hemispherical bowl is being filled with water at a uniform rate. When the height of the water is $h\ cm$ the volume is $\pi(rh^2 - {}^1/_3 h^3)\ cm^3$, r being the radius of the hemisphere. Find the rate at which the water level is rising when it is half way to the top given that $r = 6\ cm$ and that the bowl fills in a minute.



The height h of the water level is changing; however, the radius of the hemisphere r is constant. So in the formula for v, the only variable to consider is h; r is a constant.

Volume of the hemisphere;

$v = \frac{2}{3}\pi r^3$

$v = \frac{2}{3}\pi \times 6^3$

$v = 144\pi\ \ cm^3$.

The hemisphere takes 60 s to fill;

$\therefore \frac{dv}{dt} = \frac{144\pi}{60}$

$\frac{dv}{dt} = \frac{12}{5}\pi\ \ cm^3/s$.

Given that; $v = \pi\left(rh^2 - \frac{1}{3}h^3\right) = \pi r h^2 - \frac{1}{3}\pi h^3$

Volume of sphere $= \frac{4}{3}\pi r^3$

$\therefore$ *volume of hemisphere (half a sphere)*

$= \frac{1}{2} \times \frac{4}{3}\pi r^3$

$= \frac{2}{3}\pi r^3$

We divide the volume by the time it takes to fill to obtain the rate of change of the volume $^{dv}/_{dt}$.

$$\frac{dv}{dh} = 2\pi rh - \pi h^2$$

By chain rule; $\frac{dv}{dt} = \frac{dv}{dh} \times \frac{dh}{dt}$

$$\therefore \frac{dh}{dt} = \frac{dv}{dt} \Big/ \frac{dv}{dh}$$

$$= \frac{12}{5}\pi \Big/ \pi h(2r - h)$$

$$= \frac{12}{5h(2r-h)}$$

At $h = \frac{r}{2} = 3\ cm;$

$$\frac{dh}{dt} = \frac{12}{5 \times 3(2 \times 6 - 3)}$$

$$= \frac{4}{45}\ cm/s\ .$$

We differentiate with respect to h; treating r as a constant since radius is constant.

From the expression for ${}^{dv}/_{dt}$ we make ${}^{dh}/_{dt}$ the subject and substitute.

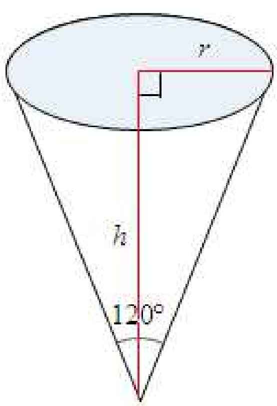
Half-way the top means $\frac{1}{2}$ *of radius* from the bottom. And we substitute that value of h into the expression for ${}^{dh}/_{dt}$, obtained above.

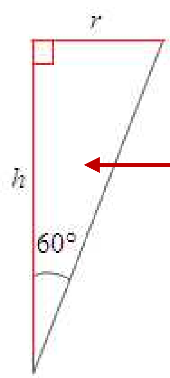
Example 8:

An inverted right circular cone of vertical angle $120°$ is collecting water from a tap at a steady rate of $18\pi\ cm^3/min$.

(a) Find the depth of the water after 12 min.

(b) Find the rate of increase of the depth at this instant.





We have extracted a longitudinal section of the cone shown. Since we are finding h and ${}^{dh}/_{dt}$, we shall use this triangle to find r in terms of h and eliminate r from the expression for volume.

Since we're looking for h and $\frac{dh}{dt}$, we express r in terms of h

$$\tan 60 = \frac{r}{h}$$

$$r = h \tan 60$$

$$r = \sqrt{3}\, h$$

From the right-angled triangle;

$$\tan 60 = \frac{opp}{adj}$$

And we express r in terms of h .

$$Volume, \quad v = \tfrac{1}{3}\pi r^2 h$$

$$putting\ r; \quad v = \tfrac{1}{3}\pi\left(\sqrt{3}h\right)^2 h$$

$$v = \pi h^3$$

$$hence \quad \frac{dv}{dh} = 3\pi h^2$$

We substitute and eliminate r from the formula for v; such that the only variable left is h.
And we differentiate to find ${}^{dh}/_{dt}$.

$$Also\ given; \quad \frac{dv}{dt} = 18\pi\ cm^3/min$$

(a) depth after t = 12 min;

$$v = \frac{dv}{dt} \times t$$

$$= 18\pi \times 12$$

$$= 216\pi\ \ cm^3$$

Volume collected is given by:

$$rate\ water\ of\ flow\ \left(\frac{dv}{dt}\right) \times time$$

$$but\ also; \quad v = \pi h^3$$

$$\therefore \quad 216\pi = \pi h^3$$

$$h = \sqrt[3]{216}$$

$$h = 6\ \ cm\ .$$

So we quote the formula for v derived above and substitute for $(v = 216\pi)$ to get the depth of the water, h.

(b) rate of increase of depth;

$$\frac{dh}{dt} = \frac{dv}{dt}\Big/\frac{dv}{dh}$$

$$= \frac{18\pi}{3\pi h^2}$$

$$= \frac{6}{h^2}$$

$$at\ \ h = 6\ cm; \quad \frac{dh}{dt} = \frac{6}{6^2}$$

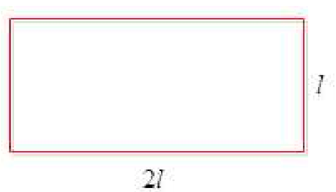
$$= \tfrac{1}{6}\ \ cm/min\ .$$

$$From \quad \frac{dv}{dt} = \frac{dv}{dh} \times \frac{dh}{dt}$$

We use the chain rule to derive ${}^{dh}/_{dt}$ and substitute $(h = 6)$ to find the rate of increase at that particular time.

Example 9:

A rectangle is twice as long as it is broad. Find the rate of change of the perimeter when the breadth of the rectangle is 1 m and its area is changing at a rate of 18 cm^2/s, assuming the expansion is uniform.



Given that length is twice the breadth.

Since we have $\frac{dA}{dt}$ *and we're looking for* $\frac{dP}{dt}$, *we need* $\frac{dA}{dp}$; *so we express A in terms of P*

Let the breadth be l.

$$A = 2l^2$$

$$P = 6l \quad \text{and} \quad l = \frac{p}{6}$$

$$\therefore A = 2\left(\frac{p}{6}\right)^2$$

$$A = \frac{p^2}{18}$$

$$\text{hence} \quad \frac{dA}{dp} = \frac{1}{9}p$$

$$\text{Given;} \quad \frac{dA}{dt} = 18 \quad cm^2/s$$

$$\text{by chain rule;} \quad \frac{dP}{dt} = \frac{dA}{dt} \Big/ \frac{dA}{dp}$$

$$= 18 \Big/ \frac{1}{9}p$$

$$= \frac{162}{p}.$$

$$\text{but when} \quad l = 1m = 100\ cm; \quad P = 6 \times 100$$

$$= 600\ cm$$

$$\text{hence} \quad \frac{dP}{dt} = \frac{162}{600}$$

$$\frac{dP}{dt} = 0.27 \quad cm/s.$$

We shall need an expression relating area A with perimeter P.

$Area = length \times width = 2l^2$
$Perimeter = 2(length + width) = 6l$
We also make l the subject.

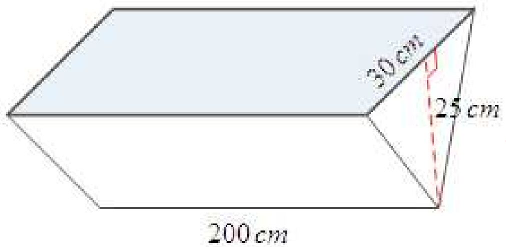
We substitute l and eliminated it from the expression for area.
We thus obtain area, A, in terms of P and find $\frac{dA}{dp}$.

We use the chain rule to get $\frac{dp}{dt}$.
Also $\frac{dA}{dt}$ was given as $18\ cm^2/s$ and $\frac{dA}{dp}$ is derived from differentiation above.

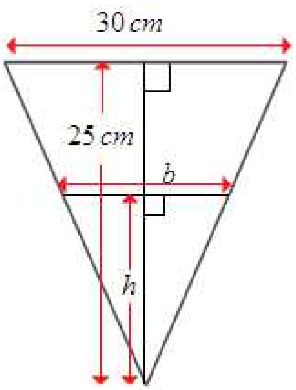
We use the length of $(1\ m)$ given to find the value of P at that time using $(P = 6l)$.
Then substitute it in $\frac{dP}{dt}$.

Example 10:

A horse-trough has a triangular cross-section of height $25cm$ and base $30cm$ and is $2\ m$ long. A horse is drinking steadily and when the water level is $5cm$ below the top it is being lowered at the rate of $1cm/min$. Find the rate of consumption of the water in litres per minute.



We need to find a relationship between the base and height and eliminate one of these variables.
Now since we are given $\frac{dh}{dt}$, we eliminate the base b.



We take a cross-section of the trough, and apply the concept of similar triangles:

$$\frac{b_1}{b_2} = \frac{h_1}{h_2}$$

where b_1 and h_1 are the base and height of the bigger triangle, and b_2 and h_2 are of the smaller triangle respectively.

$$v = cross - sectional\ area \times length$$
$$v = \tfrac{1}{2}bh \times 200$$
$$v = 100bh$$

The cross-section is the area of a triangle $\left(\frac{1}{2}bh\right)$. So we find volume in terms of base and height; and then aim at eliminating one of the variables.

Since we have $\frac{dh}{dt}$ *and we're looking for* $\frac{dv}{dt}$,

we need $\frac{dv}{dh}$ *; so we express* b *in terms of* h.

by similar triangles;

$$\frac{30}{b} = \frac{25}{h}$$
$$b = \frac{30h}{25}$$
$$b = \frac{6}{5}h$$

We shall need to eliminate b from the expression for v above. So we use the concept of similar triangles to express b in terms of h.

$$\therefore \quad v = 100h \times \frac{6}{5}h$$
$$v = 120h^2$$
$$\text{hence} \quad \frac{dv}{dh} = 240\,h$$

We substitute for b in the formula for v, thus remaining with one variable h. We also find ${dv}/{dh}$.

Aslo given; $\frac{dh}{dt} = 1\ cm/min$

$$\text{by chain rule;} \quad \frac{dv}{dt} = \frac{dv}{dh} \times \frac{dh}{dt}$$
$$\frac{dv}{dt} = 240h \times 1$$
$$= 240h\,.$$

Rate of lowering water ${dh}/{dt} = 1\ cm\ per\ min.$ We also have ${dv}/{dh}$ from above. So we use the chain rule to find ${dv}/{dt}$.

