MSA™



# Quick Math Review


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# Preface

The accumulation of skills and concepts in Mathematics through years of studying may sometimes cause overwhelming information for the students. Recall of the basic concepts may sometimes be difficult especially when one has already reached higher Mathematics.

To help students in reviewing the different Mathematical concepts from basic to higher Math, **MSA** has come up with this **MSA Quick Math Review**. Specifically, this book contains the different Math concepts in Basic Math, Algebra, Advance Algebra and Geometry. It explains in simple terms the different concepts and gives illustrative examples to aid in adding comprehension of the topic.

This concise book is a perfect companion that provides quick reference to your journey in the world of Mathematics.

-The Authors

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# MSA™
# Quick Math Review

## NUMBER PROPERTIES

**1. Integers**

Integers : { ...,−3, −2, −1, 0, 1, 2, 3, ...}
Whole Numbers : {0, 1, 2, 3, ...}
Counting Numbers : {1, 2, 3, ...}

**2. Rational/Irrational Numbers**

A rational number is a number that can be expressed as a **ratio of two integers**. Irrational numbers are real numbers - they have locations on the number line - they just **can't be expressed precisely as a fraction or terminating decimal.** The most commonly used irrational numbers are $\pi$, $\sqrt{2}$ and $e$

**3. Adding/Subtracting Signed Numbers**

**To add a positive and a negative,** first ignore the signs and find the positive difference between the number parts. Then attach the sign of the original number with the larger number part. For example, to add 45 and −67, first we ignore the minus sign and find the positive difference between 45 and 67 - that's 22. Then we attach the sign of the number with the larger number part- in this case it's the minus sign from the −67. So, $45 + (-67) = -22$.

Make **subtraction** situations simpler by turning them into addition. For example, think of $-21 - (-35)$ as $-21 + (+35)$.

**To add or subtract a string of positive and negatives,** first turn everything into addition. Then combine the positives and negatives so that the string is reduced to the sum of a single positive number and a single negative number.

**4. Multiplying/Dividing Signed Numbers**

To multiply and/or divide positives and negatives, treat the number parts as usual and **attach a minus sign if there were originally**

**an odd number of negatives.** For example, to multiply –5, –6, and –8, first multiply the number parts; 5 × 6 × 8= 240.Then go back and note that there were *three* - an *odd* number - negatives, so the product is negative : (–5) × (–6) × (–8) = –240.

5. **PEMDAS**
When performing multiple operations, remember **PEMDAS,** which means **Parentheses** first, then **Exponents,** then **Multiplication and Division** (left to right), and lastly **Addition and Subtraction** (left to right). In the expression $15 - 3 \times (8 - 5)^2 + 9 \div 3$, begin with the parenthesis : (8 – 5) = 3. Then do the exponent : $3^2 = 9$. Now the expression is: 15 – 3 × 9 + 9 ÷ 3. Next do the multiplication and division to get: 15 – 27 + 3, which gives –9.

6. **Counting Consecutive Integers**
To count consecutive integers, **subtract the smallest from the largest and add 1.** To count the integers from 15 through 51, ( 51 – 15 ) + 1 = 36 + 1 = 37 integers.

7. **Absolute Value**
The absolute value of an integer, written as $|x|$, is the **distance between 0 and *x* on the number line**. For example, $|4| = 4$ and $|-4| = 4$.

8. **Absolute Value Property**
If *x* is any number or zero, then the absolute value of *x* and the absolute value of –*x* are both equal to *x*. That is, $|-x| = x$ and $|x| = x$.

## MULTIPLES AND FACTORS

9. **Factor/Multiple**
The **factors** of integer *n* are the positive integers that divide *n* with no remainder. The **multiples** of *n* are the integers that *n* divides into with no remainder. For example, 8 is a factor of 16, and 32 is a multiple of 16. 16 is both a factor and a multiple of itself, since 16 × 1 = 16 and 16 ÷1 = 16.





10. **Prime Factorization**
To find the prime factorization of an integer, just keep breaking it up into factors until **all the factors are prime.** To find the prime factorization of 48, for example, 48 = 6 × 8 = 2 × 3 × 4 × 2 $= 2 \times 3 \times 2 \times 2 \times 2 = 2^4 \times 3$.

11. **Relative Primes**
Relative primes are integers that have no common factor other than 1. To determine whether two integers are relative primes, break them down to their prime factorizations. For example, 35 = 5 × 7, and 24 = 2 × 2 × 2 × 3. They have **no prime factors in common,** so 35 and 24 are relative primes.

12. **Common Multiple**
A common multiple is a number which is a multiple of two or more integers. You can always get a common multiple of two integers by **multiplying** them, but, unless the two numbers are relative primes, the product will not be the *least* common multiple. For example, to find a common multiple for 15 and 18, you could just multiply : 15 × 18 = 270.

13. **Least Common Multiple(LCM)**
To find the least common multiple, check out the **multiples of the larger integer** until you find one that's also a **multiple of the smaller.** To find the LCM of 15 and 18, begin by taking the multiples of 18: 18 is not divisible by 15; 36's not; nor is 54 and 72. But the next multiple of 18 which is 90, *is* divisible by 15, so it's the LCM of 15 and 18.

14. **Greatest Common Factor(GCF)**
To find the greatest common factor, break down the integers into their prime factorizations and multiply **all the prime factors they have in common.** 18 = 3 × 3 × 2 and 27 = 3 × 3 × 3. What they have in common is two three's, so the GCF is 3 × 3 = 9.

15. **Even/Odd**
To predict whether the sum, difference or product will be even or odd, just **take simple numbers like 1 and 2 and see what happens.** There are rules - "odd times even is even", for example - but there's no need to memorize them. What happens with one set of numbers generally happens with all similar sets.

**16. Multiples of 2, 4 and 8**
An integer is divisible by 2 (even) if the **last digit is even.** An integer is divisible by 4 if the **number formed by the last two digits is a multiple of 4.** An integer is divisible by 8 if the **number formed by the last three digits is a multiple of 8.** The last digit of 354 is 4, which is even, so 354 is a multiple of 2. The last two digits make 54, which is *not* divisible by 4, so 354 is not a multiple of 4. The integer 764, however is divisible by 4 because the last two digits form 64, which is a multiple of 4.

**17. Multiples of 3 and 9**
An integer is divisible by 3 if the **sum of its digits is divisible by 3.** An integer is divisible by 9 if the **sum of its digits is divisible by 9.** The sum of the digits in 759 is 21, which is divisible by 3 but not by 9, so 759 is divisible by 3 but not by 9.

**18. Multiples of 5 and 10**
An integer is divisible by 5 if the **last digit is 5 or 0.** An integer is divisible by 10 if the **last digit is 0.** The last digit of 785 is 5, so 785 is a multiple of 5 but *not* a multiple of 10.

**19. Remainders**
The remainder is the **whole number left over after division.** 569 is 4 more than 565, which is a multiple of 5, so when 569 is divided by 5, the remainder will be 4.

## FRACTIONS AND DECIMALS

**20. Reducing Fractions**
To reduce a fraction to lowest terms, **factor out and cancel** all factors common to both the numerator and denominator.

$$\frac{48}{54} = \frac{6\times 8}{6\times 9} = \frac{8}{9}$$

**21. Adding/Subtracting Fractions**
To add or subtract fractions, first **convert the fractions in**





**similar** fractions using their Least Common Denominator then add or subtract the numerators.

$$\frac{3}{10} + \frac{4}{12} = \frac{18}{60} + \frac{20}{60} = \frac{18+20}{60} = \frac{38}{60} = \frac{19}{30}$$

**22. Multiplying Fractions**
To multiply fractions, **multiply** the numerators and **multiply** the denominators and reduce the result to lowest terms.

$$\frac{7}{10} \times \frac{3}{4} = \frac{7\times 3}{10\times 4} = \frac{21}{40}$$

**23. Dividing Fractions**
To divide fractions, **invert** the divisor and **multiply.**

resulting fraction : $\frac{5}{8} \div \frac{3}{5} = \frac{5}{8} \times \frac{5}{3} = \frac{25}{24}$

**24. Converting a Mixed Number to an Improper Fraction**
To convert a mixed number to an improper fraction, **multiply** the whole number part by the denominator, then **add** the numerator. The result is the new numerator (over the same denominator). To convert $5\frac{1}{2}$, first multiply 5 by 2, then add 1, to get the new numerator of 11. Put that over the same denominator, that gives $\frac{11}{2}$.

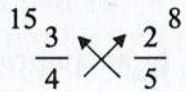
**25. Converting an Improper Fraction to a Mixed Number**
To convert an improper fraction to a mixed number, divide the denominator into the numerator to get a **whole number quotient with a remainder.** The quotient becomes the whole number part of the mixed number, and the remainder becomes the new numerator- with the same denominator. For example, to convert $\frac{251}{6}$, first divide 6 into 251,which yields 41 with a remainder of 5. Therefore, $\frac{251}{6} = 41\frac{5}{6}$.

**26. Reciprocals**

To find the reciprocal of a fraction, switch the numerator and the denominator. The reciprocal of $\frac{5}{8}$ is $\frac{8}{5}$. The product of reciprocals is 1.

**27. Comparing Fractions**

One way to compare fractions is to re-express them with a common denominator. $\frac{3}{4}=\frac{15}{20}$ and $\frac{2}{5}=\frac{8}{20}$. $\frac{15}{20}$ is greater than $\frac{8}{20}$, so $\frac{3}{4}$ is greater than $\frac{2}{5}$. Another way to compare fractions is to use cross product. Multiply the numerator of the first fraction by the denominator of the second and multiply the denominator of the first by the numerator of the second fraction. The bigger result is the bigger fraction.




The products are 5 × 3 = 15 and 4 × 2 = 8. Therefore, $\frac{3}{4} > \frac{2}{5}$
(Notice the direction of the arrows.)

**28. Converting Fractions to Decimals**

To convert a fraction to a decimal, divide top by the bottom number.

To convert $\frac{3}{8}$, divide 3 by 8, $8\overline{)3}$, yielding 0.375. When dividing the numerator by the denominator 9 of a proper fraction, the quotient is a repeating digit. This is a repeating decimal where the repeating digit is the numerator.

**29. Converting Decimals to Fractions**

To convert a decimal to a fraction, set the decimal over 1 and **multiply the numerator and denominator by ten raised to the number of digits to the right of the decimal point.** For instance, to convert 0.25 to a fraction, multiply $\frac{.25}{1}$ by $\frac{10^2}{10^2}$, or $\frac{100}{100}$. Then simplify: $\frac{25}{100}=\frac{25\times 1}{25\times 4}=\frac{1}{4}$





**30. Adding/Subtracting Decimals**

To add or subtract decimals, write each number so that the **decimal points are lined up**. Add or subtract them like whole numbers, then place the decimal point in the sum or difference so it is lined up with the other decimal points. For example, to subtact 9.6572 – 3.2542,

$$\begin{array}{r} 9.6572 \\ -\ 3.2542 \\ \hline 6.4030 \end{array}$$

**31. Multiplying Decimals**

To multiply decimals, multiply them like whole numbers. Place the decimal point in the product so that the number of decimal places in the product equals the sum of the number of decimal places in the factors. For example, 6.32 × 3.5 = 22.120.

**32. Dividing Decimals**

To divide a decimal by another decimal, first rewrite the problem in the usual long division format. Move the decimal point in the divisor and in the dividend the same number of places to the right until the divisor becomes a whole number. Then divide them like whole numbers. Place the decimal point in the quotient directly above the decimal point in the dividend. Divide 19.635 ÷ 2.1,

$$2.1\overline{)19.635} \Rightarrow 21\overline{)196.35}$$

$$\begin{array}{r} 9.35 \\ 21\overline{)196.35} \\ -\ 189 \\ \hline 73 \\ -\ 63 \\ \hline 105 \\ -\ 105 \\ \hline 0 \end{array}$$

**33. Identifying the Parts and the Whole**

The key to solving most fractions and percents story problems is to identify the part and the whole. Usually you'll find the **part** associated with the verb **"is/are"** and the **whole** associated with the word **"of"**. In the sentence, "Half of the marbles are blue", the whole is the marbles ("*of* the marbles"), and the part is the blue ("*are* blue").

## PERCENTS

**34. Convert Percents to Fractions**

To convert a percent to fraction, drop the percent symbol and multiply by $\frac{1}{100}$. Then reduce the fraction to its lowest terms.

To convert 80% to fraction, $80 \times \frac{1}{100} = \frac{80}{100} = \frac{4}{5}$.

**35. Converting Percents to Decimals**

To convert percents to decimals, drop the percent symbol and move the decimal point two places to the left. For example, 62.5% is equal to 0.625.

To convert decimals to percent, move the decimal point two places to the right and write a percent symbol.

