Factor & Remainder Theorems

[**worked solutions** included]

Exercises - **no** calculator allowed

1. Find the value(s) of c such that $x=2$ is a zero of the polynomial x^3+x^2+cx+2.

2. If $x^3+3x^2+kx+17$ has a remainder of 5 when divided by $x+3$, then find the value of k.

3. Given that $x=-1$ is a double root of the equation $4x^4-11x^2-6x+1=0$, find the other root(s) of the equation.

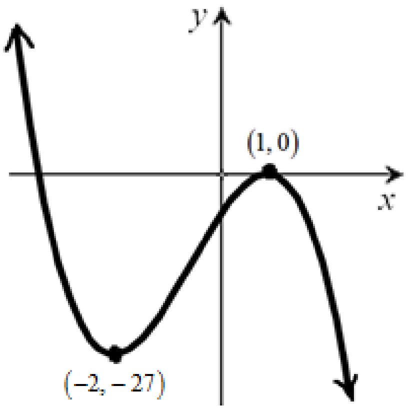
4. Find a polynomial of lowest degree with integer coefficients that has zeros of $x=2$ and $x=1+4i$.

5. When the polynomial $x^4-3x^3-3x^2+ax+b$ is divided by $(x+2)$ it leaves the same remainder as when it is divided by $(x-3)$. Find the value of a.

6. Find the value(s) of m such that $x+3$ is a factor of the polynomial $P(x)=x^3-x^2-m^2x$.

7. Given x^2-x-6 is a factor of $ax^3+x^2+bx-24$, find the value of a and the value of b.

8. The graph of a cubic function $g(x)=ax^3+bx^2-4bx+c$ is shown at right. The graph has a local minimum at $(-2,-27)$ and a local maximum at $(1,0)$. Find the values of a, b and c.



9. The polynomial $f(x)=x^3-2x^2+ax+b$ leaves the same remainder R when divided by $(x-1)$ as when divided by $(x+3)$. When $f(x)$ is divided by $(x-2)$ it leaves a remainder of 3R. Find the value of a and the value of b.

10. Given x^2+ax+b and x^2+cx+d have a common factor of $(x-k)$, show that $k=\dfrac{d-b}{a-c}$.

Factor & Remainder Theorems

WORKED SOLUTIONS

1. $2^3+2^2+c(2)+2=0 \;\Rightarrow\; 2c=-14 \;\Rightarrow\; c=-7$

2. $(-3)^3+3(-3)^2+k(-3)+17=5 \;\Rightarrow\; -3k=-12 \;\Rightarrow\; k=4$

3. if $x=-1$ is a double root, then $(x+1)^2$ is a factor of $4x^4-11x^2-6x+1$

$(x+1)^2(ax^2+bx+c)=(x^2+2x+1)(ax^2+bx+c)=4x^4-11x^2-6x+1$

therefore, $a=4$ and $c=1$

$$(x^2+2x+1)(4x^2+bx+1)=4x^4+bx^3+x^2+8x^3+2bx^2+2x+4x^2+bx+1$$
$$=4x^4+(b+8)x^3+(2b+5)x^2+(b+2)x+1$$

therefore, $c=-8$

solving $4x^2-8x+1=0 \;\Rightarrow\; x=\dfrac{8\pm\sqrt{64-16}}{8}=\dfrac{8\pm\sqrt{48}}{8}=\dfrac{8\pm4\sqrt{3}}{8}=\dfrac{2\pm\sqrt{3}}{2}$

other two roots of the equation are:

4. if $x=1+4i$ is a zero, then $x=1-4i$ (its conjugate) must also be a zero; thus the required polynomial must have the following three factors: $x-2$, $x-(1+4i)$, and $x-(1-4i)$

expanding gives: $(x-2)[x-(1+4i)][x-(1-4i)]=(x-2)[x-1-4i][x-1+4i]$

$$=(x-2)[(x-1)-4i][(x-1)+4i]$$
$$=(x-2)\left[(x-1)^2-(4i)^2\right]$$
$$=(x-2)(x^2-2x+1-16i^2)$$
$$=(x-2)(x^2-2x+1+16)$$
$$=(x-2)(x^2-2x+17)=x^3-2x^2+17x-2x^2+4x-34$$
$$=x^3-4x^2+21x-34$$

5. $(-2)^4-3(-2)^3-3(-2)^2+a(-2)+b=-2a+b+28$

$(3)^4-3(3)^3-3(3)^2+a(3)+b=3a+b-27$

$-2a+b+28=3a+b-27 \;\Rightarrow\; 5a=55 \;\Rightarrow\; a=11$

6. $P(-3)=(-3)^3-(-3)^2-m^2(-3)=0 \;\Rightarrow\; 3m^2=36 \;\Rightarrow\; m^2=12 \;\Rightarrow\; m=\pm2\sqrt{3}$

Factor & Remainder Theorems

WORKED SOLUTIONS (continued)

7. $x^2 - x - 6 = (x-3)(x+2)$ therefore, $x-3$ and $x+2$ are factors of $ax^3 + x^2 + bx - 24$

applying remainder theorem: $a(3)^3 + (3)^2 + b(3) - 24 = 0 \Rightarrow 27a + 3b = 15 \Rightarrow 9a + b = 5$

$a(-2)^3 + (-2)^2 + b(-2) - 24 = 0 \Rightarrow -8a - 2b = 20 \Rightarrow -4a - b = 10$

$$\begin{cases} 9a + b = 5 \\ -4a - b = 10 \end{cases} \Rightarrow a = 3,\ b = -22$$

8. $(x-1)^2 = x^2 - 2x + 1$

$(x^2 - 2x + 1)(ax + c) = ax^3 + (-2a + c)x^2 + (a - 2c)x + c$

$(-2, -27)$: $a(-2)^3 + (-2a + c)(-2)^2 + (a - 2c)(-2) + c = -27$

$-8a - 8a + 4c - 2a + 4c + c = -18a + 9c = -27 \Rightarrow -2a + c = -3 \Rightarrow c = 2a - 3$

$ax^3 + (-2a + 2a - 3)x^2 + (a - 2(2a-3))x + 2a - 3 = ax^3 - 3x^2 + (-3a + 6)x + 2a - 3$

therefore, $b = -3$; $ax^3 - 3x^2 + (-3a + 6)x + 2a - 3 \Rightarrow -3a + 6 = -4b = 12 \Rightarrow a = -2$

and $c = 2a - 3 = 2(-2) - 3 = -7$

thus, $g(x) = -2x^3 - 3x^2 + 12x - 7$

9. $f(1) = 1^3 - 2(1)^2 + a(1) + b = a + b - 1 = R$

$f(-3) = (-3)^3 - 2(-3)^2 + a(-3) + b = -3a + b - 45 = R$

$f(2) = (2)^3 - 2(2)^2 + a(2) + b = 2a + b = 3R$

$a + b - 1 = -3a + b - 45 \Rightarrow 4a = -44 \Rightarrow a = -11$

$2a + b = 3(a + b - 1) \Rightarrow a + 2b = 3 \Rightarrow -11 + 2b = 3 \Rightarrow 2b = 14 \Rightarrow b = 7$

10. $k^2 + ak + b = 0$ and $k^2 + ck + d = 0$; pair of simultaneous equations – subtracting first equation from second equation gives $ak - ck + b - d = 0 \Rightarrow k(a - c) = d - b \Rightarrow k = \dfrac{d-b}{a-c}$ *Q.E.D.*