When water level is 5 cm below top; $h = 25 - 5 = 20\ cm$;

$$\frac{dv}{dt} = 240 \times 20$$
$$= 4800\ cm^3/min$$
$$= 4.8\,l/min$$

Measured from the bottom, the height of the water is 20cm, so we substitute it into ${dv}/{dt}$.

Exercise 4B:

1. Some oil is spilt onto a level and spreads out in the shape of a circle. The radius of the circle is increasing at the rate of $0.5\ cms^{-1}$. At what rate is the area of the circle increasing when the radius is $5\ cm$?
2. The area of the surface of a sphere is given by $4\pi r^2$ and the volume by $\frac{4}{3}\pi r^3$, r being the radius. Find the rate of change of (a) the area (b) the volume when $r = 10cm$, given that the radius increases at a rate of $\frac{1}{4}\ cm/s$.
3. A drop of water on spilling spreads on a piece of paper forming a circular blot at a rate of $\frac{1}{8}\ cm^2/s$. Find the rate of increase of the radius of the circular blot when the radius is $\frac{1}{4}\ cm$.
4. A spherical balloon is being inflated by blowing in $2 \times 10^3\ cm^3$ of air per second. At what rate is its radius increasing when its diameter is $20\ cm$?
5. A closed cylinder is of fixed length $10cm$ but its radius is increasing at a rate of $0.5\ cm\ s^{-1}$. Find the rate of increase of its total surface area when the radius is $4\ cm$. (Leave π in your answer).
6. A hollow circular cone of radius $15\ cm$ and height $25\ cm$ is held vertex downwards. Liquid is poured into the cone at the rate of $50\ cm^3s^{-1}$. How fast is the level of the liquid rising when the radius of its surface is $10\ cm$?
7. Water is emptied from a cylindrical tank of radius $20\ cm$ at the rate of $2.5\ litres\ s^{-1}$ and fresh water is added at the rate of $2\ litres\ s^{-1}$. At what rate is the water level in the tank changing?
8. An inverted right circular cone of vertical angle $90°$ is collecting water from a tap at a steady rate of $72\pi\ cm^3/min$.
   (a) Find the depth of the water after 8 min.
   (b) The rate of increase of the depth at this instant.
9. A hollow circular cone of radius $4\ cm$ and height $20\ cm$ is held down with its axis vertical. Liquid is added at the constant rate of $20\ cm^3s^{-1}$ but leaks away through a small hole in the vertex at the constant rate of $15cm^3s^{-1}$. At what rate is the depth of the liquid in the cone changing when it is $5\ cm$?
10. A rectangle is four times as long as it is broad. Find the rate of change of the perimeter when the length of the rectangle is $1\ m$ and its area is changing at a rate of $10\ cm/s$, assuming the expansion is uniform.

4.3. Products and Quotients

Consider y a product of two functions u and v of a variable x.

$$Then;\ \ y = uv.$$

If a small increament δx in x produces increments δu in u, δv in v and δy in y:

$$y + \delta y = (u + \delta u)(v + \delta v)$$

$$y + \delta y = uv + v\delta u + u\delta v + \delta u\delta v$$

$putting\ uv = y;$

$$y + \delta y = y + v\delta u + u\delta v + \delta u\delta v$$

$$\delta y = \quad v\delta u + u\delta v + \delta u\delta v$$

$dividing\ by\ x\delta;$

$$\frac{\delta y}{\delta x} = \quad v\frac{\delta u}{\delta x} + u\frac{\delta v}{\delta x} + \frac{\delta u}{\delta x} \times \delta v$$

$as\ \delta x \to 0,\ \ \delta u,\ \ \delta v\ and\ \delta y\ \ also\ tend\ to\ 0;$

$$\frac{\delta y}{\delta x} \to \frac{dy}{dx}, \quad \frac{\delta u}{\delta x} \to \frac{du}{dx}, \quad \frac{\delta v}{\delta x} \to \frac{dv}{dx}$$

$$\therefore \frac{dy}{dx} = \quad v\frac{du}{dx} + u\frac{dv}{dx} + \frac{du}{dx} \times 0$$

The product rule is used in calculus when required to take the derivative of a function that is the multiplication of a couple or several smaller functions.

Similarly, the quotient rule is used for differentiating problems where one function is divided by another.

$$\boldsymbol{\frac{dy}{dx} = \quad v\frac{du}{dx} + u\frac{dv}{dx}}$$

is the product rule.

In other words: to differentiate the product of two functions, differentiate the first function leaving the second one alone, and then differentiate the second leaving the first one alone.

$Similarly\!:\ If\ y = {}^{u}/_{v}\ ,then;$

$$\boldsymbol{\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

is the quotient rule.

Example 11:

Differentiate:

$(a)\ \ (x+1)^2(x^2-1)$

$let\ u = (x+1)^2 \quad and \quad v = (x^2 - 1)$

$$\frac{du}{dx} = 2(x+1) \qquad \frac{dv}{dx} = 2x$$

$$\boldsymbol{\frac{dy}{dx} = \quad v\frac{du}{dx} + u\frac{dv}{dx}}$$

$$\frac{dy}{dx} = \{(x^2-1) \times 2(x+1)\} + \{(x+1)^2 \times 2x\}$$

We let one function be u and the other v and we find ${}^{du}/_{dx}$ and ${}^{dv}/_{dx}$ respectively.

We then substitute into the product rule.

$= 2(x+1)[(x^2-1) + x(x+1)]$

$= 2(x+1)[2x^2 + x - 1]$

$= 2(x+1)(x+1)(2x-1)$

$= 2(x+1)^2(2x-1).$

We have factorized out 2 and the bracket $(x+1)$, so we simplify the remaining terms.

We then factorize the term in the square bracket.

$(b)\ (x-1)\sqrt{(x^2+1)}$

Let $u = x - 1$ and $v = (x^2+1)^{1/2}$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \frac{1}{2}(x^2+1)^{-1/2} \times 2x$

$= x(x^2+1)^{-1/2}$

$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$\frac{dy}{dx} = \{(x^2+1)^{1/2} \times 1\} + \{(x-1) \times x(x^2+1)^{-1/2}\}$

$= (x^2+1)^{1/2} + x(x-1)(x^2+1)^{-1/2}$

$= (x^2+1)^{-1/2}[(x^2+1)^1 + x(x-1)]$

$= (x^2+1)^{-1/2}[x^2+1+x^2-x]$

$= \frac{2x^2-x+1}{\sqrt{(x^2+1)}}.$

We write the root as index $\left(\frac{1}{2}\right)$. Then we've used the chain rule to find $dv/_{dx}$.

We then substitute into the product rule.

The bracket (x^2+1) is common; so we factorize out the index that is least, i.e. $(x^2+1)^{-1/2}$

Note that on factorization, the remaining power here is 1.

We've written $(x^2+1)^{-1/2}$ as $\frac{1}{(x^2+1)^{1/2}} = \frac{1}{\sqrt{(x^2+1)}}$

Example 12:

Differentiate:

$(a)\ ^{x}/_{x+1}$

let $y = x(x+1)^{-1}$

let $u = x$ and $v = (x+1)^{-1}$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = -1(x+1)^{-2} \times 1$

$= -(x+1)^{-2}$

Pulling the term $(x+1)$ out of the denominator makes it powered to -1.

We've applied the chain rule here to find $dv/_{dx}$.

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = \{(x+1)^{-1} \times 1\} + \{x \times -(x+1)^{-2}\}$$

$$= (x+1)^{-1} - x(x+1)^{-2}$$

$$= (x+1)^{-2}[(x+1)^1 - x]$$

$$= (x+1)^{-2}[1]$$

$$= {}^{1}/_{(x+1)^2}.$$

We then substitute into the product rule.

The bracket $(x+1)$ is common; so we factorize out the index that is least, i.e. $(x+1)^{-2}$.
We then simplify.

Note that on factorization, the remaining power here is 1.

(b) $^{(1-x^2)}/_{(1+x^2)}$

Let $y = (1-x^2)(1+x^2)^{-1}$

Let $u = (1-x^2)$ and $v = (1+x^2)^{-1}$

$$\frac{du}{dx} = -2x \qquad \frac{dv}{dx} = -(1+x^2)^{-2} \times 2x$$

$$= -2x(1+x^2)^{-2}$$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = \{(1+x^2)^{-1} \times -2x\} + \{(1-x^2) \times -2x(1+x^2)^{-2}\}$$

$$= -2x(1+x^2)^{-1} - 2x(1-x^2)(1+x^2)^{-2}$$

$$= -2x(1+x^2)^{-2}[(1+x^2)^1 + (1-x^2)]$$

$$= -4x(1+x^2)^{-2}$$

$$= -\frac{4x}{(1+x^2)^2}.$$

Pulling the term $(1+x^2)$ out of the denominator makes it powered to -1.

We have applied the chain rule to find $^{dv}/_{dx}$.

We then substitute into the product rule and simplify.

We have factorized out the common terms $(-2x)$ and the bracket $(1+x^2)$. The bracket with the least index is the one we remove, i.e. $(1+x^2)^{-2}$.

OR using the quotient rule:

$$y = {}^{(1-x^2)}/_{(1+x^2)}$$

$u = (1-x^2)$ and $v = (1+x^2)$

$$\frac{du}{dx} = -2x \qquad \frac{dv}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{\{(1+x^2) \times -2x\} - \{(1-x^2) \times 2x\}}{(1+x^2)^2}$$

$$= \frac{-2x[1+x^2+1-x^2]}{(1+x^2)^2}$$

Alternatively, we may differentiate this as a quotient:
As previously, we simply find $^{du}/_{dx}$ and $^{dv}/_{dx}$ and apply the Quotient rule:

We then substitute into the quotient rule and simplify.
Note: $(-2x)$ can be factorized out.

$$= \frac{-4x}{(1+x^2)^2}.$$

Note that we still obtain the same answer, as with the product rule.

Example 13:

Differentiate:

(a) $\sqrt{\left(\frac{x+1}{x+2}\right)}$

Let $y = \dfrac{\sqrt{(x+1)}}{\sqrt{(x+2)}}$

Let $u = (x+1)^{1/2}$ $\quad v = (x+2)^{1/2}$; then $y = \dfrac{u}{v}$

$\dfrac{du}{dx} = \dfrac{1}{2}(x+1)^{-1/2}$ $\quad \dfrac{dv}{dx} = \dfrac{1}{2}(x+2)^{-1/2}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{\{(x+2)^{1/2} \times \frac{1}{2}(x+1)^{-1/2}\} - \{(x+1)^{1/2} \times \frac{1}{2}(x+2)^{-1/2}\}}{\{(x+2)^{1/2}\}^2}$$

$$= \frac{\frac{1}{2}(x+1)^{-1/2}(x+2)^{-1/2}[(x+2)^1 - (x+1)^1]}{(x+2)}$$

$$= \frac{\frac{1}{2}(x+1)^{-1/2}(x+2)^{-1/2} \times 1}{(x+2)}$$

$$= \frac{1}{2}(x+1)^{-1/2}(x+2)^{-\frac{1}{2}-1}$$

$$= \frac{1}{2}(x+1)^{-1/2}(x+2)^{-3/2}$$

$$= \frac{1}{2\sqrt{(x+1)(x+2)^3}}.$$

From the start, our plan is to use the quotient rule. The roots are written as index $\frac{1}{2}$.