**36. Percent Formula**

Whether you need to find the part, the whole, or the percentage, use the same formula:

**Part = Percent × Whole**

Example : What is 15% of 60?
Set-up : Part = 0.15 × 60

Example : 6 is 4% of what number?
Set-up : 6 = .04 × whole

Example : 36 is what percent of 64?
Set-up: 36 = Percent × 64





**37. Percent Increase and Decrease**

To increase a number by a percent, **add the percent to 100%,** convert to a decimal, and multiply. To increase 40 by 35%, add 35% to 100%, convert 135% to 1.35, and multiply by 40.

$$1.35 \times 40 = 54.$$

**38. Finding the Original Whole**

To find the **original whole before a percent increase or decrease,** set up an equation. Think of the result of a 20% increase over $x$ as $1.20x$.

Example : After a 20% increase, the population was 65 256. What was the population before the increase?

Set-up: $1.20x = 65\ 256 \Rightarrow x = 65\ 256 \div 1.20$

$x = 54\ 380$

**39. Combined Percent Increase and Decrease**

To determine the combined effect of multiple percents increase and/or decrease, **start with 100 and see what happens.**

Example: A price went up 10% one year, and the new price went up 30% the next year. What was the combined percent increase?

Set-up : First year : 100 + (10% of 100) = 110
Second year: 110 + (30% of 110) = 143
That's a combined 43% increase.

## RATIOS, PROPORTIONS, AND RATES

**40. Setting Up a Ratio**

To find ratio, put the number associated with the word **"of" on top** and the quantity associated with the word **"to" on the bottom** and reduce.

The ratio of 15 oranges to 6 apples is $\frac{15}{6}$, which reduces to $\frac{5}{2}$.

**41. Part-to-Part Ratios and Part-to-Whole Ratios**

If the parts add up to the whole, a part-to-part ratio can be turned into 2 part-to-whole ratios by putting **each number in the original ratio over the sum of the numbers.** If the ratio of boys to girls is 1 is to 3, then the boys-to-children ratio is $\frac{1}{1+3}=\frac{1}{4}$ and the girls-to-children ratio is $\frac{3}{1+3}=\frac{3}{4}$. In other words, $\frac{3}{4}$ of the children are girls.

**42. Proportion**

To solve a proportion, **cross-multiply:**

$$\frac{x}{6}=\frac{3}{5}\Rightarrow 5x=6\times 3\Rightarrow x=\frac{18}{5}=3.6$$

**43. Rate**

To solve for the rate, **write the units** to keep things straight and use ratio and proportion.

Example : If water level is increasing at the rate of 2 cm for every 5 hours, how many cm will the water level increase in 7 hours?

Set-up : $\frac{2\,\text{cm}}{5\,\text{hours}}=\frac{x\,\text{cm}}{7\,\text{hours}}\Rightarrow 5x=2\times 7\Rightarrow x=2.8\,\text{cm}$

**44. Average Rate**

Average rate is *not* simply the average of the rates.

$$\text{Average } A \text{ per } B=\frac{\text{total } A}{\text{total } B}$$

$$\text{Average Speed}=\frac{\text{total distance}}{\text{total time}}$$

To find the average speed for 150 miles at 50 mph and 150 miles at 75 mph, **don't just average the two speeds.** First figure out the total distance and the total time. The total distance is $150+150=300$ miles.

$t_1 = 150 \div 50 = 3$ hr. ; $t_2 = 150 \div 75 = 2$ hr.





$t_1 = 150 \div 50 = 3$ hr. ; $t_2 = 150 \div 75 = 2$ hr.

$$\text{average speed}=\frac{\text{total distance}}{\text{total time}}=\frac{300}{3+2}=\frac{300}{5}=60\text{ mph}$$

## AVERAGES

**45. Average Formula**

To find the average of a set of numbers, **add them up and divide by the number of terms.**

$$\text{Average}=\frac{\text{sum of the terms}}{\text{number of terms}}$$

To find the average of the 7 numbers 18, 22, 6, 44, 77, 25, and 11,

$$\text{average}=\frac{18+22+6+44+77+25+11}{7}=\frac{203}{7}=29$$

**46. Average of Evenly-Spaced Numbers**

To find the average of evenly-spaced numbers, just **average the smallest and the largest**. The average of all the integers from 34 through 50 is the same as the average of 34 and 50:

$$\text{average}=\frac{34+50}{2}=\frac{84}{2}=42$$

**47. Using the Average to Find the Sum**

**Sum = Average × Number of terms**

If the average of 16 number is 71, then they add up to $16\times 71=1136$.

**48. Finding the Missing Number**

To find a missing number when you're given the average, use the formula.

$$x=\left(\begin{array}{c}\text{total no.}\\ \text{of terms}\end{array}\right)\times(\text{average})-\left(\begin{array}{c}\text{sum of the remaining}\\ \text{terms}\end{array}\right)$$

Find the value of the 6th number if the average of the six numbers is 10 and the first five numbers are 5, 4, 15, 8 and 12. Using the formula,

$$x = 6 \times 10 - (5+4+15+8+12) = 60 - 44 = 16$$

**49. Median**

The median of a set of numbers is the value that falls in the middle of the set. If you have five scores, and they are 78, 91, 84, 75, and 88, you must first list the scores in increasing or decreasing order: 75, 78, 84, 88, 91. The median is the middlemost number which in this case is 84.

If there is an even number of values in a set (six scores, for instance) say 2, 4, 6, 8, 10, 12, simply take the average of the two middlemost scores.

$$\text{median} = \frac{6+8}{2} = 7$$

**50. Mode**

The mode of a set of numbers is the value that appears most often. If your test scores were 88, 84, 89, 88, 84, 84, 85, 87, and 84, the mode of the scores would be 84 because it appears more often than any other score. If there is a tie for the most common value in a set, the set has more than one mode.

## POWER AND ROOTS

**51. Multiplying and Dividing Powers**

To multiply powers with the same base, **add the exponents and keep the same base:**

$$x^5 \times x^6 = x^{5+6} = x^{11} \qquad 2^4 \times 2^6 = 2^{4+6} = 2^{10}$$

To divide powers with the same base, **subtract the exponents and keep the same base:**

$$z^6 \div z^3 = z^{6-3} = z^3 \qquad 5^8 \div 5^2 = 5^{8-2} = 5^6$$





**52. Raising Powers to Powers**

To raise a power to a power, **multiply the exponents:**

$$(x^3)^5 = x^{3 \times 5} = x^{15}$$

**53. Simplifying Square Roots**

To simplify square root, **factor out the perfect squares under** the radical, unsquare them and put the result in front.

$$\sqrt{45} = \sqrt{9 \times 5} = \sqrt{9} \times \sqrt{5} = 3\sqrt{5}$$

**54. Adding and Subtracting Radicals**

You can add or subtract radical expressions **when the part under the radicals of the same index is the same:**

$$2\sqrt{3} + 5\sqrt{3} = 7\sqrt{3} \qquad 2\sqrt[3]{3} + 3\sqrt[3]{3} + 4\sqrt{2} = 5\sqrt[3]{3} + 4\sqrt{2}$$

Don't try to add or subtract when radical parts are different. There's not much you can do with the expressions like:

$$5\sqrt{3} - 2\sqrt{5} \qquad \text{and} \qquad 5\sqrt{7} - 2\sqrt[3]{7}$$

**55. Multiplying and Dividing Roots**

The product of square roots is equal to the **square root of the product:**

$$\sqrt{5} \times \sqrt{6} = \sqrt{5 \times 6} = \sqrt{30}$$

The quotient of the square roots is equal to the **square root of the quotient:**

$$\frac{\sqrt{15}}{\sqrt{3}} = \sqrt{\frac{15}{3}} = \sqrt{5}$$

56. **Rationalize the Denominator**
To rationalize the denominator, multiply both the numerator and denominator by a multiplier that will make the denominator a rational number. Then simplify the expression. For instance,

$$\frac{5}{2\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{5\sqrt{3}}{2\sqrt{9}}=\frac{5\sqrt{3}}{6}$$

$$\frac{7}{\sqrt[3]{4}}\cdot\frac{\sqrt[3]{2}}{\sqrt[3]{2}}=\frac{7\sqrt[3]{2}}{\sqrt[3]{8}}=\frac{7\sqrt[3]{2}}{2}$$

If the expression in the denominator is a sum or difference of two square roots, multiply both the numerator and the denominator by the conjugate of the denominator. For example,

$$\frac{2}{3-\sqrt{2}}\cdot\frac{3+\sqrt{2}}{3+\sqrt{2}}=\frac{2\left(3+\sqrt{2}\right)}{9-\sqrt{4}}=\frac{2\left(3+\sqrt{2}\right)}{7}$$

If the expression in the denominator is a sum or difference of two cube roots $a$ and $b$, rationalize using this formula :

$$\frac{1}{a\pm b}\cdot\frac{a^2\mp ab+b^2}{a^2\mp ab+b^2}=\frac{a^2\mp ab+b^2}{a^3\pm b^3}$$

$$\frac{3}{2-\sqrt[3]{x}}\cdot\frac{4+2\sqrt[3]{x}+\sqrt[3]{x^2}}{4+2\sqrt[3]{x}+\sqrt[3]{x^2}}=\frac{3\left(4+2\sqrt[3]{x}+\sqrt[3]{x^2}\right)}{8-x}$$

57. **Imaginary Roots**
Imaginary roots occur when the part **when we are taking the square root of a negative number**.

We denote $i=\sqrt{-1}$ , $i^2=-1,\ i^3=-i,\ i^4=1$ .

To simplify $\sqrt{-36}$ : $\sqrt{-36}=\sqrt{36(-1)}=6\sqrt{-1}=6i$





## ALGEBRAIC EXPRESSIONS

58. **Evaluating an Expression**
To evaluate an algebraic expression, plug in the given values for the variables and calculate according to PEMDAS. To find the value of $x^2 + 2x - 5$ when $x = -1$, plug in $-1$ for $x$:

$$(-1)^2+2(-1)-5=1-2-5=-6.$$

59. **Adding and Subtracting Monomials**
To combine like terms, **keep the variable part unchanged while adding or subtracting the coefficients:**

$$5b+3b=(5+3)b=8b$$

60. **Adding and Subtracting Polynomials**
To add or subtract polynomials, **combine like terms.**

$$\begin{aligned}\left(5x^2+2x-7\right)-\left(3x+5\right)&=5x^2+2x-7-3x-5\\&=5x^2+\left(2x-3x\right)+\left(-7-5\right)\\&=5x^2-x-12\end{aligned}$$

61. **Multiplying Monomials**
To multiply monomials, **multiply the coefficients and the variables separately:**

$$3a\cdot 4a=(3\cdot 4)(a\cdot a)=12a^2$$

62. **Squaring a Binomial**
To square a binomial $(x + 4)$, square the first term : $x^2$, add twice the product of the first and the second term of the binomial: $2(4x) = 8x$, then add the square of the last term : $4^2 = 16$.

$$\begin{aligned}(x+4)^2&=x^2+2(4x)+16\\&=x^2+8x+16\end{aligned}$$

**63. Multiplying the Sum and Difference of the Same Terms**
To multiply the sum times the difference of the same terms, square the first term, write a minus sign and then square the last term.

$$(x+5)(x-5)=x^2-25$$

**64. Multiplying Binomials --FOIL**
To multiply binomials, use FOIL. To multiply $(x+4)$ by $(x+5)$, first multiply the **F**irst terms: $x \cdot x = x^2$. Next the **O**uter terms: $x \cdot 5 = 5x$. Then the **I**nner terms: $4 \cdot x = 4x$. And finally, the **L**ast terms: $4 \cdot 5 = 20$. Then add and combine like terms:

$$x^2+5x+4x+20=x^2+9x+20$$

**65. Multiplying Other Polynomials**
FOIL works only when you multiply two binomials. If you want to multiply polynomials with more than two terms, make sure you **multiply each term in the first polynomial by each term in the second.**

$$\begin{aligned}(2x^2+3x-5)(x-3) &= 2x^2(x-3)+3x(x-3)-5(x-3)\\ &= 2x^3-6x^2+3x^2-9x-5x+15\\ &= 2x^3-3x^2-14x+15\end{aligned}$$

After multiplying two polynomials together, the number of terms in your expression before simplifying should equal the number of terms in one polynomial multiplied by the number of terms in the second. In the example above, you should have $3 \times 2 = 6$ terms in the product before you simplify like terms.

**66. Dividing Polynomials by a Monomial**
To divide a polynomial by a monomial, divide each term of the polynomial by the monomial. Express the quotients as a sum and simplify. To divide $8a^2 - 6ab + 2a^3b^2$ by $2ab$,

$$\frac{8a^2}{2ab} - \frac{6ab}{2ab} + \frac{2a^3b^2}{2ab} = \frac{4a}{b} - 3 + a^2b^2$$





**67. Dividing Polynomial by a Polynomial**
To divide a polynomial by another polynomial, arrange each polynomial in descending order. Write the polynomial in a long division format. Then divide as if dividing whole numbers. For example, $(6a^3-28a+3a^2+15) \div (2a-3)$,

$$\begin{array}{r} 3a^2+6a-5 \\ 2a-3 \overline{)\, 6a^3+3a^2-28a+15} \\ (-)\ \underline{6a^3-9a^2} \\ 12a^2-28a \\ (-)\ \underline{12a^2-18a} \\ -10a+15 \\ (-)\ \underline{-10a+15} \\ 0 \end{array}$$

## FACTORING ALGEBRAIC EXPRESSIONS

**68. Factoring Out a Common Divisor**
A factor common to all terms of a polynomial can be **factored out**. All four terms in the polynomial $5x^4-15x^3+10x^2-5$ contain a factor $5x$. Pulling out the common factor yields $5x(x^3-3x^2+2x-1)$.

