Intermediate value theorem overview

The purpose of the intermediate value theorem is to allow you to prove that a function has at least one solution or root in a given interval.

The theorem states:

> Let $f(x)$ be a function which is continuous on the closed interval $[a, b]$ and let y_0 be a real number lying between $f(a)$ and $f(b)$. If $f(a) \le y_0 \le f(b)$ or $f(b) \le y_0 \le f(a)$, then there will be at least one c on the interval $[a, b]$ where $y_0 = f(c)$.

First, function $f(x)$ must be continuous on the interval $[a, b]$, which means that, in order to prove that the intermediate value theorem is valid for a particular function, we'll need to show that the function is continuous on the interval. The interval must also be a closed interval, which means that the endpoints of the interval are included. This theorem is only applicable to closed intervals.

Next, y_0 is a y-value that exists between $f(a)$ and $f(b)$. Finally, we'll need to verify that a point c could exist between the points a and b and that $y_0 = f(c)$.

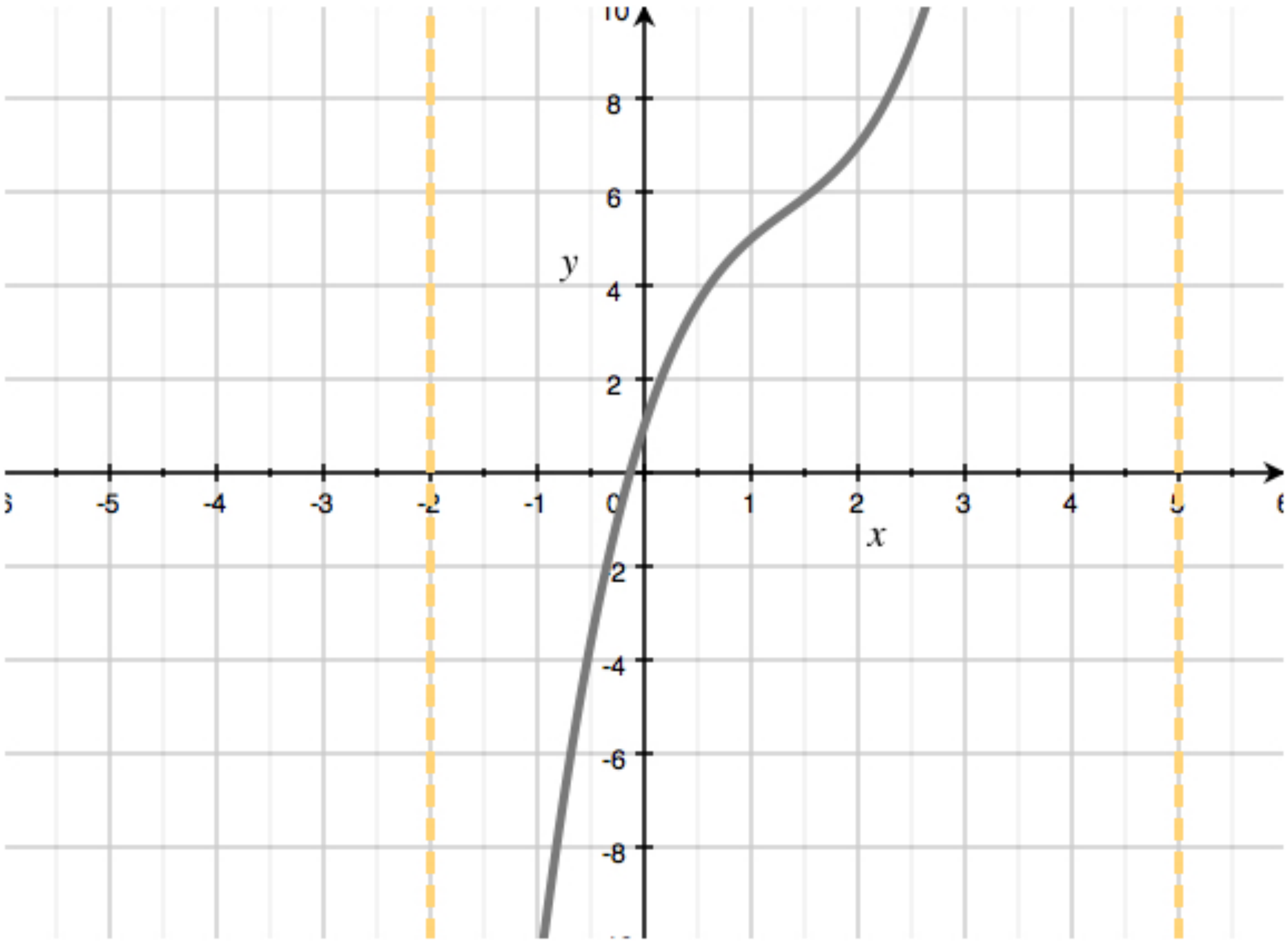
Example

Show that the function has at least one root in the interval.

$$f(x) = x^3 - 4x^2 + 7x + 1$$

$$[-2,5]$$

First, we should confirm that this function is continuous over the given interval. Continuity breaks typically occur in functions that have variables in the denominator of fractions, trigonometric functions, logarithmic functions, or functions with square roots. This function has none of these elements, so we know that it's continuous. If we want to double-check ourselves, we can graph it.



Next we will substitute into our function using our two endpoints.

$$f(-2) = (-2)^3 - 4(-2)^2 + 7(-2) + 1$$

$$f(-2) = -8 - 16 - 14 + 1$$

$$f(-2) = -37$$

and

$$f(5) = (5)^3 - 4(5)^2 + 7(5) + 1$$

$$f(5) = 125 - 100 + 35 + 1$$

$$f(5) = 61$$

Since we know that $f(x) = x^3 - 4x^2 + 7x + 1$ is continuous, and since the value of the function at the left side of the interval is negative (below the x-axis), and the value of the function at the right side of the interval is positive (above the x-axis), we know that there's at least one point c at which the function will cross the x-axis.

$$-37 \le c \le 61$$

Therefore, the function has at least one solution (root) in the given interval.