We apply the chain rule to find $^{du}/_{dx}$ and $^{dv}/_{dx}$.

We then substitute into the quotient rule.

The brackets of $(x+1)$ and $(x+2)$ are common; so we factor out the brackets with the least indices, i.e. $(x+1)^{-1/2}$ and $(x+2)^{-1/2}$.

The can subtract the indices for the $(x+2)$ brackets.

Note: $(x+2)^{-3/2} = \dfrac{1}{[(x+2)^{1/2}]^3}$
$= \dfrac{1}{\sqrt{(x+2)^3}}$

(b) $\sqrt{\left\{\frac{(x+2)^3}{x-1}\right\}}$

Let $y = \dfrac{(x+2)^{3/2}}{(x-1)^{1/2}}$

Let $u = (x+2)^{3/2}$ and $v = (x-1)^{1/2}$; then $y = {}^{u}/_{v}$

$\dfrac{du}{dx} = \dfrac{3}{2}(x+2)^{1/2}$ $\quad \dfrac{dv}{dx} = \dfrac{1}{2}(x-1)^{-1/2}$

We plan to use the quotient rule. The square root is written as index $\frac{1}{2}$.

We apply the chain rule to find $^{du}/_{dx}$ and $^{dv}/_{dx}$.

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x-1)^{1/2} \times \frac{3}{2}(x+2)^{1/2} - (x+2)^{3/2} \times \frac{1}{2}(x-1)^{-1/2}}{\{(x-1)^{1/2}\}^2}$$

$$= \frac{\frac{1}{2}(x-1)^{-1/2}(x+2)^{1/2}[3(x-1)^1 - (x+2)^1]}{(x-1)}$$

$$= \frac{\frac{1}{2}(x-1)^{-1/2}(x+2)^{1/2}(2x-5)}{(x-1)}$$

$$= \frac{(2x-5)(x+2)^{1/2}}{2(x-1)^{1/2}(x-1)^1}$$

$$= \frac{(2x-5)\sqrt{(x+2)}}{2\sqrt{(x-1)^3}}.$$

We then substitute into the quotient rule.

The brackets of $(x-1)$ and $(x+2)$ are common; so we factor out the brackets with the least indices, i.e. $(x-1)^{-1/2}$ and $(x+2)^{1/2}$.

$(x-1)^{-1/2}$ can be pulled into the denominator and $(x-1)^{1/2}(x-1)^1 = (x-1)^{3/2} = \sqrt{(x-1)^3}$.

Example 14:

Differentiate:

$$(a)\ \sqrt{\frac{x^2+1}{x^2-1}}$$

We plan to use the quotient rule. The square root can be written as index $\frac{1}{2}$.

Let $u = (x^2+1)^{1/2}$ and $v = (x^2-1)^{1/2}$; $y = {}^{u}/_{v}$

$$\frac{du}{dx} = \frac{1}{2}(x^2+1)^{-1/2} \times 2x \qquad \frac{dv}{dx} = \frac{1}{2}(x^2-1)^{-1/2} \times 2x$$

$$= x(x^2+1)^{-1/2} \qquad\qquad = x(x^2-1)^{-1/2}$$

We apply the chain rule to find ${}^{du}/_{dx}$ and ${}^{dv}/_{dx}$.

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{\{(x^2-1)^{1/2} \times x(x^2+1)^{-1/2}\} - \{(x^2+1)^{1/2} \times x(x^2-1)^{-1/2}\}}{\{(x^2-1)^{1/2}\}^2}$$

$$= \frac{x(x^2-1)^{-1/2}(x^2+1)^{-1/2}[(x^2-1)^1 - (x^2+1)^1]}{(x^2-1)}$$

$$= \frac{x(x^2-1)^{-1/2}(x^2+1)^{-1/2}[-2]}{(x^2-1)}$$

$$= \frac{-2x}{(x^2-1)^1(x^2-1)^{1/2}(x^2+1)^{1/2}}$$

$$= \frac{-2x}{(x^2-1)^{3/2}(x^2+1)^{1/2}}$$

$$= \frac{-2x}{\sqrt{(x^2-1)^3(x^2+1)}}.$$

We substitute into the quotient rule and simplify.

We factorize out x, $(x^2-1)^{-1/2}$ and $(x^2+1)^{-1/2}$ and simplify.

The negative powers have been brought down into the denominator.
The indices of (x^2-1) have been added.

(b) $\dfrac{(1-\sqrt{x})^2}{\sqrt{(x^2-1)}}$

Let $u = (1-x^{1/2})^2$ and $v = (x^2-1)^{1/2}$

$$\frac{du}{dx} = 2(1-x^{1/2}) \times -\frac{1}{2}x^{-1/2} \qquad \frac{dv}{dx} = \frac{1}{2}(x^2-1)^{-1/2} \times 2x$$

$$= -x^{-1/2}(1-x^{1/2}) \qquad = x(x^2-1)^{-1/2}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{\{(x^2-1)^{1/2} \times -x^{-1/2}(1-x^{1/2})\} - \{(1-x^{1/2})^2 \times x(x^2-1)^{-1/2}\}}{\{(x^2-1)^{1/2}\}^2}$$

$$= \frac{(x^2-1)^{-1/2}(1-x^{1/2}) \times x^{-1/2}[-(x^2-1)^1 - x^{3/2}(1-x^{1/2})]}{(x^2-1)}$$

$$= \frac{x^{-1/2}(x^2-1)^{-1/2}(1-x^{1/2})[-x^2+1-x^{3/2}+x^2]}{(x^2-1)}$$

$$= \frac{x^{-1/2}(x^2-1)^{-1/2}(1-x^{1/2})[1-x^{3/2}]}{(x^2-1)}$$

$$= \frac{(1-x^{1/2})(1-x^{3/2})}{x^{1/2}(x^2-1)^{1/2}(x^2-1)^1}$$

$$= \frac{(1-\sqrt{x})(1-\sqrt{x^3})}{\sqrt{x(x^2-1)^3}}.$$

We plan to use the quotient rule. The square roots can be written as index $\frac{1}{2}$.

We apply the chain rule to find $du/_{dx}$ and $dv/_{dx}$.

Substitute into the quotient rule and simplify.

We factorize out the common terms; $(x^2-1)^{-1/2}$, $(1-x^{1/2})$ and $x^{-1/2}$ and evaluate the remaining bracket.

The negative indices have been pulled into the denominator. And index $\left(\frac{1}{2}\right)$ can be written as square root.

Exercise 4C:

Differentiate with respect to x:

1. $x^2(x+1)$
2. $(x^2+1)(x+2)$
3. $(2x+1)^2(x+3)^2$
4. $(x^2+1)^2(x^2+2)$
5. $(x+1)^3(x^2+2)^2$
6. $(x+2)^{-1}(x+3)$
7. $(x+1)^{-1}(x+2)^{-1}$
8. $(x^2+2x)^{-1}(2-x)^{-2}$
9. $(1+x)(1-x)^{\frac{1}{2}}$
10. $(x-1)^{\frac{1}{2}}(x-2)^{\frac{1}{2}}$
11. $(2-x)^{-\frac{1}{2}}(1-2x)^{-\frac{1}{3}}$
12. $\sqrt{(1+x^2)}\sqrt{(2x^2+1)}$
13. $\frac{x}{x+2}$
14. $\frac{x}{x^2+1}$
15. $\frac{1+x^2}{1-x^2}$
16. $\frac{x}{(1+x)^2}$
17. $\frac{(x^2+1)}{(1-x^2)^2}$
18. $\frac{(1-x)^2}{(x^2+2)^2}$

19. $\frac{(x^2+2)^2}{(x^2-2)^3}$

20. $\frac{x}{\sqrt{(x+1)}}$

21. $\frac{\sqrt{x}}{\sqrt{1-x^2}}$

22. $\sqrt{\frac{1-x}{1+x}}$

23. $\sqrt{\left(\frac{x^2+1}{x^2-1}\right)}$

24. $\sqrt[3]{\left(\frac{x+1}{x-1}\right)}$

25. If $y = \frac{x}{x+1}$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Hence show that $(1+x)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$

26. If $y = \frac{x}{2x-1}$, show that $\frac{dy}{dx} = -\frac{1}{(2x-1)^2}$. Hence or otherwise find $\int_1^5 \frac{dx}{(2x-1)^2}$.

26. Given that $y = \frac{x}{2x+3}$, find $\frac{dy}{dx}$. Hence or otherwise evaluate $\int_1^6 \frac{dx}{(2x+3)^2}$

27. Given that $y = \frac{1+x^2}{1-x^2}$, find $\frac{dy}{dx}$ and hence show that $\int_{-4}^{-5} \frac{x}{(1-x^2)^2}dx = \frac{1}{80}$.

28. (a) Given that $y = \frac{x+a}{x+2}$ and that $\frac{dy}{dx} = -\frac{1}{25}$ when $x = 3$, find the value of a.

(b) Show that $(x+2)^3\frac{d^2y}{dx^2} + (x+2)^2\frac{dy}{dx} + 1 = 0$.

4.4. Implicit functions

An expression of y in terms of x is an explicit function e.g. $y = x^2 + 2x - 5$. However,

$x = y^2 + 2x^2 - 5xy$ is an implicit function because we cannot easily express y in terms of x.

In differentiating implicitly, we shall make use of the chain rule;

$$\frac{d}{dx} = \frac{d}{dy} \times \frac{dy}{dx}$$

Example 15:

Differentiate with respect to x:

(a) y

$$\frac{d}{dx}(y) = \frac{d}{dy}(y) \times \frac{dy}{dx}$$

$$= 1\frac{dy}{dx}$$

$$= \frac{dy}{dx}$$

We apply the chain rule:
Note that we first differentiate y with respect to y as $^{d}/_{dy}(y)$; then simply multiply by $^{dy}/_{dx}$.

(b) y^2

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \times \frac{dy}{dx}$$

$$= 2y\frac{dy}{dx}$$

Applying the chain rule; first we differentiate y^2 with respect to y as $^{d}/_{dy}(y^2)$ to give $2y$; then multiply by $^{dy}/_{dx}$.