**69. Factor by Grouping**
To factor by grouping, **group together terms that have a common factor or groups that are special forms**. Factor the resulting groups. Then factor out the GCF. To simplify $2xy+3xz-10y-15z$, group together the first two terms since they have a common factor which is $x$ and the last two since they have a common factor which is –5,

$$\begin{aligned}(2xy+3xz)-(10y+15z) &\Rightarrow x(2y+3z)-5(2y+3z)\\ &\Rightarrow (2y+3z)(x-5)\end{aligned}$$

**70. Factoring the Difference of Two Squares**
One of the testmaker's favorite factorables is the **difference of two squares.**

$$a^2 - b^2 = (a-b)(a+b)$$

$x^2 - 49$, for example, factors to $(x-7)(x+7)$.

**71. Factoring the Square of a Binomial**
Learn to recognize polynomials that are squares of binomials:

$$a^2 + 2ab + b^2 = (a+b)^2$$

$$a^2 - 2ab + b^2 = (a-b)^2$$

For example, $4x^2 - 28x + 49$ factors to $(2x-7)^2$ and $b^2 + 4b + 4$ factors to $(b+2)^2$.

**72. Factoring the Sum or Difference of Two Cubes**
To factor the sum of two cubes $a^3 + b^3$, write the cube root of $a^3$ plus the cube root of $b^3$. This is the binomial factor $a + b$. Using this, find the trinomial factor ( $a^2 - ab + b^2$ ).

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

For example, factor $8x^3 + 125y^6$,

$$8x^3 + 125y^6 \Rightarrow (2x+5y^2)\left[(2x)^2 - (2x)(5y^2) + (5y^2)^2\right]$$
$$\Rightarrow (2x+5y^2)(4x^2 - 10xy^2 + 25y^4)$$

To factor the difference of two cubes $a^3 - b^3$, write the cube root of $a^3$ minus the cube root of $b^3$. This is the binomial factor $a - b$.





Using this, find the trinomial factor $(a^2 + ab + b^2)$.

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

For example, factor $64 - 125b^3$,

$$64 - 125b^3 \Rightarrow (4-5b)\left[(4)^2 + (4)(5b) + (5b)^2\right]$$
$$\Rightarrow (4-5b)(16 + 20b + 25b^2)$$

**73. Factoring $x^2$ + px + q -- FOIL in Reverse**

To factor a quadratic expression, think about what **binomials you could use FOIL on to get that quadratic expression.** To factor $x^2 + 7x + 10$, think about what **F**irst terms will produce $x^2$, what **L**ast terms will produce +10, and what **O**uter and **I**nner terms will produce $+7x$. Some common sense -- and a little trial and error--lead you to $(x+2)(x+5)$.

**74. Factoring ax² + bx + c --- by *ac* method**
To factor a trinomial using the *ac* method, **compute *a* × *c*.** Next, find **two integers *m* and *n* that have a product of *ac* and a sum of *b*.** Rewrite the $ax^2 + bx + c$ as $ax^2 + mx + nx + c$, and factor by grouping. For example, factor out $12x^2 + 33x - 9$. The product $ac$ is $12 \times 9 = 108$. The two factors of 108 which has a sum of 33 is 36 and –3. Now we rewrite the original problem as $12x^2 + 36x - 3x - 9$. Applying factor by grouping,

$$12x^2 + 36x - 3x - 9 \Rightarrow (12x^2 + 36x) + (-3x - 9)$$
$$\Rightarrow 12x(x+3) - 3(x+3)$$
$$\Rightarrow (x+3)(12x-3)$$
$$\Rightarrow 3(x+3)(4x-1)$$

## RATIONAL EXPRESSIONS

**75. Simplifying an Algebraic Fraction**

Simplifying an algebraic fraction is a lot like simplifying a numerical fraction. The general idea is to **find factors common to both numerator and denominator and cancel them.** Thus, simplifying an algebraic fraction begins with factoring. For example, $\frac{x^2-2x-15}{x^2-25}$, first factor the numerator and the denominator:

$$\frac{x^2-2x-15}{x^2-25}=\frac{\cancel{(x-5)}(x+3)}{\cancel{(x-5)}(x+5)}=\frac{x+3}{x+5}$$

**76. Multiplying Rational Expressions**

To multiply rational expressions, factor out all the polynomials. Write the product of the numerators over the product of the denominators. Then reduce if possible.

$$\frac{x^3-2x^2}{5x^2+20x}\cdot\frac{x^2-16}{x^2-x-12}\Rightarrow\frac{x^{\cancel{2}}(x-2)}{5\cancel{x}\cancel{(x+4)}}\cdot\frac{\cancel{(x-4)}\cancel{(x+4)}}{\cancel{(x-4)}(x+3)}$$

$$\Rightarrow\frac{x(x-2)}{5(x+3)}$$

**77. Dividing Rational Expressions**

To divide rational expressions, factor out all the polynomials then multiply the dividend by the reciprocal of the divisor.

$$\frac{y^2-9y}{4y^2+28y}\div\frac{y^2-7y-18}{y^2+9y+14}\Rightarrow\frac{y(y-9)}{4y(y+7)}\div\frac{(y-9)(y+2)}{(y+7)(y+2)}$$

$$\Rightarrow\frac{\cancel{y}\cancel{(y-9)}}{4\cancel{y}\cancel{(y+7)}}\cdot\frac{\cancel{(y+7)}\cancel{(y+2)}}{\cancel{(y-9)}\cancel{(y+2)}}$$

$$\Rightarrow\frac{1}{4}$$





**78. Finding the LCD of Rational Expressions**

To find the LCD, factor each denominator. The LCD is the product of the different factors. Each factor is used the greatest number of times it occurs in any one factorization.

**79. Adding/Subtracting Rational Expressions**

To add or subtract a rational expression, factor out all the polynomials then find the LCD. Change each rational expression to an equivalent expression using the LCD as denominator. Then perform the indicated operation and reduce if possible. For example, simplify $\frac{9x+2}{3x^2-2x-8}+\frac{7}{3x^2+x-4}$.

$$\frac{9x+2}{3x^2-2x-8}+\frac{7}{3x^2+x-4}\Rightarrow\frac{9x+2}{(3x+4)(x-2)}+\frac{7}{(3x+4)(x-1)}$$

$$\Rightarrow\frac{(9x+2)(x-1)}{(3x+4)(x-2)(x-1)}+\frac{7(x-2)}{(3x+4)(x-2)(x-1)}$$

$$\Rightarrow\frac{(9x+2)(x-1)+7(x-2)}{(3x+4)(x-2)(x-1)}\Rightarrow\frac{9x^2-7x-2+7x-14}{(3x+4)(x-2)(x-1)}$$

$$\Rightarrow\frac{9x^2-16}{(3x+4)(x-2)(x-1)}\Rightarrow\frac{\cancel{(3x+4)}(3x-4)}{\cancel{(3x+4)}(x-2)(x-1)}\Rightarrow\frac{3x-4}{(x-2)(x-1)}$$

**80. Complex Fraction**

A complex fraction is a rational expression that has a fraction in the numerator, the denominator or both. For example,

$$\frac{\frac{x+5}{3}}{\frac{2x}{x-7}}$$

**81. Simplifying Complex Fractions**

To simplify complex fractions, multiply the numerator and the denominator of the complex fraction by the LCD of all fractions appearing in the complex fraction, then simplify.

$$\frac{\frac{1}{xy}+\frac{2}{yz}+\frac{3}{xz}}{\frac{2x+3y+z}{xyz}} \Rightarrow \frac{\left[\frac{1}{xy}+\frac{2}{yz}+\frac{3}{xz}\right]\cdot xyz}{\left[\frac{2x+3y+z}{xyz}\right]xyz}$$

$$\Rightarrow \frac{\frac{xyz}{xy}+\frac{2xyz}{yz}+\frac{3xyz}{xz}}{\left(\frac{2x+3y+z}{xyz}\right)xyz} \Rightarrow \frac{z+2x+3y}{2x+3y+z}$$

$$\Rightarrow \frac{2x+3y+z}{2z+3y+z} \Rightarrow 1$$

## SOLVING EQUATIONS AND INEQUALITIES

**82. Addition/Subtraction Property of Equality**
If the same quantity is added or subtracted to both sides of the equation, the resulting equation is equivalent to the original equation. If A = B, then A + C = B + C. (A – C = B – C)

**83. Multiplication Property of Equality**
If the same nonzero quantity is multiplied to both sides of an equation, the resulting equation is equivalent to the original equation. If A = B, then AC = BC.

**84. Division Property of Equality**
If both sides of an equation is divided by the same nonzero quantity, then the resulting equation is equivalent to the original equation.

If A = B, then $\frac{A}{C}=\frac{B}{C}$.

**85. Solving a Linear Equation**
To solve an equation, simplify both sides of the equation by eliminating fractions, grouping symbols and combining like terms. Use the addition or subtraction property of equality to **isolate the variable** on one side and the constant terms on the other.





Use the multiplication or division property of equality to solve the resulting equation. To solve the equation $6x + 25 = 3x + 19$, first get all the $x$'s on one side by adding $-3x$ to both sides: $3x + 25 = 19$. Then add $-25$ on both sides: $3x = -6$. Then divide both sides by 3: $x = -2$.

**86. Conditional Equations**
A conditional equation is an equation that is true for only certain values of the variable. For example, $2x + 8 = 10$ is a conditional equation since it is only true if $x = 1$.

**87. Identity Equation**
An equation is an identity if it is a true statement for every value of the variable. For example, $3x = x + 2x \Rightarrow 3x = 3x$ is always true.

**88. Inconsistent Equation**
An equation that is a false statement for every value of the variable. For example, $2x = 4 + 2x \Rightarrow 2x - 2x = 4 \Rightarrow 0 = 4$ is always false.

**89. Solving "In terms Of"**
To solve an equation for one variable **in terms of** another means to **isolate the variable we want to solve for on one side of the equation,** leaving an expression containing the other variable or variables on the other side. To solve the equation $4x + 7y = 3x + 15y$ for $x$ in terms of $y$, isolate $x$:

$$4x+7y=3x+15y \Rightarrow 4x-3x=15y-7y \Rightarrow x=8y$$

**90. Absolute Value Equations**
To solve equations involving absolute value $|ax+b|=c$, solve the two equations $ax+b=c$ and $ax+b=-c$ only if $c>0$. For example, $|3x-4|=15$ :

$$3x-4=15 \qquad 3x-4=-15$$
$$3x=19 \qquad 3x=-11$$
$$x=\frac{19}{3} \quad \text{and} \quad x=-\frac{11}{3}$$

If $c = 0$, just solve the equation $ax + b = 0$. And if $c < 0$, the equation has no solution or the solution set is $\varnothing$.

91. **Translating from English to Algebra**
To translate from English into Algebra, look for the key words and systematically turn phrases into algebraic expressions and sentences into equations. Be careful about order, especially when subtraction is called for.

Example: The charge for a phone call is $r$ pesos for the first 3 minutes and $s$ pesos for each minute thereafter. What is the cost, in pesos, of a phone call lasting exactly $t$ minutes?

Set-up: The charge begins with $r$, and then something more is added, depending on the length of the call. The amount added is $s$ times the number of minutes past 3 minutes. If the total number of minutes is $t$, then the minutes past 3 is $t - 3$. So the charge is $r + s(t - 3)$.

92. **Addition/Subtraction Property of Inequality**
If the same quantity is added or subtracted on both sides of an inequality, the resulting inequality is equivalent to the original inequality. If $A < B$, then $A + C < B + C$.

93. **Multiplication Property of Inequality**
If the same positive quantity is multiplied to both sides of an inequality, the resulting inequality is equivalent to the original inequaltiy. If $A > B$ and $C > 0$, then $AC > BC$.

If the same negative quantity is multiplied to both sides of an inequality, the direction of the inequality should be reversed. Then the resulting inequality is equivalent to the original inequality. If $A > B$ and $C < 0$, then $AC < BC$.

94. **Division Property of Inequality**
If both sides of an inequality is divided by the same positive quantity, then the resulting inequality is equivalent to the original inequality. If $A > B$ and $C > 0$, then $\frac{A}{C} > \frac{B}{C}$.

If both sides of an inequality is divided by the same negative quantity, the direction of the inequality should be reversed. Then





the resulting inequality is equivalent to the original inequality.

If $A > B$ and $C < 0$, then $\frac{A}{C} < \frac{B}{C}$.

95. **Solving Inequalities**
To solve an inequality, apply the same procedures and properties used in solving linear equations except when multiplying or dividing both sides by a negative number. If both sides are **multiplied or divided by a negative number, reverse the direction of the inequality symbol**. To solve $-6x + 15 < -9$, subtract $-15$ from both sides to get: $-6x < -24$. Now divide both sides by $-6$, remember to reverse the sign: $x > 4$.

96. **Conditional Inequality**
A inequality that is a true statement for only some specific values of the variable. For example, $7x - 3 > 11$. This is only true if $x > 2$.

97. **All Real Numbers Solution Inequality**
An inequality that is true for every value of the variable. For example, $x < x + 10$. This is true for all values of $x$ since $0 < 10$ is always true.

98. **Inconsistent Inequality**
An inequality that is false for every value of the variable. For example, $x < x - 6$. This is false for all values of $x$ since $0 < -6$ is false.

99. **Absolute Value Inequalities**
Solve an inequality involving absolute value of the form $|ax + b| < c$, where $a \neq 0$ and $c > 0$ by solving the inequality $-c < ax + b < c$. For example, solve for the inequality $|4x + 3| < 2$ :

$$-2 < 4x + 2 < 2$$
$$-2 - 2 < 4x < 2 - 2$$
$$\frac{-4}{4} < \frac{4x}{4} < \frac{0}{4}$$
$$-1 < x < 0$$

Solve an inequality involving absolute value of the form $|ax+b| \geq c$, where $a \neq 0$ and $c > 0 \Rightarrow ax+b \geq c$ or $ax+b \leq -c$

and for $|ax+b| \leq c, \Rightarrow -c \leq ax+b \leq c$

Examples: $|2x-1|>5$

$$2x-1>5 \quad \text{or} \quad 2x-1<-5$$
$$2x>6 \qquad 2x<-4$$
$$x>3 \qquad x<-2$$

$|2x-1| \leq 5$

$$-5 \leq 2x-1 \leq 5$$
$$-5+1 \leq 2x \leq 5+1$$
$$-4 \leq 2x \leq 6$$
$$-2 \leq x \leq 3$$

## 100. Solving Radical Equations

To solve a radical equation, rewrite the equation so that one radical expression is isolated on one side of the equation. Square both sides and solve the resulting equation. Then check your solution by substituting your answer to the original equation. To solve $2\sqrt{x} = \sqrt{2x+8}$, square both sides to get $\left(2\sqrt{x}\right)^2 = \left(\sqrt{2x+8}\right)^2$ $\Rightarrow 4x = 2x+8$. Then solve for $x$: $4x-2x=8 \Rightarrow 2x=8 \Rightarrow x=4$.

* Checking of the roots obtained is a MUST because not every root obtained can be a solution.

## 101. Solving Quadratic Equations by Factoring

To solve a quadratic equation, put it in the "$ax^2+bx+c=0$" form, **factor** the left side (if you can), and set each factor equal to 0 separately to get the two solutions. To solve $x^2+15=8x$, first rewrite it as $x^2-8x+15=0$. Then factor the left side:

$$(x-5)(x-3)=0$$
$$x-5=0 \quad \text{or} \quad x-3=0$$
$$x=5 \quad \text{or} \quad x=3$$





## 102. Solving Quadratic Equations by Completing the Square

To solve a quadratic equation by completing the square, put it in the "$x^2+\frac{b}{a}x=\frac{c}{a}$" form. Then add $\left(\frac{b}{2a}\right)^2$ on both sides of the equation. Then factor the perfect square on one side. Get the square root of both sides to solve for the two solutions. To solve for $x^2+6x-7=0$. Express it as $x^2+6x=7$, where $a=1$, $b=6$, and c = 7, then add $\left(\frac{b}{2a}\right)^2 = \left(\frac{6}{2(1)}\right)^2 = 9$ on both sides :

$$x^2+6x+9 = 7+9$$
$$\sqrt{(x+3)^2} = \sqrt{16}$$
$$x+3 = \pm 4$$
$$x+3=4 \Rightarrow x=1 \quad \text{or} \quad x+3=-4 \Rightarrow x=-7$$

## 103. Solving Quadratic Equations by Quadratic Formula

To solve a quadratic equation, put it in the "$ax^2+bx+c=0$" form. Then substitute your $a$, $b$ and $c$ in the quadratic formula to solve for the two solutions.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

To solve for $3x^2-5x-6=0$, $a=3$, $b=-5$ and $c=-6$ :

$$x = \frac{-(-5) \pm \sqrt{(-5)^2-4(3)(-6)}}{2(3)} = \frac{5 \pm \sqrt{25+72}}{6} = \frac{5 \pm \sqrt{97}}{6}$$

The roots are $x_1 = \frac{5+\sqrt{97}}{6}$ and $x_2 = \frac{5-\sqrt{97}}{6}$.

### 104. Discriminant

The number represented by $b^2 - 4ac$ is called **discriminant** of the quadratic equation. It describes the character of the roots whether it is real, double root or imaginary.

$b^2 - 4ac = 0$ roots are real and equal; double root

$b^2 - 4ac > 0$ roots are real and unequal

$b^2 - 4ac < 0$ roots are imaginary and unequal

### 105. Sum and Product of Roots

To find the sum and product of roots of a quadratic equation $ax^2 + bx + c = 0$ without having to solve for the roots, the **sum** is given by $-\frac{b}{a}$ and the **product** is given by $\frac{c}{a}$. Find sum and product of the roots of $2x^2 + 8x - 9 = 0$, $a = 2$, $b = 8$, $c = -9$:

$$\text{sum} = \frac{-b}{a} = \frac{-8}{2} = -4 \text{ and product} = \frac{c}{a} = \frac{-9}{2} = -\frac{9}{2}.$$

### 106. Solving a System of Equations

You can solve for 2 variables only if you have 2 distinct equations. Two forms of the same equation will not be adequate. **Combine the equations** in such a way that **one of the variables cancels out.** To solve the two equations $2x - 4y = 8$ and $x - 3y = 10$, make the coefficients of the variable you wish to eliminate equal. To elimimate $x$, multiply both sides the second equation by –2 to get $-2x + 6y = -20$. Now add the two equations; the $2x$ and $-2x$ cancel out, leaving $2y = -12$. Dividing both sides by 2 yields $y = -6$. Plug that back into either one of the original equations and you'll find that $x = -8$.

### 107. Consistent-Independent Equations

A system is consistent-independent if the graph of the lines intersect at exactly one point or the system has a unique solution.





### 108. Inconsistent Equations

A system is inconsistent if the graph of the lines do not intersect or the system has no solution.

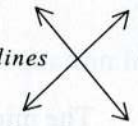
### 109. Dependent Equations

A system is dependent if the graph of the lines are the same or the system has infinitely many solution.

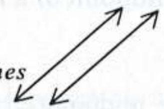
### 110. Determine if Consistent, Inconsistent or Dependent

To determine if a system of linear equations $\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$ is consistent, inconsistent or dependent without solving for the system,

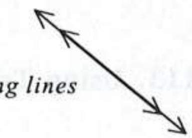
$\frac{a}{d} \neq \frac{b}{e} \Rightarrow$ consistent and independent *intersecting lines*



$\frac{a}{d} = \frac{b}{e} \neq \frac{c}{f} \Rightarrow$ inconsistent *parallel lines*



$\frac{a}{d} = \frac{b}{e} = \frac{c}{f} \Rightarrow$ consistent and dependent *overlapping lines*



## COORDINATE GEOMETRY

### 111. Finding the Distance Between Two Points

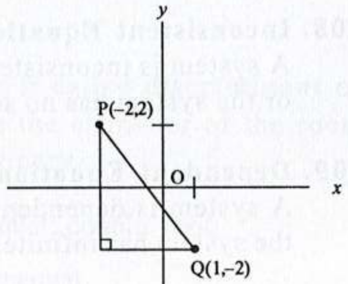
To find the distance between points, **use the Pythagorean theorem.** The difference between the $x$'s is one leg and the difference between the $y$'s is the other.

**Distance formula**:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

To find the distance between $P(-2,2)$ and $Q(1,-2)$

$$d=\sqrt{(-2-1)^2+(2-(-2))^2}$$
$$=\sqrt{(-3)^2+(4)^2}$$
$$=\sqrt{9+16}$$
$$=\sqrt{25}=5$$




**112. Midpoint of a Line**
To find the midpoint of a line A( $x_1, y_1$ ) and B( $x_2, y_2$ ),

$$(x_m, y_m)=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

The midpoint of a line connecting points A(2, 3) and B(4, –1) is

$$\text{midpoint}=\left(\frac{2+4}{2},\frac{3-1}{2}\right)=\left(\frac{6}{2},\frac{2}{2}\right)=(3,1)$$

**113. Using Two points to Find the Slope**

$$\text{Slope}=\frac{\text{Change in } y}{\text{Change in } x}=\frac{\text{Rise}}{\text{Run}}$$

The slope of the line that contains the points A(2,4) and B(–2,5) is

$$\text{slope}=\frac{y_A-y_B}{x_A-x_B}=\frac{4-5}{2-(-2)}$$
$$=\frac{-1}{4}=-\frac{1}{4}$$

The slope of a **horizontal line is equal to 0** and the slope of a **vertical line is undefined.**





**114. Using a Point and a Slope to Find the Equation of a Line**
To find the equation of a line given a point and a slope, use the **Point-Slope** form:

$$(y-y_1)=m\,(x-x_1)$$

Substitute the slope $m$ and the given point $(x_1, y_1)$. To find the equation of a line passing through (–4, 1) with slope 5,

$$y-1=5\,(x-(-4))$$
$$y-1=5\,(x+4)$$
$$y-1=5x+20$$
$$y=5x+21$$

**115. Using an Equation to Find the Slope**
To find the slope of a line from an equation, put the equation into the **slope-intercept** form:

$$\mathbf{y = mx + b}$$

The **slope is *m*.** To find the slope of the equation $4x + 6y = 3$, isolate $y$ :

$$4x+6y=3$$
$$6y=-4x+3$$
$$y=-\frac{4}{6}x+\frac{3}{6}$$
$$y=-\frac{2}{3}x+\frac{1}{2}$$

The slope is $-\frac{2}{3}$

**116. Using an Equation to Find an Intercept**
To find the $y$-intercept, you can either put the equation into **$y = mx + b$ (slope-intercept)** form --in which case ***b* is the *y*-intercept**--or you can just **plug $x = 0$** into the equation and **solve for *y*.**

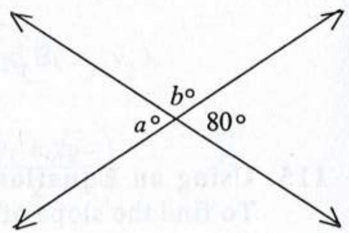
**117. Parallel and Perpendicular Lines**

Two lines are **parallel if they have the same slope** and **perpendicular if the product of their slopes is –1**.

## LINES AND ANGLES

**118. Intersecting Lines**

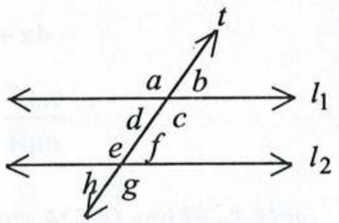
When two lines intersect, **adjacent angles are supplementary and vertical angles are equal.**




In the figure at the right, the angles marked $a^\circ$ and $b^\circ$ are adjacent and supplementary, so $a + b = 180^\circ$. Furthermore, the angles marked $a^\circ$ and $80^\circ$ are vertical and equal, so $a^\circ = 80^\circ$ and $b^\circ = 100^\circ$.

**119. Parallel Lines**

A transversal across two parallel lines forms **four equal acute angles and four equal obtuse angles.**




In the figure shown, $l_1$ is parallel to $l_2$. Angles $a, c, e,$ and $g$ are obtuse, and they are all equal. Angles $b$, $d$, $f$, and $h$ are acute, and they are all equal.

Furthermore, **any of the acute angles is supplementary to any of the obtuse angles.** Angles $a$ and $b$ are supplementary so are $b$ and $e$, $c$ and $f$, and so on.

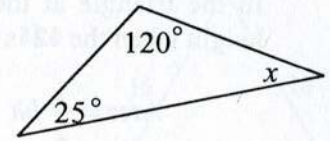
## TRIANGLES

**120. Interior Angles of a Triangle**

The three angles of any triangle **add up to 180°.**



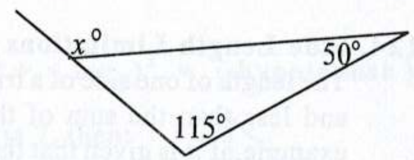





In the figure at the right, $x^\circ + 25^\circ + 120^\circ = 180^\circ$, therefore : $x = 35^\circ$.

**121. Exterior Angles of a Triangle**

An exterior angle of a triangle is equal to the **sum of the remote interior angles.**




In the figure at the right, the exterior angle labeled $x^\circ$ is equal to the sum of the remote angles: $x = 115^\circ + 50^\circ = 165^\circ$.