(c) xy

using the product rule;

$$\frac{d}{dx}(x_u\ y_v) = y \times 1 + x \times \frac{dy}{dx}$$

$$= y + x\frac{dy}{dx}$$

We apply the product rule here:
Fix y and differentiate x to get 1;
Then fix x and differentiate y to give $^{dy}/_{dx}$ (as obtained in (a) above).

(d) x^2y

$$\frac{d}{dx}({x^2}_u\ y_v)$$

$$= y \times 2x + x^2 \times \frac{dy}{dx}$$

$$= 2xy + x^2\frac{dy}{dx}$$

Fix y and differentiate x^2 to give $2x$;
Then fix x^2 and differentiate y to give $^{dy}/_{dx}$ (as obtained in (a) above).

(e) xy^2

$$\frac{d}{dx}(x_u\ {y^2}_v)$$

$$= y^2 \times 1 + x \times 2y\frac{dy}{dx}$$

$$= y^2 + 2y\frac{dy}{dx}$$

Fix y^2 and differentiate x to give 1;
Then fix x and differentiate y^2 to give $2y\,^{dy}/_{dx}$ (as obtained in (b) above).

Example 16:

Find $\frac{dy}{dx}$ if $x^2 + y^2 - 6xy + 3x - 2y + 5 = 0.$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) - \frac{d}{dx}(6x_u\ y_v) + \frac{d}{dx}(3x) - \frac{d}{dx}(2y) + \frac{d}{dx}(5) = \frac{d}{dx}(0)$$

$$2x + 2y\frac{dy}{dx} - \left(y \times 6 + 6x \times \frac{dy}{dx}\right) + 3 - 2\frac{dy}{dx} + 0 = 0$$

$$2x + 2y\frac{dy}{dx} - 6y - 6x\frac{dy}{dx} + 3 - 2\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} - 6x\frac{dy}{dx} - 2\frac{dy}{dx} + 2x - 6y + 3 = 0$$

$$\frac{dy}{dx}(2y - 6x - 2) + 2x - 6y + 3 = 0$$

We differentiate the individual terms. For $6xy$ we apply the product rule:

Fix y and differentiate $6x$ to get 6;
Then fix $6x$ and differentiate y to give $^{dy}/_{dx}$.

We then collect the $^{dy}/_{dx}$ terms together and pull out $^{dy}/_{dx}$.

$$\frac{dy}{dx}(2y - 6x - 2) = -(2x - 6y + 3)$$

$$\therefore \quad \frac{dy}{dx} = -\frac{(2x - 6y + 3)}{(2y - 6x - 2)}$$

$$\text{Hence} \quad \frac{dy}{dx} = \frac{2x - 6y + 3}{6x - 2y + 2}.$$

We take the independent terms on the right and then divide to make ${}^{dy}/_{dx}$ the subject.

We've preferably incorporated the negative into the denominator.

Example 17:

Find $\frac{dy}{dx}$ *in terms* x *and* y *when* $x^2 + y^2 - 2xy + 3y - 2x = 7$.

$$x^2 + y^2 - 2xy + 3y - 2x = 7$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) - \frac{d}{dx}(2xy) + \frac{d}{dx}(3y) - \frac{d}{dx}(2x) = \frac{d}{dx}(7)$$

$$2x + 2y\frac{dy}{dx} - \left(y \times 2 + 2x \times \frac{dy}{dx}\right) + 3\frac{dy}{dx} - 2 = 0$$

$$2x + 2y\frac{dy}{dx} - 2y - 2x\frac{dy}{dx} + 3\frac{dy}{dx} - 2 = 0$$

$$(2y - 2x + 3)\frac{dy}{dx} + 2x - 2y - 2 = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{-(2x - 2y - 2)}{(2y - 2x + 3)}$$

$$= \frac{2(x - y - 1)}{2x - 2y - 3}.$$

Differentiate the individual terms. For $2xy$ we apply the product rule as shown below:

Fix y and differentiate $2x$ to get 2;
Then fix $2x$ and differentiate y to give ${}^{dy}/_{dx}$.

We then group the ${}^{dy}/_{dx}$ terms and the independent terms, and divide to make $\frac{dy}{dx}$ the subject.

The negative sign in front has been incorporated into the denominator.

Example 18:

Find ${}^{dy}/_{dx}$ *in terms of* x, y *when* $3(x - y)^2 = 2xy + 1$.

$$3(x - y)^2 = 2xy + 1$$

$$3(x^2 - 2xy + y^2) = 2xy + 1$$

$$3x^2 - 6xy + 3y^2 = 2xy + 1$$

$$3x^2 - 8xy + 3y^2 = 1$$

We expand the squared bracket. Then we collect the like terms of $6xy$ and $2xy$.

$$\frac{d}{dx}(3x^2) - \frac{d}{dx}(8xy) + \frac{d}{dx}(3y^2) = \frac{d}{dx}(1)$$

$$6x - \left(y \times 8 + 8x \times \frac{dy}{dx}\right) + 6y\frac{dy}{dx} = 0$$

$$6x - 8y + (6y - 8x)\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{8y - 6x}{6y - 8x}$$

We differentiate the individual terms. For $8xy$ we apply the product rule:

Fix y and differentiate $8x$ to get 8; Then fix $8x$ and differentiate y to give ${}^{dy}/_{dx}$.

We then group the ${}^{dy}/_{dx}$ terms and the independent terms, pull out ${}^{dy}/_{dx}$ and then divide.

Example 19:

Find the gradient of the curve $x^2 - 3xy + 2y^2 - 2x = 4$ *at the point* $(1, -1)$.

$$x^2 - 3xy + 2y^2 - 2x = 4$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(3xy) + \frac{d}{dx}(2y^2) - \frac{d}{dx}(2x) = \frac{d}{dx}(4)$$

$$2x - \left(y \times 3 + 3x \times \frac{dy}{dx}\right) + 4y\frac{dy}{dx} - 2 = 0$$

$$2x - 3y - 3x\frac{dy}{dx} + 4y\frac{dy}{dx} - 2 = 0$$

$$\frac{dy}{dx}(4y - 3x) = 3y - 2x + 2$$

$$\frac{dy}{dx} = \frac{3y - 2x + 2}{4y - 3x}$$

$At\ (1, -1); \quad y = -1, \ x = 1$

$$\therefore \frac{dy}{dx} = \frac{(3 \times -1) - (2 \times 1) + 2}{(4 \times -1) - (3 \times 1)}$$

$$= \frac{-3 - 2 + 2}{-4 - 3}$$

$$= \frac{3}{7}.$$

Differentiate the individual terms. For $3xy$ we apply the product rule:

Fix y and differentiate $8x$ to get 8; Then fix $8x$ and differentiate y to give ${}^{dy}/_{dx}$.

We group the ${}^{dy}/_{dx}$ terms and the independent terms, pull out ${}^{dy}/_{dx}$ and then divide.

We substitute x and y into the expression for ${}^{dy}/_{dx}$ and evaluate,

Example 20:

Find the gradient of the ellipse $2x^2 + 3y^2 = 14$ *at the points where* $x = 1$.

$$\frac{d}{dx}(2x^2) + \frac{d}{dx}(3y^2) = \frac{d}{dx}(14)$$

$$4x + 6y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{3y}$$

$$At\ x = 1; \quad 2(1^2) + 3y^2 = 14$$

$$2 + 3y^2 = 14$$

$$y^2 = 4$$

$$y = \pm 2$$

$$points\ (1,\ 2)\ and\ (1, -2)$$

$$At\ (1,\ 2); \quad \frac{dy}{dx} = \frac{-2 \times 1}{3 \times 2}$$

$$= -\frac{1}{3}$$

$$At\ (1, -2); \quad \frac{dy}{dx} = \frac{-2 \times 1}{3 \times -2}$$

$$= \frac{1}{3}.$$

$$Hence\ at\ (1,2),\ \frac{dy}{dx} = -\frac{1}{3}\ and\ at\ (1,-2),\ \frac{dy}{dx} = \frac{1}{3}.$$

We differentiate the individual terms with respect to x.

We then make ${}^{dy}/_{dx}$ the subject of the expression.

We put $x = 1$ into the equation of the curve to obtain the y-coordinates.

We substitute x and y at the respective points into the expression for ${}^{dy}/_{dx}$.

Example 21:

Find the x –coordinates of the stationary points of the curve represented by the equation

$x^3 - y^3 - 4x^2 + 3y = 11x + 4$.

Finding ${}^{dy}/_{dx}$ **:**

$$\frac{d}{dx}(x^3) - \frac{d}{dx}(y^3) - \frac{d}{dx}(4x^2) + \frac{d}{dx}(3y) = \frac{d}{dx}(11x) + \frac{d}{dx}(4)$$

$$3x^2 - 3y^2\frac{dy}{dx} - 8x + 3\frac{dy}{dx} = 11 + 0$$

$$3\frac{dy}{dx} - 3y^2\frac{dy}{dx} + 3x^2 - 8x = 11$$

$$(3 - 3y^2)\frac{dy}{dx} = 11 + 8x - 3x^2$$

$$\frac{dy}{dx} = \frac{11 + 8x - 3x^2}{3 - 3y^2}$$

We differentiate the individual terms with respect to x.

We group the ${}^{dy}/_{dx}$ terms and the independent terms, pull out ${}^{dy}/_{dx}$ and then divide.

$$\textit{At the stationary points, } {}^{dy}/_{dx} = 0$$

$$\therefore \quad \frac{11 + 8x - 3x^2}{3 - 3y^2} = 0$$

$$-3x^2 + 8x + 11 = 0$$

$$3x^2 - 8x - 11 = 0$$

$$(3x - 11)(x + 1) = 0$$

$$x = \frac{11}{3} \quad or \quad x = -1\,.$$

For the fraction to be equal to zero, the numerator must be equal to zero. [Alternatively, one may simply cross-multiply.]

We factorize the quadratic equation and solve.

Example 22:

At what points are the tangents to the circle $x^2 + y^2 - 6y - 8x = 0$ parallel to the $y - axis$.

$$x^2 + y^2 - 6y - 8x = 0$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) - \frac{d}{dx}(6y) - \frac{d}{dx}(8x) = 0$$

$$2x + 2y\frac{dy}{dx} - 6\frac{dy}{dx} - 8 = 0$$

$$\frac{dy}{dx}(2y - 6) = 8 - 2x$$

$$\therefore \frac{dy}{dx} = \frac{8 - 2x}{2y - 6}$$

The $y - axis$ *is line* $x = 0$ *with gradient* ∞;

hence the parallel tangents also have ∞ *gradient*.

$$\therefore \frac{8 - 2x}{2y - 6} = \infty$$

$$thus \quad 2y - 6 = 0$$

$$y = 3$$

$$when \ y = 3; \quad x^2 + 3^2 - 6(3) - 8x = 0$$

$$x^2 - 8x - 9 = 0$$

$$(x - 9)(x + 1) = 0$$

$$x = 9 \quad or \quad x = -1\,.$$

$$points \ (9, 3) \ and \ (-1, 3)\,.$$

We differentiate the individual terms with respect to x.

We group the ${}^{dy}/_{dx}$ terms and the independent terms, pull out ${}^{dy}/_{dx}$ and then divide.

For the fraction to be equal to ∞, the denominator must be zero,

$$since; \quad \frac{any\ number}{0} = \infty$$

So we obtain y .

We put $y = 3$ into the equation of the curve to find the x-coordinates.

Exercise 4D:

Differentiate the following with respect to x:

1. $x^2 + y^2 = 8$
2. $x^2 + 2y^2 = 3x$
3. $4x^2 + 3y^3 = 4x + 6y$
4. $2x^3 + 3y^2 - 6x + 8y = 12$
5. $3x^2 + 3y^2 + 4y = 10x + 12$
6. $x^2 + y^2 + xy = 12$
7. $2x^2 + 3y^2 + 4xy + 8 = 0$
8. $x^3 + 2x^2y + 2xy^2 - y^3 = 16$
9. $x^3 - y^3 + 4x^2y - 2xy^2 + 12 = 0$
10. $x^3y + 2x^2y^2 - xy^3 = 24$
11. Find the gradient of the curve $4x^2 + 3y^3 = 4x + 6y$ at the point $(1, \sqrt{2})$.
12. Find the gradient of the curve with equation $x^3 + y^2 = 2(x^2 + y^2)$ at the point with coordinates $(0, 0)$.
13. Find the gradient of the ellipse $2x^2 + 3y^2 = 14$ at the points where $x = 1$.
14. Find the gradient of the ellipse $3x^2 + 3y^2 + 4y = 10x + 12$ at the points where $x = -1$.
15. Find the x —coordinates of the stationary points of the curve represented by the equation $2x^3 + 3y^2 - 6x + 8y = 12$
16. The curve C has equation $5x^2 + 2xy - 3y^2 + 3 = 0$. The point P on the curve C has coordinates $(1, 2)$. Find the gradient of the curve at P.

    Find the equation of the normal to the curve at P in the form $y = ax + b$, where a and b are constants to be stated.
17. A curve is described by the equation $3x^2 + 4y^2 - 2x + 6xy - 5 = 0$. Find an equation of the tangent to the curve at the point $(1, -2)$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.
18. Find the coordinates of the turning points on the curve with equation $y^3 + 3xy^2 - x^3 = 3$.
19. A curve is described by the equation $x^3 + xy - 4x + y^2 - 3 = 0$.

    Find an equation of the normal to the curve at the point $(1, 2)$ giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

4.5. Parameters

At times both x and y are given in terms of another variable termed a **parameter**, for instance

$$x = t^2 + 1, \quad y = t^2 - 1;$$

$$\text{the parameter being} \quad \boldsymbol{t}.$$

First we find ${}^{dx}/_{dt}$ and ${}^{dy}/_{dt}$ independently and then relate them using the Chain rule.

In such cases the gradient is obtained also in terms of this parameter.

$$\text{By the chain rule;} \quad \frac{\boldsymbol{dy}}{\boldsymbol{dx}} = \frac{\boldsymbol{dy}}{\boldsymbol{dt}} \times \frac{\boldsymbol{dt}}{\boldsymbol{dx}}$$

Example 23:

Find $^{dy}/_{dx}$ *in terms of* t, *when;*

(a) $x = t^2 + 1$, $y = t^2 - 1$

$x = t^2 + 1$ $\qquad$ $y = t^2 - 1$

$\frac{dx}{dt} = 2t$ $\qquad$ $\frac{dy}{dt} = 2t$

First we find $^{dx}/_{dt}$ and $^{dy}/_{dt}$ independently.

by the chain rule; $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ *also* $\frac{dt}{dx} = {}^{1}\Big/\frac{dx}{dt}$

$$\frac{dy}{dx} = 2t \times \frac{1}{2t}$$

$$= 1 .$$

Then we relate them using the chain rule, to find $^{dy}/_{dx}$.

(b) $x = at^2$, $y = 2at$

$x = at^2$ $\qquad$ $y = 2at$

$\frac{dx}{dt} = 2at$ $\qquad$ $\frac{dy}{dt} = 2a$

First we find $^{dx}/_{dt}$ and $^{dy}/_{dt}$ independently.

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 2a \times {}^{1}/_{2at}$$

$$= \frac{1}{t}$$

Then we relate them using the chain rule, to find $^{dy}/_{dx}$.

Example 24:

If $x = {}^{t}/_{1-t}$ *and* $y = {}^{t^2}/_{1-t}$ *find* $\frac{dy}{dx}$ *in terms of* t.

$x = \frac{t}{(1-t)}$ $\qquad$ $y = {}^{t^2}/_{1-t}$

By quotient rule; $\frac{dx}{dt} = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$

Since x and y are expressed as quotients, we shall use the quotient rule to find $^{dx}/_{dt}$ and $^{dy}/_{dt}$.

$$\frac{dx}{dt} = \frac{[(1-t)\times 1]-[t\times -1]}{(1-t)^2} \qquad \frac{dy}{dt} = \frac{[(1-t)\times 2t]-[t^2\times -1]}{(1-t)^2}$$

$$\frac{dx}{dt} = \frac{1-t\ +\ t}{(1-t)^2} \qquad \frac{dy}{dt} = \frac{2t\ -2t^2+\ t^2}{(1-t)^2}$$

$$= \frac{1}{(1-t)^2} \qquad = \frac{2t-t^2}{(1-t)^2}$$

By the chain rule; $\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx}$

$$\frac{dy}{dx} = \frac{2t-t^2}{(1-t)^2}\times\frac{(1-t)^2}{1}$$

$$\frac{dy}{dx} = 2t\ -t^2\ .$$

For $x = \frac{t}{1-t}$; $u = t,\ v = 1-t$

$\frac{du}{dt} = 1,\ \frac{dv}{dt} = -1$

For $y = \frac{t^2}{1-t}$; $u = t^2,\ v = 1-t$

$\frac{du}{dt} = 2t,\ \frac{dv}{dt} = -1$

We substitute into the formula.

We have found ${}^{dx}/_{dt}$ and ${}^{dy}/_{dt}$ independently.
We then relate them using the chain rule.

Example 25:

If $x = {}^{t}/_{(1-t)}$, $y = {}^{(1-2t)}/_{(1-t)}$, *find* ${}^{dy}/_{dx}$.

As above, since x and y are expressed as quotients, we shall use the quotient rule to find ${}^{dx}/_{dt}$ and ${}^{dy}/_{dt}$.

$$x = {}^{t}/_{(1-t)} \qquad y = {}^{(1-2t)}/_{(1-t)}$$

$$\frac{dx}{dt} = \frac{[(1-t)\times 1]-[t\times -1]}{(1-t)^2} \qquad \frac{dy}{dt} = \frac{[(1-t)\times -2]-[(1-2t)\times -1]}{(1-t)^2}$$

$$= \frac{1-t+t}{(1-t)^2} \qquad = \frac{-2+2t+1-2t}{(1-t)^2}$$

$$= \frac{1}{(1-t)^2} \qquad = -\frac{1}{(1-t)^2}$$

By the chain rule; $\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx}$

$$= -\frac{1}{(1-t)^2}\times\frac{(1-t)^2}{1}$$

$$\frac{dy}{dx} = -1$$

For $x = \frac{t}{1-t}$; $u = t,\ v = 1-t$

$\frac{du}{dt} = 1,\ \frac{dv}{dt} = -1$

For $y = \frac{1-2t}{1-t}$; $u = 1-2t,\ v = 1-t$

$\frac{du}{dt} = -2,\ \frac{dv}{dt} = -1$

We substitute into the quotient formula.

We have found ${}^{dx}/_{dt}$ and ${}^{dy}/_{dt}$ independently.
We then relate them using the chain rule.

Example 26:

If $x = {}^{1}/_{\sqrt{1+t^2}}$, $y = {}^{t}/_{\sqrt{1+t^2}}$ find ${}^{dy}/_{dx}$ in terms of t .

$x = (1+t^2)^{-1/2}$ $\qquad y = {t}/{(1+t^2)^{\frac{1}{2}}}$

$$\frac{dy}{dt} = \frac{[(1+t^2)^{1/2} \times 1] - [t \times (\frac{1}{2}(1+t^2)^{-1/2} \times 2t)]}{((1+t^2)^{1/2})^2}$$

$$\frac{dx}{dt} = -\frac{1}{2}(1+t^2)^{-3/2} \times 2t \qquad \frac{dy}{dt} = \frac{(1+t^2)^{1/2} - t^2(1+t^2)^{-1/2}}{(1+t^2)}$$

$$\frac{dx}{dt} = -t(1+t^2)^{-3/2} \qquad \frac{dy}{dt} = \frac{(1+t^2)^{-1/2}\{(1+t^2)^1 - t^2\}}{(1+t^2)}$$

$$= \frac{(1+t^2)^{-1/2}}{(1+t^2)}$$

$$= (1+t^2)^{-3/2}$$

Using chain rule: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$$= (1+t^2)^{-3/2} \times \frac{1}{-t(1+t^2)^{-3/2}}$$

$$= -{1}/{t}\,.$$

We shall use the quotient rule to find ${dy}/{dt}$. For ${dx}/{dt}$ we shall simply use the chain rule.

$u = t, \quad v = (1+t^2)^{1/2}$

$\frac{du}{dt} = 1; \quad \frac{dv}{dt} = \frac{1}{2}(1+t^2)^{-\frac{1}{2}} \times 2t$

We substitute into the quotient rule.

We factorize by pulling out the bracket with the least index $(1+t^2)^{-1/2}$ and simplify.

We finally relate the two expressions by the chain rule.