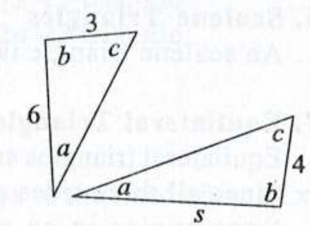
**122. Similar Triangles**

Similar triangles have the same shape: **corresponding angles are equal and corresponding sides are proportional.** The triangles below are similar because they have the same angles. The side measuring 3 corresponds to the side measuring 4 and the side measuring 6 corresponds to the side measuring $s$.

$$\frac{3}{4} = \frac{6}{s}$$

$$3s = 24$$

$$s = 8$$

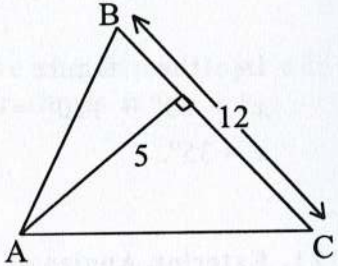



**123. Area of a Triangle**

$$\textbf{Area of Triangle} = \frac{1}{2}(\textbf{base})(\textbf{height})$$

The height is the perpendicular distance between the side that's chosen as the base and the opposite vertex.

In the triangle at the right, 5 is the height when the 12 is the chosen base.




$$\text{Area} = \frac{1}{2}bh$$
$$= \frac{1}{2}(12)(5)$$
$$= 30$$

*The height may not always be inside the triangle.

**124. Side Length Limitations of a Triangle**
The length of one side of a triangle must be greater than the difference and less than the sum of the lengths of the other two sides. For example, if it is given that the length of one side is 4 and the length of another side is 9, then you know that the length of the third side must be greater than 9 – 4 = 5 and less than 9 + 4 = 13.

**125. Isosceles Triangles**
An isosceles triangle is a triangle that has two equal sides. Not only are two sides equal, but the angles opposite the equal sides, called the base angles, are also equal.

**126. Scalene Triangles**
An scalene triangle is a triangle that has no equal sides.

**127. Equilateral Triangles**
Equilateral triangles are triangles in which all three sides are equal. Since all three sides are equal, all the angles are also equal. All three angles in an equilateral triangle measure 60 degrees, regardless of the length of sides.

**128. Area of an Equilateral Triangle**

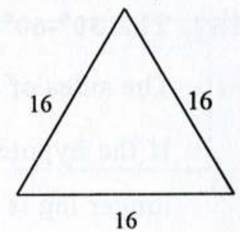
$\textbf{Area} = \frac{s^2\sqrt{3}}{4}$ , $s$ - the length of the equal sides

$\frac{s\sqrt{3}}{2}$ - the height of the equilateral triangle





In the equilateral triangle at the right, the length of the sides is 16.

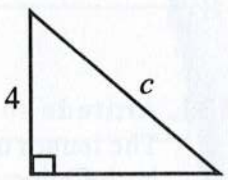



$$\text{Area} = \frac{(16)^2\sqrt{3}}{4} = \frac{256\sqrt{3}}{4} = 64\sqrt{3}$$

## RIGHT TRIANGLES

**129. Pythagorean Theorem**
For all right triangles: $(\text{leg}_1)^2 + (\text{leg}_2)^2 = (\text{hypotenuse})^2$

If one leg is 4 and the other leg is 7, then:



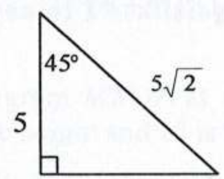

$$4^2 + 7^2 = c^2$$
$$c^2 = 16 + 49$$
$$c = \sqrt{65}$$

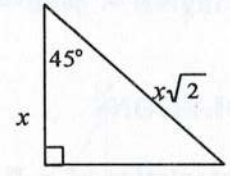
**130. Some Pythagorean Triples**

a. 3-4-5 triangle
b. 9-40-41 triangle
c. 8-15-17 triangle
d. 5-12-13 triangle
e. 7-24-25 triangle

**131. The 45°-45°-90° Triangle**

The sides of a 45°-45°-90° triangle are in the ratio of $x : x : x\sqrt{2}$.



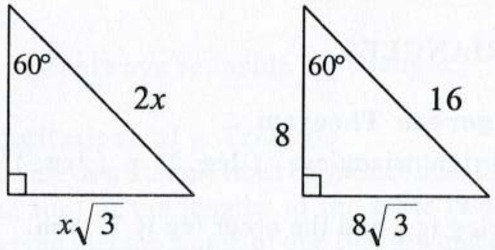




If one leg is 5, then the other leg is also 5, and the hypotenuse is equal to a leg times $\sqrt{2}$ , or $5\sqrt{2}$.

## 132. The 30°-60°-90° Triangle

The sides of a 30°-60°-90° triangle are in a ratio of $x : x\sqrt{3} : 2x$.

If the hypotenuse is 18, then the leg is half that, or 9; and the longer leg is equal to the short leg times $\sqrt{3}$, or $9\sqrt{3}$.

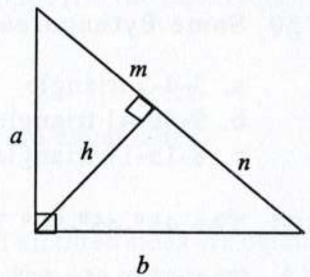



## 133. Altitude to the Hypotenuse

The length of the **altitude to the hypotenuse of a right triangle** is the geometric mean between the lengths of the segments of the hypotenuse.

$$h = \sqrt{mn}$$

The length of each leg of a right triangle is equal to the geometric mean between the length of its adjacent segment of the hypotenuse and the length of the hypotenuse.




$$a = \sqrt{m(m+n)} \text{ and } b = \sqrt{n(m+n)}$$

## OTHER POLYGONS

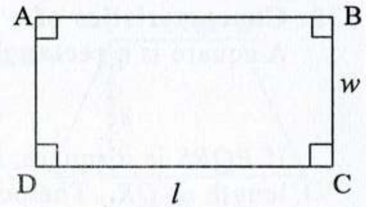
## 134. Characteristics of a Rectangle

A rectangle is a **4-sided figure with 4 right angles**. Opposite sides are equal. Diagonals are equal.





Quadrilateral $ABCD$ is shown at the right to have four right angles. The perimeter of a rectangle is equal to the sum of the lengths of the four sides, which is equivalent to 2 (length + width).

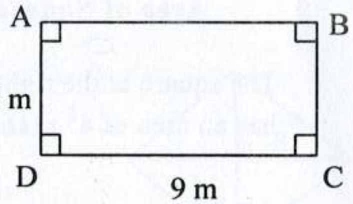



$$P = 2(l + w)$$

## 135. Area of a Rectangle

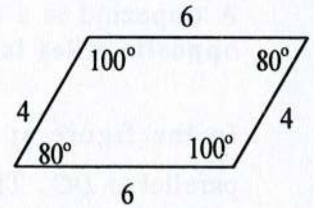
**Area of Rectangle = Length × Width**

The area of a 9-by-4 rectangle is $9 \text{ m} \times 4 \text{ m} = 36 \text{ m}^2$




## 136. Characteristics of a Parallelogram

A parallelogram has **two pairs of parallel sides.** Opposite sides are equal. Opposite angles are equal. Consecutive angles add up to 180°.

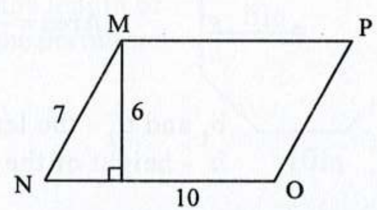



## 137. Area of a Parallelogram

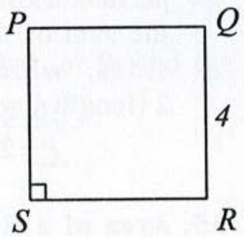
**Area of Parallelogram = Base × Height**

In parallelogram $MNOP$ at the right, 6 is the height and 10 is the base.

$A = bh = 10 \times 6 = 60$ square units

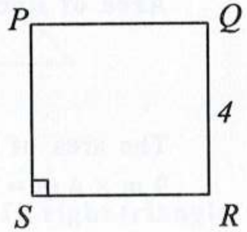


**138. Characteristics of a Square**
A square is a **rectangle with 4 equal sides.**




If $PQRS$ is a square, all sides are the same length as $QR$. The perimeter of a square is equal to 4 times the length of one side.

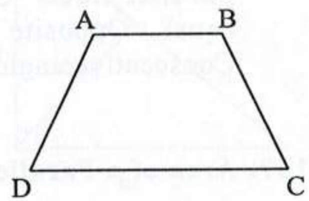
**139. Area of a Square**




**Area of Square =( Side )²**

The square at the right, with sides of length 4, has an area of $4^2 = 16\text{ m}^2$.

**140. Characteristics of a Trapezoid**
A trapezoid is a 4-sided figure in which exactly **one pair of opposite sides is parallel.**




In the figure at the right, $\overline{AB}$ is parallel to $\overline{DC}$. The parallel sides are called the bases of the trapezoid.

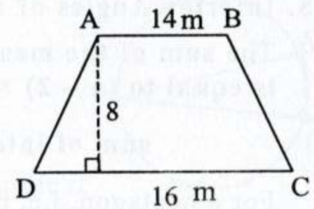
**141. Area of a Trapezoid**

$$\textbf{Area} = \frac{1}{2}(b_1 + b_2)h$$

$b_1$ and $b_2$ - the lengths of the parallel sides
h - height of the trapezoid



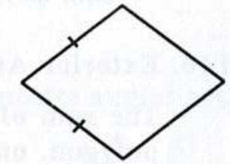





In the trapezoid at the right, the length of the bases are 14 m and 16 m, and the height is 8 m.

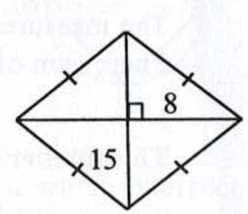
$$\text{Area} = \frac{1}{2}(14+16)(8) = 120\text{m}^2$$

**142. Characteristics of a Rhombus**
A rhombus is a 4-sided polygon with all sides equal and no angle is equal to 90°.



**143. Area of a Rhombus**

$$\textbf{Area} = \frac{1}{2} d_1 \cdot d_2$$




$d_1$ and $d_2$ are the lengths of the diagonals

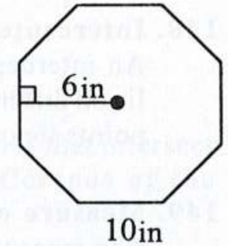
The area of the rhombus at the right is

$$\text{Area} = \frac{1}{2}(8)(15) = 60 \text{ square units}$$

**144. Area of Regular Polygon**

$$\textbf{Area} = \frac{1}{2} a P$$

$a$ - apothem - the perpendicular distance from the center of the polygon to a side
$P$ - perimeter of the polygon




In the octagon shown at the right, the length of each equal side is 10 in, therefore the perimeter is 8 × 10 = 80.

$$A = \frac{1}{2}ap = \frac{1}{2}(6)(80)$$

$$A = 240\text{ in}^2$$

**145. Interior Angles of a Polygon**

The **sum of the measures of the interior angles of a polygon is equal to (*n* – 2) × 180**, where *n* is the number of sides.

**sum of interior angles = (*n* – 2) × 180**

For a heptagon, i.e. n = 7
sum of the interior angles = ( 7 – 2 ) × 180 = 900°.

**146. Exterior Angles of a Polygon**

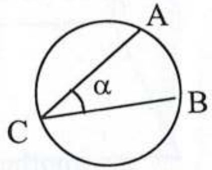
The **sum of the measures of the exterior angle of a convex polygon, one angle at each vertex, is equal to 360°.**

The measure of an angle of a regular octagon is 135°.
The measure of the exterior angle is equal to 180° – 135° = 45°.
Then, sum of the exterior angles = 45° × 8 = 360°

**The number of diagonals** in a convex polygon of *n*-sides is given by the formula $\frac{n(n-3)}{2}$. For a hexagon the number of diagonals is 9.

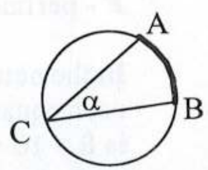
## CIRCLES

**147. Inscribed Angle**




An angle inscribed in an arc is an angle whose sides contain the endpoints of the arc and whose vertex is a point on the arc other than the endpoints.

**148. Intercepted Arc**



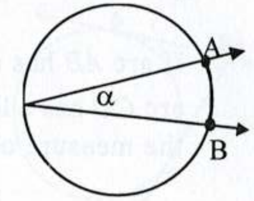

An intercepted arc is an arc whose endpoints lie on different rays of an angle and whose other points lie on the interior of the angle.

**149. Measure of the Inscribed Angle**

The measure of an inscribed angle is **one-half the measure of** its intercepted arc.





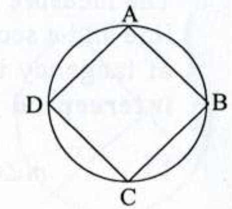



$$m\angle\alpha = \frac{1}{2}\, m\widehat{AB}$$

If arc AB is 68°, then the measure of angle α is 68 ÷ 2 = 34°

**150. Quadrilaterals Inscribed in a Circle**

If a quadrilateral is inscribed in a circle, its **opposite angles are supplementary.**

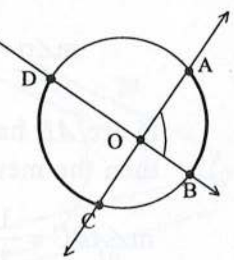



$$m\angle A + m\angle C = 180°$$
$$m\angle B + m\angle D = 180°$$

**151. Angles Formed by Two Secant Lines**

The measure of an angle formed by two secants which intersect in the interior of a circle is **one-half the sum of the measure of the arcs intercepted by the angle and its vertical angle.**

$$m\angle AOB = \frac{1}{2}\left(m\widehat{AB} + m\widehat{CD}\right)$$



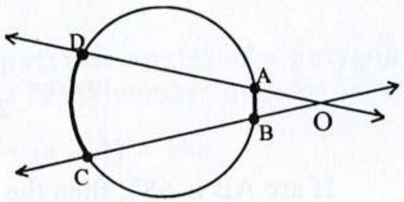

If arc *AB* has a length of 36° and arc *CD* has a length of 82°, then the measure of angle *AOB* is

$$m\angle AOB = \frac{1}{2}(36° + 82°) = \frac{1}{2}(118°) = 59°$$

The measure of an angle formed by two secant lines that intersect in the exterior of a circle is **one-half the difference of the measure of the intercepted arcs.**

$$m\angle\alpha = \frac{1}{2}\left(m\widehat{CD} - m\widehat{AB}\right)$$

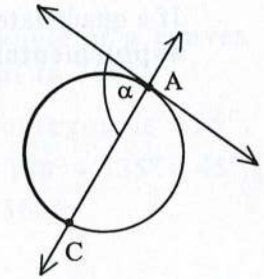
If arc $AB$ has a length of 42° and arc $CD$ has a length of 184°, then the measure of angle $AOB$ is




$$m\angle AOB = \frac{1}{2}(184° - 42°) = \frac{1}{2}(142°) = 71°$$

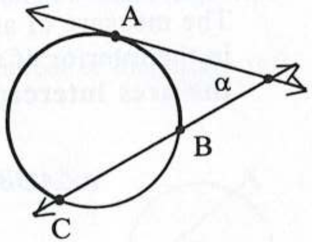
**152. Angles Formed by a Tangent and a Secant**

The measure of an angle formed by a tangent line and a secant line that intersect at a point of tangency is **one-half the measure of the intercepted arc.**




$$m\angle\alpha = \frac{1}{2}m\widehat{AC}$$

The measure of an angle formed by a tangent line and a secant line that intersect in the exterior of a circle is **one-half the difference of the measures of the intercepted arcs.**




$$m\angle\alpha = \frac{1}{2}(m\widehat{AC} - m\widehat{AB})$$

If arc $AB$ has a length of 89° and arc $AC$ has a length of 159°, then the measure of angle 1 is

$$m\angle ABC = \frac{1}{2}(159° - 89°) = \frac{1}{2}(70°) = 35°$$

**153. Angles Formed by Two Tangent Lines**

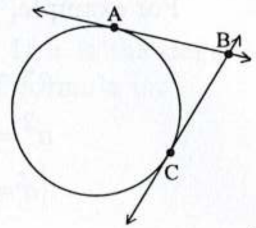
The measure of an angle formed by two tangent lines which intersect in the exterior of a cirlce is **one-half the difference of the measures of the intercepted arcs.**

$$m\angle ABC = \frac{1}{2}(m\widehat{AC} - m\widehat{CA})$$





If arc $AC$ has a length of 228° and arc $CA$ has a length of 132°, then

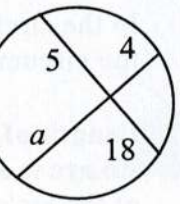



$$m\angle ABC = \frac{1}{2}(228° - 132°) = \frac{1}{2}(96°) = 48°$$

**154. Segments Formed by Two Chords**

If two chords intersect in a circle, then the product of the lengths of the segments on one chord is equal to the product of the lengths of the segments of the other.

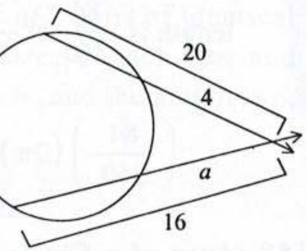
$$4a = 5 \cdot 18$$
$$4a = 90$$
$$a = 22.5$$




**155. Segments Formed by Two Secants**

If two secants are drawn to a circle from an exterior point, the product of the lengths of one secant and its external secant segment is equal to the product of the lengths of the other secant and its external secant segment.

$$16a = 20 \cdot 4$$
$$16a = 80$$
$$a = 5$$

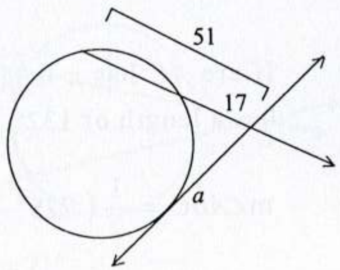



**156. Segments Formed by a Tangent and a Secant**

If a tangent and a secant are drawn to a circle from an exterior point of the circle, the square of the length of the tangent segment is equal to the product of the lengths of the secant segment and its external secant segment.

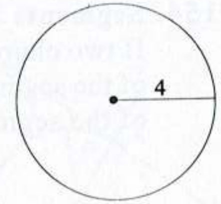
For example,

$$a^2 = 51 \cdot 17$$
$$a^2 = 867$$
$$a = 17\sqrt{3}$$




## 157. Circumference of a Circle

**Circumference = $2\pi r$**



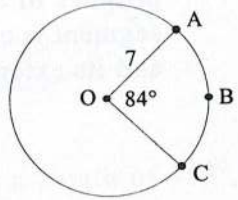

In the circle at the right, the radius is 4, and so the circumference is $2\pi(4) = 8\pi$.

## 158. Length of an Arc

An **arc** is a piece of the circumference. If $n$ is the degree measure of the arc's central angle, then the formula is:

$$\textbf{Length of an Arc} = \left(\frac{n}{360}\right)(2\pi r)$$

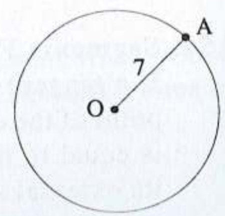
In the figure at the right, the radius is 7 and the measure of the central angle is 84°. The arc length is $\frac{84}{360}$ or $\frac{7}{30}$ of the circumference:




$$\left(\frac{84}{360}\right)(2\pi)(7) = \left(\frac{7}{30}\right)(14\pi) = \frac{49}{15}\pi$$

## 159. Area of a Circle

**Area of a Circle = $\pi r^2$**




The area of the circle is $\pi(7)^2 = 49\pi$ sq. units





## 160. Area of a Sector

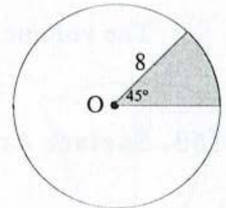
A **sector** is a piece of the area of a circle. If $n$ is the degree measure of the sector's central angle, then the formula is:

$$\textbf{Area of a Sector} = \left(\frac{n}{360}\right)(\pi r^2)$$

In the figure below, the radius is 8 and the measure of the sector's central angle is 45°. The sector has $\frac{45}{360}$ or $\frac{1}{8}$ of the area of the circle:

$$A = \left(\frac{45}{360}\right)(\pi)(8^2) = \left(\frac{1}{8}\right)(64\pi)$$

$A = 8\pi$ square units




## SOLIDS

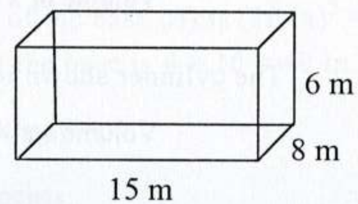
## 161. Surface Area of a Rectangular Solid

The surface of a rectangular solid consists of 3 pairs of identical faces. To find the surface area, find the area of each face and add them up. If the length is $l$, the width is $w$, and the height is $h$, the formula is:

**Surface Area = $2lw + 2wh + 2lh$**

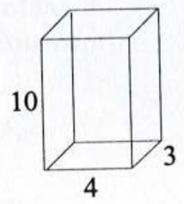
For instance, the surface area of the box shown is:

$(2 \times 15 \times 8) + (2 \times 8 \times 6) + (2 \times 15 \times 6)$
$= 240 + 96 + 180 = 516\ m^2$.



## 162. Volume of a Rectangular Solid

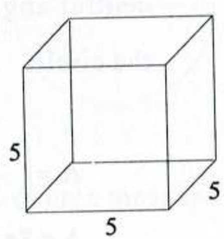
**Volume of a Rectangular Solid = $lwh$**




The volume of a 3-by-4-by-10 box is :
$3 \times 4 \times 10 = 120$ cubic units

A cube is a rectangular solid with all its dimensions equal. If $s$ is the length of an edge of a cube, the volume formula is:

**Volume of a Cube = $s^3$**

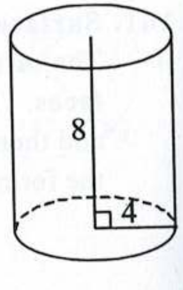



The volume of this cube is $5^3 = 125$ cu. units.

## 163. Surface Area of a Cylinder

$$\textbf{Surface Area} = 2\pi\left(r^2 + rh\right)$$

Find the surface area of the cylinder in the figure.




$$\text{Surface Area} = 2\pi\left(4^2 + (4)(8)\right)$$
$$= 2\pi(16 + 32) = 96\pi \text{ square units}$$

## 164. Volume of a Cylinder

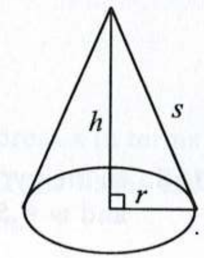
**Volume of a Cylinder = $\pi r^2 h$**

The cylinder shown in the figure, $r = 4$ and $h = 8$.

$$\text{Volume} = \pi\left(4^2\right)(8) = 128\pi \text{ cu. units}$$



## 165. Surface Area of a Cone



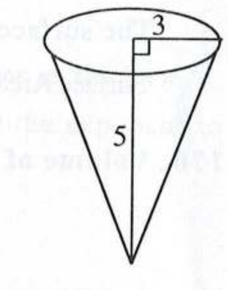

$$\textbf{Surface Area} = \pi\left(r^2 + rs\right)$$

A cone with height ($h$) 12 cm, radius 5 cm and a slant height ($s$) 13 cm.

$$\text{Surface Area} = \pi\left((5)^2 + (5)(13)\right)$$
$$= \pi(25 + 65)$$
$$= 90\pi \text{ square units}$$

## 166. Volume of a Cone

$$\textbf{Volume} = \frac{1}{3}\pi r^2 h$$

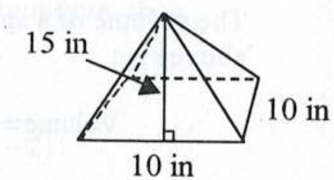



The volume of the cone shown at the right :

$$\text{Volume} = \frac{1}{3}\pi(3)^2(5) = 15\pi \text{ cu. units}$$

## 167. Surface Area of a Regular Pyramid

$$\text{Surface Area} = \frac{1}{2}LP + B$$

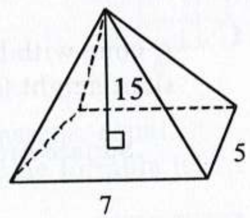



The figure above is a square-based pyramid with side = 10 in and the slant height ($L$) is 15in. The area of the base ($B$) is $(10 \text{ in})^2 =$ 100 sq. inches. The perimeter ($P$) of the base is $4 \times 10 = 40$ in.

$$\text{Surface Area} = \frac{1}{2}(15)(40) + 100$$
$$= 400 \text{ square inches}$$

$$\text{Volume} = \frac{1}{3}Bh$$

**168.** In the pyramid at the right, $h = 15$, $l = 7$ and $w = 5$, so :



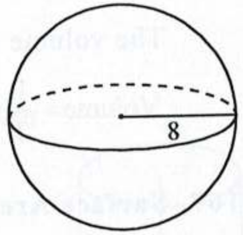

$$\text{Volume} = \frac{1}{3}(7\times5)(15) = 175 \text{ cu. units}$$

**169. Surface Area of a Sphere**

$$\textbf{Surface Area} = 4\pi r^2$$

The surface area of the sphere at the right with radius 8,
Surface Area $= 4\pi(8)^2 = 256\pi$ sq. units

**170. Volume of a Sphere**




$$\textbf{Volume} = \frac{4}{3}\pi r^3$$

The volume of a sphere with radius 8 (figure above) is :

$$\text{Volume} = \frac{4}{3}\pi(8)^3 = \frac{2048}{3}\pi \text{ cu. units}$$

**171. Areas and Volumes of Similar Solids**
If two solids are similar and their corresponding parts are in the ratio $a : b$, then the ratios of base areas, lateral areas and surface areas are all equal to $a^2 : b^2$ and the ratio of their volumes is $a^3 : b^3$. For example, two spheres are similar and the ratio of their radius is 5 : 8, then the ratio of their volumes is $5^3 : 8^3$ or 125 : 512.