Exercise 4E:

Find the derivates of the following parametric equations:

1. $x = 2t\,, \quad y = t^2$
2. $x = t^2\,, \quad y = t^3$
3. $x = 1 + t^2, \ y = 1 - t^2$
4. $x = 3t^2 + 2, \ y = t^3 + 2$
5. $x = t^3 + 1, \ y = 2t^3 - 1$
6. $x = \frac{1}{t}, \ y = \frac{1}{t^2}$
7. $x = a(t^2 - 1)\,, \quad y = 2a(t+1)$
8. $x = (1+t^2)^2, \ y = (1-t^2)^2$
9. $x = \frac{t}{t+1}, \ y = \frac{t^2}{t+1}$
10. $x = \frac{2t}{1-t}, y = \frac{1}{(1-t)^2}$
11. $x = \frac{1}{\sqrt{1-t^2}}, y = \frac{t}{\sqrt{1-t^2}}$
12. $x = \frac{t}{\sqrt{1+t^2}}\,, \ y = \frac{t^2}{\sqrt{1+t^2}}$
13. If $x = {t}/{1-t^2}$ and $y = {t^2}/{1-t^2}$ find $\frac{dy}{dx}$ at $t = 1$.
14. A curve is given by the parametric equations $x = 2t^2 - 3t, y = t^2 - 8t + 1$.
    (a) Find $\frac{dy}{dx}$ in terms of t.
    (b) Find the value of t at the point where the tangent is parallel to the line $y = 2x + 3$.
15. Find $\frac{dy}{dx}$ in terms of t for the curve $x = \frac{t^2}{1+t}, \ y = \frac{1-t}{1+t}$. Hence find the equation of the tangent and the normal where $t = 2$.

16. The parametric equations of a curve are $x = \frac{1}{1+t}, y = (2t-1)^2$. Find the equation of the normal where $t = 1$.

17. A curve is given by $x = t - 1, y = t^2 + t$.

(a) Find the value of t where the normal to the curve is parallel to the line $3x + y = 5$.

(b) Find also the turning point of the curve.

4.6. Second derivative

$If\ \frac{dy}{dx}\ is\ found\ in\ terms\ of\ a\ parameter\ \ t,\ \frac{d^2y}{dx^2}\ requires\ differentiation\ with\ respect$

$to\ \ x;$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

We shall apply the chain rule to ${}^{dy}/_{dx}$ to obtain the second derivative

Example 26:

$If\ \ x = a(t^2 - 1)\ and\ \ \ y = 2a(t+1)\ \ find\ \ \frac{dy}{dx}\ \ and\ \ \frac{d^2y}{dx^2}\ \ in\ terms\ of\ \ t.$

$x = a(t^2 - 1)$ $\qquad$ $y = 2a(t+1)$

$\frac{dx}{dt} = 2at$ $\qquad$ $\frac{dy}{dt} = 2a$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 2a \times {}^{1}/_{2at}$$

$$= {}^{1}/_{t}\ .$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

$$= \frac{d}{dt}(t^{-1}) \times \frac{1}{2at}$$

$$= -t^{-2} \times \frac{1}{2at}$$

$$= -{}^{1}/_{2at^3}\ .$$

As seen above, we find ${}^{dx}/_{dt}$ and ${}^{dy}/_{dt}$ independently; and then relate them using the chain rule, to find ${}^{dy}/_{dx}$.

We apply the chain rule again to ${}^{dy}/_{dx}$ to obtain the second derivative, (still differentiating expressions with t).

We differentiate ${}^{dy}/_{dx}$ (obtained above), then simply multiply it with ${}^{dt}/_{dx}$.

Example 28:

$If\ \ \ x = (1+t^2)^2\ \ and\ \ y = \ (1-t^2)^2,\ \ find\ \ \frac{dy}{dx}\ \ and\ \ \frac{d^2y}{dx^2}\ \ in\ terms\ of\ t\ .$

$x = (1+t^2)^2 \qquad y = (1-t^2)^2$

$\frac{dx}{dt} = 4t(1+t^2) \qquad \frac{dy}{dt} = -4t(1-t^2)$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= -4t(1-t^2) \times \frac{1}{4t(1+t^2)}$

$= -\frac{(1-t^2)}{(1+t^2)}$

$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$

$= \frac{d}{dt}\left[-\frac{(1-t^2)_u}{(1+t^2)_v}\right] \times \frac{dt}{dx}$

$= -\left[\frac{\{(1+t^2) \times -2t\} - \{(1-t^2) \times 2t\}}{(1+t^2)^2}\right] \times \frac{1}{4t(1+t^2)}$

$= -\left[\frac{-2t - 2t^3 - 2t + 2t^3}{(1+t^2)^2}\right] \times \frac{1}{4t(1+t^2)}$

$= \frac{4t}{(1+t^2)^2} \times \frac{1}{4t(1+t^2)}$

$= \frac{1}{(1+t^2)^3}$.

We find ${}^{dx}/_{dt}$ and ${}^{dy}/_{dt}$ independently; and then relate them using the chain rule, to find ${}^{dy}/_{dx}$.

We differentiate ${}^{dy}/_{dx}$ (obtained above), then simply multiply it with ${}^{dt}/_{dx}$.

$u = 1 - t^2, \quad v = 1 + t^2$

$\frac{du}{dt} = -2t, \quad \frac{dv}{dt} = 2t$

We apply the quotient rule:

$\frac{d}{dt} = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$

Notice that $4t$ cancels out and the indices in the denominator can be added as $(1+t^2)^{2+1}$.

Example 29:

A curve is given parametrically by $x = (t^2-1)^2$ *and* $y = t^3$. *Find* $\frac{dy}{dx}$ *and* ${}^{d^2y}/_{dx^2}$ *in terms of* t.

$x = (t^2-1)^2 \qquad y = t^3$

$\frac{dx}{dt} = 2(t^2-1) \times 2t \qquad \frac{dy}{dt} = 3t^2$

$= 4t(t^2-1)$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= 3t^2 \times \frac{1}{4t(t^2-1)}$

$= \frac{3t}{4(t^2-1)}$

We find ${}^{dx}/_{dt}$ and ${}^{dy}/_{dt}$ independently; and then relate them using the chain rule, to find ${}^{dy}/_{dx}$.

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

$$= \frac{d}{dt}\left\{\frac{3t_u}{4(t^2-1)_v}\right\} \times \frac{1}{4t(t^2-1)}$$

$$= \frac{3}{4}\left[\frac{\{(t^2-1)\times 1\} - \{t \times 2t\}}{(t^2-1)^2}\right] \times \frac{1}{4t(t^2-1)}$$

$$= \frac{3}{4}\left[\frac{t^2-1-2t^2}{(t^2-1)^2}\right] \times \frac{1}{4t(t^2-1)}$$

$$= \frac{3(-t^2-1)}{4(t^2-1)^2} \times \frac{1}{4t(t^2-1)}$$

$$= -\frac{3(t^2+1)}{16t(t^2-1)^3}.$$

We differentiate ${}^{dy}/_{dx}$ (obtained above), then simply multiply it with ${}^{dt}/_{dx}$.

$u = t, \quad v = t^2 - 1$

$\frac{du}{dt} = 1, \quad \frac{dv}{dt} = 2t$

We apply the quotient rule:

$$\frac{d}{dt} = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$$

The negative has been extracted out. Also, indices in the denominator can be added as $(t^2-1)^{2+1}$.

Example 30:

If $x^2 + 3xy - y^2 = 3$, *find* $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *at the point* $(1, 1)$.

Finding dy/dx at point (1, 1)

$$x^2 + 3xy - y^2 = 3$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(3xy) - \frac{d}{dx}(y^2) = \frac{d}{dx}(3)$$

$$2x + \left(3y + 3x\frac{dy}{dx}\right) - 2y\frac{dy}{dx} = 0$$

$$(3x - 2y)\frac{dy}{dx} = -(2x + 3y)$$

$$\frac{dy}{dx} = \frac{2x+3y}{2y-3x}$$

At $(1, 1)$ $\quad x = 1$ *and* $y = 1$;

$$\therefore \frac{dy}{dx} = \frac{2(1)+3(1)}{2(1)-3(1)} = -5.$$

Finding d²y/dx² at point (1, 1)

$$\frac{dy}{dx} = \frac{2x+3y}{2y-3x}\begin{matrix} u \\ v \end{matrix}$$

We differentiate implicitly. Note that for $3xy$ we use the product rule as shown below:

$$\frac{d}{dx}(3xy) = y \times 3 + 3x \times \frac{dy}{dx}$$

[Fix y and differentiate $3x$ to get 3; Then fix $3x$ and differentiate y to give ${}^{dy}/_{dx}$.]

We group the ${}^{dy}/_{dx}$ terms and the independent terms, pull out ${}^{dy}/_{dx}$ and then divide.

We substitute x and y into ${}^{dy}/_{dx}$ to obtain the gradient at the point.

We need to apply the quotient rule:

$$\frac{d^2y}{dx^2} = \frac{\left\{(2y-3x)\times\left(2+3\frac{dy}{dx}\right)\right\} - \left\{(2x+3y)\times\left(2\frac{dy}{dx}-3\right)\right\}}{(2y-3x)^2}$$

$$= \frac{\left\{2(2y-3x)+3(2y-3x)\frac{dy}{dx}\right\} - \left\{2(2x+3y)\frac{dy}{dx} - 3(2x+3y)\right\}}{(2y-3x)^2}$$

$$= \frac{4y-6x+(6y-9x)\frac{dy}{dx} - (4x+6y)\frac{dy}{dx} + 6x + 9y}{(2y-3x)^2}$$

$$= \frac{13y + \frac{dy}{dx}[(6y-9x)-(4x+6y)]}{(2y-3x)^2}$$

$$= \frac{13y - (13x)\frac{dy}{dx}}{(2y-3x)^2}$$

$$= \frac{13\left(y - x\frac{dy}{dx}\right)}{(2y-3x)^2}$$

$$At\ (1,\ 1);\ \frac{d^2y}{dx^2} = \frac{13(1-(1)(-5))}{(2-3)^2}$$

$$= 78\ .$$

$u = 2x+3y,\ \ v = 2y-3x$

$\frac{du}{dx} = 2 + 3\frac{dy}{dx},\quad \frac{dv}{dx} = 2\frac{dy}{dx} - 3$

We apply: $\frac{d}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

We've opened the brackets. [Notice how the bracket with $\frac{dy}{dx}$ multiplies with the other bracket.]

We pull ${}^{dy}/_{dx}$ out and evaluate the terms in the bracket. We also get $13y$ from $4y + 9y$ and the $6x$ cancel out.

Again, we substitute x and y into ${}^{dy}/_{dx}$ to obtain ${}^{d^2y}/_{dx^2}$ at the point.