## LOGARITHMS

**172. Exponential Equation**
To solve for an **exponential equation** $x = b^y$, express $x$ in terms of $b$ raised to a power $n$. It becomes $b^n = b^y$. If the bases are the same, then the exponents must be equal, $y = n$.

To solve for the exponential equation $2^n = 32$, express 32 in an exponential form with base 2. You now have $2^n = 2^5$. Therefore, n = 5.

**173. Logarithmic Function with Base $b$**
The logarithmic function with base $b$ is the inverse of the exponential function with base $b$.

$$y = \log_b x \Rightarrow x = b^y$$

**174. A number raised to a Logarithm with the number as the base**
If a logarithm is an exponent, that is $\log_b x$ is the exponent to which $b$ must be raised yields $x$.

$$b^{\log_b x} = x$$

**175. Product Rule for Logarithm**
If $b > 0$, $b \neq 1$, and $u$ and $v$ are positive numbers, then

$$\log_b uv = \log_b u + \log_b v$$

To simplify the expression $\log_3 x + \log_3(x-2)$:

$$\log_3 x + \log_3(x-2) = \log_3 x(x-2)$$
$$= \log_3(x^2 - 2x)$$

**176. Quotient Rule for Logarithm**
If $b > 0$, $b \neq 1$, and $u$ and $v$ are positive numbers, then

$$\log_b\left(\frac{u}{v}\right) = \log_b u - \log_b v$$

To simplify the expression $\log_7\left(27x^3-1\right)-\log_7(3x-1)$ :

$$\log_7\left(27x^3-1\right)-\log_7(3x-1)=\log_7\left(\frac{27x^3-1}{3x-1}\right)$$

$$=\log_7\left(\frac{(3x-1)\left(9x^2+3x+1\right)}{3x-1}\right)$$

$$=\log_7\left(9x^2+3x+1\right)$$

## 177. Power Rule For Logarithm

If $b>0$, $b\neq 1$, and $u$ and $v$ are positive numbers, then

$$\log_b u^n = n\log_b u$$

## 178. Logarithmic Function with Base $e$

The logarithmic function with base $e$ is known as the natural logarithmic function.

$$\log_e x = \ln x$$

## 179. Change of Base Formula

If the base of a logarithmic expression is other than *10* or *e*, then it can be simplified using the **base-changing** formula :

$$\log_b x = \frac{\log x}{\log b} = \frac{\ln x}{\ln b}$$

To simplify the expression $\log_4 3\cdot\log_3 32$ ,

$$\frac{\log 3}{\log 4}\cdot\frac{\log 32}{\log 3}=\frac{\log 32}{\log 4}=\frac{\log 2^5}{\log 2^2}=\frac{5\log 2}{2\log 2}=\frac{5}{2}$$

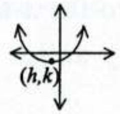
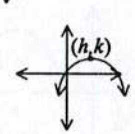
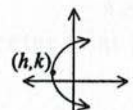
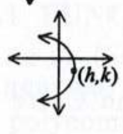
or $$\frac{\ln 3}{\ln 4}\cdot\frac{\ln 32}{\ln 3}=\frac{\ln 32}{\ln 4}=\frac{\ln 2^5}{\ln 2^2}=\frac{5\ln 2}{2\ln 2}=\frac{5}{2}$$





# CONICS

## 180. Equation of a Parabola

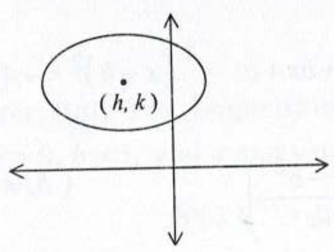
| For a parabola which opens | | General Equation | Standard Equation |
|---|---|---|---|
| upward |  | $y=ax^2+bx+c$ | $(x-h)^2=4p(y-k)$ |
| downward |  | $y=-ax^2+bx+c$ | $(x-h)^2=-4p(y-k)$ |
| vertex is at the point | | $\left(\frac{-b}{2a},\frac{4ac-b^2}{4a}\right)$ | $(h, k)$ |
| to the right |  | $x=ay^2+by+c$ | $(y-k)^2=4p(x-h)$ |
| to the left |  | $x=-ay^2+by+c$ | $(y-k)^2=-4p(x-h)$ |
| vertex is at the point | | $\left(\frac{4ac-b^2}{4a},\frac{-b}{2a}\right)$ | $(h, k)$ |
| | | $a>0$ | $p>0$ |

## 181. Equation of an Ellipse

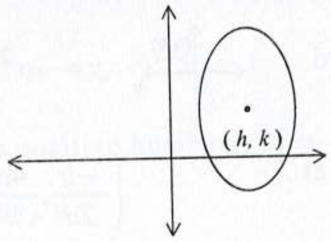
The equation $Ax^2+Bx+Cy^2+Dy+E=0$ is an equation of an **ellipse if $A \neq C$ and the signs of A and C are the same.** The standard equation is given by

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

where the point ( $h$ , $k$ ) are the coordinates of its center. If $a > b$, the axis of the ellipse is parallel to the $x$-axis and if $a < b$, the axis of the ellipse is parallel to the $y$-axis.




$a > b$




$a < b$

## 182. Equation of a Circle

The equation $Ax^2 + Bx + Cy^2 + Dy + E = 0$ is an equation of a **circle** if $A = C$ and the signs of A and C are the same. The standard equation is given by

$$(x-h)^2 + (y-k)^2 = r^2$$

where the point ( $h$ , $k$ ) are the coordinates of its center and $r$ is the length of its radius.

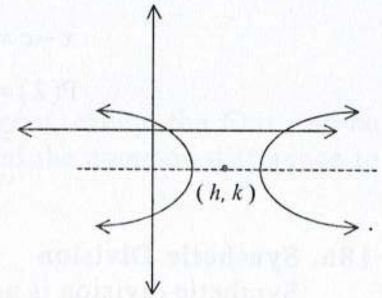
## 183. Equation of a Hyperbola

The equation $Ax^2 + Bx + Cy^2 + Dy + E = 0$ is an equation of a **hyperbola** if the signs of A and C are opposite. The standard equation is given by



$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

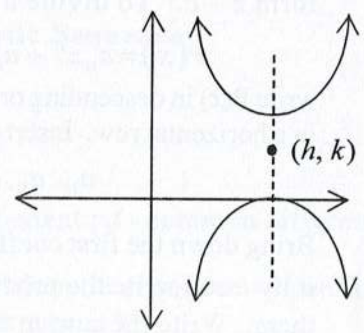
if the axis of the hyperbola is parallel to the $x$-axis




and

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

if the axis of the hyperbola is parallel to the $y$-axis




The center is at the point ( $h$ , $k$ ).

# POLYNOMIAL FUNCTIONS

## 184. Factor Theorem

If $P(x)$ is a polynomial and c is a real number, then $P(x)$ has $(x - c)$ as a factor if and only if $P(c) = 0$. To show that ( $x + 2$ ) is a factor of $P(x) = x^4 + 2x^3 - 12x^2 - 11x + 26$,

$$x - c = x + 2 \Rightarrow c = -2$$

$$\begin{aligned} P(-2) &= (-2)^4 + 2(-2)^3 - 12(-2)^2 - 11(-2) + 26 \\ &= 16 + 2(-8) - 12(4) + 22 + 26 \\ &= 16 - 16 - 48 + 48 = 0 \end{aligned}$$

## 185. Remainder Theorem

If $P(x)$ is a polynomial and c is a real number, then if $P(x)$ is divided by $(x - c)$, the remainder is $P(c)$. For example, to find the remainder when $P(x) = 2x^3 + 4x^2 - 3x + 5$ is divided by $(x - 2)$,

$$x - c = x - 2 \Rightarrow c = 2$$

$$\begin{aligned} P(2) &= 2(2)^3 + 4(2)^2 - 3(2) + 5 \\ &= 2(8) + 4(4) - 6 + 5 \\ &= 16 + 16 - 1 = 31 \end{aligned}$$

**186. Synthetic Division**

Synthetic division is used only if the divisor is a binomial of the form $x - c$. To divide a polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + ..... + ax + a_0 \text{ by } x + c,$$

write P($x$) in descending order and write the coefficients of P($x$) in order in a horizontal row. Insert zero coefficient for any missing term.

$$a_n \quad a_{n-1} \quad a_{n-2} \; ..... a \quad a_0$$

Bring down the first coefficient to the bottom row (1). Then multiply it by $c$ and write the product under the coefficient $a_{n-1}$(2). Then add them. Write the sum in the bottom row (3). Continue this until the last term which is $a_0$.

| | | | | |
|---|---|---|---|---|
| $a_n$ | $a_{n-1}$ | $a_{n-2}$ ..... $a$ | $a_0$ | $\lfloor c$ |
| (1) ↓ | $ca_n$ (2) | | | |
| $a_n$ | $a_{n-1} + ca_n$(3) | | | |

The last number in the bottom row is the remainder and the preceding numbers are the coefficients of the quotient which is a polynomial of degree one less than that of P($x$). For example, divide $P(x) = 2x^3 - x^2 + 3x + 12$ by $x - 4$.

| | | | | |
|---|---|---|---|---|
| 2 | −1 | 3 | 12 | $\lfloor 4$ |
| | 8 | 28 | 124 | |
| 2 | 7 | 31 | 136 | |

The quotient is $2x^2 + 7x + 31 + \dfrac{136}{x-4}$





## SEQUENCES AND SERIES

**187. Arithmetic Sequence**

It is a sequence for which any element, except the first, can be obtained by adding a constant called the common difference to the preceding element.

$$a_{n+1} = a_n + d$$

**188. The *n*th element of an Arithmetic Sequence**

The nth element is given by

$$a_n = a_1 + (n-1)d$$

where $a_1$ - first element ; $a_n$ - nth element ; $d$ - common difference

To find the 100th element of an arithmetic sequence whose first element is 4 and the common difference is 3,

$$a_{100} = a_1 + (100-1)d = 4 + (99)(3) = 4 + 297 = 301$$

**189. Arithmetic Series**

An arithmetic series is the sum of the elements of an arithmetic sequence.

If the first element, the nth element and the common difference are known,

$$S_n = \frac{n}{2}(a_1 + a_n) \qquad (1)$$

If the first element and the common difference are known,

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right] \qquad (2)$$

To find the sum of the first 50 positive multiples of 4,

$$4, 8, 12, 16, \ldots, a_{50}$$

The common difference is $d = 8 - 4 = 4$ but the 50th element is not given, use equation (2),

$$S_{50} = \frac{50}{2}\left[2(4)+(50-1)(4)\right]$$
$$= 25\left[8+(49)(4)\right] = 25\left[8+196\right]$$
$$= 25(204) = 5\,100$$

## 190. Geometric Sequence

It is a sequence for which any element, except the first, can be obtained by multiplying a constant called the common ratio to the preceding element.

## 191. The *n*th element of a Geometric Sequence

The nth element is given by

$$a_n = a_1 r^{n-1}$$

To find the 15th element of the geometric sequence for which the first element is 4 and the common ratio is 2.

$$a_{15} = a_1 r^{15-1} = 4(2)^{14}$$
$$= 2^2(2)^{14} = 2^{16}$$
$$= 65\,536$$

## 192. Geometric Mean

The **geometric mean** between two numbers $a$ and $b$ is

$\sqrt{ab}$ if $a$ and $b$ are positive

$-\sqrt{ab}$ if $a$ and $b$ are negative



The geometric mean of 8 and 12 is

$$\sqrt{8\cdot 12} = \sqrt{96} = \sqrt{16\cdot 6} = 4\sqrt{6}$$

The **geometric mean of a set of numbers** is

$$\text{GM} = \sqrt[n]{a_1 a_2 a_3 \ldots a_n}$$

## 193. Geometric Series

Geometric series is the sum of the elements of the geometric sequence.

$$S_n = \frac{a_1\left(1-r^n\right)}{1-r}, \quad r \neq 1$$

The sum of geometric sequence with $r = 5$, $a_1 = 2$ and $n = 7$:

$$S_7 = \frac{2\left(1-5^7\right)}{1-5} = \frac{2(1-78\,125)}{-4} = \frac{2(-78\,124)}{-4}$$
$$= \frac{-156\,248}{-4} = 39\,062$$

## 194. Infinite Geometric Series

The sum is given by

$$S_\infty = \frac{a_1}{1-r}, \quad |r| < 1$$

The sum of an infinite geometric series with $a_1 = 3$, $r = \frac{2}{3}$

$$S_\infty = \frac{3}{1-\frac{2}{3}} = \frac{3}{\frac{1}{3}} = 9$$

## PROBABILITIES

### 195. $n!$

$n!$ (n factorial) is defined as $n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot\ldots\cdot 3\cdot 2\cdot 1$