Exercise 4F:

1. If $x = \frac{t}{1+t}$ and $y = \frac{t^2}{1+t}$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t.
2. If $x = \frac{t^2}{1+t}$ and $y = \frac{1-t}{1+t}$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t.
3. If $x = {}^{t}/_{1-t^2}$ and $y = {}^{t^2}/_{1-t^2}$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t.
4. If $x = \frac{t}{\sqrt{1+t^2}}$ and $y = \frac{t^2}{\sqrt{1+t^2}}$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t.
5. If $x = \frac{2t}{1-t}$ and $y = \frac{1}{(1-t)^2}$ show that $\frac{d^2y}{dx^2}$ *is* $\frac{1}{2}$.
6. If $x = \frac{1}{1+t}$, $y = (2t-1)^2$ find $\frac{d^2y}{dx^2}$ at $t = -\frac{1}{2}$.
7. If $x = \frac{t}{\sqrt{1+t^2}}$ and $y = \frac{t^2}{\sqrt{1+t^2}}$ find $\frac{d^2y}{dx^2}$ at $t = \sqrt{3}$.
8. If $x^2 + y^2 = 10$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the points where $x = 1$.
9. If $3x^2 + 3y^2 + 4y = 10x + 12$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the points where $x = -1$.

Examination questions:

1. Find $\frac{dy}{dx}$ in terms of x and y if $2xy^2 + y + 2x = 8$. Hence find the gradient of the curve at the points where $x = 1$.
2. A curve is defined parametrically by the equation;

$$x = t^3 - 6t + 4, \quad y = t - 3 + {}^{2}/_{t}.$$

   (a) Find the equations of the normal to the curve at the points where the curve meets the x −axis.

   (b) Find the coordinates of their point of intersection.
3. A curve is described by the equation $x^2 - 4y^2 = 2xy$.

   (a) Find the coordinates of the two points on the curve where $x = -8$.

   (b) Find the gradient of the curve at each of these points.
4. A curve is given by the parametric equations:

$$x = {}^{(1-t)}/_{(1+t)}, \qquad y = (1-t)(1+t)^2$$

   (a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t.

   (b) Find the equation of the tangent to the curve at the point where $t = -2$.
5. Find the equations of the tangents to the curve $y^2 + 3xy + 4x^2 = 14$ at the points where $x = 1$.
6. Given that $y^2 - 5xy + 8x^2 = 2$, prove that $\frac{dy}{dx} = \frac{5y-16x}{2y-5x}$.

   The distinct points P and Q on the curve $y^2 - 5xy + 8x^2 = 2$ each have x −coordinate 1. The tangents to the curve at P and Q meet at the point N. Calculate the coordinates of N.
7. A curve is defined parametrically by the equations $x = \frac{t}{1+t}$, $y = \frac{t^2}{1+t}$.

   (a) Show that $\frac{dy}{dx} = t(t+2)$.

   (b) Find the value of t at the point where the tangent is parallel to the tangent at the point where $t = -3$.

   (c) Find the equation of the normal at the point where $t = -3$.
8. A curve is given parametrically by $x = t^2, y = t^3$.

   (a) Show that the equation of the normal at the point $(4,\ 8)$ is $x + 3y - 28 = 0$.

   (b) Also show that the tangent to the curve at $(4,\ 8)$ meets the curve again at point $(1, -1)$.
9. The parametric equations of a curve are $x = t^2$, $y = t - \frac{1}{3}t^3$. Find:

   (a) $\frac{dy}{dx}$ in terms of t.

   (b) The coordinates of the turning points of the curve.

   (c) The values of t where the tangent to the curve is parallel to the line $3y = 4x + 2$.

   (d) The Cartesian equation of the curve.

10. The pressure P $units$ and the volume V cm^3 of a quantity of gas stored at a constant temperature in a cylinder are related by Boyle's Law $PV = k$ (a constant). At a certain time, the volume of gas in the cylinder is $30\ m^3$ and its pressure is 20 $units$. If the gas is being compressed at the rate of $6\ m^3 s^{-1}$, at what rate is the pressure changing?

11. The equation of the curve is $y - x^2 + xy = 8$.

    (a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.

    (b) Find the equation of the normal to the curve at the point $\left(1,\ 4\frac{1}{2}\right)$.

    (c) Find the coordinates of the stationary points on the curve.

12. A curve has the equation $3x^2 - 2x + xy + y^2 - 11 = 0$. The point P on the curve has coordinates $(-1,\ 3)$.

    (a) Show that the normal to the curve at P has the equation $y = 2 - x$.

    (b) Find the coordinates of the point where the normal to the curve at P meets the curve again.

13. Sand falls on to level ground at a rate of 1200 $cm^3 s^{-1}$ and piles up in the form of a circular cone whose vertical angle is $60°$.

    (a) Given that $\tan 30 = \frac{1}{\sqrt{3}}$, show that the radius of the base is given by $r = \frac{h}{\sqrt{3}}$.

    (b) Show that the volume of the pile is $\frac{\pi h^3}{9}$.

    (c) Hence find the rate at which the height of the pile is increasing when $h = 24\ cm$.

14. A curve has equation $7x^2 + 48xy - 7y^2 + 75 = 0$.

    At two distinct points on the curve the gradient of the curve is equal to $\frac{2}{11}$.

    (a) Using implicit differentiation, show that $x + 2y = 0$ at each of these points.

    (b) Find the coordinates of the two points.

15. The equation of a curve is $2x^2 - xy + y^2 = 56$.

    (a) Find $\frac{dy}{dx}$ in terms of x and y.

    (b) Find the equation of the normal to the curve at the point $(2, -6)$.

    (c) Find the coordinates of the two points on the curve at which the tangents to the curve are parallel to the x −axis.

16. A water trough $100\ cm$ long has a triangular cross-section of base $30\ cm$ and a vertex angle of $60°$. The trough is placed on a level ground and is being filled at the rate of $10\ litres\ s^{-1}$.

    (a) Given that $\tan 30 = \frac{1}{\sqrt{3}}$, show that the volume $V\ cm^3$ of water in the trough when it is $h\ cm$ deep is given by $V = \frac{50\sqrt{3}h^2}{3}$.

    (b) Hence calculate the rate at which the water level is rising when $h = 50\ cm$.

17. A curve has parametric equations;

$$x = \frac{t}{2-t}, \quad y = \frac{1}{1+t}, -1 < t < 2\,.$$

(a) Show that $\frac{dy}{dx} = -\frac{1}{2}\left(\frac{2-t}{1+t}\right)^2$.

(b) Find the equation of the normal to the curve at the point where $t = 1$.

(c) Show that the Cartesian equation of the curve can be written in the form

$y = \frac{1+x}{1+3x}$.

18. A curve has the equation $x^2 - 4xy + 2y^2 = 1$.

(a) Find an expression for $\frac{dy}{dx}$ in terms of y in its simplest form.

(b) Show that the tangent to the curve at the point $P(1,\ 2)$ has the equation $3x - 2y + 1 = 0$.

(c) The tangent to the curve at the point Q is parallel to the tangent at P. Find the coordinates of Q.

19. A vessel containing liquid is in the form of an inverted hollow cone with a semi-vertical angle of $30°$. The liquid is running out of a small hole at the vertex of the cone, at a constant rate of 3 cm^3/s. Find the rate at which the surface area which is in contact with the liquid is changing, at the instant when the volume of the liquid left in the vessel is $81\pi\ cm^3$.

20. A straight metal bar, of square cross-section, is expanding due to heating. After t seconds the bar has dimensions $x\ cm$ by $x\ cm$ by $10x\ cm$. Given that the area of the cross-section is increasing at $0.024\ cm^2s^{-1}$ when $x = 6$. Find the rate of increase of the side of the cross-section at this instant. Find also the rate of increase of the volume when $x = 6$.

21. A curve has parametric equations

$$x = t(t-1),\ \ y = \frac{4t}{1-t}, \qquad t \neq 1.$$

(a) Find $\frac{dy}{dx}$ in terms of t.

(b) The point P on the curve has the parameter $t = -1$. Show that the tangent to the curve at P has the equation $x + 3y + 4 = 0$.

(c) The tangent to the curve at P meets the curve again at the point Q. Find the coordinates of Q.

22. The curve has parametric equations $x = a\sqrt{t},\ y = at(1-t),\ t \geq 0$,

(a) Find $\frac{dy}{dx}$ in terms of t.

(b) The curve meets the x −axis at the origin O and the point A. The tangent to the curve at A meets the y −axis at the point B. Show that the area of the triangle OAB is a^2.

Exercise 4A:

1.

(a) $2(x+1)$

(b) $4(2x+3)$

(c) $9(3x+2)^2$

(d) $24(4x-1)^2$

(e) $-4(5-8x)^3$

(f) $3(4x+3)^3$

(g) $-\frac{10}{(5x+2)^3}$

(h) $\frac{1}{(2x+3)^4}$

(i) $\frac{1}{\sqrt{2x+1}}$

(j) $\frac{6}{(3x+5)^{3/4}}$

(k) $9\sqrt{6x+5}$

(l) $\frac{12}{(3-8x)^{5/2}}$

2.

(a) $(x+1)^3+C$

(b) $(2x+3)^4+C$

(c) $\frac{1}{(4-3x)^2}+C$

(d) $\frac{(2x+3)^{3/2}}{3}+C$

(e) $-\frac{\sqrt{5-4x}}{2}+C$

(f) $-\frac{2}{3\sqrt{3x+4}}+C$

3.

(a) $-\frac{2}{(2x+3)^2}$

(b) $-\frac{6}{(3x+4)^3}$

(c) $-\frac{2}{(4x-1)^{\frac{3}{2}}}$

(d) $-\frac{2}{(6x+1)^{4/3}}$

(e) $-\frac{8}{(12x+5)^{5/3}}$

(f) $-\frac{9}{4(3x-4)^{7/4}}$

4.

(a) $-\frac{1}{x+2}+C$

(b) $-\frac{1}{4(2x+1)^2}+C$

(c) $\sqrt{2x+3}+C$

(d) $\frac{(3x+4)^{2/3}}{2}+C$

(e) $\frac{1}{3\sqrt{4-6x}}+C$

(f) $\frac{\sqrt[3]{12x+1}}{4}+C$

5.

(a) $8x(2x^2+1)$

(b) $18x(3x^2+2)^2$

(c) $6x^2(x^3+4)$

(d) $24x^3(2x^4+3)^2$

(e) $2x(3x+1)(6x+1)$

(f) $18x^5(x-1)(2x-3)^2$

(g) $\frac{2(6x-5)}{(5-4x)^2x^3}$

(h) $\frac{4(2x^3-3)}{3(6x-x^4)^{5/3}}$

6.

(a) $-\frac{2x}{(x^2+1)^2}$

(b) $-\frac{2x}{(2x^2+3)^{3/2}}$

(c) $-\frac{3x^2+1}{2(x^3+x)^{3/2}}$

(d) $\frac{x(9x-4)}{3(2x^2-3x^3)^{4/3}}$

(e) $-\frac{4x(3x+1)}{3(2x^3+x^2)^{5/3}}$

(f) $-\frac{2x+1}{4(x^2+x)^{5/4}}$

7.

(a) $\frac{1}{2\sqrt{(x+1)}\sqrt{(x+2)^2}}$

(b) $\frac{(2x-5)\sqrt{(x+2)}}{2(x-1)^{3/2}}$

(c) $-\frac{2x}{\sqrt{\{(x^2-1)^3(x^2+1)\}}}$

(d) $\frac{(1-\sqrt{x})(1-x\sqrt{x})}{\sqrt{x}(x^2-1)^{3/2}}$

Exercise 4B:

1. $5\pi\ cm^2s^{-1}$

2. $(a)\ 20\pi\ cm^2s^{-1}$ $(b)\ 100\pi\ cm^3s^{-1}$

3. $\frac{1}{4\pi}\ cms^{-1}$

4. $\frac{5}{\pi}\ cms^{-1}$

5. $18\pi\ cm^2s^{-1}$

6. $\frac{1}{2\pi}\ cms^{-1}$

7. $\frac{5}{4\pi}\ cms^{-1}$

8. $(a)\ 12\ cm$

$(b)\ \frac{1}{2}\ cms^{-1}$

9. $\frac{5}{\pi}\ cms^{-1}$

10. $\frac{1}{2}\ cms^{-1}$

Exercise 4C:

1. $x(3x+2)$

2. $(x+1)(3x+1)$

3. $2(x+3)(2x+1)(4x+7)$

4. $2x(x^2+1)(3x^2+5)$

5. $(x+1)^2(x^2+2)(7x^2+4x+6)$

6. $-\frac{1}{(x+2)^2}$

7. $-\frac{(2x+3)}{(x+1)^2(x+2)^2}$

8. $\frac{2(2x^2+x-2)}{(x^2+2x)^2(2-x)^3}$

9. $\frac{1-3x}{2\sqrt{1-x}}$

10. $\frac{2x-3}{2\sqrt{(x-2)(x-1)}}$

11. $\frac{(11-10x)}{6(2-x)^{\frac{3}{2}}(1-2x)^{4/3}}$

12. $\frac{x(4x^2+3)}{\sqrt{(2x^2+1)(x^2+1)}}$

13. $\frac{2}{(x+2)^2}$

14. $\frac{1-x^2}{(x^2+1)^2}$

15. $\frac{4x}{(1-x^2)^2}$

16. $\frac{(1-x)}{(1+x)^3}$

17. $\frac{2x(x^2+3)}{(1-x^2)^3}$

18. $-\frac{2(x-1)(x^2-2x-2)}{(x^2+2)^3}$

19. $-\frac{2x(x^2+2)(x^2+10)}{(x^2-2)^4}$

20. $\frac{2}{\sqrt{(x+1)^3}}$

21. $\frac{(1+x^2)}{2\sqrt{x(1-x^2)^3}}$

22. $-\frac{1}{\sqrt{(1+x)^3(1-x)}}$

23. $-\frac{2x}{\sqrt{(x^2+1)(x^2-1)^3}}$

24. $-\frac{2}{3\sqrt[3]{(x-1)^4(x+1)^2}}$

25. $\frac{dy}{dx}=\frac{1}{(x+1)^2}$;

$\frac{d^2y}{dx^2} = -\frac{2}{(x+1)^3}$

26. $\frac{4}{9}$

27. $\frac{1}{15}$

28. $\frac{dy}{dx} = \frac{4x}{(1-x^2)^2}$

29. $(a)\ a = 3$

Exercise 4D:

1. $\frac{dy}{dx} = -\frac{x}{y}$

2. $\frac{dy}{dx} = \frac{3-2x}{4y}$

3. $\frac{dy}{dx} = \frac{4-8x}{9y^2-6}$

4. $\frac{dy}{dx} = \frac{3(1-x^2)}{3y+4}$

5. $\frac{dy}{dx} = \frac{5-3x}{3y+2}$

6. $\frac{dy}{dx} = -\frac{2x+y}{2y+x}$

7. $\frac{dy}{dx} = -\frac{2(x+y)}{3y+2}$

8. $\frac{dy}{dx} = -\frac{3x^2+4xy+2y^2}{2x^2+4xy-2y}$

9. $\frac{dy}{dx} = -\frac{3x^2+8xy-2y^2}{4x^2-3y^2-4xy}$

10. $\frac{dy}{dx} = -\frac{3x^2+4xy^2-y^3}{x^3+4x^2y-3xy^2}$

11. $-\frac{1}{3}$

12. 0

13. $-\frac{1}{3}$

14. $At\ (-1,\ -1),\ \frac{dy}{dx} = -8;$

$At\ \left(-1, -\frac{1}{3}\right), \frac{dy}{dx} = 8$

15. $x = -1\ or\ x = 1$

16. $\frac{7}{5};\ \ y = -\frac{5}{7}x + \frac{19}{7};$

$a = -\frac{5}{7},\ \ b - \frac{19}{7}$

17. $4x + 5y + 6 = 0$

18. $(1,\ \ 1)$

19. $5x - y - 3 = 0$

Exercise 4E:

1. t

2. $\frac{3t}{2}$

3. -1

4. $\frac{t}{2}$

8. $-\frac{(1-t^2)}{(t^2+1)}$

9. $t\ (t + 2)$

10. $\frac{1}{(1-t)}$

11. $\frac{1}{t}$

12. $t\ (t^2 + 2)$

13. 1

14. $(a)\ \frac{dy}{dt} = \frac{2\,t-8}{4\,t-3}$

$(b)\ t = -\frac{1}{3}$

5. 3

6. $\frac{2}{t}$

7. $\frac{1}{t}$

15. $\frac{dy}{dx} = -\frac{2}{t^2+2t};$

16. $y = -\frac{x}{4};$

$y = 4\,x - \frac{17}{3}$

17. $y = \frac{x}{16} + \frac{31}{32}$

18. $(a)\ t = -\frac{1}{3}$

$(b)\ \left(-\frac{3}{2}, -\frac{1}{4}\right)\ mini$

Exercise 4F:

1. $\frac{dy}{dx} = t\,(t+2)$; $\frac{d^2y}{dx^2} = 2\,(t+1)^3$
2. $\frac{dy}{dx} = -\frac{2}{t^2+2t}$; $\frac{d^2y}{dx^2} = \frac{4\,(t+1)^3}{(t^2+2\,t)^3}$
3. $\frac{dy}{dx} = \frac{2\,t}{t^2+1}$; $\frac{d^2y}{dx^2} = \frac{2\,(1-t^2)^3}{(t^2+1)^3}$
4. $\frac{dy}{dx} = t\,(t^2+2)$;

   $\frac{d^2y}{dx^2} = (3\,t^2+2)\,(t^2+1)^{\frac{3}{2}}$
5. $\frac{d^2y}{dx^2} = \frac{1}{2}$
6. $-\frac{3}{2}$
7. 88
8. $At\;\;(1,-3), \frac{dy}{dx} = \frac{1}{3}, \frac{d^2y}{dx^2} = \frac{10}{27}$;

   $at\;\;(1,\;\;3),\; \frac{dy}{dx} = -\frac{1}{3},\; \frac{d^2y}{dx^2} = -\frac{8}{27}$
9. $At\;\;(-1,-1),\; \frac{dy}{dx} = -8,\; \frac{d^2y}{dx^2} = 195$;

   $at\;\;\left(-1,\;-\frac{1}{3}\right), \frac{dy}{dx} = 8,\; \frac{d^2y}{dx^2} = -195$

Examination questions

1. $\frac{dy}{dx} = -\frac{(2y^2+2)}{(4xy+1)}$; $at\;(1,-2)$,

   $\frac{dy}{dx} = \frac{10}{7}$; $at\;\;\left(1,\;\frac{3}{2}\right),\; \frac{dy}{dx} = -\frac{13}{14}$
2. $(a)\;y = -3\,x-3$; $y = -12\,x$

   $(b)\;\left(\frac{1}{3}, -4\right)$
3. $(a)\;\left(-8,\;\;2-2\,\sqrt{5}\right), \left(-8,\;\;2+2\,\sqrt{5}\right)$

   $(b)\;\frac{\sqrt{5}-1}{4}$; $\frac{-\sqrt{5}-1}{4}$
4. $(a)\;\frac{dy}{dx} = \frac{1}{2}(1+t)^3(3t-1)$;

   $\frac{d^2y}{dx^2} = -3\,t\,(t+1)^4$

   $(b)\;y = \frac{7\,x+27}{2}$
5. $At\;(1,-5),\;\; y = \frac{-2\,x-33}{7}$;

   $at\;\;(1,\;\;2),\;\; y = \frac{26-12\,x}{7}$
6. $N\left(\frac{8}{7},\;\frac{20}{7}\right)$
7. $(b)\;t = 1$

   $(c)\;y = -\frac{x}{3} - 4$
8. $(a)\;x + 3y - 28 = 0$
9. $(a)\;\frac{dy}{dx} = \frac{(1-t^2)}{2t}$

   $(b)\;\left(1, -\frac{2}{3}\right), \left(1,\;\;\frac{2}{3}\right)$

   $(c)\;t = -3\;or\;t = \frac{1}{3}$

   $(d)\;y^2 = x - \frac{2}{3}x^2 + \frac{1}{9}x^3$
10. $4\;units\;per\;s$
11. $(a)\;\frac{dy}{dx} = \frac{2x-y}{x+1}$

    $(b)\;y = \frac{4\,x}{5} + \frac{37}{10}$

    $(c)\;(2,\;\;4)\;\;and\;\;(-4,\;\;-8)$
12. $(a)\;\left(\frac{7}{3},\;\;-\frac{1}{3}\right)$
13. $(c)\;\frac{25}{4\,\pi}\;cms^{-1}$
14. $(b)\;(2,-1)\;and\;\;(-2,\;\;1)$
15. $(a)\;\frac{dy}{dx} = \frac{y-4x}{2y-x}$

    $(b)\;y = -x - 4$

    $(c)(-2,-8)\;and\;\;(2,\;\;8)$
16. $(b)\;2\,\sqrt{3}\;cms^{-1}$
17. $(b)\;y = 8\,x - \frac{15}{2}$
18. $(a)\;\frac{dy}{dx} = \frac{(2y-x)}{2(y-x)}$

    $(b)\;Q(-1,-2)$
19. $-\frac{4}{3}\;cm^2s^{-1}$
20. $0.002\;cm\;s^{-1}$; $2.16\;cm^3s^{-1}$
21. $(a)\;\frac{dy}{dx} = \frac{4}{(2t-1)(1-t)^2}$

    $(b)\;Q\left(12, -\frac{16}{3}\right)$
22. $(a)\;\frac{dy}{dx} = 2\sqrt{t}(1-2t)$

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