For example, $5!=5\cdot 4\cdot 3\cdot 2\cdot 1=120$

Note : $1! = 1$ and $0! = 1$

### 196. Permutation

Permutation is a set of **ordered arrangement** of $n$ objects taken $r$ at a time.

$$_{n}P_{r}=\frac{n!}{(n-r)!}$$

To find the number of permutation of 8 objects taken 3 at a time,

$$_{8}P_{3}=\frac{8!}{(8-3)!}=\frac{8!}{5!}=\frac{8\cdot 7\cdot 6\cdot 5!}{5!}=336$$

If $r=n$ , then $_{n}P_{n}=n!$

To arrange 5 books on a bookshelf, $5! = 120$ ways

If you are given $n$ elements of which $a_1$ are alike of one kind, exactly $a_2$ are alike, ... , and exactly $a_k$ are alike of one kind, and if $n=a_1+a_2+\ldots+a_k$, then the number of permutations of the $n$ elements taking them all is

$$\frac{n!}{a_1!a_2!\ldots a_k!}$$

For example, in the word *Mississippi*, how many permutations can you make using all the letters? There are 11 letters in all, 1 $M$, 4 $s$, 4 $i$'s, 2 $p$'s, then the number of permutation is

$$\frac{11!}{1!\,4!\,4!\,2!}=34\,650$$



### 197. Combination

Combination is a set of **unordered arrangement** of $n$ objects taken $r$ at a time.

$$_{n}C_{r}=\binom{n}{r}=\frac{n!}{(n-r)!r!}$$

To choose 5 students in a class of 15 to sit in the back row,

$$_{15}C_{5}=\frac{15!}{(15-5)!\,5!}=\frac{15!}{10!5!}=3\,003 \text{ ways}$$

***How many ways can 5 students in a class of 15 sit in the back row? This is a permutation problem because arrangement is considered and therefore can be solved using the formula

$$_{15}P_{5}=\frac{15!}{(15-5)!}=\frac{15!}{10!}=\frac{15\cdot 14\cdot 13\cdot 12\cdot 11\cdot 10!}{10!}=360360$$

### 198. Counting the Possibilities

The fundamental counting principle: If there are ***m*** **ways** one event can happen and ***n*** **ways** a second event can happen, then there are $\boldsymbol{m\times n}$ **ways** for the two events to happen. For example, with 10 polo shirts and 5 pairs of pants to choose from, you can put together $10\times 5=50$ different outfits.

### 199. Probability

$$\text{Probability}=\frac{\text{favorable outcomes}}{\text{total possible outcomes}}$$

For example, if you have 15 shirts in a drawer and 6 of them are blue, the probability of picking a blue shirt at random is $\frac{6}{15}=\frac{2}{5}$

This probability can also be expressed as 0.40 or 40%.

## BINOMIAL EXPANSION

### 200. Pascal's Triangle

The Pascal's triangle gives the pattern for the coefficients in the terms of the Binomial expansion.

```
                1
              1   2   1
            1   3   3   1
          1   4   6   4   1
        1   5  10  10   5   1
      1   6  15  20  15   6   1
    1   7  21  35  35  21   7   1
  1   8  28  56  70  56  28   8   1
```

Arranging the coefficients in this way shows that each number in the triangle is the sum of the two numbers just above it (one to the right and one to the left)

### 201. Binomial Theorem

The expansion of $(x+y)^n$, for any positive integer $n$,

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3$$

$$+ \dots + \binom{n}{n-1}xy^{n-1} + y^n$$

$$= x^n + \frac{n!}{(n-1)!\,1!}x^{n-1}y + \frac{n!}{(n-2)!\,2!}x^{n-2}y^2$$

$$+ \frac{n!}{(n-3)!\,3!}x^{n-3}y^3 + \dots + \frac{n!}{1!(n-1)!}xy^{n-1} + y^n$$





For example, expand $(x+y)^8$

$$(x+y)^8 = x^8 + \binom{8}{1}x^{8-1}y + \binom{8}{2}x^{8-2}y^2 + \binom{8}{3}x^{8-3}y^3$$

$$+ \binom{8}{4}x^{8-4}y^4 + \binom{8}{5}x^{8-5}y^5$$

$$+ \binom{8}{6}x^{8-6}y^6 + \binom{8}{7}x^{8-7}y^7 + y^8$$

$$= x^8 + \frac{8!}{7!1!}x^7y + \frac{8!}{6!2!}x^6y^2 + \frac{8!}{5!3!}x^5y^3 + \frac{8!}{4!4!}x^4y^4$$

$$+ \frac{8!}{3!5!}x^3y^5 + \frac{8!}{2!6!}x^2y^6 + \frac{8!}{1!7!}xy^7 + y^8$$

$$(x+y)^8 = x^8 + 8x^7y + 28x^6y^2 + 56x^5y^3 + 70x^4y^4$$

$$+ 56x^3y^5 + 28x^2y^6 + 8xy^7 + y^8$$

### 202. The $r$th term of the Binomial Expansion

To find the $r$th term of a binomial expansion of $(x+y)^n$ where $r \le n$,

$$\frac{n!}{[n-(r-1)]!\,(r-1)!}x^{n-(r-1)}y^{(r-1)}$$

In the expansion of $(x+3y)^7$, the fourth term is :

$$r = 4 \text{ and } n = 7$$

$$\frac{7!}{[7-(4-1)]!\,(4-1)!}x^{7-(4-1)}(3y)^{(4-1)} = \frac{7!}{4!3!}x^4(3y)^3$$

$$= 35\left(27x^4y^3\right)$$

$$= 945x^4y^3$$

## MEASUREMENT

### 203. Metric Units of Measure

Metric units use different prefixes such as:

kilo - 1000 times
hecto - 100 times
deka - 10 times
deci - $\frac{1}{10}$ times

centi - $\frac{1}{100}$ times
milli - $\frac{1}{1000}$ times

### 204. English Units of Measure

Volume :
1 gallon (gal) = 4 quarts (qt)
1 quart = 2 pints (pt)
1 $yard^3 = 27\ ft^3$
1 cup = 16 Tablespoons

Length :
1 foot (ft) = 12 inches (in)
1 yard (yd) = 3 feet
1 mile (mi) = 5 280 feet = 1 760 yards

Area :
1 acre = 43 560 $ft^2$
1 sq. m = 640 acres

Weight :
1 pound (lb) = 16 ounces (oz)
1 ton = 2 000 pounds

Time :
1 minute (min) = 60 seconds (s)
1 hour (hr) = 60 minutes
1 day (da) = 24 hours
1 year (yr) = 365 days
1 decade = 10 years
1 century = 100 years
1 millennium = 1 000 years





### 205. Approximate English-Metric Conversion

Volume :
1 liter (L) = 1.06 quarts
1 gallon = 3.8 liters
1 mL = 1 $cm^3$

Weight :
1 kilogram (kg) = 2.2 pounds
1 ounce = 28.4 grams (g)

Length :
1 inch = 2.54 centimeters
1 foot = 0.3 meters
1meter = 1.1 yards
1 mile = 1.6 kilometers

### 206. Convert from one unit to another

To convert one unit of measure to another unit, express the given unit as a fraction with denominator of 1. Write a unit fraction that has the same unit of measure in the denominator that appears in the numerator of the preceding fraction. Repeat until the desired units appear in the numerator. Then multiply or divide out common units of measure.

Convert 1 520 ml to quarts.

$$\frac{1\,520\cancel{mL}}{1} \times \frac{1\cancel{L}}{1\,000\cancel{mL}} \times \frac{1.06\text{ qt}}{1\cancel{L}} = 1.6112\text{ qt}$$

Convert 2 miles to inches.

$$\frac{2\cancel{mi}}{1} \times \frac{5\,280\cancel{ft}}{1\cancel{mi}} \times \frac{12\text{ in}}{1\cancel{ft}} = 126\,720\text{ inches}$$

Convert 212°F to °C

$$°C = \frac{5}{9}(°F - 32) \Rightarrow \frac{5}{9}(212 - 32) = \frac{5}{9}(180) = 5(20) = 100°C$$

Convert 80°C to °F

$$°F = \frac{9}{5}°C + 32 \Rightarrow \frac{9}{5}(80) + 32 = 9(16) + 32 = 144 + 32 = 176°F$$

## ADDITIONAL FORMULAS

**207. The number of terms $n$ :** $n = \dfrac{a_n - a_1}{d} + 1$

$$1 + 2 + 3 \ldots + 100 \Rightarrow n = \frac{100 - 1}{1} + 1 = 99 + 1 = 100$$

$$2 + 4 + 6 \ldots + 100 \Rightarrow n = \frac{100 - 2}{2} + 1 = 49 + 1 = 50$$

$$1 + 3 + 5 + \ldots + 99 \Rightarrow n = \frac{99 - 1}{2} + 1 = \frac{98}{2} + 1 = 49 + 1 = 50$$

**208. The sum of consecutive integers from 1 to $n$:**

$$s = \frac{n(n+1)}{2}$$

$$1 + 2 + 3 \ldots + 100 = \frac{100(101)}{2} = 50\,(101) = 5\,050$$

**209. The sum of consecutive even integers from 2 to $2n$:**

$$s = n(n+1)$$

Example : $2 + 4 + 6 \ldots + 100 = 50(51) = 2\,550$

**210. The sum of consecutive odd integers from 1 to $(2n - 1)$:**

$$s = \frac{1}{2}\left(\frac{a_n - 1}{2} + 1\right)(a_n + 1)$$

$$1 + 3 + 5 + 7 + 9 \ldots + 97 + 99$$

$$s = \frac{1}{2}\left(\frac{a_n - 1}{2} + 1\right)(a_n + 1)$$

$$s = \frac{1}{2}\left(\frac{99 - 1}{2} + 1\right)(99 + 1) = \frac{1}{2}(49 + 1)(100) = 2500$$



### Fractional Equivalent of Special Percents

| | | | | |
|---|---|---|---|---|
| $11\frac{1}{9}\% = \frac{1}{9}$ | $16\frac{2}{3}\% = \frac{1}{6}$ | $12\frac{1}{2}\% = \frac{1}{8}$ | $10\% = \frac{1}{10}$ | $70\% = \frac{7}{10}$ |
| $22\frac{2}{9}\% = \frac{2}{9}$ | $33\frac{1}{3}\% = \frac{1}{3}$ | $25\% = \frac{1}{4}$ | $20\% = \frac{1}{5}$ | $80\% = \frac{4}{5}$ |
| $44\frac{4}{9}\% = \frac{4}{9}$ | $66\frac{2}{3}\% = \frac{2}{3}$ | $37\frac{1}{2}\% = \frac{3}{8}$ | $30\% = \frac{3}{10}$ | $90\% = \frac{9}{10}$ |
| $55\frac{5}{9}\% = \frac{5}{9}$ | $83\frac{1}{3}\% = \frac{5}{6}$ | $62\frac{1}{2}\% = \frac{5}{8}$ | $40\% = \frac{2}{5}$ | $100\% = 1$ |
| $77\frac{7}{9}\% = \frac{7}{9}$ | $133\frac{1}{3}\% = \frac{4}{3}$ | $75\% = \frac{3}{4}$ | $50\% = \frac{1}{2}$ | $120\% = \frac{6}{5}$ |
| $88\frac{8}{9}\% = \frac{8}{9}$ | $166\frac{2}{3}\% = \frac{5}{3}$ | $87\frac{1}{2}\% = \frac{7}{8}$ | $60\% = \frac{3}{5}$ | $180\% = \frac{9}{5}$ |

### Squares and Cubes of Numbers

| | | | | |
|---|---|---|---|---|
| $11^2 = 121$ | $21^2 = 441$ | $31^2 = 961$ | $41^2 = 1\,681$ | $1^3 = 1$ |
| $12^2 = 144$ | $22^2 = 484$ | $32^2 = 1\,024$ | $42^2 = 1\,764$ | $2^3 = 8$ |
| $13^2 = 169$ | $23^2 = 529$ | $33^2 = 1\,089$ | $43^2 = 1\,849$ | $3^3 = 27$ |
| $14^2 = 196$ | $24^2 = 576$ | $34^2 = 1\,156$ | $44^2 = 1\,936$ | $4^3 = 64$ |
| $15^2 = 225$ | $25^2 = 625$ | $35^2 = 1\,225$ | $45^2 = 2\,025$ | $5^3 = 125$ |
| $16^2 = 256$ | $26^2 = 676$ | $36^2 = 1\,296$ | $46^2 = 2\,116$ | $6^3 = 216$ |
| $17^2 = 289$ | $27^2 = 729$ | $37^2 = 1\,369$ | $47^2 = 2\,209$ | $7^3 = 343$ |
| $18^2 = 324$ | $28^2 = 784$ | $38^2 = 1\,444$ | $48^2 = 2\,304$ | $8^3 = 512$ |
| $19^2 = 361$ | $29^2 = 841$ | $39^2 = 1\,521$ | $49^2 = 2\,401$ | $9^3 = 729$ |
| $20^2 = 400$ | $30^2 = 900$ | $40^2 = 1\,600$ | $50^2 = 2\,500$ | $10^3 = 1\,000$ |